Digital Signal Processing

Assignment 1

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Contents

Abstract—**This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
    -scipy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/amaan28/EE3900/raw/
    main/codes/filter_codes_Sound_Noise.
    wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?

Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.

Solution:

```
import soundfile as sf
from scipy import signal
#read .wav file
input_signal,fs = sf.read('
    filter_codes_Sound_Noise.wav')
#sampling frequency of Input signal
sampl_freq=fs
#order of the filter
order=4
#cutoff frquency 4kHz
cutoff_freq=4000.0
#digital frequency
Wn=2*cutoff_freq/sampl_freq
# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')
#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output signal = signal.lfilter(b, a,
    input_signal)
#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?

Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.+

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

Solution: The following code to sketch $x(n)$, i.e, Fig. 3.1.

```
wget https://github.com/amaan28/EE3900/
    blob/main/codes/Ex_3.1.py
```

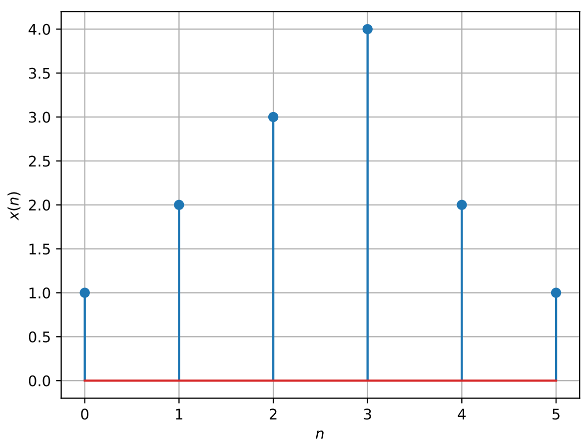

Fig. 3.1

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.

Solution: The following code yields Fig. 3.2.

```
wget https://github.com/gadepall/EE1310/raw/
    master/filter/codes/xnyn.py
```

3.3 Repeat the above exercise using a C code.

Solution: The following C code yields Fig. 3.3.

```
https://github.com/amaan28/EE3900/blob/main
    /codes/Ex_3.3.c
```

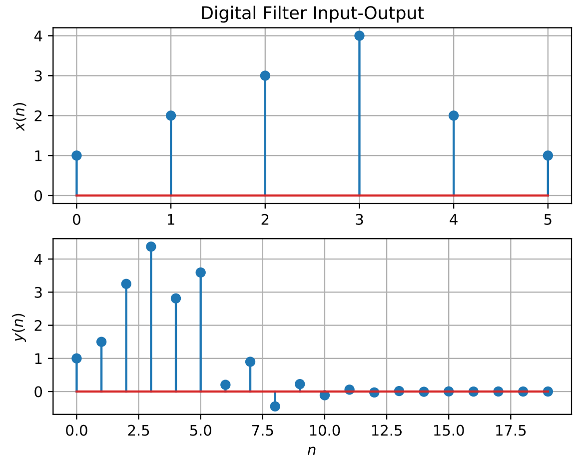

Fig. 3.2

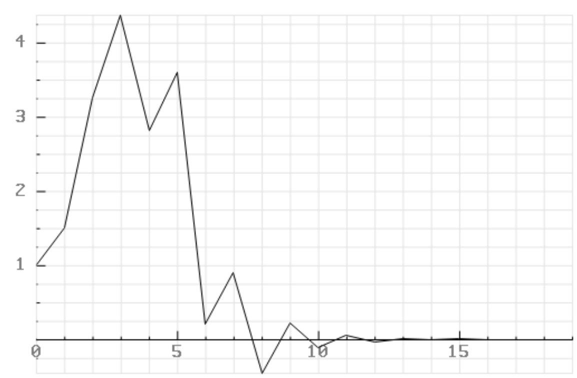

Fig. 3.3

4 $\mathcal{Z}$-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.

Solution: Applying (4.1) in (3.1),

$$X(z) = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \frac{2}{z^5} + \frac{1}{z^6} \tag{4.7}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.8}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.9}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.10}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.11}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.12}$$

is

$$U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.13}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.14}$$

and from (4.12),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.15}$$

$$= \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.16}$$

using the formula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}} \quad |z| > |a| \tag{4.17}$$

Solution: From (4.12), we have,

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{n=\infty} a^n z^{-n} = \frac{1}{1-az^{-1}} \tag{4.18}$$

using the formula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \tag{4.19}$$

Plot $|H(e^{j\omega})|$. Is it periodic? If so, find the period. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.
Solution: The following code plots Fig. 4.6.

```
wget https://github.com/amaan28/EE3900/
    blob/main/codes/dtft.py
```

Using (4.10), we observe that $|H(e^{j\omega})|$ is given by

$$|H(e^{j\omega})| = \left|\frac{1+e^{-2j\omega}}{1+\frac{1}{2}e^{-j\omega}}\right| \tag{4.20}$$

$$= \sqrt{\frac{(1+\cos 2\omega)^2 + (\sin 2\omega)^2}{\left(1+\frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}} \tag{4.21}$$

$$= \sqrt{\frac{2(1+\cos 2\omega)}{\frac{5}{4}+\cos\omega}} \tag{4.22}$$

$$= \sqrt{\frac{2(2\cos^2\omega)}{\frac{5}{4}+\cos\omega}} \tag{4.23}$$

$$= \frac{4|\cos\omega|}{\sqrt{5+4\cos\omega}} \tag{4.24}$$

Thus,

$$\left|H\left(e^{j(\omega+2\pi)}\right)\right| = \frac{4|\cos(\omega+2\pi)|}{\sqrt{5+4\cos(\omega+2\pi)}} \tag{4.25}$$

$$= \frac{4|\cos\omega|}{\sqrt{5+4\cos\omega}} \tag{4.26}$$

$$= |H(e^{j\omega})| \tag{4.27}$$

and so its fundamental period is 2π.

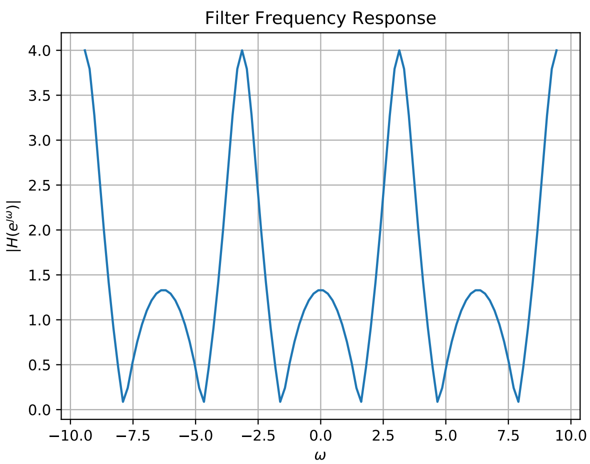

Fig. 4.6: $\left|H\left(e^{\jmath\omega}\right)\right|$

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$.
Solution: We have,

$$H(e^{\jmath\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-\jmath\omega k} \tag{4.28}$$

However,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}d\omega = \begin{cases} 2\pi & n = k \\ 0 & \text{otherwise} \end{cases} \tag{4.29}$$

and so,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{j\omega n}d\omega \tag{4.30}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} h(k)e^{\jmath\omega(n-k)}d\omega \tag{4.31}$$

$$= \frac{1}{2\pi}2\pi h(n) = h(n) \tag{4.32}$$

which is known as the Inverse Discrete Fourier Transform. Thus,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega \tag{4.33}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1+e^{-2\jmath\omega}}{1+\frac{1}{2}e^{-\jmath\omega}}e^{\jmath\omega n}d\omega \tag{4.34}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.10).
Solution:

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$

$$1+z^{-2} = \left(1+\frac{1}{2}z^{-1}\right)*\left(2z^{-1}-4\right)+5 \tag{5.3}$$

$$H(z) = \frac{\left(1+\frac{1}{2}z^{-1}\right)*\left(2z^{-1}-4\right)+5}{1+\frac{1}{2}z^{-1}} \tag{5.4}$$

$$= 2z^{-1}-4+\frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.5}$$

Now,

$$\frac{5}{1+\frac{1}{2}z^{-1}} = 5\left(1-\frac{z^{-1}}{2}+\frac{z^{-2}}{4}-\frac{z^{-3}}{8}+\dots\right) \tag{5.6}$$

$$= 5-\frac{5}{2}z^{-1}+\frac{5}{4}z^{-2}-\frac{5}{8}z^{-3}+\dots \tag{5.7}$$

$$= \sum_{n=0}^{\infty}5\left(\frac{-z^{-1}}{2}\right)^n \tag{5.8}$$

$$H(z) = 2z^{-1}-4+\frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.9}$$

$$= 2z^{-1}-4+\sum_{n=0}^{\infty}5\left(\frac{-z^{-1}}{2}\right)^n \tag{5.10}$$

As $n < 5$,

$$H(z) = 2z^{-1}-4+\sum_{n=0}^{4}5\left(\frac{-z^{-1}}{2}\right)^n \tag{5.11}$$

$$H(z) = 1-\frac{1}{2}z^{-1}+\frac{5}{4}z^{-2}-\frac{5}{8}z^{-3}+\frac{5}{16}z^{-4} \tag{5.12}$$

$$\implies h(n) = \left(1, \frac{-1}{2}, \frac{5}{4}, \frac{-5}{8}, \frac{5}{16}\right) \tag{5.13}$$

for general n,

$$h(n) = \begin{cases} 1 & n = 0 \\ -\frac{1}{2} & n = 1 \\ \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} & n \geq 2 \end{cases} \tag{5.14}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.15}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.10),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.16}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.17}$$

using (4.17) and (4.6).
This solution is valid for the ROC,

$$|z| > \frac{1}{2} \tag{5.18}$$

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. 5.3.

```
wget https://github.com/amaan28/EE3900/
    blob/main/codes/hn.py
```

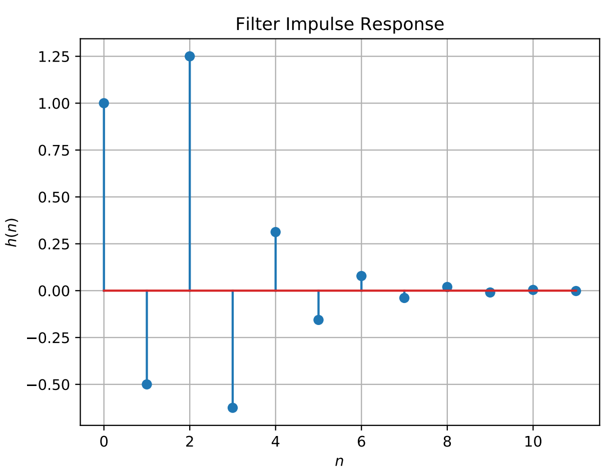

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

$$|u(n)| \leq 1 \tag{5.19}$$

$$\left|\left(-\frac{1}{2}\right)^{n}\right| \leq 1 \tag{5.20}$$

$$\implies \left|\left(-\frac{1}{2}\right)^{n} u(n)\right| \leq 1 \tag{5.21}$$

Similarly,

$$\left|\left(-\frac{1}{2}\right)^{n-2} u(n-2)\right| \leq 1 \tag{5.22}$$

$$\implies |h(n)| \leq 2 \tag{5.23}$$

Hence, $h(n)$ is bounded.

5.4 Convergent? Justify using the ratio test.
Solution: For $n > 2$,

$$h(n) = \left(-\frac{1}{2}\right)^{n} + \left(-\frac{1}{2}\right)^{n-2} \tag{5.24}$$

$$h(n) = 5\left(-\frac{1}{2}\right)^{n} \tag{5.25}$$

$$\left|\frac{h(n+1)}{h(n)}\right| = \frac{1}{2} < 1 \tag{5.26}$$

Hence, $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.27}$$

Is the system defined by (3.2) stable for the impulse response in (5.15)?
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n} u(n) + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.28}$$

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n} + \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^{n-2} \tag{5.29}$$

These are both sums of infinite geometric progressions with first terms 1 and common ratios $-\frac{1}{2}$

$$\sum_{n=-\infty}^{\infty} h(n) = \frac{1}{1 - \left(-\frac{1}{2}\right)} + \frac{1}{1 - \left(-\frac{1}{2}\right)} \tag{5.30}$$

$$= \frac{4}{3} < \infty \tag{5.31}$$

Therefore, the system is stable.

5.6 Verify the above result using a python code.
Solution:

```
wget https://github.com/amaan28/EE3900/
    blob/main/codes/Ex_5.6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.32}$$

This is the definition of $h(n)$.

Solution:

$$h(0) = 1 \tag{5.33}$$

Now, for $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) = 0 \tag{5.34}$$

$$\implies h(1) = -\frac{1}{2}h(0) = -\frac{1}{2} \tag{5.35}$$

For $n = 2$,

$$h(2) + \frac{1}{2}h(1) = \delta(2) + \delta(0) = 1 \tag{5.36}$$

$$\implies h(2) = 1 - \frac{1}{2}h(1) = \frac{5}{4} \tag{5.37}$$

For $n > 2$, the right hand side of the equation is always zero. Thus,

$$h(n) = -\frac{1}{2}h(n-1) \qquad n > 2 \tag{5.38}$$

$$h(3) = \frac{5}{4}\left(-\frac{1}{2}\right) \tag{5.39}$$

$$h(4) = \frac{5}{4}\left(-\frac{1}{2}\right)^2 \tag{5.40}$$

$$\vdots \tag{5.41}$$

$$h(n) = \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} \tag{5.42}$$

Therefore,

$$h(n) = \begin{cases} 1 & n = 0 \\ -\frac{1}{2} & n = 1 \\ \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} & n \geq 2 \end{cases} \tag{5.43}$$

Thus, it is bounded and convergent to 0

$$\lim_{n \to \infty} h(n) = 0 \tag{5.44}$$

The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://github.com/amaan28/EE3900/
    blob/main/codes/hndef.py
```

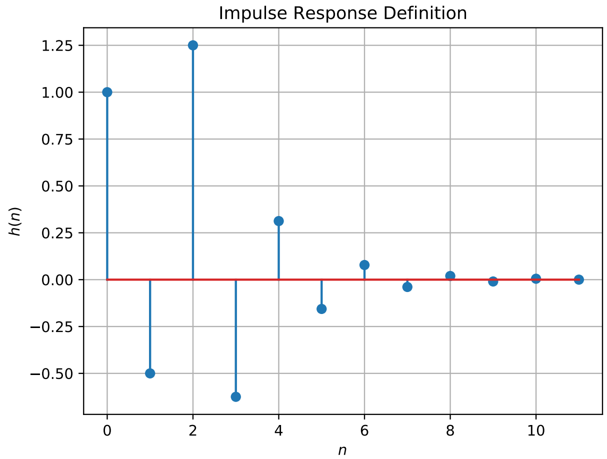

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.45}$$

Comment. The operation in (5.45) is known as *convolution*.

Solution:

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.46}$$

$$= \sum_{k=0}^{5} x(k)h(n-k) \tag{5.47}$$

The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/amaan28/EE3900/
    blob/main/codes/ynconv.py
```

5.9 Express the above convolution using a Teoplitz matrix.

Solution:

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \tag{5.48}$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_1 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & h_3 & h_2 & h_1 \\ 0 & . & . & . & h_2 & h_1 \\ 0 & . & . & . & 0 & h_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{5.49}$$

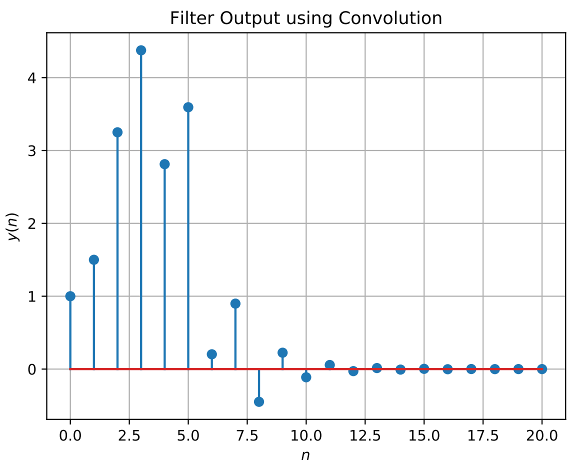

Fig. 5.8: $y(n)$ from the definition of convolution

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.50}$$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.51}$$

Solution: Substitute $k = n - i$

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{n-i=-\infty}^{\infty} x(n-i)h(n-(n-i)) \tag{5.52}$$

$$= \sum_{i=\infty}^{-\infty} x(n-i)h(i) \tag{5.53}$$

$$= \sum_{i=-\infty}^{\infty} x(n-i)h(i) \tag{5.54}$$

since the order of limits does not matter for a summation. Thus,

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.55}$$

$$\implies x(n) * h(n) = h(n) * x(n) \tag{5.56}$$

Therefore, convolution is commutative.

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://github.com/amaan28/EE3900/
    blob/main/codes/yndft.py
```

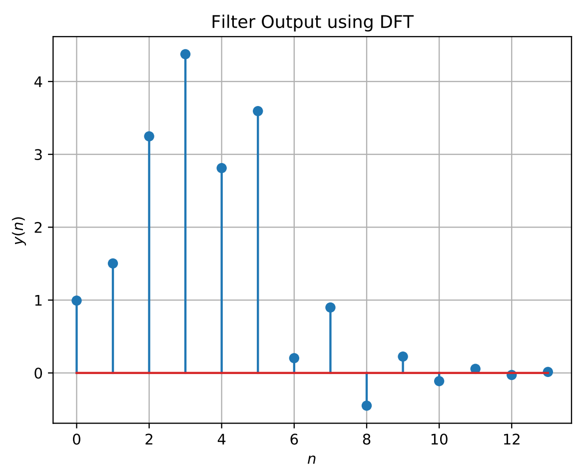

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Download the code from

```
wget https://github.com/amaan28/EE3900/
    blob/main/codes/Ex_6.4,py
```

Observe that Fig. (6.4) is the same as $y(n)$ in Fig. (3.2).

7 FFT

7.1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

7.2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

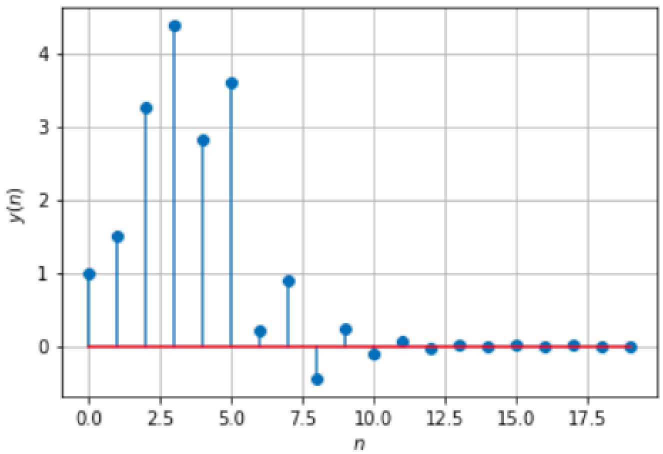

Fig. 6.4: $y(n)$ using FFT and IFFT

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = \left[W_N^{mn}\right], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

7.3. Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

7.4. The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$

7.5. Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution: We write

$$W_N^2 = \left(e^{-\frac{\jmath 2\pi}{N}}\right)^2 = e^{-\frac{\jmath 2\pi}{N/2}} = W_{N/2} \tag{7.8}$$

7.6. Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.9}$$

Solution: Observe that for $n \in \mathbb{N}$, $W_4^{4n} = 1$ and $W_4^{4n+2} = -1$. Using (7.7),

$$\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \tag{7.10}$$

$$= \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{bmatrix} \tag{7.11}$$

$$= \begin{bmatrix} W_4^0 & W_4^0 \\ W_4^1 & W_4^3 \end{bmatrix} \tag{7.12}$$

$$\implies -\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} W_4^2 & W_4^6 \\ W_4^3 & W_4^9 \end{bmatrix} \tag{7.13}$$

$$\mathbf{F}_2 = \begin{pmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{pmatrix} \tag{7.14}$$

$$= \begin{pmatrix} W_4^0 & W_4^0 \\ W_4^0 & W_4^2 \end{pmatrix} \tag{7.15}$$

Hence,

$$\mathbf{W}_4 = \begin{pmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^2 & W_4^1 & W_4^3 \\ W_4^0 & W_4^4 & W_4^2 & W_4^6 \\ W_4^0 & W_4^6 & W_4^3 & W_4^9 \end{pmatrix} \tag{7.16}$$

$$= \begin{bmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2 F_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2 F_2 \end{bmatrix} \tag{7.17}$$

$$= \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & \mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \tag{7.18}$$

Multiplying (7.18) by $\mathbf{P}_4$ on both sides, and noting that $\mathbf{W}_4\mathbf{P}_4 = \mathbf{F}_4$ gives us (7.9).

7.7. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.19}$$

Solution: Observe that for even N and letting $\mathbf{f}_N^i$ denote the i^{th} column of $\mathbf{F}_N$, from (7.12) and (7.13),

$$\begin{pmatrix} \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^2 & \mathbf{f}_N^4 & \dots & \mathbf{f}_N^N \end{pmatrix} \tag{7.20}$$

and

$$\begin{pmatrix} \mathbf{I}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{I}_{N/2}\mathbf{F}_{N/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_N^1 & \mathbf{f}_N^3 & \dots & \mathbf{f}_N^{N-1} \end{pmatrix} \tag{7.21}$$

Thus,

$$\begin{bmatrix} \mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix}$$

$$= \begin{pmatrix} \mathbf{f}_N^1 & \dots & \mathbf{f}_N^{N-1} & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N \end{pmatrix} \tag{7.22}$$

and so,

$$\begin{bmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2}\\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{N/2} & 0\\ 0 & \mathbf{F}_{N/2}\end{bmatrix}\mathbf{P}_N$$
$$= \begin{pmatrix}\mathbf{f}_N^1 & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N\end{pmatrix} = \mathbf{F}_N \tag{7.23}$$

7.8. Find

$$\mathbf{P}_4\mathbf{x} \tag{7.24}$$

Solution: We have,

$$\mathbf{P}_4\mathbf{x} = \begin{pmatrix}\mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4\end{pmatrix}\begin{pmatrix}x(0)\\x(1)\\x(2)\\x(3)\end{pmatrix} = \begin{pmatrix}x(0)\\x(2)\\x(1)\\x(3)\end{pmatrix} \tag{7.25}$$

7.9. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.26}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution: Writing the terms of X,

$$X(0) = x(0) + x(1) + \ldots + x(N-1) \tag{7.27}$$

$$X(1) = x(0) + x(1)e^{-\frac{\jmath 2\pi}{N}} + \ldots + + x(N-1)e^{-\frac{\jmath 2(N-1)\pi}{N}} \tag{7.28}$$

$$\vdots$$

$$X(N-1) = x(0) + x(1)e^{-\frac{\jmath 2(N-1)\pi}{N}} + \ldots + + x(N-1)e^{-\frac{\jmath 2(N-1)(N-1)\pi}{N}} \tag{7.29}$$

Clearly, the term in the m^{th} row and n^{th} column is given by ($0 \le m \le N-1$ and $0 \le n \le N-1$)

$$T_{mn} = x(n)e^{-\frac{\jmath 2mn\pi}{N}} \tag{7.30}$$

and so, we can represent each of these terms as a matrix product

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.31}$$

where $\mathbf{F}_N = \left[e^{-\frac{-\jmath 2mn\pi}{N}}\right]_{mn}$ for $0 \le m \le N-1$ and $0 \le n \le N-1$.

7.10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix}X(0)\\X(1)\\X(2)\\X(3)\end{bmatrix} = \begin{bmatrix}X_1(0)\\X_1(1)\\X_1(2)\\X_1(3)\end{bmatrix} + \begin{bmatrix}W_8^0 & 0 & 0 & 0\\ 0 & W_8^1 & 0 & 0\\ 0 & 0 & W_8^2 & 0\\ 0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0)\\X_2(1)\\X_2(2)\\X_2(3)\end{bmatrix} \tag{7.32}$$

$$\begin{bmatrix}X(4)\\X(5)\\X(6)\\X(7)\end{bmatrix} = \begin{bmatrix}X_1(0)\\X_1(1)\\X_1(2)\\X_1(3)\end{bmatrix} - \begin{bmatrix}W_8^0 & 0 & 0 & 0\\ 0 & W_8^1 & 0 & 0\\ 0 & 0 & W_8^2 & 0\\ 0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0)\\X_2(1)\\X_2(2)\\X_2(3)\end{bmatrix} \tag{7.33}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix}X_1(0)\\X_1(1)\end{bmatrix} = \begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} + \begin{bmatrix}W_4^0 & 0\\ 0 & W_4^1\end{bmatrix}\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} \tag{7.34}$$

$$\begin{bmatrix}X_1(2)\\X_1(3)\end{bmatrix} = \begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} - \begin{bmatrix}W_4^0 & 0\\ 0 & W_4^1\end{bmatrix}\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} \tag{7.35}$$

$$\begin{bmatrix}X_2(0)\\X_2(1)\end{bmatrix} = \begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} + \begin{bmatrix}W_4^0 & 0\\ 0 & W_4^1\end{bmatrix}\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} \tag{7.36}$$

$$\begin{bmatrix}X_2(2)\\X_2(3)\end{bmatrix} = \begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} - \begin{bmatrix}W_4^0 & 0\\ 0 & W_4^1\end{bmatrix}\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} \tag{7.37}$$

$$P_8\begin{bmatrix}x(0)\\x(1)\\x(2)\\x(3)\\x(4)\\x(5)\\x(6)\\x(7)\end{bmatrix} = \begin{bmatrix}x(0)\\x(2)\\x(4)\\x(6)\\x(1)\\x(3)\\x(5)\\x(7)\end{bmatrix} \tag{7.38}$$

$$P_4\begin{bmatrix}x(0)\\x(2)\\x(4)\\x(6)\end{bmatrix} = \begin{bmatrix}x(0)\\x(4)\\x(2)\\x(6)\end{bmatrix} \tag{7.39}$$

$$P_4\begin{bmatrix}x(1)\\x(3)\\x(5)\\x(7)\end{bmatrix} = \begin{bmatrix}x(1)\\x(5)\\x(3)\\x(7)\end{bmatrix} \tag{7.40}$$

Therefore,

$$\begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} = F_2\begin{bmatrix}x(0)\\x(4)\end{bmatrix} \tag{7.41}$$

$$\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} = F_2\begin{bmatrix}x(2)\\x(6)\end{bmatrix} \tag{7.42}$$

$$\begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} = F_2\begin{bmatrix}x(1)\\x(5)\end{bmatrix} \tag{7.43}$$

$$\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} = F_2\begin{bmatrix}x(3)\\x(7)\end{bmatrix} \tag{7.44}$$

Solution: We write out the values of performing an 8-point FFT on $\mathbf{x}$ as follows.

$$X(k) = \sum_{n=0}^{7} x(n) e^{-\frac{\jmath 2kn\pi}{8}} \quad (7.45)$$

$$= \sum_{n=0}^{3} \left(x(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \quad (7.46)$$

$$= X_1(k) + e^{-\frac{\jmath 2k\pi}{4}} X_2(k) \quad (7.47)$$

where $\mathbf{X}_1$ is the 4-point FFT of the even-numbered terms and $\mathbf{X}_2$ is the 4-point FFT of the odd numbered terms. Noticing that for $k \geq 4$,

$$X_1(k) = X_1(k-4) \quad (7.48)$$

$$e^{-\frac{\jmath 2k\pi}{8}} = -e^{-\frac{\jmath 2(k-4)\pi}{8}} \quad (7.49)$$

we can now write out $X(k)$ in matrix form as in (7.32) and (7.33). We also need to solve the two 4-point FFT terms so formed.

$$X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-\frac{\jmath 2kn\pi}{8}} \quad (7.50)$$

$$= \sum_{n=0}^{1} \left(x_1(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x_2(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \quad (7.51)$$

$$= X_3(k) + e^{-\frac{\jmath 2k\pi}{4}} X_4(k) \quad (7.52)$$

using $x_1(n) = x(2n)$ and $x_2(n) = x(2n+1)$. Thus we can write the 2-point FFTs

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \quad (7.53)$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \quad (7.54)$$

Using a similar idea for the terms X_2,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \quad (7.55)$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \quad (7.56)$$

But observe that from (7.25),

$$\mathbf{P}_8 \mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \quad (7.57)$$

$$\mathbf{P}_4 \mathbf{x}_1 = \begin{pmatrix} \mathbf{x}_3 \\ \mathbf{x}_4 \end{pmatrix} \quad (7.58)$$

$$\mathbf{P}_4 \mathbf{x}_2 = \begin{pmatrix} \mathbf{x}_5 \\ \mathbf{x}_6 \end{pmatrix} \quad (7.59)$$

where we define $x_3(k) = x(4k)$, $x_4(k) = x(4k+2)$, $x_5(k) = x(4k+1)$, and $x_6(k) = x(4k+3)$ for $k = 0, 1$.

7.11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad (7.60)$$

compte the DFT using (7.26)

Solution: Download the Python code from

```
$ wget https://github.com/amaan28/EE3900/
    blob/main/codes/Ex_7.11.py
```

7.12. Repeat the above exercise using the FFT after zero padding $\mathbf{x}$.

7.13. Write a C program to compute the 8-point FFT.

Solution: The C code for the above two problems can be downloaded from

```
$ wget https://github.com/amaan28/EE3900/
    blob/main/codes/Ex_7.13.c
```

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m) y(n-m) = \sum_{k=0}^{N} b(k) x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

Solution: The implementation is at

```
$ wget https://raw.githubusercontent.com/
    prajwal−3−14159/
    EE3900_ma20btech11013/main/filter/
    codes/problem_8−1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b. **Solution:** For the given values, the difference equation is

$$\begin{aligned} &y(n) - (4.44)\,y(n-1) + (8.78)\,y(n-2) \\ &- (9.93)\,y(n-3) + (6.90)\,y(n-4) \\ &- (2.93)\,y(n-5) + (0.70)\,y(n-6) \\ &- (0.07)\,y(n-7) = \left(5.02 \times 10^{-5}\right) x(n) \\ &+ \left(3.52 \times 10^{-4}\right) x(n-1) + \left(1.05 \times 10^{-3}\right) x(n-2) \\ &+ \left(1.76 \times 10^{-3}\right) x(n-3) + \left(1.76 \times 10^{-3}\right) x(n-4) \\ &+ \left(1.05 \times 10^{-3}\right) x(n-5) + \left(3.52 \times 10^{-4}\right) x(n-6) \\ &+ \left(5.02 \times 10^{-5}\right) x(n-7) \end{aligned} \tag{8.2}$$

From (8.1), we see that the transfer function can be written as follows

$$H(z) = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{\sum_{k=0}^{M} a(k)z^{-k}} \tag{8.3}$$

$$= \sum_i \frac{r(i)}{1 - p(i)z^{-1}} + \sum_j k(j)z^{-j} \tag{8.4}$$

where $r(i)$, $p(i)$, are called residues and poles respectively of the partial fraction expansion of $H(z)$. $k(i)$ are the coefficients of the direct polynomial terms that might be left over. We can now take the inverse z-transform of (8.4) and get using (4.17),

$$h(n) = \sum_i r(i)[p(i)]^n u(n) + \sum_j k(j)\delta(n-j) \tag{8.5}$$

Substituting the values,

$$\begin{aligned} h(n) &= [(2.76)\,(0.55)^n \\ &+ (-1.05 - 1.84\jmath)\,(0.57 + 0.16\jmath)^n \\ &+ (-1.05 + 1.84\jmath)\,(0.57 - 0.16\jmath)^n \\ &+ (-0.53 + 0.08\jmath)\,(0.63 + 0.32\jmath)^n \\ &+ (-0.53 - 0.08\jmath)\,(0.63 - 0.32\jmath)^n \\ &+ (0.20 + 0.004\jmath)\,(0.75 + 0.47\jmath)^n \\ &+ (0.20 - 0.004\jmath)\,(0.75 - 0.47\jmath)^n]u(n) \\ &+ \left(-6.81 \times 10^{-4}\right) \delta(n) \end{aligned} \tag{8.6}$$

The values $r(i)$, $p(i)$, $k(i)$ and thus the impulse response function are computed and plotted at

```
$ wget https://raw.githubusercontent.com/
    prajwal−3−14159/
    EE3900_ma20btech11013/main/filter/
    codes/problem_8−2−1.py
```

The filter frequency response is plotted at

```
$ wget https://raw.githubusercontent.com/
    prajwal−3−14159/
    EE3900_ma20btech11013/main/filter/
    codes/problem_8−2−2.py
```

Observe that for a series $t_n = r^n$, $\frac{t_{n+1}}{t_n} = r$. By the ratio test, t_n converges if $|r| < 1$. We note that observe that $|p(i)| < 1$ and so, as $h(n)$ is the sum of convergent series, we see that $h(n)$ converges. From Fig. (8.2), it is clear that $h(n)$ is bounded. From (4.1),

$$\sum_{n=0}^{\infty} h(n) = H(1) = 1 < \infty \tag{8.7}$$

Therefore, the system is stable. From Fig. (8.2), $h(n)$ is negligible after $n \geq 64$, and we can apply a 64-bit FFT to get y(n). The following code uses the DFT matrix to generate $y(n)$ in Fig. (8.2).

```
$ wget https://raw.githubusercontent.com/
    prajwal−3−14159/
    EE3900_ma20btech11013/main/filter/
    codes/problem_8−2−3.py
```

Fig. 8.2: Plot of $h(n)$

Fig. 8.2: Filter frequency response

Fig. 8.2: Plot of $y(n)$

8.3 What is the sampling frequency of the input signal? **Solution:** Sampling frequency f_s = 44.1 kHZ.

8.4 What is type, order and cutoff frequency of the above Butterworth filter? **Solution:** The given Butterworth filter is low pass with order 4 and cutoff frequency 4 kHz.

8.5 Modifying the code with different input parameters and to get the best possible output. **Solution:** A better filtering was found on setting the order of the filter to be 7.