Punto #3 Taller SyS.

Encuentre la función de Densidad espectral (Transformada de Fourier) para las siguientes señales (Sin aplicar propiedades)

Transformada de Fourier continua de una señal $x(t)$ es definida como:

$$\boxed{X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dt}$$

a.) $x(t) = e^{-a|t|}$, $a \in \mathbb{R}^+$

La señal es par

$$x(t) = \begin{cases} e^{-at} & , t \geq 0 \\ -e^{at} & , t < 0 \end{cases}$$

$$X(w) = \int_{-\infty}^{0} \left(e^{at} - e^{-jwt}\right) dt + \int_{0}^{\infty} \left(e^{-jwt}\, e^{-at}\right) dt$$

$$\int_{-\infty}^{0} (e^{at - jwt})\, dt = \int_{-\infty}^{0} (e^{(a-jw)t})\, dt = \left[\frac{e^{(a-jw)t}}{a-jw} \Big|_{\infty}^{0} \right]$$

$$= \left[\frac{1}{a-jw} \right] - 0 = \boxed{\frac{1}{a-jw}}$$

$$\int_{0}^{\infty} (e^{-(a+jw)t})\, dt = \left[\frac{e^{-(a+jw)t}}{-(a+jw)} \Big|_{0}^{\infty} \right]$$

$$= -\left[-\frac{1}{(a+jw)} \right] = \boxed{\frac{1}{a+jw}}$$

$$X(w) = \frac{1}{a-jw} + \frac{1}{a+jw} = \boxed{\frac{2a}{a^2+w^2}}$$

b) $\cos(w_c t)$, $w \in \mathbb{R}$

$$X(w) = \int_{-\infty}^{\infty} \cos(w_c t)\, e^{-jwt}\, dt$$

$$\cos(W_c t) = \int_{-\infty}^{\infty} \left(\frac{e^{jw_c t} + e^{-jw_c t}}{2} \right) dt$$

$$X(w) = \int_{-\infty}^{\infty} \left(\frac{e^{jw_c t} + e^{-jw_c t}}{2} * e^{-jwt} \right) dt$$

$$X(w) = \frac{1}{2} \int_{-\infty}^{\infty} (e^{j(W_c - w)t} dt + \frac{1}{2} \int_{-\infty}^{\infty} (e^{-j(W_c + w)t}$$

Recordemos: La integral de una exponencial compleja es un Delta de Dirac:

$$\boxed{\int_{-\infty} e^{-j(a)t} dt = 2\pi \delta(a)}$$

Aplicando esto cada termino tiene:

$$X(w) : \frac{1}{2} \left(2\pi \delta(w - W_c) + 2\pi \delta(w + W_c) \right)$$

$$\boxed{X(w) = \pi [\delta(w - W_c) + \delta(w + W_c)]}$$

c.) $x(t) = \text{sen}(W_s t)$, $W_s \in \mathbb{R}$.

Aplicando la definición de Transforma de Fourier se tiene:

$$X(w) = \int_{-\infty}^{\infty} \text{Sin}(W_s t) * e^{-jwt}\, dt.$$

⇒ Sabemos que: $\text{sen}\,\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$$X(w) = \int_{-\infty}^{\infty} \left[\frac{e^{jW_s t} - e^{-jW_s t}}{2t} * \left(e^{-jwt}\right) \right] dt$$

$$= \frac{1}{2j} \int_{-\infty}^{\infty} \left(e^{j(W_s - w)t} - e^{-j(W_s + w)t} \right) dt$$

$$X(w) = \frac{1}{2j} \left(\int_{-\infty}^{\infty} e^{j(W_s - w)t}\, dt - \int_{-\infty}^{\infty} e^{-j(W_s + w)t}\, dt \right)$$

La integral compleja es un Delta de Dirac:

$$\int_{-\infty}^{\infty} e^{j(\alpha)t} \, dt = 2\pi\,\delta(\alpha)$$

* Primera Integral
$\alpha = W_s - W \rightarrow 2\pi\,\delta(W - W_s)$

* Segunda Integral.
$\alpha = -(W_s + W) \rightarrow 2\pi\,\delta(W + W_s)$

Entonces

$$X(w) = \frac{1}{2j}\left(2\pi\,\delta(w - w_s) - 2\pi\,\delta(w + w_s)\right)$$

$$X(w) = \pi\left[\delta(w - w_s) - \delta(w + w_s)\right)$$

$$\boxed{X(w) = \left[\delta(w + w_s) - \delta(w + w_s)\right]}$$

d.) $x(t) = f(t) * \cos(w_c t)$ $\quad W_c \in \mathbb{R}$
$f(t) \in R,$

$$X(w) = \int_{-\infty}^{\infty} f(t) * \cos(w_c t) * e^{-jw_c t} \, dt$$

$$\cos\theta = \frac{1}{2}\left(e^{jw_c t} + e^{-jw_c t}\right)$$

$$X(w) = \int_{-\infty}^{\infty} f(t) \cdot \frac{1}{2}\left(e^{jw_ct} + e^{-jw_ct}\right) * e^{-jw_ct}\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} F(t) \left(e^{j(w_c-w)t)} + e^{-j(w_c+w)t}\right) dt$$

$$= \frac{1}{2} \left[\int_{-\infty}^{\infty} F(t) * e^{-j(w-w_c)t}\, dt + \int_{-\infty}^{\infty} f(t) * e^{-j(w+w_c)t}\, dt \right]$$

cada integral corresponde a la transformada de Fourier de f(t) evaluada en las frecuencias especificadas

$$\boxed{\begin{array}{c} X(w) = \frac{1}{2}\left[F(w-w_c) + F(w+w_c)\right) \\ F(w) = \mathcal{F}(t) \end{array}}$$

e). $x(t) = e^{-a|t|^2}$, $a \in \mathbb{R}^+$ señal Gaussiana.

$$X(w) : \int_{-\infty}^{\infty} e^{-a|t|^2} * e^{-jwt}\, dt$$

$|t|^2 = t^2$

para todo $t \in \mathbb{R}$.

$$X(w) = \int_{-\infty}^{\infty} e^{-at^2} * e^{-jwt}\, dt = \int_{-\infty}^{\infty} e^{-at^2 - jwt}\, dt$$

si $-at - jwt$ factorizado $= (t^2 - bt)$.

$$-at^2 - jwt = -a\left(\frac{-at^2}{-a} + \frac{-jwt}{-a}\right)$$

completando cuadrados de $t^2 \; + \frac{jw}{a} t$

$$\boxed{= -a\left(t^2 \; \frac{+jwt}{a}\right)} \Rightarrow (x-1)^2 \circ (x+y)^2$$

$x = t$

$= x^2 + 2xy + y^2$

$2yt = \frac{jw}{a}\, t$

$$t^2 + \frac{jw}{a} t + \left(\frac{jw}{2a}\right)^2 - \left(\frac{jw}{2a}\right)^2 \qquad \boxed{Y = jw/2a}$$

Entonces:

$$-at^2 - jwt = -a\left[\left(t + \frac{jw}{2a}\right)^2 - \left(\frac{jw}{2a}\right)^2\right]$$

$$= -a\left(t + \frac{jw}{2a}\right)^2 \; \frac{-j^2 w^2}{4a^2}$$

Recordemos: $j^2 = -1$ queda

$$-a\left(\left(a\frac{t+jw}{2a}\right)^2 - \frac{(-1)w^2}{4a^2}\right) = a\left(t + \frac{jw}{2a}\right)^2 + \frac{w^2}{4(a)^2}(-a)$$

$$= -a\left(t + \frac{jw}{2a}\right)^2 - \frac{w^2}{4a}$$

sustituimos y hacemos simplificación

$$X(w) = \int_{-\infty}^{\infty} e^{\left[-a\left(t+\frac{jw}{2a}\right)^2 - \frac{w^2}{4a}\right]} dt$$

$$X(w) = e^{-\frac{w^2}{4a}} \int_{-\infty}^{\infty} e^{-a\left(t+\frac{jw}{2a}\right)^2} dt$$

$u = t + \frac{jw}{2a}$ $\quad du = dt$

* $t \to -\infty$ $u \to -\infty \ldots$
  $\to -\infty + \frac{jw}{2a} \simeq \infty$

* $t \to +\infty$ $u \to \infty \ldots$
  $\to \infty + \frac{jw}{2a} \simeq \infty$

$$X(w) = e^{-\frac{w^2}{4a}} \int_{-\infty}^{\infty} e^{-au} du$$ Intergal gaussianna.

$$\boxed{\int_{-\infty}^{\infty} e^{-cx^2}\,dx = \sqrt{\frac{\pi}{c}}}$$

$c = a$

y la variable es u por ende

$$\int_{-\infty}^{\infty} e^{-au^2}\,du = \sqrt{\frac{\pi}{a}} \rightarrow \boxed{X(w) = e^{\frac{-w^2}{4a}} * \sqrt{\frac{\pi}{a}}}$$

$t) = A\,\mathrm{rect}_d(t),\quad A, d \in \mathbb{R}.$

$$\mathrm{rect}_d(t) = \begin{cases} 1 & \text{si } |t| \leq d/2 \\ 0 & \text{o.c} \end{cases} \qquad x(t) = A \quad \text{si } |t| \leq$$

$$X(w) = \int_{-\infty}^{\infty} A\,\mathrm{rect}_d(t) * e^{-jwt}\,dt$$

$$X(w) = A\int_{-d/2}^{d/2} e^{-jwt}\,dt = A\left[\frac{e^{-jwt}}{-jw}\Bigg|_{-d/2}^{d/2}\right]$$

$$= A\cdot\frac{1}{-jw}\left(e^{-jw(d/2)} - e^{-jw(-d/2)}\right)$$

$$e^{-j\theta} - e^{j\theta} = -2j \,\text{sen}\,\theta$$

$$X(w) = \frac{A}{jw} \left(-2j \,\text{sen}\,(w\,d/2)\right)$$

$$= \frac{A * 2 \,\text{sen}\,(w\,d/2)}{w}$$

$$\text{sinc}(x) = \frac{\sin(x)}{x} \qquad X(w) = A \cdot d \,\frac{\text{sen}\,(w\,d/2)}{(w\,d/2)}.$$

$$\boxed{X(w) = A \cdot d \,\text{sinc}\left(\frac{wd}{2}\right)}$$

Punto # 4.

a.) $F\{e^{-jw_1 t} \cos(w_c t)\}$, $w_1, w_c \in \mathbb{R}$

recordemos.

$$\cos(w_c t) = \frac{1}{2}\left(e^{jw_c t} + e^{-jw_c t}\right)$$

$$e^{-jwt}\cos(w_c t) = e^{-jw_1 t} * \frac{1}{2}\left(e^{jw_c t} + e^{-jw_c t}\right)$$

$$= \frac{1}{2}\left(e^{-j(w_1 - w_c)} + e^{-j(w_1 + w_c)t}\right)$$

Aplicamos TF $\mathcal{F}\left\{e^{-jw_0 t}\right\} = 2\pi\delta(w - w_0)$

$$F\left\{e^{-jw_1 t} * \cos(w_c t)\right\} = \frac{1}{2}\left[\mathcal{F}\left\{e^{-j(w_1 - w_c)t}\right\} + \dots\right.$$

$$\left. \dots \mathcal{F}\left\{e^{-j(w_1 + w_c)t}\right\}\right]$$

$$= \frac{1}{2}\left[2\pi\delta\left(w - (w_1 - w_c)\right) + 2\pi\delta\left(w - (w_1 + w_c)\right)\right]$$

$$\boxed{F\left\{e^{-jw_1 t} * \cos(w_c t)\right\} = \pi\left[\delta\left(w - (w_1 - w_c)\right) + \delta\left(w - (w_1 + w_c)\right)\right]}$$

b.) $\mathcal{F}\left\{u(t)\cos^2(w_c t)\right\}$ $w_c \in \mathbb{R}$ $u(t)$ Función Escalon.

$$u(t)\cos^2(w_c t) = u(t)\left(\frac{1}{2} + \frac{1}{2}\cos(2w_c t)\right)$$

$$= \frac{1}{2}u(t) + \frac{1}{2}u(t)\cos(2w_c t)$$

$$\mathcal{F}\{u(t)\cos^2(w_c t)\} = \frac{1}{2}\mathcal{F}\{u(t)\} + \frac{1}{2}\mathcal{F}\{u(t)\cos(2w_c t)\}$$

$$\boxed{\mathcal{F}\{u(t)\} = \pi\,\delta(w) + \frac{1}{jw}}$$

$u(t)\cos(2w_c t)$ se utiliza la propiedad de modulación

$$\mathcal{F}\{(u(t) * \cos(w_0 t))\} = \frac{1}{2}\left[\mathcal{F}\{u(t)e^{jw_0 t}\} + \mathcal{F}\{u(t)e^{-jw_0 t}\}\right]$$

$$\boxed{\mathcal{F}\{u(t)e^{\pm j w_0 t}\} = \pi\,\delta(w \mp w_0) + \frac{1}{j(w \mp w_0)}}$$

$$\mathcal{F}\{u(t) * \cos^2(w_c t)\} = \frac{1}{2}\left(\pi\delta(w) + \frac{1}{jw}\right) + \frac{1}{4}\left[\pi\delta(w-2w_c) + \ldots\right.$$

$$\left.\ldots + \frac{1}{j(w-2w_c)} + \pi\delta(w+2w_c) + \frac{1}{j(w+2w_c)}\right]$$

$$\boxed{\begin{aligned}\mathcal{F}\{u(t) * \cos^2(w_c t)\} &= \frac{\pi}{2}\delta(w) + \frac{1}{2jw} + \frac{\pi}{4}\left[\delta(w-2w_c) + \ldots\right. \\ &\left.\ldots\ \delta(w+2w_c)\right] + \frac{1}{4j}\left[\frac{1}{w-2w_c} + \frac{1}{w+2w_c}\right]\end{aligned}}$$

c) $\mathcal{F}^{-1}\left\{\dfrac{7}{w^2+6w+45} * \dfrac{10}{\left(8+\frac{jw}{3}\right)^3}\right\}$

$$\mathcal{F}^{-1}\{F(w) * G(w)\} = 2\pi * f(t) * g(t)$$

$$f(t) = \mathcal{F}^{-1}\{F(w)\} \quad , \quad g(t) = \mathcal{F}^{-1}\{G(w)\}$$

$$F(w) = \frac{7}{w^2+6w+45} \qquad w^2+6w+45 = (w^2+6w+9)+36$$

$$\boxed{= (w+3)^2 + 6^2}$$

$$F(w) = \frac{7}{(w+3)^2+6^2} \qquad \mathcal{F}\{e^{-a|t|}\} = \frac{2a}{a^2+w^2}$$

$$H(w) = \frac{7}{w^2+6^2} \quad \rightarrow \quad \boxed{a = 6}$$

$$\mathcal{F}^{-1}\left\{\frac{2(6)}{6^2+w^2}\right\} = e^{-6|t|}$$

Ajustando las constantes.

$$H(w) = \frac{7}{12} * \frac{12}{w^2} \qquad \boxed{h(t) = \mathcal{F}^{-1}\{H(w)\} = \frac{7}{12}e^{-6|t|}}$$

Luego aplicamos la propiedad de desplazamiento de la frecuencia

$$H(w_0+3) \rightarrow w_0+3=0 \qquad f(t)=\mathcal{F}^{-1}\{H(w+3)\}$$

$$w_0=-3 \qquad = e^{-j3t}\,h(t)$$

$$\boxed{f(t)=\frac{7}{12}e^{-j3t}e^{-6|t|}}$$

* Calculo de la segunda función $g(t)$

$$G(w)=\frac{10}{(8+jw/3)^2} \Rightarrow \mathcal{F}\{t e^{-at}u(t)\}=\frac{1}{(a+jw)^2}$$

$$G(w)=\frac{10}{\left(\frac{1}{3}(24+jw)\right)^2}=\frac{10}{\frac{1}{9}(24+jw)^2}=\boxed{\frac{90}{(24+jw)^2}}$$

$$g(t)=\mathcal{F}^{-1}\left\{\frac{90}{(24+jw)^2}\right\}=90*\mathcal{F}^{-1}\left\{\frac{1}{(24+jw)^2}\right\}$$

$$\boxed{g(t)=90\,t\,e^{-24t}\,u(t)}$$

convolución: $Y(t) = 2\pi * F(t) * g(t)$.

$$Y(t) = 2\pi\left(\frac{7}{12}\, e^{-j3t}\, e^{-6|t|}\right) * \left(90t\, e^{-24t}\, u(t)\right)$$

$$Y(t) = 105\pi\, e^{-j3t}\, e^{-6|t|}\, t\, e^{-24t}\, u(t) \Rightarrow$$ $u(t)$ 0 para $t<0$
$|t|$ t para $t \geq 0$

$$e^{-6|t|} * e^{-24t} = e^{-6t} * e^{-24t} = e^{-30t} \quad (t \geq 0)$$

$$\boxed{Y(t) = 105\pi * t * e^{-(30+j3)t}\, u(t)}$$

d.) $F\{3t^3\}$ Propiedad Linealidad.

$$F\{3t^3\} = 3\, F\{t^3\}$$

$$F\{t^n x(t)\} = j^n \frac{d^n}{dw^n} X(w)$$ $n = 3$
$x(t) = 1$

$$X(w) = F\{x(t) = 1\}$$

$$\boxed{F\{1\} = 2\pi\, \delta(w)}$$

$$F\{t^3\} = j^3 \frac{d^3}{dw^3}[2\pi\delta(w)] \rightarrow j^3 = j^2 \cdot j$$

$$(-1)j = -j$$

$$F\{t^3\} = -j\, 2\pi \frac{d^3}{dw^3}\delta(w)$$

$$\boxed{\frac{d^3}{dw^3}\delta(w) = \delta^{(3)}(w)}$$

$$F\{3t^3\} = 3 * (-j 2\pi\, \delta^{(3)}(w))$$

$$\boxed{F\{3t^3\} = -j 6 2\pi \delta^{(3)}(w)}$$

e) $\frac{B}{T}\sum_{n=-\infty}^{\infty}\left(\frac{1}{a^2+(w-nw_0)^2} + \frac{1}{a+j(w-nw_0)}\right)$

$n \in \{0, \pm 1, \pm 2, \ldots\}$

$w_0 = \frac{2\pi}{T}$ y $B, T \in \mathbb{R}^+$

$$X(w) = c\sum_{n-\infty}^{\infty} F(w-nw_0) \quad \text{si } x(t) = f(t)\sum_{k=-\infty}^{\infty}\delta(t-kT)$$

$$X(w) = \frac{1}{T}\sum_{\infty}^{+\infty} F(w-nw_0)$$

$$F(w) = \frac{1}{a^2+w^2} + \frac{1}{a+j(w)} \quad \Big| \quad F(t) = \mathcal{F}^{-1}\{F(w)\}$$

$$\mathcal{F}^{-1}\left\{\frac{1}{a^2+w^2}\right\} + \mathcal{F}^{-1}\left\{\frac{1}{a+jw}\right\}$$

Propiedad para par de trasformadas.

$$\mathcal{F}^{-1}\{e^{-a|t|}\} = \frac{2a}{a^2+w^2} \rightarrow \mathcal{F}^{-1}\left\{\frac{1}{a^2+w^2}\right\} = \frac{1}{2a} e^{-a|t|}$$

$$\mathcal{F}^{-1}\{e^{-a|t|} u(t)\} = \frac{1}{a+jw} \rightarrow \mathcal{F}^{-1}\left\{\frac{1}{a+jw}\right\} = e^{-at} u(t)$$

f(t) es: $\boxed{f(t)\ \frac{1}{2a} e^{-a|t|} + e^{-at} u(t).}$

$$x(t) = B * f(t) \left(\sum_{n=-\infty}^{+\infty} \delta(t-KT)\right)$$

$$x(t) = B\left(\frac{1}{2a} e^{-a|t|} + e^{-at} u(t)\right) \sum_{n-\infty}^{+\infty} \delta(t-KT)$$

$$x(t) = B \sum_{n-\infty}^{\infty} f(KT) - \delta(t-KT) \quad \Big| \quad f(KT) = \frac{1}{2a} e^{-a|KT|} + e^{-aKT} u(KT)$$

$$\boxed{x(t) = B \sum_{n=-\infty}^{\infty} \left(\frac{1}{2a} e^{-a|KT|} + e^{-aKT} u(KT)\right) \delta(t-KT)}$$