Studies in Quantum Field Theory

On techniques for calculating Feynman Integrals

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INTRODUCTION

Need for Feynman Integrals?

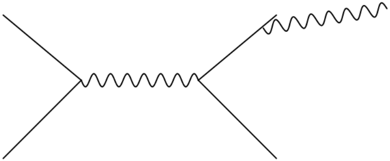


Figure: $e^+e^- \to \mu^+\mu^-\gamma$

Observables $\to$ Scattering cross section $\to$

$$\sigma = \frac{1}{2s}\int |\mathcal{M}|^2 d\phi_n \tag{1}$$

Amplitude/Matrix Element($\mathcal{M}$) $\to$ Feynman Integral

Amplitude expressions for $e^+e^- \rightarrow \mu^+\mu^-$

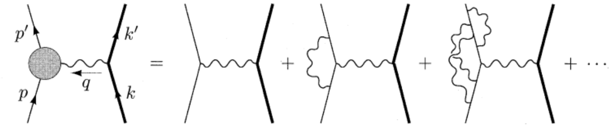
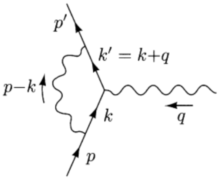


Figure: Lowest order $e^- - \gamma$ vertex, ($Court$: $Peskin$)



Applying the Feynman rules, we find, to order α, that $\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu$, where

$$\begin{aligned}
&\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) \\
&\quad = \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\, \bar{u}(p')(-ie\gamma^\nu)\frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}\,\gamma^\mu\, \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}(-ie\gamma^\rho)u(p) \\
&\quad = 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\left[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right]u(p)}{((k-p)^2+i\epsilon)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}. \qquad (6.38)
\end{aligned}$$

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- We have a lot of these!

- These are hard to calculate!
- We have a lot of these
- Way out? Differential equations

Method of Differential Equations

Feynman Integrals in most general form :

$$I(\nu_1, \ldots, \nu_N) = \int \prod_{i=1}^{L} d^D k_i \frac{\mathcal{N}(k_i.k_j, p_i.p_j)}{(q_1^2 - m^2 + i\epsilon)^{\nu_1} \ldots (q_N^2 - m^2 + i\epsilon)^{\nu_N}}$$

where D : Spacetime dimension, $\nu_i \in \mathbb{Z}$ and,

$$q_i = \sum_{j=1}^{L} \alpha_j k_j + \sum_{j=1}^{E} \beta_j p_j \;\; ; \alpha_i, \beta_i \in \{\pm 1, 0\}$$

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- Are these all independent?

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- Are these all independent? NO
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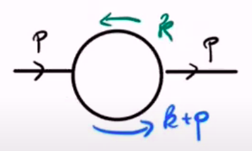
- Amplitude calculation might require the computation of 1000+ integrals.
- Are these all independent?
- Can we find a basis of integrals? $\implies$ Yes! Use IBP relations

Idea :

- $I(\nu_1, \ldots, \nu_N)$ defines a point $(\nu_1, \ldots, \nu_N)$ on $\mathbb{Z}$.
- Find recursive relations on this lattice using differential operators called **IBP operators**.

IBP Example : Massless 1-loop bubble



$$Bub(\nu_1, \nu_2) = \int \frac{d^D k}{(k^2)^{\nu_1}(k+p)^{\nu_2}}$$

IBP Operators :

$$\frac{\partial}{\partial k^{\mu}}(N^{\mu}) \ \ ; N = k, p$$

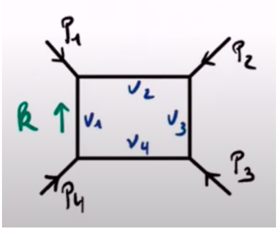
Applying IBP operators on this family,

$$Bub(v_1, v_2) = \frac{v_1 + v_2 - 1 - D}{p^2(v_2 - 1)} Bub(v_1, v_2 - 1) + \frac{1}{p^2} Bub(v_1 - 1, v_2); \quad v_2 \neq 1$$

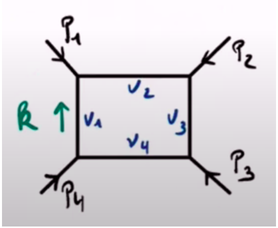
$$Bub(v_1, v_2) = \frac{v_1 + v_2 - 1 - D}{p^2(v_1 - 1)} Bub(v_1 - 1, v_2) + \frac{1}{p^2} Bub(v_1, v_2 - 1); \quad v_1 \neq 1$$

All integrals of this family can be reduced to $Bub(0,1)$ and $Bub(1,1)$

1-loop Massless Box



$$Box(\nu_1, \nu_2, \nu_3, \nu_4) = \int \frac{d^D k}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4}}$$



$$Box(\nu_1, \nu_2, \nu_3, \nu_4) = \int \frac{d^D k}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4}} \qquad : D = 4 - 2\epsilon$$

Form of Propagators :

$$D_1 = k^2, \quad D_4 = (k + p_1 + p_2 + p_3)^2$$
$$D_2 = (k + p_1)^2, \quad D_3 = (k + p_1 + p_2)^2$$

- 4-Momentum conservation

$$p_1 + p_2 + p_3 + p_4 = 0$$

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$$p_1 + p_2 + p_3 + p_4 = 0$$

- Mandelstam Variables

$$\begin{aligned} s_{ij} = (p_i + p_j)^2 = 2p_i.p_j \\ s = s_{12}, \quad t = s_{23}, \quad u = s_{13} \\ s + t + u = 0 \end{aligned}$$

Goal : Sequence of change of variables

$$Box(p_1, p_2, p_3) = B(s_{12}, s_{23}, s_{13}) = F(s, t) = (-s)^{2-\epsilon} I(x)$$

where,

$$x = t/s$$

Start with :

$$D_{ij} = p_i^{\mu} \frac{\partial}{\partial p_j^{\mu}}$$

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$$D_{ij} = p_i^{\mu} \frac{\partial}{\partial p_j^{\mu}}$$

Apply D_{33} on $Box(1,1,1,1)$:

$$D_{33} Box(1,1,1,1) = \frac{2(D-3)}{t^2} Box(0,1,0,1) - Box(1,1,1,1)$$

Chain Rule Step 1 :

$$
\begin{aligned}
s_{12}\frac{\partial}{\partial s_{12}} &= \frac{1}{2}(D_{11} + D_{22} - D_{33}) \\
s_{23}\frac{\partial}{\partial s_{23}} &= \frac{1}{2}(-D_{11} + D_{22} + D_{33}) \\
s_{13}\frac{\partial}{\partial s_{13}} &= \frac{1}{2}(D_{11} - D_{22} + D_{33})
\end{aligned}
$$

Chain Rule Step 2 :

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial s_{12}} - \frac{\partial}{\partial s_{13}}$$
$$\frac{\partial}{\partial t} = \frac{\partial}{\partial s_{23}} - \frac{\partial}{\partial s_{13}}$$

Chain Rule Step 3 :

$$\frac{\partial}{\partial x} = -\sigma\frac{\partial}{\partial t}$$
$$\frac{\partial}{\partial \sigma} = -\frac{\partial}{\partial s} - x\frac{\partial}{\partial t}$$

Current basis, $\vec{f} = \{f_1, f_2, f_3\}^T$:

$$f_1 = G_{0,1,0,1}$$
$$f_2 = G_{1,0,1,0}$$
$$f_3 = G_{1,1,1,1}$$

Differential equation :

$$\partial_s \vec{f}(s,t;\epsilon) = A_s(s,t,\epsilon)\vec{f}(s,t;\epsilon)\,,$$
$$\partial_t \vec{f}(s,t;\epsilon) = A_t(s,t,\epsilon)\vec{f}(s,t;\epsilon)\,.$$

$$A_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\epsilon}{s} & 0 \\ \frac{-2(1-2\epsilon)}{st(s+t)} & \frac{2(1-2\epsilon)}{s^2(s+t)} & -\frac{s+t+\epsilon t}{s(s+t)} \end{pmatrix}, \qquad A_t = \begin{pmatrix} -\frac{\epsilon}{t} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-2(1-2\epsilon)}{t^2(s+t)} & \frac{-2(1-2\epsilon)}{st(s+t)} & -\frac{s+\epsilon s+t}{t(s+t)} \end{pmatrix}$$

New Basis, $\vec{g} = \{g_1, g_2, g_3\}^T$:

$$\begin{aligned} g_1 &= c(-s)^{\epsilon} G_{0,1,0,2} \\ g_2 &= c(-s)^{\epsilon} G_{1,0,2,0} \\ g_3 &= c\epsilon(-s)^{\epsilon} st G_{1,1,1,1} \end{aligned}$$

Canonical Differential equation :

$$\partial_x \vec{g}(x;\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1+x}\right] \vec{g}(x,\epsilon)\,,$$

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{pmatrix}.$$

HQET Integral Family at 2 loops

Cross Ladder Integral Family

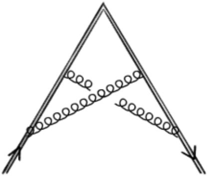

Figure: 2-loop Web Diagram

$$T^{(2)}[a_1, \ldots, a_7] = \int \frac{d^D k_1}{i\pi^{D/2}} \frac{d^D k_2}{i\pi^{D/2}} \frac{D_7^{-a_7}}{D_1^{a_1} \ldots D_6^{a_6}}$$

For soft gluons emitted by heavy quarks,

$$D_1 = -2k_1.v_1 + \delta, \quad D_2 = -2(k_1 + k_2).v_1 + \delta,$$
$$D_3 = -2(k_1 + k_2).v_2 + \delta, \quad D_4 = -2k_2.v_2 + \delta,$$
$$D_5 = -k_1^2, \quad D_6 = -k_2^2, \quad D_7 = k_1.k_2$$

For admissible integrals with numerators,

$$D_8 = D_3 - D_4 = -2k_1.v_2$$
$$D_9 = D_2 - D_1 = -2k_2.v_1$$

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In Auxillary Notation,

$$t^{(2)}[a_1, \ldots, a_9] = \int \frac{d^D k_1}{i\pi^{D/2}} \frac{d^D k_2}{i\pi^{D/2}} \frac{D_7^{-a_7} D_8^{-a_8} D_9^{-a_9}}{D_1^{a_1} \ldots D_6^{a_6}}$$

Admissibile Integrals :

- The integrand has the overall scaling dimension $-4L$ for the transformation $k_i \to \lambda k_i, i = 1, 2, \ldots, L$.
- No UV divergence in any subloop subdiagram.
- No infrared (IR) divergences. This forbids higher powers of massless propagators.

Admissible Integrals : Top sector with 6 propagators

$$
\begin{aligned}
A_1 &= T^{(2)}[1,1,1,1,1,1,0] \\
A_2 &= t^{(2)}[2,1,1,1,1,1,0,-1,0] \\
A_3 &= t^{(2)}[1,1,1,2,1,1,0,0,-1]
\end{aligned}
$$

Admissible Integrals : Sector with 5 propagators

$$
\begin{aligned}
B_1 &= T^{(2)}[1,2,1,0,1,1,0] \\
B_2 &= T^{(2)}[1,1,2,0,1,1,0] \\
B_3 &= t^{(2)}[2,1,2,0,1,1,0,-1,0] \\
B_4 &= t^{(2)}[2,2,1,0,1,1,0,-1,0] \\
B_5 &= T^{(2)}[1,2,0,1,1,1,0] \\
B_6 &= T^{(2)}[1,3,-1,1,1,1,0] \\
B_7 &= t^{(2)}[1,2,0,2,1,1,0,0,-1] \\
B_8 &= t^{(2)}[2,2,0,1,1,1,0,-1,0]
\end{aligned}
$$

Admissible Integrals : Sector with 4 propagators

$$
\begin{aligned}
C_1 &= T^{(2)}[1,0,3,0,1,1,0] \\
C_2 &= T^{(2)}[1,-1,4,0,1,1,0] \\
C_3 &= t^{(2)}[2,0,3,0,1,1,0,-1,0] \\
C_4 &= T^{(2)}[1,3,0,0,1,1,0] \\
C_5 &= T^{(2)}[1,4,-1,0,1,1,0] \\
C_6 &= t^{(2)}[2,3,0,0,1,1,0,-1,0]
\end{aligned}
$$

IBP Vectors :

$$\boldsymbol{P_1} = \partial_{1\mu} k_1^{\mu} \tag{2}$$

$$\boldsymbol{P_2} = \partial_{1\mu} k_{1\nu} v_1^{[\mu} v_2^{\nu]} \tag{3}$$

$$\boldsymbol{P_3} = (\partial_{1\mu} k_{1\nu} + \partial_{2\mu} k_{2\nu}) v_1^{[\mu} v_2^{\nu]} \tag{4}$$

IBP relations : Top Sector

$$\boldsymbol{P_2} A_1 = 0$$
$$\boldsymbol{P_3} A_1 = 0$$

IBP relations : Top Sector

$$\boldsymbol{P_2} A_1 = 0 \implies A_2 + B_5 - 2B_1 - 2(v_1.v_2)B_2 = 0 \tag{5}$$

$$\boldsymbol{P_3} A_1 = 0 \implies A_3 - A_2 = 0 \tag{6}$$

IBP relations : 5 Propagator Sector

$$\boldsymbol{P_3} B_1 = 0$$
$$\boldsymbol{P_3} B_2 = 0$$
$$\boldsymbol{P_1} B_5 = 0$$
$$\boldsymbol{P_1} B_6 = 0$$
$$\boldsymbol{P_1} B_8 = 0$$
$$\boldsymbol{P_1} B_7 = 0$$

IBP relations : 5 Propagator Sector

$$\boldsymbol{P_3} B_1 = 0 \implies B_4 - 2(v_1.v_2)B_1 - B_2 + 2C_4 = 0 \tag{7}$$

$$\boldsymbol{P_3} B_2 = 0 \implies B_3 + B_1 - 2C_1 = 0 \tag{8}$$

$$\boldsymbol{P_1} B_5 = 0 \implies B_5 - 2C_1 = 0 \tag{9}$$

$$\boldsymbol{P_1} B_6 = 0 \implies 2B_6 - C_4 - 3C_2 = 0 \tag{10}$$

$$\boldsymbol{P_1} B_8 = 0 \implies B_8 - 2B_6 + 2C_4 = 0 \tag{11}$$

$$\boldsymbol{P_1} B_7 = 0 \implies B_7 - 2C_3 = 0 \tag{12}$$

IBP relations : 4 Propagator Sector (Important Direct[1] results)

$$\boldsymbol{P_2} C_1 = 0$$
$$\boldsymbol{P_3} C_1 = 0$$
$$\boldsymbol{P_1} C_6 = 0$$

[1] Not really direct

IBP relations : 4 Propagator Sector (Important Direct results)

$$\boldsymbol{P_2}\, C_1 = 0 \implies C_3 - C_4 = 0$$

$$\boldsymbol{P_3}\, C_1 = 0 \implies C_3 - 3C_2 + 2(v_1.v_2)C_1 = 0$$

$$\boldsymbol{P_1}\, C_6 = 0 \implies 2(v_1.v_2)C_4 - C_6 + 3C_5 = 0$$

Combining all these results, admissible integrals can be decomposed as,

$$\begin{aligned} C_2 &= \frac{2(v_1.v_2)}{3} C_1 + \frac{1}{3} C_4 \\ C_3 &= C_4 \\ C_5 &= -\frac{(v_1.v_2)}{3} C_4 \\ C_6 &= (v_1.v_2) C_4 \end{aligned}$$

For top sector intrgrals,

$$A_2 = A_3 = 2B_1 + 2(v_1.v_2)B_2 - 2C_1$$

For 5 propagator sector integrals :

$$\begin{aligned}
B_3 &= -B_1 + 2C_1 \\
B_4 &= 2(v_1.v_2)B_1 + B_2 - 2C_4 \\
B_5 &= 2C_1 \\
B_6 &= C_4 + (v_1.v_2)C_1 \\
B_7 &= 2C_4 \\
B_8 &= 2(v_1.v_2)C_1
\end{aligned}$$

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B_5 &= 2C_1 \\
B_6 &= C_4 + (v_1.v_2)C_1 \\
B_7 &= 2C_4 \\
B_8 &= 2(v_1.v_2)C_1
\end{aligned}
$$

$\implies A_1, B_1, B_2, C_1, C_4$ are the Master Integrals!

Deriving Differential equations

We use the differential operator d on $\vec{f}$,

$$\boldsymbol{d} \equiv \frac{x^2-1}{2}\frac{d}{dx} \equiv \frac{(v_1.v_2)v_1^\mu - v_1^2 v_2^\mu}{\sqrt{v_1^2 v_2^2}}\frac{\partial}{\partial v_1^\mu}$$

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where,

$$\vec{f} = \{A_1, B_1, B_2, C_1, C_4\}^T$$

Applying on master integrals,

$$
\begin{aligned}
\boldsymbol{d} A_1 + 2(v_1.v_2)A_1 - A_2 - B_5 &= 0\\
\boldsymbol{d} B_1 + 3(v_1.v_2)B_1 - B_4 - 2C_4 &= 0\\
\boldsymbol{d} B_2 + 2(v_1.v_2)B_2 - B_3 - B_1 &= 0\\
\boldsymbol{d} C_1 + (v_1.v_2)C_1 - C_4 &= 0\\
\boldsymbol{d} C_4 &= 0
\end{aligned}
$$

Simplifying,

$$\frac{d}{dx}\vec{f} = \boldsymbol{A}(x)\vec{f}$$

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$$\frac{d}{dx}\vec{f} = \boldsymbol{A}(x)\vec{f}$$

where,

$$\boldsymbol{A} = \frac{1+x^2}{1-x^2}\begin{pmatrix} \frac{2}{x} & -\frac{4}{1+x^2} & -\frac{2}{x} & 0 & 0 \\ 0 & \frac{1}{x} & -\frac{2}{1+x^2} & 0 & 0 \\ 0 & 0 & \frac{2}{x} & -\frac{4}{1+x^2} & 0 \\ 0 & 0 & 0 & \frac{1}{x} & -\frac{2}{1+x^2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

To convert the DE to canonical form, choose :

$$F_1 = \frac{(x^2-1)^2}{2x^2} A_1$$
$$F_2 = \frac{x^2-1}{x}(B_1 + \frac{1+x^2}{2x} B_2)$$
$$F_3 = \frac{2(x^2-1)}{x} C_1$$
$$F_4 = C_4$$

Canonical form :

$$\frac{d}{dx}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{x} & 0 & 0 \\ 0 & 0 & \frac{1+x^2}{x(1-x)(1+x)} & 0 \\ 0 & 0 & 0 & \frac{4}{x} \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}$$

This system can be solved iteratively!

References

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- Introduction to QFT by Peskin and Schroeder