Little Groups of Timelike and Lightlike vectors

Project for Midsem evaluation

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October 11, 2023

One particle states

Definitions and Important relations

- Poincare Group : Group of Inhomogeneous Lorentz Transformations, $U(\Lambda, a)$.
- Physical state vector in terms of eigenvectors of 4-momentum operator.

$$P^\mu |p, \sigma\rangle = p^\mu |p, \sigma\rangle \tag{1}$$

The label σ denotes all other degrees of freedom. We define it to be discrete.

- Effect of translations on $|p, \sigma\rangle$,

$$U(1, a)|p, \sigma\rangle = e^{-\iota p.a}|p, \sigma\rangle \tag{2}$$

- Effect of Homogeneous LT on $|p, \sigma\rangle$,

$$P^\mu \left[U(\Lambda)|p, \sigma\rangle\right] = (\Lambda p)^\mu \left[U(\Lambda)|p, \sigma\rangle\right] \tag{3}$$

One particle states

Definitions and Important relations contd.

- $U(\Lambda)|p,\sigma\rangle$ must be a Linear Combination of $|\Lambda p,\sigma'\rangle$,

$$U(\Lambda)|p,\sigma\rangle = \sum_{\sigma'} C_{\sigma\sigma'}(\Lambda, p)|\Lambda p,\sigma'\rangle \tag{4}$$

  We try to figure out the structure of the $C_{\sigma\sigma'}(\Lambda, p)$ in terms of irreducible representations of the inhomogeneous Lorentz group

- Corresponding to each value of p^2 and every sign of p^0 (if $p^2 \leq 0$), we choose a 'standard four-momentum', k^μ and express any p^μ of this class as,

$$p^\mu = L^\mu{}_\nu(p)k^\nu \tag{5}$$

- L is some 'standard' LT that depends on p and implicitly on k as well. Next we define, the states $|p,\sigma\rangle$ of momentum by,

$$|p,\sigma\rangle \equiv N(p)U(L(p))|k,\sigma\rangle \tag{6}$$

Little Groups

- Operate $U(\Lambda)$ on $|p, \sigma\rangle$

$$U(\Lambda)|p, \sigma\rangle = N(p)U(\Lambda L(p))|k, \sigma\rangle \tag{7}$$

Manipulating a little,

$$U(\Lambda)|p, \sigma\rangle = N(p)U(\Lambda L(p))\left[U(L^{-1}(\Lambda p)\Lambda L(p))\right]|k, \sigma\rangle \tag{8}$$

The LT in square bracket leaves the state invariant, it belongs to the subgroup of Homogeneous LTs that leave k^μ invariant,

$$W^{\mu}{}_{\nu}k^{\nu} = k^{\mu} \tag{9}$$

This subgroup is called the Little group.

Little Groups contd.

For any LT satisfying (9), we have,

$$U(W)|k,\sigma\rangle = \sum_{\sigma'} D_{\sigma'\sigma}(W)|k,\sigma'\rangle \tag{10}$$

The coefficients $D(W)$ furnish a representation of the little group. Now, Applying (10) on (6),

$$U(\Lambda)|p,\sigma\rangle = N(p)\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))U(L(\Lambda p))|k,\sigma'\rangle \tag{11}$$

Using the definition (7),

$$U(\Lambda)|p,\sigma\rangle = \left(\frac{N(p)}{N(\Lambda p)}\right)\sum_{\sigma'} D_{\sigma'\sigma}(W(\Lambda,p))|\Lambda p,\sigma'\rangle \tag{12}$$

	Standard k^μ	Little Group
(a) $p^2 = -M^2 < 0, p^0 > 0$	$(0,0,0,M)$	$SO(3)$
(b) $p^2 = -M^2 < 0, p^0 < 0$	$(0,0,0,-M)$	$SO(3)$
(c) $p^2 = 0, p^0 > 0$	$(0, 0, \kappa, \kappa)$	$ISO(2)$
(d) $p^2 = 0, p^0 < 0$	$(0, 0, \kappa, -\kappa)$	$ISO(2)$
(e) $p^2 = N^2 > 0$	$(0,0,N,0)$	$SO(2,1)$
(f) $p^\mu = 0$	$(0, 0, 0, 0)$	$SO(3,1)$

Figure: Standard momenta and corresponding little group for various classes of 4-momenta

Normalization Constant $N(p)$

Moreover, we have the dot products, given by,

$$\langle p', \sigma' | p, \sigma \rangle = |N(p)|^2 \delta_{\sigma' \sigma} \delta^3(k' - k) \tag{13}$$

$f(p)$ can be written as Lorentz Invariant Integral over four-momenta with $-p^2 = M^2 \geq 0$ and $p^0 > 0$ for case (a) or (c),

$$\int d^4p \; \delta(p^2 + m^2)\theta(p^0) f(p) \tag{14}$$

$$= \int d^3\mathbf{p} \; dp^0 \; \delta((p^0)^2 - \mathbf{p}^2 - M^2)\theta(p^0) f(\mathbf{p}, p^0) \tag{15}$$

$$= \int d^3\mathbf{p} \frac{f(\mathbf{p}, \sqrt{p^2 + M^2})}{2\sqrt{p^2 + M^2}} \tag{16}$$

References

- Section 2.5, The Quantum Theory of Fields Vol 1, S. Weinberg