Problem Set I

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Fundamentals

Introduction to Sets

1. (a) (i) $\{x \in \mathbb{Z} : |2x| < 5\} \;=\; \{-2, -1, 0, 1, 2\}$

   (ii) $\{X : X \subseteq \{3, 2, a\} \text{ and } |X| = 2\} \;=\; \{\{3, 2\}, \{3, a\}, \{2, a\}\}$

   (iii) $\{X \subseteq \mathbb{N} : |X| \leq 1\} \;=\; \{\emptyset, \{x\}\},\ x \in \mathbb{N}$

   (b) (i) $\{0, 1, 4, 9, 16, 25, 36\}$

   (1) $\{x^2 : x \in \mathbb{Z}_{\geq 0}\}$

   (2) $\{x^2 : x \in \mathbb{Z} \text{ and } x \geq 0\}$

   (ii) $\{3, 4, 5, 6, 7, 8\}$

   (1) $\{x \in \mathbb{N} : 3 \leq x \leq 8\}$

   (2) $\{x \in \mathbb{Z} : 3 \leq |x| \leq 8\}$

   (iii) $\{\ldots, -\pi, \frac{-\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, \ldots\}$

   (1) $\{\frac{x\pi}{2} : x \in \mathbb{Z}\}$

   (2) $\{x \in \mathbb{R} : sin(x) = sign(x)\}$

2. (a) $|\mathcal{P}(\mathcal{P}(\mathcal{P}(A)))| = 2^{2^{2^m}}$

   (b) $|\mathcal{P}(A) \times \mathcal{P}(B)| = 2^{m+n}$

(c) $|\{X \in \mathcal{P}(A) : |X| \leq 1\}| = m + 1$

(d) $|\{X \subseteq \mathcal{P}(A) : |X| \leq 1\}| = 2^m + 1$

Introduction to Mathematical Logic

1. (a)
   - Proposition: 0 is an integer.
     The statement contains the word "is", which is equivalent to the "=" logical operator indicating a truth value; the statement is true because 0 is an integer.
   - Not a Proposition: To be or not to be?
     The statement indicates no truth value and is posing a philisophical question from Hamlet.

   (b) (i) $P:\ x = 0$ and $Q:\ y = 0$
   $P \wedge \neg Q$

   (ii) P : *matrix is invertible* and Q : *determinent is not zero*
   $Q \implies P$

   (iii) P : *function is continuous* and Q : *function is differentiable*
   $Q \implies P$

   (iv) P : *you fail* and Q : *stop writing*
   $P \implies Q$

2. (a)

P	Q	$P \implies Q$	$P \wedge \neg Q$	$Q \wedge \neg Q$	$P \wedge \neg Q \implies Q \wedge \neg Q$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	F	T
F	F	T	F	F	T

Equivalent

(b)

P	Q	R	$P \vee Q$	$(P \vee Q) \wedge R$	$Q \wedge R$	$P \vee (Q \wedge R)$
F	F	F	F	F	F	F
F	F	T	F	F	F	F
F	T	F	T	F	F	F
F	T	T	T	T	T	T
T	F	F	T	F	F	T
T	F	T	T	T	F	T
T	T	F	T	F	F	T
T	T	T	T	T	T	T

Not Equivalent

(c)

P	Q	$\neg(P \implies Q)$	$P \wedge \neg Q$
F	F	T	F
F	T	F	F
T	F	T	T
T	T	F	F

Equivalent

3. (a)
   - Symbolic: $a + x = x$; $\exists a, \forall x \in \mathbb{R}$
   - Negation: There exists a real number x for which $a + x = x$ is false for every real number a.

   (b)
   - Symbolic: $|f(x) - b| < \epsilon$ and $x > M$; $\forall \epsilon, \exists M > 0$
   - Negation: For every positive number M there is a positive number ϵ for which $|f(x) - b| \geq \epsilon$ whenever $x \leq M$

   (c)
   - Symbolic: $f = \textit{polynomial}$ and $\textit{deg}(f) > 2 \implies f'' \neq 0$
   - Negation: If f is not a polynomial or its degree is less than or equal to 2, then f' is a constant.

Advanced

1. (a) i. Let $a \in A$ and $r \in \mathbb{Z}$. By definition of A, $a = 6r + 12 = 3(2r + 4)$. Because $r \in \mathbb{Z}$ and $2r + 4 \in Z$, $a \in B \implies A \subset B$.

   ii. $A \neq B$ holds true if there is an element in B not in A. For instance, given that $n \in B$, lets say $n = 3$; then, $s = 1 \in \mathbb{Z}$. However, for $m \in A$, m cannot be 3 because there is no such $r \in \mathbb{Z}$ that satisfies $3 = 6r + 12$.

iii. Thus, A is a proper subset of B because $A \subset B$ and $A \neq B$.

(b) False.
Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$, $C = \{2, 3, 4\}$, and $D = \{2, 3, 4, 5, 6\}$. Then, $A \subseteq B$ and $C \subseteq D$. $A \setminus C = \{1, 4\}$ and $B \setminus D = \{1, 6\} \implies A \setminus C \subsetneq B \setminus D$.

(c) Given that $\bigcup_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} A_i$, then all A_i's must be equal since the the unions and intersections of A_i's are equal.