Problem Set III

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Fundamentals

Relations, Functions, and Cardinality

1. (a) Prove $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Z}\}$ on $\mathbb{R}$ is an equivalence relation.

   By definition, a relation is defined as an **equivalence relation** given that it exibits reflexive, symmetric, and transitive properties.

   i. Suppose $x, y \in \mathbb{Z}$. By definition of R and when $x = y$, $xRy = xRx \implies x - y = x - x = 0 \in \mathbb{Z}$. Thus, R is reflexive.
   ii. Suppose $x, y \in \mathbb{Z}$ such that $x-y \in \mathbb{Z}$. Then, $y-x = -(x-y)$. $-(x - y) \in \mathbb{Z} \implies y - x \in \mathbb{Z}$. Thus, $xRy \implies yRx$, where $y - x$ means yRx by definition of R.
   iii. Suppose $x, y, z \in \mathbb{Z}$ such that there are the following relations: xRy and yRz. Let $a, b \in \mathbb{Z}$ such that $x - y = a$ and $y - z = b$. Then, the following statement holds true: $a + b = (x - y) + (y - z) = x - z$, then $x - z \in \mathbb{Z}$. Therefore, R is transitive because xRz is the result from xRy and yRz.
   iv. Hence, R is an equivalence relation of integer differences.

   (b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the formula $f(x, y) = (xy, x^3)$. Is f injective, surjective, or bijective? Explain.

   i. By definition, an injective function has for every $y \in Y$ at most one $x \in X$, where Y is the codomain and X is the domain. By example, let $(x,y) = (0,0)$ and $(x,y) = (0,1)$. Then, $f(0,0) = (0,0)$ and $f(0,1) = (0,0)$, which violates the definition of an injective function. Therefore, f is not injective.

ii. By definition, a surjective function has every element $y \in Y$ mapped by at least one element $x \in X$, where Y is the codomain and X is the domain. However, by example, lets say $f(x,y) = (xy, x^3) = (1,0)$. Then, $x^3 = 0 \implies x = 0 \implies xy = 0$. But, based on the exampled, $xy = 1 \neq 0$. Therefore, f is not surjective.

iii. Because f is not injective and not surjective, f is also not bijective by definition.

2. (a) Lets state the relation of $(x-y)^2 < 1$ on $\mathbb{R}$. When xRy is restricted to $\mathbb{Z}$, then the relation holds only when $x = y$. If $x \geq y$, then $x = y$ for the relation to hold. If $x \leq y$, then $x = y$ for the relation to hold as well. However, the relation is still valid on $\mathbb{R}$ and isn't simply equality on $\mathbb{R}$. For example, $(1 - 0.5)^2 = 0.25 < 1$.

   (b) Let $A = \{a,b,c,d\}$ and $B = \{a,b\}$

   i. $\{(a,a),(a,b),(b,a),(c,b),(d,a)\}$

   ii. $f = \{(a,a),(b,b),(c,b),(d,b)\}$

   iii. $f = \{(a,a),(b,a),(c,a),(d,a)\}$

   iv. $|A| \neq |B| \because 4 \neq 2$

Basics of Algorithms

1.
```
function(seq,key):
      for (i = N; i >= 1; i = i - 1):
           if (seq[i] == key):
                return (i + 1)
return 0
```

2.

3. seq = [s_1, s_2, ..., s_N]

```
/* i from index 0 and j from index N*/
function(seq,i,j):
     if (i == 1):
          return (s[i].s[j]=s[j],s[i])
          function(seq, i + 1, j - 1)
```

Advanced