5. Sea la señal gaussiana $x(t)=e^{-at^2}$

$$x(t)=e^{-at^2} \quad a \in \mathbb{R}^+$$

sistema A: $y_A(t) = x^2(t)$

sistema B un SLIT con respuesta impulso

$$h_B(t) = Be^{-bt^2}$$

a) Salida serie

$$x(t) \xrightarrow[h_B(t)]{} y(t) \xrightarrow{\text{cuadrado}} y_A(t)$$

1.) $x(t) * h_B(t) \rightarrow y(t)$

2.) $y_A(t) = y^2(t)$

convolucion de $x(t) * h_B(t)$

$$y(t) = x(t) * h_B(t) = \int_{-\infty}^{\infty} x(\tau) h_B(t-\tau)\, d\tau$$

$$x(\tau) = e^{-a\tau^2} \quad * \quad h_B(t-\tau) = Be^{-b(t-\tau)^2}$$

$$y(t) = \int_{-\infty}^{\infty} e^{a\tau^2} - Be^{-b(t-\tau)^2}$$

$$y(t) = \int_{\infty}^{\infty} e^{-a\tau^2} - Be^{-b(t-\tau)^2}\, d\tau$$

$$y(t) = B \int_{-\infty}^{\infty} e^{-a\tau^2} \cdot Be^{-b(t-\tau)^2}\, d\tau$$

$$\boxed{(t-\tau)^2 = t^2 - 2t\tau + \tau^2}$$

Expandiendo el producto Notable

sustituyendo

$$\gamma(t) = B\int_{-\infty}^{\infty} e^{-a\tau^2} e^{-b\left(t^2 - 2\left(\tau T + \tau^2\right)\right)} d\tau$$

$$\gamma(t) = B e^{-bt^2} \int_{-\infty}^{\infty} e^{-(a+b)\tau^2 + 2bz\tau} d\tau$$

$$\gamma(t) = Be^{-bt^2} \int_{-\infty}^{\infty} e^{-(a+b)\left[\tau^2 - \frac{2bt}{a+b}\tau\right]} d\tau$$

completando

$$\tau^2 - \frac{2bt}{a+b}\tau = \left(\tau - \frac{bt}{a+b}\right)^2 - \left(\frac{bt}{a+b}\right)^2$$

Sustituyendo

$$\gamma(t) = Be^{-bt^2} \int_{-\infty}^{\infty} e^{-(a+b)\left[\left(\tau - \frac{bt}{a+b}\right)^2 - \left(\frac{bt}{a+b}\right)^2\right]} d\tau$$

$$= Be^{-bt^2} e^{(a+b)\left(\frac{bt}{a+b}\right)^2} \int_{-\infty}^{\infty} e^{-(a+b)\left(\tau - \frac{bt}{a+b}\right)^2} d\tau$$

$$= B e^{-bt^2} e^{\frac{b^2t^2}{a+b}} \int_{\infty}^{\infty} e^{-(a+b)(\tau-\mu)^2} d\tau$$

$$\boxed{\mu = \frac{bt}{a+b}}$$

La integral gaussiana y su resultado es

$$\int_{-\infty}^{\infty} e^{-k(\tau-\mu)^2} d\tau = \sqrt{\frac{\pi}{k}} \qquad \boxed{k = a+b}$$

$$y(t) = B\sqrt{\frac{\pi}{a+b}}\, e^{-bt^2 + \frac{b^2t^2}{a+b}}$$

simplificamos la exponente

$$-bt^2 + \frac{b^2t^2}{a+b} = t^2\left(\frac{-b(a+b)+b^2}{a+b}\right) = -\frac{abt^2}{a+b}$$

$$y(t) = B\sqrt{\frac{\pi}{a+b}}\, e^{\frac{-abt^2}{a+b}}$$

Aplicar $y_A(t) = y^2(t)$

$$y_A(t) = \left(B\sqrt{\frac{\pi}{a+b}}\, e^{\frac{-abt^2}{a+b}}\right)^2$$

$$\boxed{y(t) = B^2\, \frac{\pi}{a+b}\, e^{-2\frac{abt^2}{a+b}}}$$

→ Salida del sistema serie

$$x(t) \longrightarrow y_A(t) = x^2(t) \xrightarrow{h_B(t)} y(t)$$

→ Aplicar A directamente

$$y_A(t) = x^2(t) = (e^{-at^2})^2$$

$$\boxed{y_A(t) = e^{-2at^2}}$$

Convolucion con $h_B(t) = Be^{-bt}$

$$Y(t) = Y_A(t) * h_B(t)$$
$$= \int_{-\infty}^{\infty} e^{-2a\tau^2} \cdot Be^{-b(t-\tau)^2} \, d\tau$$

$$Y(t) = B\int_{-\infty}^{\infty} e^{-2a\tau^2} \cdot e^{-b(t-\tau)^2} \, d\tau$$

$$\boxed{Y(t) = B\sqrt{\frac{\pi}{2a+b}} * e^{-2\frac{abt^2}{2a+b}}}$$

8. Demuestre las siguientes propiedades sin utilizar tablas de propiedades

A] $\mathcal{L}\{x(t-t_0) = e^{-st_0} X(s)$

$$\mathcal{L}\{x(t-t_0)\} = \int_0^{\infty} x(t-t_0)e^{-st} dt$$

$u = t - t_0 \quad \therefore \quad t = u + t_0. \quad t \to \infty = u \to \infty$
$t \to \infty = u \to -\infty$

$$\boxed{\mathcal{L}\{x(t-t_0)\} = e^{-st_0}\int_{-\infty}^{\infty} x(u)\, e^{-su} du = e^{-st_0} X(s)}$$

B] $\mathcal{L}\{x(at)\} = \frac{1}{|a|} x(s/a)$

$$\mathcal{L}\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-st}\, dt$$

si $a>0$

$u = at \therefore t = u/a \qquad t \to +\infty \quad u \to +\infty$

$\qquad\qquad\qquad\qquad t \to -\infty \quad u \to -\infty$

$$\frac{1}{a}\int_{-\infty}^{\infty} x(u) e^{-su/a} = 1/a\, x(s/a)$$

$a<0$

$$(-1)\,\frac{1}{a}\int_{+\infty}^{-\infty} x(u)\, e^{-s\frac{u}{a}} = -1/a\, x(s/a$$

$$\boxed{\mathcal{L}\{x(at)\} = 1/|a|\, x(s/a)}$$

c] $\mathcal{L}\left\{\dfrac{dx(t)}{dt}\right\} = sX(s) = \displaystyle\int_{-\infty}^{\infty} x'(t) e^{-st} dt$

$u = e^{-st} \qquad\qquad du = -s\, e^{-st}$

$dv = x'(t) \qquad\qquad v = x(t)$

$$= e^{-st} x(t)\Big|_0^{\infty} \int_{-\infty}^{\infty} s e^{-st} x(t)\, dt = 0 - X(0) + s\mathcal{L}\{x(t)\}$$

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = s\,X(s) - X(0)$$

$$\boxed{\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s)}$$

D] $\mathcal{L}\{x(t) * y(t)\} = X(s)\,Y(s)$

$$\mathcal{L}\{x(t) * y(t)\} = \int_{-\infty}^{\infty} (x(t) * y(t))\, e^{-st}\, dt$$

$$\mathcal{L}\{x(t) * y(t) = \int_{-\infty}^{\infty} x\tau\, y(\tau - t)\, dt$$

$$\mathcal{L}\{x(t) * y(t)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, y(\tau - t)\, d\tau\, e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} y(\tau - t)\, e^{-st}\, dt\, d\tau$$

$$u = \tau - t \qquad du = -dt \qquad t = \tau + u$$

$$= \int_{-\infty}^{\infty} x(\tau)(+1) \int_{-\infty}^{\infty} y(u)\, e^{-s(\tau + u)}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau}\, d\tau \int_{-\infty}^{\infty} y(u)\, e^{-su}\, du$$

$$\boxed{= X(s) : Y(s)}$$

9. Encuentre la trasformada de laplace, dibuje el esquema de ceros y polos y la region de convergencia (ROC) de:

A] $e^{-2t}u(t) + e^{-3t}u(t)$

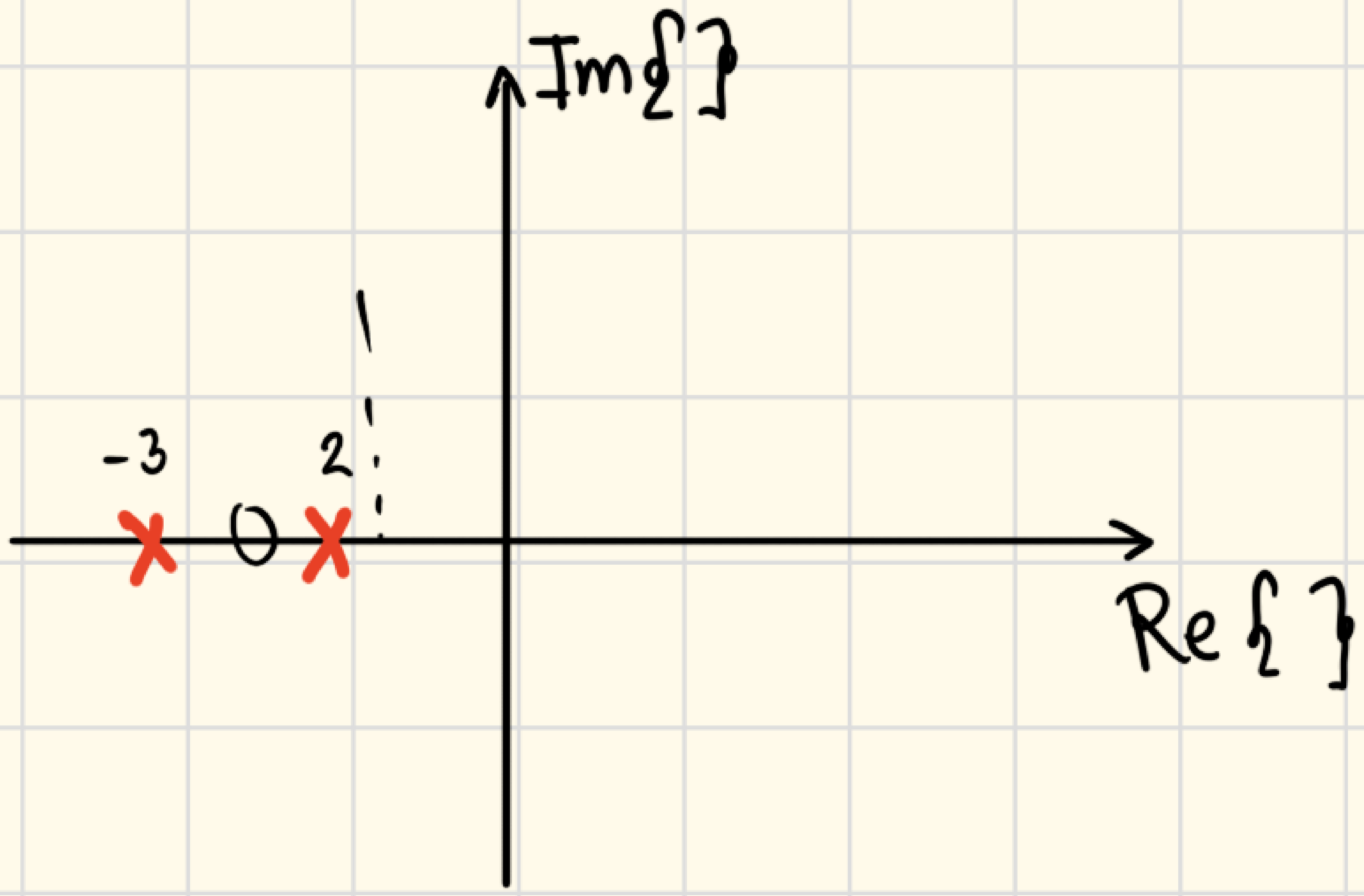
$$\mathcal{L}\{e^{-2t}u(t)\} + \mathcal{L}\{e^{-3t}u(t)\} = \int_{-0}^{\infty} e^{-2t}\, e^{-st}\, dt + \int_{0}^{\infty} e^{-3t}\, e^{-st}\, dt$$

$$= \int_0^{\infty} e^{-(s+2)t}\, dt + \int_0^{\infty} e^{-(s+3)t}\, dt = -\frac{e^{-(s+2)t}}{s+2}\Bigg|_0^{\infty} - \frac{e^{-(s+3)t}}{s+3}\Bigg|_0^{\infty}$$

ROC $s > -2$ y $s > -3$ $\therefore$ $s > -3$

$$\frac{1}{s+2} + \frac{1}{s+3} = \frac{2s+5}{(s+2)(s+3)}$$

donde polos igual a -2 y -3 cero $= -5/2$



B] $e^{2t}u(t) + e^{-3t}u(-t)$

$\mathcal{L}\{e^{2t}u(t)\} + \mathcal{L}\{e^{-3t}u(-t)\}$

$$= \int_0^{\infty} e^{2t} e^{-st}\, dt + \int_{\infty}^{0} e^{3t} e^{-st}\, dt$$

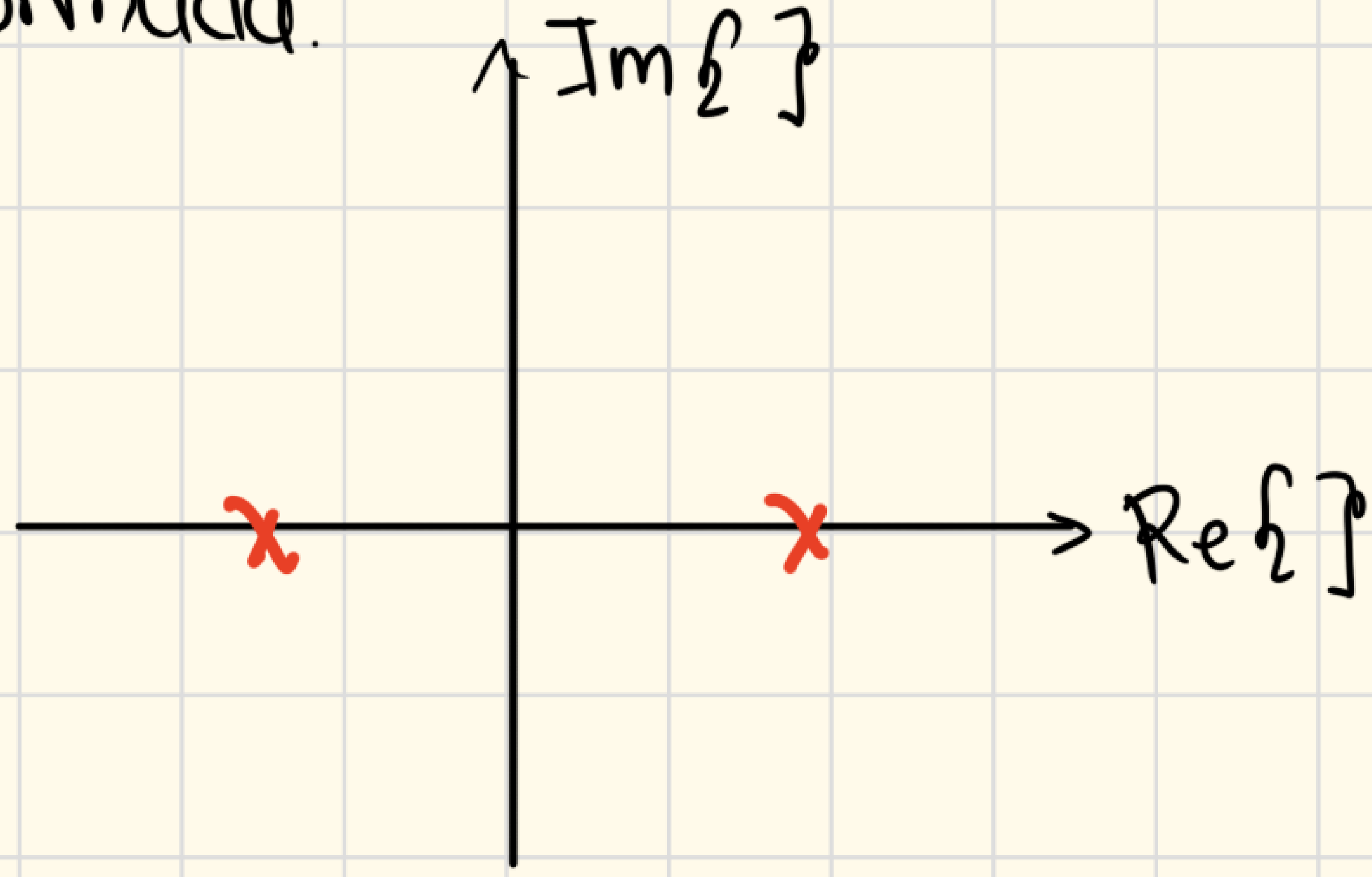
$$= \int_0^{\infty} e^{-(s-2)t}\, dt + \int_{-\infty}^{0} e^{-(3+s)t}\, dt$$

$$= -\frac{e^{-(s-2)t}}{(s-2)}\Bigg|_0^{\infty} - \frac{e^{(-3-s)}}{(3+s)}\Bigg|_{-\infty}^{0}$$

ROC $s>2$ y $s<-3$

$$= \frac{1}{s-2} - \frac{1}{3+s} = \frac{5}{(s-2)(3+s)}$$

2 y -3 no hay ceros si fuera posible su trasformada.



c] $e^{-a|t|}$

$$\mathcal{L}\{e^{-a|t|}\} = \int_{-\infty}^{0} e^{-a(-t)} e^{-st} dt + \int_{0}^{\infty} e^{-at} e^{st} dt$$

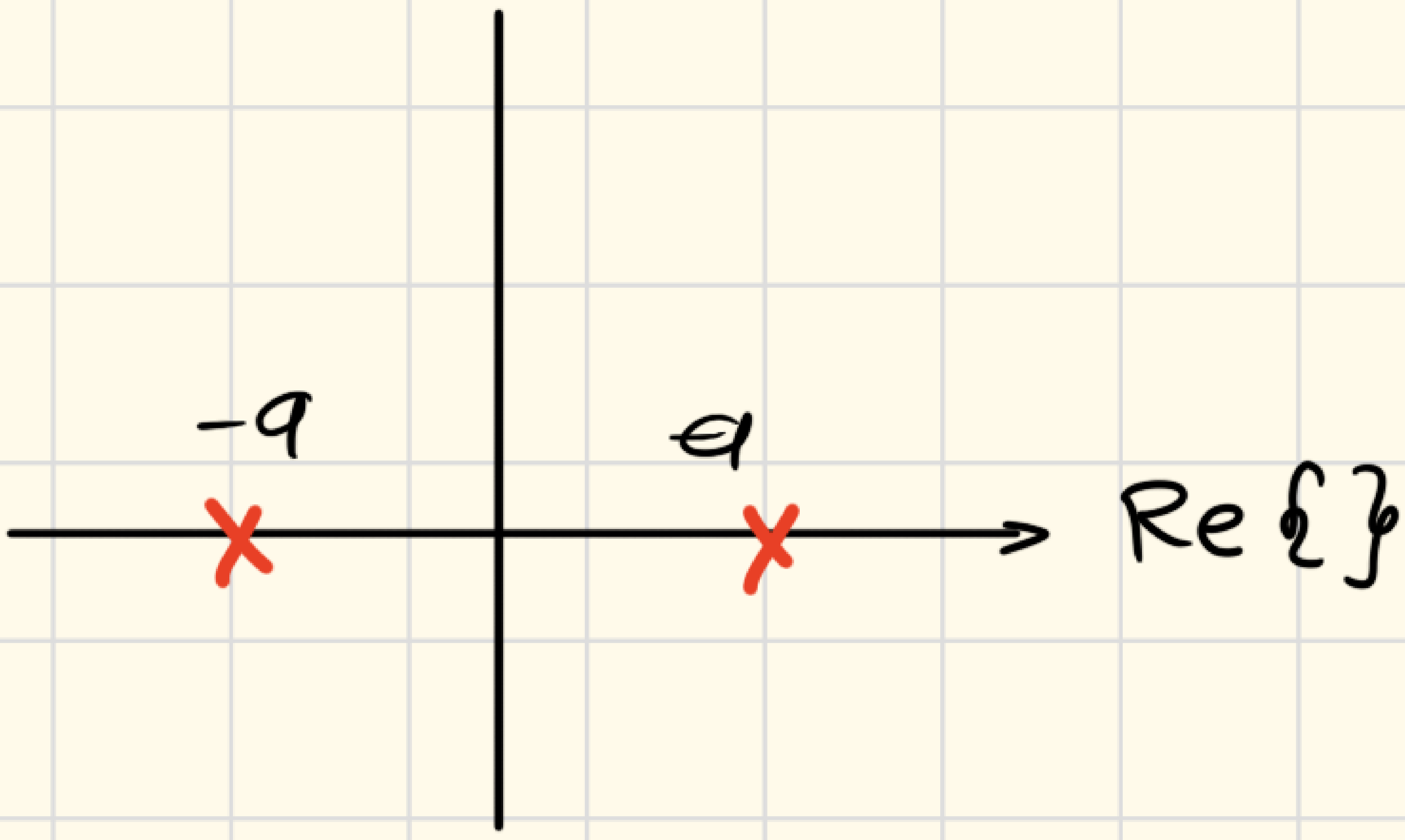
$$= \int_{-\infty}^{0} e^{(a-s)t} dt + \int_{0}^{\infty} e^{-(a+s)t} dt$$

$$= \frac{e^{(a-s)t}}{a-s}\Bigg|_{-\infty}^{0} - \frac{e^{-(a+s)t}}{a+s}\Bigg|_{0}^{\infty}$$

ROC : $s<a$ y $s>-a$ por lo tanto la region de convergencia $t(-a,a)$

$$= \frac{1}{a-s} + \frac{1}{a+s} = \frac{2a}{a^2 - s^2}$$

por lo tanto $-a$ y a ningun cero



D] $e^{-2t}\left[u(t) - u(t-5)\right] \quad t \in [0,5]$

$$\mathcal{L}\{e^{-2t}\,u(t) - u(t-5)\} = \int_0^5 e^{-2t} e^{-st}\,dt$$

$$= \int_0^5 e^{-(2+s)t}\,dt = -\left.\frac{e^{-(2+s)t}}{2+s}\right|_0^5$$

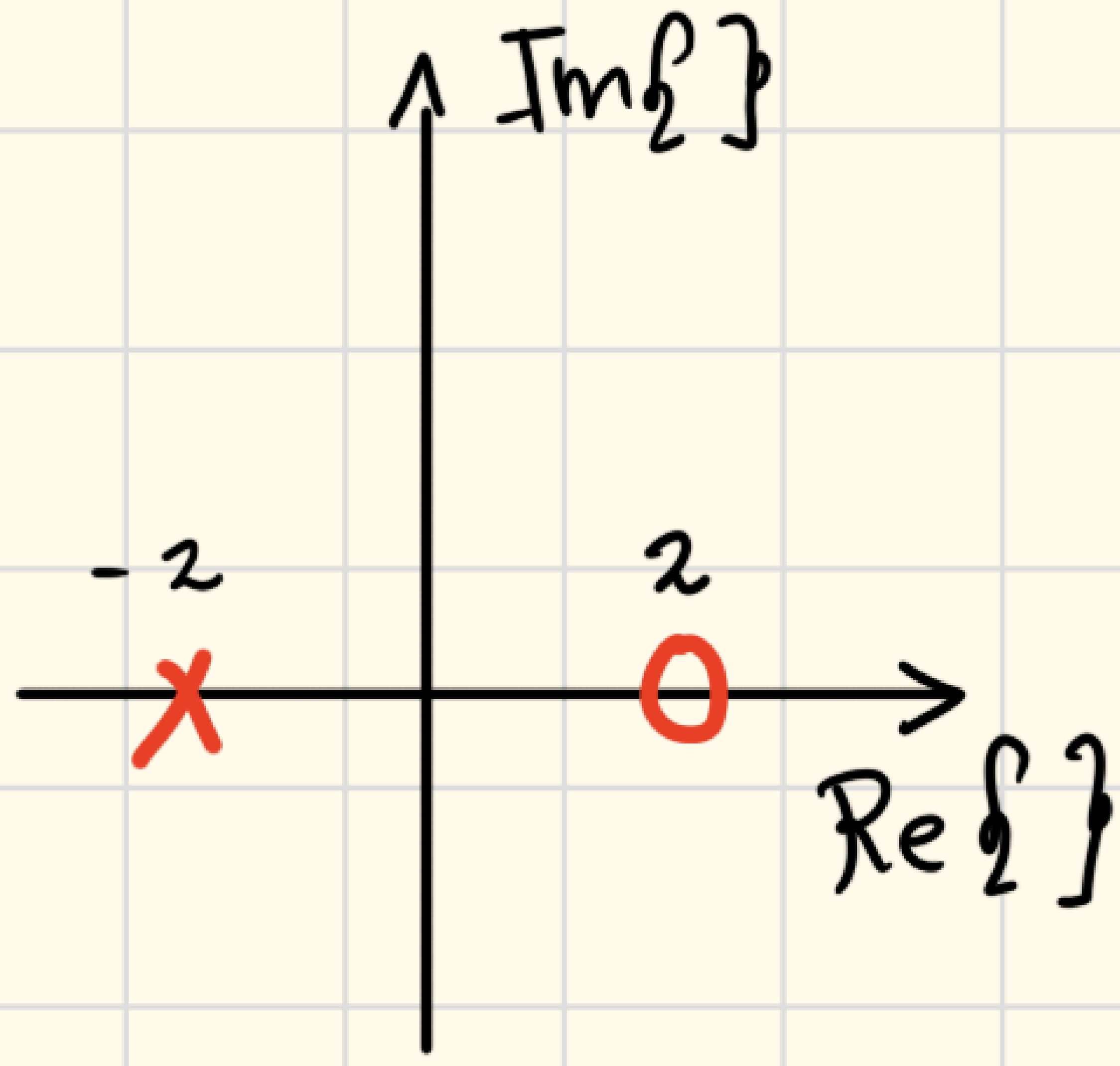
$$= \frac{1 - e^{-(5s+10)}}{s+2} \qquad \text{ROC } s > -2$$

$$1 - e^{-(5s+10)} = 0$$

$$\ln(1) = -5s + 10$$

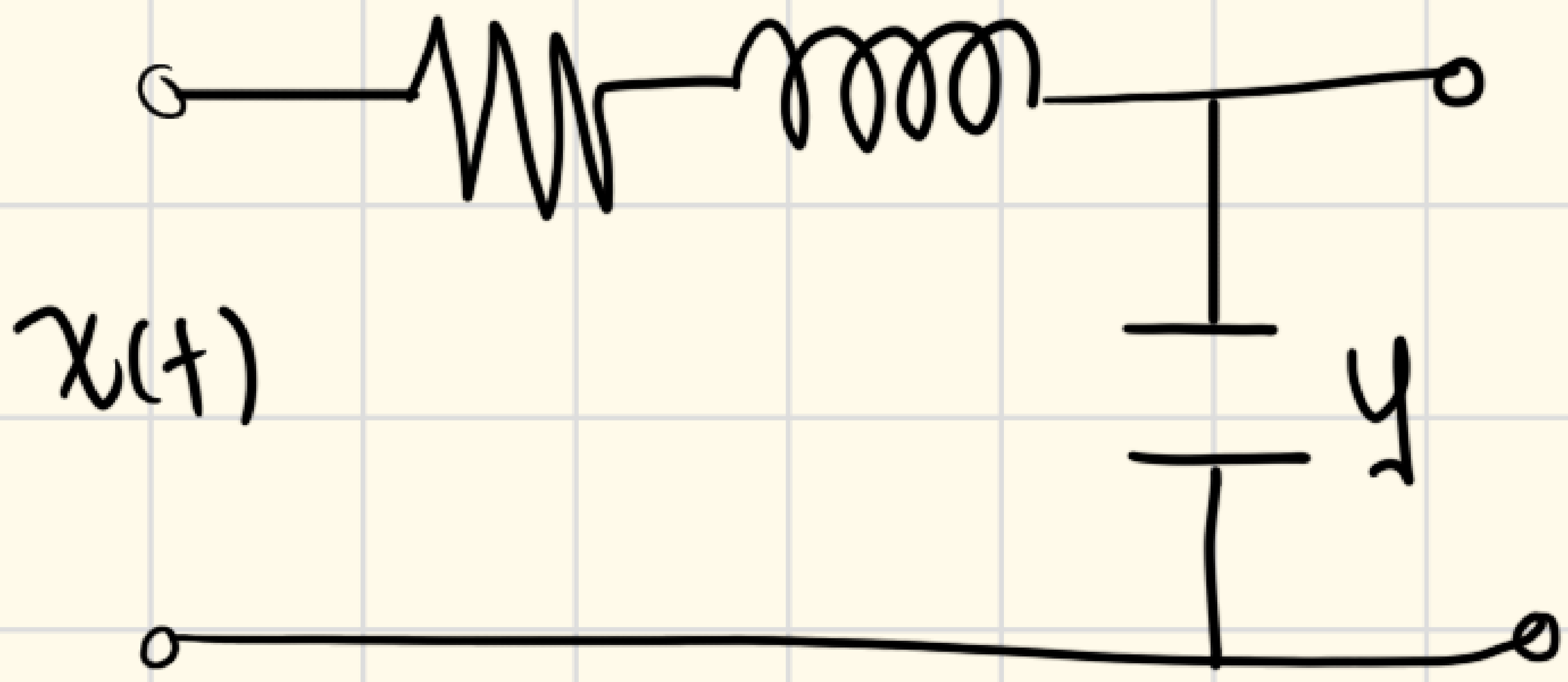
$$0 = -5s + 10$$

$$s = 2$$



10. La función de transferencia en lazo abierto par RLC serie y circuito RCL paralelo.

RLC serie



$$i(t) = \frac{C\,dv_c}{dt} = \frac{c\,dy(t)}{dt}$$

$$i(t)R_L + \frac{L\,di(t)}{dt} + \frac{1}{c}\int i(t)dt = x(t)$$

$$Rc'\frac{dy(t)}{dt} + L\frac{d}{dt}\left(\frac{c\,dy(t)}{dt}\right) + \frac{1}{c}\int c\frac{dy(t)}{dt} = x(t)$$

$$x(t) = LC\frac{d^2y(t)}{dt} + Rc\frac{dy(t)}{dt} + y(t)$$

$$X(s) = LC\,(s^2y(s)) - sy(0) - y'(0)) + Rc\,(sY(s) - Y(0)) + Y(s)$$

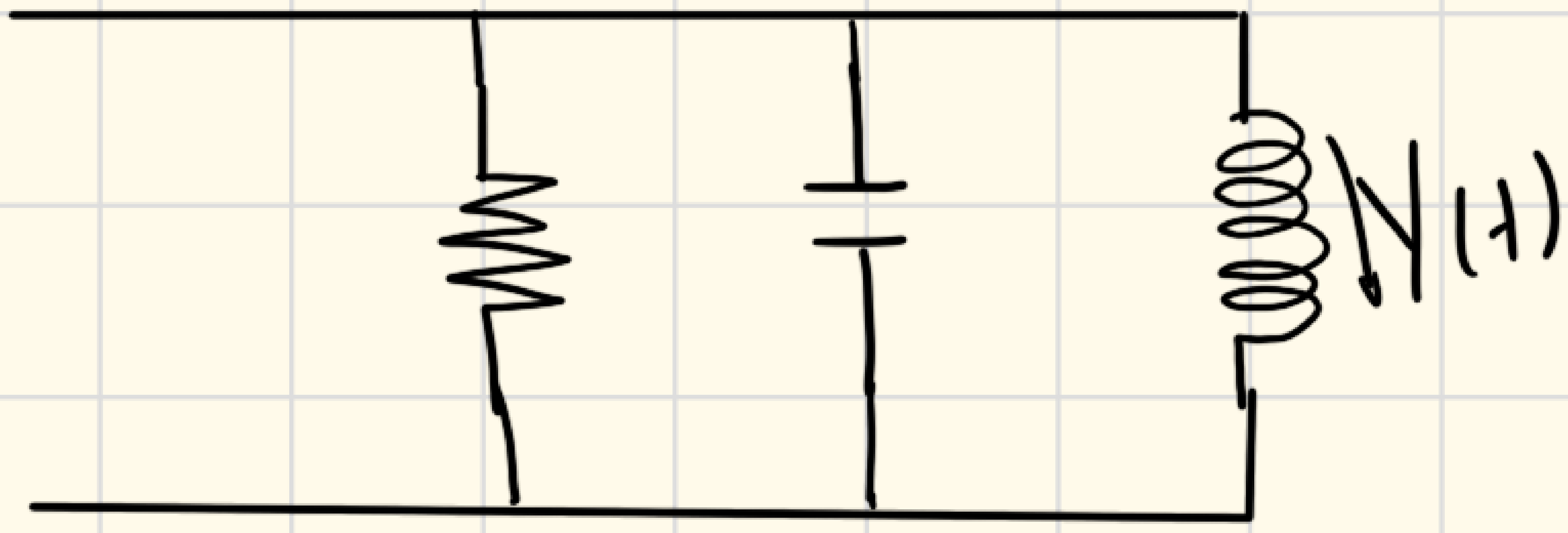
$$X(s) = LCs^2Y(s) - LSsY(0) - LCY'(0) + RCsY(s) - RCY(0) + Y(s)$$

$$X(s) = Y(s)(LCs^2 + RCs + 1) - Y(0)(LCs + RC) - LCY'(0)$$

$$Y(s) = \frac{X(s) + Y(0)(LCs + RC) + LCY'(0)}{LCs^2 + RCs + 1}$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{X(s) + Y(0)\,(LCs + RC) + LCY'(0)}{X(s)\ (LCs^2 + RCs + 1)}}$$

RLC Paralelo.



$$V_L = L\frac{di(t)}{dt} = L\frac{dY(t)}{dt}$$

$$x(t) = i_R(t) + i_C(t) + Y(t)$$

$$x(t) = \frac{V_L}{R} + C\frac{dV_L}{dt} + Y(t)$$

$$x(t) = \frac{L}{R}\frac{dy(t)}{dt} + LC\frac{d^2}{dt^2}y(t) + y(t)$$

$$R\,x(s) = L(sY(s) - Y(0)) + RCL(Y(s)s^2 - Y(0)s - Y'(0)) + Y(s)\cdot R$$

$$R\,x(s) = Y(s)(RLCs^2 + Ls + R) - Y(0)(RCLs + L) - Y'(0)RLC$$

$$Y(s) = \frac{Rx(s) + Y(0)(RLCs + L) + Y'(0)RLC}{RLCs^2 + Ls + R}$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{Rx(s) + Y(0)(RLCs + L) + Y'(0)RLC}{x(s)(RLCs^2 + Ls + R)}}$$

13. Encuentre la expresion de salida en el tiempo para una configuración en lazo cerrado del sistema en función R.L.C :

A] IMPULSO:

$$x(t) = \delta(t) \qquad \mathcal{L}\{\delta(t)\} = 1 = X(s)$$

$$x(0) = 1 \qquad x'(0) = 1$$

$$Y(s) = \frac{1}{LCs^2 + RCs + 1} + \frac{(LCs + RC)\,Y(0)}{LCs^2 + RCs + 1} + \frac{LC\,Y'(0)}{LCs^2 + RCs + 1}$$

$$Y(s) = \frac{1}{LC} * \frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} + \frac{s\,Y(0)}{s^2 + \frac{R}{L}s + \frac{1}{LC}} + \frac{R\,Y(0)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$+ \frac{Y'(0)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

$$Y(s) = \frac{1}{LC} * \frac{1}{\left(s + \frac{R}{2L}\right)^2 + \frac{4L - CR^2}{4L^2C}} + \frac{Y(0)\left(s - \frac{R}{L} + \frac{R}{2L}\right)}{\left(s + \frac{R}{2L}\right)^2 + \frac{4L - CR^2}{4L^2C}}$$

$$\frac{+\,R\,Y(0)}{\left(s + \frac{R}{2L}\right)^2 + \frac{4L - CR^2}{4LC}} + \frac{Y'(0)}{\left(s + \frac{R}{2L}\right)^2 + \frac{4LCR^2}{4L^2C}}$$

$$Y(t) = \left(\frac{\frac{1}{LC} + R\,Y(0) + Y'(0)\;\; -\frac{R}{2L}}{\sqrt{\frac{4L - CR^2}{4L^2C}}}\right) e^{-\frac{R}{2L}t} \sin\left(\sqrt{\frac{4L - CR^2}{4L^2C}}\, t\right)$$

$$+ e^{-\frac{R}{2L}t} \cos\left(\sqrt{\frac{4L - CR^2}{4L^2C}}\, t\right)$$

$$Y(s) = \frac{R + Y(0)\,RLCs + Y(0)L + Y'(0)\,RLC}{RLCs^2 + Ls + R}$$

$$Y(s) = \frac{\left(R + Y(0)L + Y'(0)\,RLC - Y(0)\,1/_{RLC\,2RC}\right)}{RLC}$$

$$* \frac{1}{\left(s + \frac{1}{2RC}\right)^2 + \frac{4R^2C - 1}{4R^2LC^2}} + Y(0)\,\frac{\left(s + 1/_{2RC}\right)}{\left(s + \frac{1}{2RC}\right)^2 + \frac{4R^2C - 1}{4R^2LC^2}}$$

$$Y(t) = \frac{2R + RLY(0) - LY(0)}{L\sqrt{4R^2C - L}}\, e^{-1/_{2RC}t} * \sin\left(\sqrt{\frac{4R^2C - L}{4R^2LC^2}}\,t\right)$$

$$+ Y(0)\, e^{-1/_{2RC}t} \cos\left(\sqrt{\frac{4R^2C - L}{4R^2LC^2}}\,t\right)$$

B] ESCALON UNITARIO.

$$\mathcal{L}\{u(t)\} = 1/s$$

$$Y(s) = \frac{1/s + Y(0)(LCs + RC) + LCY'(0)}{LCs^2 + RCs + 1}$$

$$Y(s) = \frac{1 + Y(0)\,LCs^2 + RCs + LCsY'(0)}{s\,(LCs^2 + RCs + 1)}$$

$$Y(s) = \frac{1}{s\,(LCs^2+RCs+1)} + \frac{sY(0)LC}{(LCs^2+RCs+1)} + \frac{Y_0 RC + LCY'(0)}{LCs^2+RCs+1}$$

$$\frac{1}{s\,(LCs^2+RCs+1)} = \frac{a}{s} + \frac{bs+d}{LCs^2+RCs+1}$$

$$a\,(LCs^2+RCs+1)+bs^2+ds+1$$

$$a=1 \qquad b=-LC \qquad d=-RC$$

$$Y(s) = \frac{1}{s} - \frac{LCs+RC}{LCs^2+RCs+1} + \frac{Y(0)\,LCs}{LCs^2+RCs+1} + \frac{7C+LCY'(0)}{LCs^2+RCs+1}$$

$$Y(s) = \frac{1}{s} + (Y(0)-1)\,\frac{s+R/2L}{\left(s+\frac{R}{2L}\right)^2+\frac{4L-CR^2}{4L^2C}} + \left(RC+LCY'(0)-\frac{RLC}{2L}+RLC\,Y(0)\right)\left(\frac{1}{LCs^2+RCs+1}\right)$$

con calculos anteriores

$$Y(t) = 1 + (Y(0)-1)\,e^{-\frac{R}{2L}t}\cos\left(\sqrt{\frac{4L-CR^2}{4L^2C}}\,t\right)$$

$$+\left(\frac{RC+LCY'(0)-\frac{RC}{2}+\frac{RC}{2}Y(0)}{LC\sqrt{4L-CR^2/4L^2C}}\,t\right)\left(e^{-\frac{R}{2L}t}\sin\sqrt{\frac{4L-CR^2}{4L^2C}}\right)$$

$$Y(t) = 1 + (Y(0) - 1)e^{-\frac{R}{2L}t}\cos\left(\frac{\sqrt{4L-CR^2}}{4L^2C}t\right) + \left(\frac{\sqrt{C}(R+2LY'(0)+R(0))\sqrt{4L-R^2C}}{4L-R^2C}\right)$$

$$\left(e^{-\frac{R}{2L}t}\sin\left(\sqrt{\frac{4L-CR^2}{4L^2C}}\,t\right)\right)$$

AHORRA EN EL CIRCUITO EN PARALELO

$$Y(s) = \frac{R/s + Y(0)RLCS + Y(0)L + Y'(0)LRC}{RLCS^2 + LS + R}$$

$$Y(s) = \frac{Ra}{S^2(RLCS^2+LS+R)} + \frac{Y(0)RCS}{RLCS^2+LS+R} + \frac{LY(0)+Y(0)RLC}{RLCS^2+LS+R}$$

Fraciones parciales

$$\frac{Ra}{S^2(RLCS^2+LS+R)} = \frac{bs+d}{S^2} + \frac{es+f}{RLCS^2+LS+R}$$

$$Ra = bRLCS^3 + bs^2L + bRs + dRLCs^2 + dLS + dR + es^3 + Fs^3$$

$$Ra = dr \qquad bR + dL = 0$$

$$\boxed{d = a} \qquad \boxed{b = -aR/R}$$

$$bL + dRLC + f = 0 \qquad e + bRLC = 0$$

$$f = \frac{aL^2}{R} - aRLC \qquad e = aL^2C$$

ahora queda:

$$Y(s) = \frac{\frac{-aL}{R}s + a}{s^2} + \frac{aL^2Cs + \frac{aL^2}{R} - aRLC}{RLCs^2 + Ls + R} + \frac{Y(0)RCs}{RLCs^2+Ls+R}$$

$$+ \frac{LY(0) + Y(0)RLC}{RLCs^2 + Ls + R}$$

$$Y(s) = \frac{-aL}{Rs} + \frac{a}{s^2} + (Y(0)RC + aL^2C)\frac{s}{RLCs^2+Ls+R}$$

$$+ \left(2Y(0) + Y(0)RLC + \frac{aL^2}{R} - aRLC\right)\frac{1}{RLCs^2+Ls+R}$$

$$Y(s) = \frac{-aL}{Rs} + \frac{a}{s^2} + \left(\frac{Y(0)RC + aL^2C}{RLC}\right)\frac{s + R/2L}{\left(s + \frac{R}{2L}\right)^2 + \frac{4L - CR^2}{4L^2C}}$$

$$+ \frac{\left(LY(0) + Y(0)RLC + aL^2/R - aRLC + 2\cancel{L}L\,Y(0)(RC) + \frac{R}{2L}aL^2C\right)}{RLC\sqrt{\frac{4L - CR^2}{4L^2C}}}$$

$$Y(t) = \frac{-aL}{R} + at + \left(\frac{Y(0)}{L} + \frac{aL}{R}\right)e^{-\frac{R}{2L}t}\cos\left(\sqrt{\frac{4L - CR^2}{4L^2C}}\,t\right)$$

$$+ \frac{2L\sqrt{C}\left(\frac{Y(0)}{RC} + Y(0) + \frac{aL}{R^2} - a + \frac{RY(0)}{2L^2} + \frac{a}{2}\right)}{\sqrt{4L - CR^2}}e^{\frac{-Rt}{2L}}\operatorname{sen}\left(\sqrt{\frac{4L-CR^2}{4L^2C}}\,t\right)$$