KONTSEVICH'S CHARACTERISTIC CLASSES

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Contents

Introduction

The goal of this note is to construct characteristic classes for framed sphere bundles. We follow the overview of these classes in [3] and [4]. The main results of [3] and [4] show that many of these characteristic classes are nontrivial. Watanabe constructs framed odd-dimensional sphere bundles over spheres for which Kontsevich's characteristic classes from graphs with vertices of valence 3 do not vanish.

The characteristic classes are defined roughly as follows, with terms to be defined later. Let $\pi\colon E \to B$ be a framed sphere bundle, say with fiber M a homology d-sphere, with d odd. We will define a characteristic class for π given the data of an oriented graph Γ with n vertices and m edges whose vertices have valence at least 3. We want to define a closed form in $\Omega^*_{\mathrm{dR}}(B)$. Consider the bundle $\pi_n\colon EC_n(\pi) \to B$ associated to π whose fiber is the configuration space of n points in M, $C_n(M)$. We would like to say that integration along the fibers gives us a map from the deRham complex of $EC_n(\pi)$ to the deRham complex of B, but the fibers of π_2 are not compact. To remedy this, we instead consider the bundle $E\overline{C}_n(\pi) \to B$ with fiber the Fulton-MacPherson-Kontsevich compactification of the configuration space. Each edge e of Γ determines a map $\phi_e\colon E\overline{C}_n(\pi) \to E\overline{C}_2(\pi)$ which, on the fibers, picks out the two points of the configuration corresponding to the labels of the vertices of the edge e. Wedging together all of the maps, we obtain a map $\Omega^*_{\mathrm{dR}}(E\overline{C}_n(\pi)) \to \Omega^*_{\mathrm{dR}}(E\overline{C}_2(\pi))$. In §4, we will define a form ω on $E\overline{C}_n(\pi)$ called a "propagator." The deRham form on B of π corresponding to Γ is then the image of ω under the composite

$$\Omega^{d-1}_{\mathrm{dR}}(E\overline{C}_2(\pi)) \xrightarrow{\bigwedge_e \phi_e^*} \Omega^{m(d-1)}_{\mathrm{dR}}(E\overline{C}_n(\pi)) \xrightarrow{\int_{\mathrm{fiber}}} \Omega^{m(d-1)-nd}_{\mathrm{dR}}(B).$$

In §1 we describe the classifying space for such sphere bundles; i.e., the space whose cohomology contains the classes we will construct. We will construct characteristic classes for each oriented graph of valence at least 3. Such graphs fit together into a chain complex which we describe in §2. In §3, we describe a compactification of configuration spaces originally defined by Fulton-MacPherson and Kontsevich [1], [2]. In §4 we define propagators on families of configuration spaces. Finally, in

§5, we define the characteristic classes and prove Kontsevich's theorem (Thm. 5.1) showing that the classes are closed, do not depend on the choice of propagator, and are natural with respect to bundle maps of framed bundles.

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1. Classifying Space

We begin by giving precise definitions of the type of bundles we want to study. Characteristic classes of such bundles should live in the cohomology of the classifying space of such bundles. We end this section by giving a model for such a classifying space.

Definition 1. A *homology d-sphere* is a smooth closed d-dimensional manifold M so that

$$H_i(M;\mathbb{R}) \cong H_i(S^d;\mathbb{R})$$

for all i.

Let M be a homology d-sphere. Fix a point $\infty \in M$. Then $M \setminus \infty$ is acyclic and hence parallelizable. Fix $U_\infty \subset M$ a small disk around $\infty \in M$ and also fix a diffeomorphism $h_\infty\colon U_\infty \to V_\infty \subset \mathbb{R}^d \cup \{\infty\}$ of balls centered at ∞, sending ∞ to ∞.

Definition 2. Say a trivialization $\kappa\colon T(M\setminus\infty) \xrightarrow{\cong} \mathbb{R}^d \times (M\setminus\infty)$ is *standard near* ∞ if over $U_\infty\setminus\infty$, the trivialization κ is the pullback along h_∞ of the standard framing ι on $T\mathbb{R}^d$. Equivalently, if the diagram

$$\begin{tikzcd}
T(M\setminus\infty)|_{U_\infty\setminus\infty} \arrow[r, "dh_\infty"] \arrow[d, "\kappa"] & (T\mathbb{R}^d)|_{V_\infty\setminus\infty} \arrow[d, "\iota"] \\
\mathbb{R}^d \times (U_\infty\setminus\infty) \arrow[r, "\mathrm{Id}\times h_\infty"'] & \mathbb{R}^d \times (V_\infty\setminus\infty)
\end{tikzcd}$$

commutes.

Fix such a trivialization κ that is standard near ∞. Let $\mathrm{Diff}(M,U_\infty)$ denote the group of diffeomorphisms of M that are the identity on U_∞.

Definition 3. An (M,U_∞)-bundle on a smooth manifold B is a smooth fiber bundle $\pi\colon E \to B$ with fiber M and structure group $\mathrm{Diff}(M,U_\infty)$.

Remark. Let $D_M = M\setminus U_\infty$. Then $\mathrm{Diff}(M,U_\infty) \cong \mathrm{Diff}(D_M,\partial D_M)$. We will alternate between the two descriptions. Note that Watanabe uses (M,U_∞) in [3] and (D_M,∂) in [4].

A (M,U_∞)-bundle $\pi\colon E\to B$ has a section at infinity, $s\colon B\to E$. Let $\overline{U}_\infty$ denote a tubular neighborhood of $s(B)$ in E. Note that Watanabe continues to denote this tubular neighborhood by U_∞.

Recall that if B is a manifold, the vertical tangent bundle $T^{\mathrm{fib}}(E)$ of a bundle $\pi\colon E\to B$ is the kernel of $d\pi\colon TE\to TB$.

Definition 4. Say a trivialization τ of $(T^{\mathrm{fib}}E)|_{U_\infty\setminus s(B)}$ is *induced from* κ if the following diagram commutes

$$\begin{tikzcd}
(T^{\mathrm{fib}}E)|_{\overline{U}_\infty\setminus s(B)} \arrow[r, "\tau", "\cong"'] \arrow[dr, "\pi"'] & \big(\overline{U}_\infty\setminus s(B)\big)\times\mathbb{R}^d\times B \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & (M\setminus\infty)\times\mathbb{R}^d\times B \arrow[d, "\kappa\times 1", "\cong"'] \\
& \overline{U}_\infty\setminus s(B) \arrow[r, "\cong"'] & (U_\infty\setminus\infty)\times B \arrow[r, "0\text{-section}"'] & T(M\setminus\infty)\times B
\end{tikzcd}$$

Definition 5. A *vertical framing* of a (M, U_∞)-bundle $\pi\colon E \to B$ is a trivialization τ of $(T^{\mathrm{fib}}E)|_{E\setminus s(B)}$ of the vertical tangent bundle outside the image of s, such that τ is induced by κ on $\overline{U}_\infty \setminus s(B)$.

Let $\mathsf{Fr}(M)$ be the space of framings on $T(M \setminus \infty)$ that agree with κ near ∞.

Proposition 1.1. *Let B be a manifold, and M, U_∞ as above. Homotopy classes of maps*

$$B \to E\mathsf{Diff}(M, U_\infty) \times_{\mathrm{Diff}(M,U_\infty)} \mathsf{Fr}(M)$$

are in bijection with equivalence classes of vertically framed (M, U_∞)-bundles.

In other words, $\widetilde{B\mathsf{Diff}}(M, U_\infty) := E\mathsf{Diff}(M, U_\infty) \times_{\mathrm{Diff}(M,U_\infty)} \mathsf{Fr}(M)$ is a model for the classifying space of vertically framed (M, U_∞)-bundles. We will prove Proposition as a special case of the following lemma.

Lemma 1.2. *Let G be a topological group and X a G-space. For a manifold B, homotopy classes of maps $[B, EG \times_G X]$ are in bijection with concordance classes of triples (P, π, f) where $\pi\colon P \to B$ is a principle G-bundle and $f\colon P \to X$ is a G-equivariant map.*

Proof. Given such a triple (P, π, f), consider the classifying map $\psi\colon B \to BG$ for π,

$$\begin{tikzcd} P \arrow[r, "\hat\psi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & EG \arrow[d] \\ B \arrow[r, "\psi"] & BG \end{tikzcd}$$

define a map $\sigma\colon B \to EG \times_G X$ by

$$\sigma(\pi(x)) = \hat\psi(x) \times_G f(x) \tag{1}$$

This is well-defined since π is surjective and $\hat\psi$ and f are G-equivariant. Conversely, given a map $\sigma\colon B \to EG \times_G X$, define $\pi\colon P \to B$ to be the pullback of $EG \to BG$ along $\psi := \pi_X \circ \sigma$ where $\pi_X\colon EG \times_G X \to BG$ collapses X. Define f again by the formula 1. $\square$

Proof of Proposition 1. Take $G = \mathrm{Diff}(M, U_\infty)$ and $X = \mathsf{Fr}(M)$. By Lemma 1.2, homotopy classes of maps

$$B \to E\mathrm{Diff}(M, U_\infty) \times_{\mathrm{Diff}(M,U_\infty)} \mathsf{Fr}(M)$$

are in bijective correspondence with concordance classes of triples (P, π, f) where $\pi\colon P \to B$ is a principle $\mathrm{Diff}(M, U_\infty)$-bundle and $f\colon P \to \mathsf{Fr}(M)$ is a $\mathrm{Diff}(M, U_\infty)$-equivariant map. Associated to $\pi\colon P \to B$, we have a (M, U_∞)-bundle $E \to B$. The map $f\colon P \to \mathsf{Fr}(M)$ corresponds to a vertical framing on E by taking the composition

$$T(M \setminus \infty) \times_G P \xrightarrow{1 \times_G f} T(M \setminus \infty) \times_G \mathsf{Fr}(M) \xrightarrow{\mathrm{ev}} \mathbb{R}^d \times (M \setminus \infty)$$

where $\mathrm{ev}((m, v) \times_G \tau) = \tau(m, v)$ is the evaluation map. $\square$

More generally, given smooth compact manifolds $X \subset W$ and a trivialization τ of TW near X, the space

$$\widetilde{B\mathsf{Diff}}(W, X; \tau) := E\mathsf{Diff}(W, X) \times_{\mathsf{Diff}(W,X)} \mathsf{Fr}(W, X; \tau)$$

is a model for the classifying space of framed (W, X)-bundles. Here $\mathsf{Fr}(W, X; \tau)$ is the space of framings on W that agree with $\tau|_X$ on X.

2. Graph Complex

Later (Theorem 5.1), we will see that the cycle condition for an element in the graph complex corresponds to the cocycle condition in the deRham complex for the associated characteristic class.

All graphs are assumed to be finite, connected, and of valence at least 3. Let Γ be such a graph and denote by $V(\Gamma)$ the set of vertices of Γ and $E(\Gamma)$ the set of edges. For each edge $e \in E(\Gamma)$ let $H(e) = \{e^+, e^-\}$ be the set of half-edges of e. Choose a labeling of the vertices $v_1, \ldots, v_n$ of Γ and of the edges $e_1, \ldots, e_m$.

Definition 6. An *orientation* of Γ is an orientation of the vector space

$$\mathbb{R}^{V(\Gamma)} \oplus \bigoplus_{e \in E(\Gamma)} \mathbb{R}^{H(e)}$$

An orientation of Γ is therefore a choice of ordering on the vertices $V(\Gamma)$ together with a choice of order of e^+ and e^- for each $e \in E(\Gamma)$.

Definition 7. The *graph complex* is the chain complex $(\mathcal{G}_{n,m}, d)$ where

• $\mathcal{G}_{n,m}$ is the $\mathbb{Q}$ vector space generated by oriented labeled graphs $(\Gamma, \mathfrak{o})$ with n vertices and m edges, under the equivalence relation $(\Gamma, -\mathfrak{o}) \sim -(\Gamma, \mathfrak{o})$ where $-(\Gamma, \mathfrak{o})$ is obtained from $(\Gamma, -\mathfrak{o})$ by reversing one of the edge orientations.

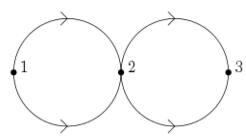
• the differential $d\colon \mathcal{G}_{n,m} \to \mathcal{G}_{n-1,m-1}$ is given by the formula

$$d(\Gamma, \mathfrak{o}) := \sum_{e \in E'(\Gamma)} (\Gamma/e, \mathfrak{o}_e)$$

where Γ/e is the graph obtained from Γ by collapsing the edge e and $\mathfrak{o}_e$ is the induced orientationm (see §2.2 of [4] for details) and the sum is taken over those edges $e \in E'(\Gamma)$ between distinct vertices; i.e., not loops.

Example. Any graph Γ with a simple loop is identified with 0 in $\mathcal{G}_{n,m}$.

Example. The graph



has $d(\Gamma) = 0$ since Γ/e has a simple loop for every edge e of Γ.

3. Compactifications of Configurations Spaces

Recall that the ordered configuration space $C_n(M)$ of n points in M is a the complement of the fat diagonal in M^n,

$$C_n(M) = \{(x_1, \ldots, x_n) \in M^n : x_i \neq x_j, i \neq j\}$$

From the inclusion $M \setminus \infty \hookrightarrow M$ we have maps

$$C_n(M, \infty) := C_n(M \setminus \infty) \to C_n(M) \subset M^n$$

We can identify $C_n(M, \infty)$ as the complement of a subspace Σ in M^n where

$$\Sigma = \{(x_1, \ldots, x_n) \in M^n : x_i = x_j \text{ for some } i \neq j \text{ or } x_i = \infty \text{ for some } i\}$$

There is a natural filtration on Σ by how far a point $(x_1, \ldots, x_n) \in \Sigma$ is from being a configuration. Explicitly, we have a filtration

$$\Sigma = \Sigma_n \supset \Sigma_{n-1} \supset \cdots \supset \Sigma_1 \supset \Sigma_0$$

with

$$\Sigma_j = \left\{ (x_1, \dots, x_n) \in M^n : \left| \bigcup_{i=1}^{n} \{x_i\} \cup \{\infty\} \right| \leq j+1 \right\}$$

Example. The subspace Σ_0 is a single point $\{(\infty, \dots, \infty)\}$.

Example. The subspace Σ_1 is the union over $A \subset \{1, \dots, n\}$ of the subspaces

$$\{(x_1, \dots, x_n) \in M^n : x_i = x_j \text{ if } i, j \in A \text{ and } x_k = \infty \text{ if } k \notin A\}$$

The Fulton-MacPherson-Kontsevich compactification $\overline{C}_n(M, \infty)$ of $C_n(M, \infty)$ is obtained by taking successive blow-ups of M^n along this filtration. Recall that a blow-up replaces a submanifold with the total space of its normal sphere bundle, viewed as the boundary of a tubular neighborhood. We denote the blow-up of a manifold A along a submanifold B by $Bl(A, B)$.

Example. The compactification $\overline{C}_2(M, \infty)$ is obtained by blowing up M^2 at (∞, ∞) and then blowing up the result along the image $\overline{\Sigma_M}$ of the subset

$$\Sigma_M = M \times \{\infty\} \cup \{\infty\} \times M \cup \Delta_M$$

in $Bl(M^2, \{(\infty, \infty)\})$ where Δ is the diagonal. Since $Bl(X, Y) \simeq X \setminus Y$, we have

$$\begin{aligned}
\overline{C}_2(M, \infty) &= Bl(Bl(M^2, \{(\infty, \infty)\}), \overline{\Sigma}_M) \\
&\simeq Bl(M^2 \setminus \{(\infty, \infty)\}, \overline{\Sigma}_M) \\
&\simeq M^2 \setminus \{(\infty, \infty)\} \setminus \Sigma_M \\
&= C_2(M, \infty)
\end{aligned}$$

The following will be used in §4.

Lemma 3.1. *The space $C_2(M, \infty)$ has the homology of S^{d-1}. Hence $\overline{C}_2(M, \infty)$ has the homology of S^{d-1}.*

Note that $C_2(M, \infty)$ is not a homology $(d-1)$-sphere since it is not compact and $\overline{C}_2(M, \infty)$ is not a homology $(d-1)$-sphere since it has nonempty boundary.

Proof Sketch. The projection map $C_2(M, \infty) \to M \setminus \infty$ sending $(x_1, x_2) \to x_2$ is a fibration with fiber $M \setminus \{\infty, p\}$ over $p \in M \setminus \infty$. One may show that the fundamental group $\pi_1(M \setminus \infty)$ acts trivially on $H_*(M \setminus \{\infty, p\})$ and that the Serre spectral sequence collapses at the E^2-term. Thus

$$H_*(C_2(M, \infty)) \cong H_*(M \setminus \{\infty, p\}) \otimes H_*(M \setminus \infty) \cong H_*(S^{d-1})$$

□

3.0.1. *Stratification of the Boundary.* In the proof of Theorem 5.1 below, we will need an explicit description of a stratification of $\partial\overline{C}_n(M, \infty)$. The stratification is given by assigning to each point x in $\partial\overline{C}_n(M, \infty)$ a bracketing of $\{1, \dots, n\}$ with i brackets where numbers l, k are bracketed together if x (viewed as a point in the normal sphere bundle over Σ_i) lives over a point $(x_1, \dots, x_n) \in \Sigma_i$ with $x_l = x_k$.

Example. The stratification on $\partial\overline{C}_2(M, \infty)$ is into faces $S_{\infty\times\infty}$, $S_{\infty\times M}$, $S_{M\times\infty}$, and S_{Δ_M} corresponding to the brackets (12∞), $(1\infty)2$, $(2\infty)1$, and $(12)\infty$, respectively.

The codimension one strata of this stratification come from points in the boundary living in the normal sphere bundle of Σ_1. These correspond to bracketings with a single bracket. Codimension one strata of $\partial\overline{C}_n(M, \infty)$ are in bijection with nonempty subsets A of $\{1, \dots, n, \infty\} =: [n]^+$. For example, if $A \subset [n]^+$ does not contain ∞, then A corresponds to the blow-up along the closure of the subset

$$\Delta_A := \left\{(x_1, \dots, x_n) \in (M \setminus \infty)^{\times n} : x_k = x_j \text{ if } k, j \in A, \text{ otherwise distinct}\right\}.$$

If $Z \subset [n]^+$ does contain ∞, then Z corresponds to the the blow-up along the closure of the subset

$$\Delta_Z^\infty := \left\{(x_1, \dots, x_n) \in M^{\times n} : x_j = \infty \text{ if } k \in Z, \text{ otherwise distinct}\right\}.$$

For $A \subset [n]^+$, let $\mathcal{S}_A \subset \partial\overline{C}_n(M, \infty)$ denote the face corresponding to A. Let $\overline{C}^{\text{local}}_{|A|}(\mathbb{R}^d)$ denote the fiber of $\mathcal{S}_A$ over a point of Δ_A.

Lemma 3.2. *The space $\overline{C}^{\text{local}}_{|A|}(\mathbb{R}^d)$ is the Fulton-MacPherson-Kontsevich compactification of the configuration space*

$$C^{\text{local}}_{|A|}(\mathbb{R}^d) := C_{|A|}(\mathbb{R}^d)/(\textit{translation, dilation})$$

Moreover, $\overline{C}^{\text{local}}_{|A|}(\mathbb{R}^d)$ is homotopy equivalent $C_{|A|}(R^d)$. In particular, if $|A| = 2$, the Gauss map defines a homotopy equivalence $p\colon \overline{C}^{\text{local}}_{|A|}(\mathbb{R}^d) \to S^k$ where $k = (|A|-1)d - 1$.

4. Propagators

Given a (D_M, ∂)-bundle $\pi\colon E \to B$, let $\overline{C}_n(\pi)\colon E\overline{C}_n(\pi) \to B$ denote the associated $\mathrm{Diff}(D_M, \partial)$-bundle with fiber $\overline{C_n}(M \setminus \infty)$. Explicitly, if $P \to B$ is the principal $\mathrm{Diff}(M, U_\infty)$-bundle associated to π, then

$$E\overline{C}_n(\pi) := P \times_{\mathrm{Diff}(M, U_\infty)} \overline{C}_n(\pi)$$

Similarly, for $Y(M, \infty)$ a $\mathrm{Diff}(M, U_\infty)$-space, let $EY(\pi)$ denote the $Y(M, \infty)$-bundle associated to π. We would like to construct a map

$$\partial^{\text{fib}} E\overline{C}_2(\pi) \to S^{d-1}$$

Given a vertical framing τ_E of π, we can construct such a map (which we will denote $p(\tau_E)$) by constructing a map $p(\kappa)\colon \partial\overline{C}_2(M, \infty) \to S^{d-1}$ on the fibers and gluing using the trivialization. We define the map $p(\kappa)$ componentwise using the stratification of $\partial\overline{C}_2(M, \infty)$ into the blow-up faces $S_{\infty\times\infty}$, $S_{\infty\times M}$, $S_{M\times\infty}$, and S_{Δ_M}. Details of this construction can be found in §2.2 of [3] where τ_M is used for κ.

On $S_{\infty\times\infty}$, define $p(\kappa)$ by the Gauss map $(x_1, x_2) \mapsto \frac{x_2 - x_1}{\|x_2 - x_1\|}$. Explicitly, a point x in $S_{\infty\times\infty}$ lies on the sphere bundle of the normal bundle of $(\infty, \infty) \in M^2$. Since the blow-up only changes things near (∞, ∞), we can view x as a point in $\partial Bl(U_\infty \times U_\infty, (\infty, \infty))$. Under the diffeomorphism

$$h_\infty \times h_\infty \colon U_\infty \times U_\infty \to V_\infty \times V_\infty$$

we can view x as a point in $\partial Bl((\mathbb{R}^d \sqcup \{\infty\})^{\times 2}, (\infty, \infty))$. Now we have an identification

$$\partial Bl((\mathbb{R}^d \sqcup \{\infty\})^{\times 2}, (\infty, \infty)) \cong ((\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(0,0)\})/\mathbb{R}_{>0}$$

where $\mathbb{R}_{>0}$ acts diagonally. Thus, we can view $x \in S_{\infty\times\infty}$ as the class of a point $(x_1, x_2) \neq (0,0)$ in $\mathbb{R}^d \times \mathbb{R}^d$. Under this identification, we define $p(\kappa)\colon S_{\infty\times\infty} \to S^{d-1}$ by

$$p(\kappa)(x) = \frac{x_2 - x_1}{\| x_2 - x_1 \|}$$

On $S_{\infty\times M}$, we define $p(\kappa)$ to be the composite

$$S_{\infty\times M} \cong \partial Bl(M, \infty) \times Bl(M, \infty) \xrightarrow{\text{pr}_1} \partial Bl(M, \infty) \xrightarrow{h_\infty} \partial Bl(S^d, \infty) \cong S^{d-1}$$

The map $p(\kappa)$ is defined similarly on $S_{M\times\infty}$.

Lastly, we define $p(\kappa)$ on S_{Δ_m} by the composite

$$S_{\Delta_M} \cong \partial Bl(\mathbb{R}^d, \{0\}) \times Bl(M, \infty) \xrightarrow{\mathrm{pr}_1} \partial Bl(\mathbb{R}^d, \{0\}) \cong S^{d-1}$$

Definition 8. The *volume form* $\mathrm{vol}_{S^{d-1}} \in \Omega^{d-1}_{\mathrm{dR}}(S^{d-1})$ on S^{d-1} is the $SO(d)$-invariant closed $(d-1)$-form on $S^{d-1} \subset \mathbb{R}^d$ defined by

$$\mathrm{vol}_{S^{d-1}} := \frac{1}{\mathrm{vol}(S^{d-1})} \sum_{i=1}^{d} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_d$$

Definition 9. A *propagator* is a closed form $\omega \in \Omega^{d-1}_{\mathrm{dR}}(E\overline{C}_2(\pi))$ such that

$$\omega|_{\partial^{\mathrm{fib}} E\overline{C}_2(\pi)} = p(\tau_E)^* \mathrm{vol}_{S^{d-1}}$$

The following lemma says that propagators exist in the context we care about.

Lemma 4.1. *A propagator $\omega \in \Omega^{d-1}_{\mathrm{dR}}(E\overline{C}_2(\pi))$ exists for every vertically framed (D_m, ∂)-bundle $\pi\colon E \to B$. Such a propagator is unique up to homotopy.*

Proof. It suffices to show that the inclusion induced map

$$H^{d-1}(E\overline{C}_2(\pi)) \to H^{d-1}(\partial^{\mathrm{fib}} E\overline{C}_2(\pi)) \tag{2}$$

is an isomorphism. We have a relative fibration

$$(\overline{C}_2(M, \infty), \partial\overline{C}_2(M, \infty)) \to (E\overline{C}_2(\pi), \partial^{\mathrm{fib}}(E\overline{C}_2(\pi)) \to B$$

By Poincaré-Lefshetz duality and Lemma 3.1, we have

$$H^q(\overline{C}_2(M, \infty), \partial\overline{C}_2(M, \infty)) \cong H_{2d-q}(\overline{C}_2(M, \infty)) \cong H_{2d-p}(S^{d-1})$$

which vanishes for $0 \le q \le d$. Thus the E_2-terms $E_2^{p,q}$ in the Serre spectral sequence for the above relative fibration vanish for $0 \le p + q \le d$. Hence

$$H^n(E\overline{C}_2(\pi), \partial^{\mathrm{fib}} E\overline{C}_2(\pi)) = 0$$

for $n = d-1, d$. Using the long exact sequence for the pair $(E\overline{C}_2(\pi), \partial^{\mathrm{fib}}(E\overline{C}_2(\pi))$, we see that the map (2) is an isomorphism.

For the uniqueness statement, see [3, Lemma 3]. □

5. Kontsevich's Characteristic Classes

We give a definition of Kontsevich's characteristic classes for framed sphere bundles as in §1. More explicitly, let M be an homology d-sphere with d odd and $\pi\colon E \to B$ be a (D_M, ∂)-bundle. We will define classes $I(\Gamma) \in \Omega^*_{\mathrm{dR}}(B)$ for each oriented graph Γ.

The definition we will give uses a fixed propagator ω; however, we will see (Theorem 5.1) that the resulting class $I(\Gamma)$ does not depend on the choice of ω. Consider the associated bundle $\overline{C}_n(\pi)$ with fiber $\overline{C}_n(M \setminus \infty)$. The propagator ω is a deRham form on $E\overline{C}_2(\pi)$, not on $E\overline{C}_n(\pi)$. We therefore need way to get from $E\overline{C}_n(\pi)$ to $E\overline{C}_2(\pi)$. Such a map will be defined as a projection in terms of Γ. For each edge e of an oriented graph $(\Gamma, \mathfrak{o}) \in \mathcal{G}_{n,m}$, choose an orientation of $\mathbb{R}^{H(e)}$ so that the vector space

$$\mathbb{R}^{V(\Gamma)} \oplus \bigoplus_{e \in E(\Gamma)} \mathbb{R}^{H(e)}$$

with this orientation on $\mathbb{R}^{H(e)}$ and the total ordering ρ of the vertices, is $\mathfrak{o}$.

Definition 10. For an edge e of Γ, define $\phi_e\colon E\overline{C}_n(\pi) \to E\overline{C}_2(\pi)$ to be the projection

$$(b, (x_1, \ldots, x_n)) \mapsto (b, (x_k, x_j))$$

where e is the edge between vertices v_k and v_j using the labels from ρ.

For each oriented graph Γ with n vertices and m edges, we obtain an $m(d-1)-nd$ form on B by looking at the image of ω under the following composite

$$\Omega^*_{\mathrm{dR}}(E\overline{C}_2(\pi)) \xrightarrow{\bigwedge_{e\in E(\Gamma)} \phi_e^*} \Omega^*_{\mathrm{dR}}(E\overline{C}_n(\pi)) \xrightarrow{\int_{\text{fiber}}} \Omega^*_{\mathrm{dR}}(B) \longrightarrow H^*B$$

Definition 11. Let $\omega(\Gamma) = \bigwedge_{e\in E(\Gamma)} \phi_e^*(\omega)$ and $I(\Gamma) = \int_{\text{fiber}} \omega(\Gamma)$.

Theorem 5.1 (Kontsevich)**.** *Let Γ be a cycle in $\mathcal{G}_{n,m}$.*

(i) *I is a chain map: $dI(\Gamma) = (-1)^{|I(d\Gamma)|} I(d\Gamma)$. In particular, $dI(\Gamma) = 0$.*
(ii) *The class $[I(\Gamma)] \in H^*(B;\mathbb{R})$ is independent of the choice of propagator ω.*
(iii) *The class $[I(\Gamma)]$ is natural with respect to bundle maps of framed bundles.*
(iv) *Evaluation of $I(\Gamma)$ on base spaces gives a homomorphism*

$$\Omega_{m(d-1)-nd}(\widetilde{B\mathrm{Diff}}(D_M,\partial)) \to \mathbb{R}$$

where $\Omega_*(-)$ is the oriented bordism group functor.

Remark. The class $I(\Gamma)$ may depend on the choice of vertical framing τ_E of the (D_M,∂)-bundle $\pi\colon E \to B$. See, for example, [3, Rmk 2].

Proof of (iii) and (iv). This follows from the facts that integration along the fibers is natural with respect to bundle maps and that propagators pullback to propagators along bundle maps of framed bundles. Part (iv) is obvious. □

The proofs of parts (i) and (ii) of Theorem 5.1 will use the following. Recall from §3.0.1, that codimension one strata of $\partial\overline{C}_n(M,\infty)$ are in bijection with subsets $A \subseteq [n]^+ := \{1,\dots,n,\infty\}$ of size at least 2. Let $[n]$ denote the subset $\{1,\dots,n\}$ of $[n]^+$.

Definition 12. For $\Gamma \in \mathcal{G}_{n,m}$ and $A \subseteq [n]$ with $|A| \geq 2$, let Γ_A be the subgraph of Γ consisting of vertices v_a for $a \in A$ and all of the edges connecting them. Let Γ/A be the quotient graph $\Gamma/(\Gamma_A)$ where all vertices v_a, $a \in A$ are collapsed to a single vertex $[*]$ and all edges $e \in \Gamma_A$ are collapsed to $[*]$.

Example. Consider the Θ graph Γ with two vertices v_1 and v_2 and three edges connecting them. Let $A = \{1,2\}$. Then $\Gamma_A = \Gamma$ and Γ/A is a point.

For $A \subset [n]^+$ let A^c denote the complement of A in $[n]^+$. Note that if $\infty \in A$ then $A^c \subset [n]$. Let $\pi_A : E\mathcal{S}_A(\pi) \to B$ be the $\mathcal{S}_A$-bundle corresponding to π. The framing τ_E on π determines a natural diffeomorphism

$$f_A\colon E\mathcal{S}_A(\pi) \to \overline{C}^{\text{local}}_{|A|}(\mathbb{R}^d) \times E\overline{C}_{n-|A|+1}(\pi)$$

Let $\Gamma \in \mathcal{G}_{n,m}$ and $\omega \in \Omega^{d-1}_{\mathrm{dR}}(E\overline{C}_2(\pi))$ a propagator. If $A \subset [n]$, we have

$$\omega(\Gamma)|_{E\mathcal{S}_A(\pi)} = f_A^*\big(\mathrm{pr}_1^*\hat{\omega}(\Gamma_A) \wedge \mathrm{pr}_2^*\omega(\Gamma/A)\big) \tag{3}$$

If $A \subset [n]^+$ and $\infty \in A$, then we have

$$\omega(\Gamma)|_{E\mathcal{S}_A(\pi)} = f_A^*\big(\mathrm{pr}_1^*\hat{\omega}(\Gamma/A^c) \wedge \mathrm{pr}_2^*\omega(\Gamma_{A^c})\big) \tag{4}$$

Here $\omega(\Gamma/A) \in \Omega^*_{\mathrm{dR}}(E\overline{C}_{n-|A|+1}(\pi))$ is defined by Definition 11 and $\hat{\omega}(\Gamma_A) \in \Omega^*_{\mathrm{dR}}(\overline{C}^{\text{local}}_{|A|}(\mathbb{R}^d))$ is defined as follows: Let $p\colon \overline{C}^{\text{local}}_2(\mathbb{R}^d) \to S^{d-1}$ be the Gauss map. Write $\hat{\omega} = p^*(\mathrm{vol}_{S^{d-1}})$ for the pullback of the volume form.

Definition 13. Define $\hat{\omega}(\Gamma_A) \in \Omega^*_{\mathrm{dR}}(\overline{C}^{\mathrm{local}}_{|A|}(\mathbb{R}^d))$ to be the wedge of forms

$$\hat{\omega}(\Gamma_A) = \bigwedge_{e \in E(\Gamma_A)} \hat{\phi}_e^* \hat{\omega}$$

where $\hat{\phi}_e \colon \overline{C}^{\mathrm{local}}_{|A|}(\mathbb{R}^d) \to \overline{C}^{\mathrm{local}}_2(\mathbb{R}^d)$ picks out the two coordinates corresponding to the vertices of e.

Remark. We explain the difference in the formulas (3) and (4). If $A \subset [n]^+$ does not contain ∞, then the $n - |A| + 1$ points in $M \setminus \infty$ in a fiber of $E\overline{C}_{n-|A|+1}(\pi)$ come from picking out the $n - |A|$ points x_k for $k \in [n]^+ \setminus A \setminus \{\infty\}$ and an additional point x_j chosen to represent the $|A|$ points x_j, $j \in A$ that are all equal. The corresponding graph therefore has a vertex v_k for each $k \in [n] \setminus A$ and a single vertex $[*]$ representing all of the vertex v_j, $j \in A$. This is exactly Γ/A. On the other hand, if $A \subset [n]^+$ contains ∞, then the $n - |A| + 1$ points in $M \setminus \infty$ in a fiber of $E\overline{C}_{n-|A|+1}(\pi)$ come from picking out the the $n - |A| + 1$ points x_k for $k \in [n]^+ \setminus A$. The corresponding graph is therefore Γ_{A^c}, which does not contain a point representing all of v_j, $j \in A$. Heuristically, if $\infty \in A$, then not only did the x_j, $j \in A$ all collide, they also disappeared at ∞.

For the proof of (i), we need to compute $dI(\Gamma)$. Since $I(\Gamma) = \int_{\mathrm{Fiber}} \omega(\Gamma)$ is an integral of a form, we would like to use Stoke's theorem to compute $dI(\Gamma)$. Stoke's theorem for integration along the fibers takes the following form. Let $\rho \colon X \to Y$ be a bundle. For $\gamma \in \Omega^*_{\mathrm{dR}}(Y)$ a form, let $J(\gamma) = (-1)^{\deg(\gamma)}\gamma$. Integration along the fiber satisfies the following generalized Stoke's theorem:

$$d \int_{\mathrm{Fib}(\rho)} \alpha = \int_{\mathrm{Fib}(\rho)} d\alpha + J\Big(\int_{\partial \mathrm{Fib}(\rho)} \alpha \Big)$$

In particular, if α is a closed form, we have

$$d \int_{\mathrm{Fib}(\rho)} \alpha = J\Big(\int_{\partial \mathrm{Fib}(\rho)} \alpha \Big) \tag{5}$$

Proof of (i). Since $\omega(\Gamma) = \bigwedge_{e \in E(\Gamma)} \phi_e^*(\omega)$ is the wedge of closed form, it itself is closed. We apply (5) to the form $\alpha = \omega(\Gamma)$ to get

$$dI(\Gamma) = d \int_{\mathrm{Fib}(\overline{C}_n(\pi))} \omega(\Gamma) = \int_{\mathrm{Fib}(\overline{C}_n(\pi))} d\omega(\Gamma) + J\Big(\int_{\partial \mathrm{Fib}(\overline{C}_n(\pi))} \omega(\Gamma) \Big)$$

The second term in the sum can be rewritten as

$$J\Big(\int_{\partial \mathrm{Fib}(\overline{C}_n(\pi))} \omega(\Gamma) \Big) = J\Big(\sum_{\substack{A \subset [n] \\ |A| \geq 2}} \int_{\mathrm{Fib}(\pi_A)} \omega(\Gamma) + \sum_{\substack{Z \subset [n]^+ \\ |Z| \geq 2, \infty \in Z}} \int_{\mathrm{Fib}(\pi_Z)} \omega(\Gamma) \Big)$$

using the description of the boundary of the fiber in terms of its strata,

$$\partial \mathrm{Fib}(\overline{C}_n(\pi)) = \partial \overline{C}_n(M, \infty) = \Big(\coprod_{\substack{A \subset [n]^+ \\ |A| \geq 2}} \mathrm{Fib}(\pi_A) \Big) \sqcup (\text{ lower dimensional strata})$$

To show that $dI(\Gamma) = (-1)^{|I(d\Gamma)|} I(d\Gamma)$ we are therefore left with showing that

$$\sum_{\substack{A \subset [n]^+ \\ |A| \geq 2}} \int_{\mathrm{Fib}(\overline{C}_n(\pi_A))} \omega(\Gamma) = I(d\Gamma) \tag{6}$$

We have defined $d\Gamma$ to be the formal sum over edges of Γ of the oriented graphs $(\Gamma/e, \mathfrak{o}_e)$ obtained by collapsing an edge e of Γ with the induced orientation. Using the labeling of vertices of Γ by $\{1, \dots, n\}$, we can rewrite $d\Gamma$ as

$$d\Gamma = \sum_{A \in \mathcal{E}(\Gamma)} (\Gamma/A, \mathfrak{o}_A)$$

where the sum is taken over the collection $\mathcal{E}(\Gamma)$ of all subsets $A \subseteq [n]$ with $|A| = 2$ and so that the corresponding pair of vertices in Γ are connected by a *single* edge. For $A \in \mathcal{E}(\Gamma)$, we have used A to denote both a pair of integers $1, \dots, n$ and the corresponding edge of Γ.

By Lemmas 5.2, 5.6, 5.4 and Corollaries 5.3, and 5.5 below, the integral along a fiber of π_A of $\omega(\Gamma)$ vanishes unless $|A| = 2$ and

(A) if $\infty \in A$, then Γ/A^c is a graph with two vertices and at most 1 edge, or
(B) if $\infty \notin A$, then Γ_A is a graph with two vertices and a single edge.

However, situation (A) cannot occur. Indeed, say $|A| = 2$ with $\infty \in A$. Then $A = \{k, \infty\}$ for some $k \in \{1, \dots, n\}$. The graph Γ/A^c consists of the vertex v_k and a vertex $[*]$ coming from the collapse of A^c. Edges of Γ/A^c are in bijection with edges of Γ involving v_k. By assumption, every vertex of Γ has valence at least 3. Thus Γ/A^c has at least 3 edges. Thus the only contribution to the integral of $\omega(\Gamma)$ along $\partial \mathrm{Fib}(\overline{C}_n(\pi))$ comes from situation (B). We therefore just need to show that for $A \in \mathcal{E}(\Gamma)$, the following equality holds

$$\int_{\mathrm{Fib}(\pi_A)} \omega(\Gamma) = I(\Gamma/A, \mathfrak{o}_A)$$

for $A \in \mathcal{E}(\Gamma)$. Assuming for a moment that the orientations work out, we have from (3) and Lemma 3.2,

$$\begin{aligned}
\int_{\mathrm{Fib}(\pi_A)} \omega(\Gamma) &= \int_{\overline{C}_2^{\mathrm{local}}(\mathbb{R}^d)} \hat{\omega}(\Gamma_A) \wedge \int_{\mathrm{Fib}(\overline{C}_{n-1}(\pi))} \omega(\Gamma/A) \\
&= \int_{S^{d-1}} \mathrm{vol}_{S^{d-1}} \wedge \int_{\mathrm{Fib}(\overline{C}_{n-1}(\pi))} \omega(\Gamma/A) \\
&= \int_{\mathrm{Fib}(\overline{C}_{n-1}(\pi))} \omega(\Gamma/A) \\
&= I(\Gamma/A)
\end{aligned}$$

Details on the orientations can be found in the proof of Lemma A.2 of [4]. □

There are three ways a nonempty subset $A \subseteq [n]^+$ can fail to be in $\mathcal{E}(\Gamma)$:
- every vertex of Γ_A has valence ≥ 2,
- $|A| \geq 3$ and every vertex of Γ_A has valence 1, or
- $|A| = 2$ and the corresponding vertices of Γ are not connected.

Lemma 5.2. *Let $A \subseteq [n]$ with $|A| \geq 2$. If every vertex of Γ_A has valence ≥ 2, then the integral along the fiber of π_A of $\omega(\Gamma)$ vanishes,*

$$\int_{\mathrm{Fib}(\pi_A)} \omega(\Gamma) = 0$$

Proof. We break the proof into two cases:

A) All vertices of Γ_A have valence at least 3.

B) Γ_A has a vertex of valence 2.

Case A. By (3), it suffices to show that the integral

$$\int_{\overline{C}^{\mathrm{local}}_{|A|}(\mathbb{R}^d)} \hat{\omega}(\Gamma_A)$$

vanishes. To do this, we will show that $\omega(\Gamma_A)$ is not a top form on $\overline{C}^{\mathrm{local}}_{|A|}(\mathbb{R}^d)$:

$$\dim\left(\overline{C}^{\mathrm{local}}_{|A|}(\mathbb{R}^d)\right) \neq \deg \omega(\Gamma_A)$$

The dimension of $\overline{C}^{\mathrm{local}}_{|A|}(\mathbb{R}^d)$ is $(|A|-1)d-1$ and the degree of $\omega(\Gamma_A)$ is $m'(d-1)$ where m' is the number of edges of Γ_A. Since each of the $|A|$ vertices of Γ_A has valence at least 3, we have $2m' \geq 3|A|$. In particular, $m' \geq |A| > |A|-1$. Thus

$$(|A|-1)d-1 > m'(d-1)$$

and therefore the integral vanishes.

Case B. Let a be a vertex of Γ_A of valence 2. Say Γ_A has two edges e_b and e_c with boundary vertices $\{a,b\}$ and $\{a,c\}$, respectively. Define an automorphism ι of $E\mathcal{S}_A(\pi)$ sending

$$\iota(x_k) = \begin{cases} x_b + x_c - x_a & k = a \\ x_k & k \neq a \end{cases}$$

Then $\iota^*\omega(\Gamma) = \omega(\Gamma)$ and ι reverses the orintation of $\overline{C}^{\mathrm{local}}_j(\mathbb{R}^d)$. Thus the integral vanishes. ☐

Using (4) instead of (3), the exact same argument applied to Γ/Z^c for $Z \subset [n]^+$ containing ∞ gives the following:

Corollary 5.3. *Let $Z \subseteq [n]^+$ with $|Z| \geq 2$ and $\infty \in Z$. If every vertex of Γ/Z^c has valence ≥ 2, then the integral along the fiber of π_Z of $\omega(\Gamma)$ vanishes,*

$$\int_{\mathrm{Fib}(\pi_Z)} \omega(\Gamma) = 0$$

Lemma 5.4. *Let $A \subseteq [n]$ with $|A| \geq 3$. If Γ_A has a vertex of valence 1, then the integral*

$$\int_{\mathrm{Fib}(\pi_A)} \omega(\Gamma) = 0$$

vanishes.

Proof. Say b is a vertex of Γ_A of valence 1. Let e be the unique edge of Γ_A that is connected to b. Say $\partial e = \{a,b\}$. Write $j = |A|$. Let Y be the subset of $\overline{C}^{\mathrm{local}}_j(\mathbb{R}^d)$ of points whose x_a and x_b coordinates are equal. Consider the map

$$q\colon \overline{C}^{\mathrm{local}}_j(\mathbb{R}^d) \to Y$$

defined by

$$q(x_k) = \begin{cases} x_k & k \neq a,b \\ \frac{x_b - x_a}{\|x_b - x_a\|} & k = a,b \end{cases}$$

Then there exists some form α in $\Omega^{(d-1)m'}_{\mathrm{dR}}(Y)$ so that

$$q^*\alpha = \omega(\Gamma_A)|_{\overline{C}^{\mathrm{local}}_j(\mathbb{R}^d)}$$

Since Y has dimension strictly less than the dimension of $\overline{C}^{\mathrm{local}}_j(\mathbb{R}^d)$, the integral vanishes. ☐

Again we can apply the same argument to Γ/Z^c for $Z \subset [n]^+$, $|Z| \geq 3$ containing ∞.

Corollary 5.5. *Let $Z \subset [n]^+$ containing ∞. If $|Z| \geq 3$ has a vertex of valence 1, then the integral*

$$\int_{\mathrm{Fib}(\pi_Z)} \omega(\Gamma) = 0$$

vanishes.

Lemma 5.6. *Let $A \subseteq [n]$ with $|A| = 2$. If the corresponding two vertices of Γ are not connected by any edge then the integral along the fiber of $\omega(\Gamma)$ vanishes,*

$$\int_{\mathrm{Fib}(\pi_A)} \omega(\Gamma) = 0$$

Proof. By (3) it suffices to show that $\hat{\omega}(\Gamma_A) = 0$. Since the vertices of Γ corresponding to A are not connected, the graph Γ_A has no edges. By definition, $\hat{\omega}(\Gamma_A)$ is a wedge of forms,

$$\hat{\omega}(\Gamma_A) = \bigwedge_{e \in E(\Gamma_A)} \hat{\phi}_e^* \hat{\omega}$$

Since Γ_A has no edges, this is a wedge over an empty set, and hence zero. $\square$

We now proof part (ii) of Theorem 5.1.

Proof of (ii). Let ω_0 and ω_1 be two propagators corresponding to the same homotopy class of the framing τ_E. We want to construct a form α so that the following equality holds

$$\int_{\text{fiber}} \omega_0(\Gamma) - \int_{\text{fiber}} \omega_1(\Gamma) = d\alpha$$

By Lemma 4.1, there exists a form $\tilde{\omega}$ on $[0,1] \times E\overline{C}_2(\pi)$ restricting to ω_i, $i = 0, 1$ on $\{i\} \times E\overline{C}_2(\pi)$. Write $\tilde{\omega}(\Gamma)$ for $(\mathrm{Id} \otimes \bigwedge_{e \in E(\Gamma)} \phi_e^*)(\tilde{\omega})$. By the generalized Stoke's theorem (5) for closed forms such as $\tilde{\omega}(\Gamma)$, we have

$$d \int_{[0,1] \times \mathrm{Fib}(\overline{C}_n(\pi))} \tilde{\omega}(\Gamma) = J \int_{\partial([0,1] \times \mathrm{Fib}(\overline{C}_n(\pi)))} \tilde{\omega}(\Gamma)$$

Now the boundary $\partial([0,1] \times \mathrm{Fib}(\overline{C}_n(\pi)))$ can be rewritten as the disjoint union

$$\big(\{0\} \times \mathrm{Fib}(\overline{C}_n(\pi))\big) \sqcup \big(\{1\} \times \mathrm{Fib}(\overline{C}_n(\pi))\big) \sqcup \big([0,1] \times \partial \mathrm{Fib}(\overline{C}_n(\pi))\big)$$

where the second term has the opposite orientation. Thus

$$d \int_{[0,1] \times \mathrm{Fib}(\overline{C}_n(\pi))} \tilde{\omega}(\Gamma) = J\Big(\int_{\mathrm{Fib}(\overline{C}_n(\pi))} \omega_0(\Gamma) - \int_{\mathrm{Fib}(\overline{C}_n(\pi))} \omega_1(\Gamma) + \int_{[0,1] \times \partial \mathrm{Fib}(\overline{C}_n(\pi))} \tilde{\omega}(\Gamma)\Big)$$

It suffices to show that the third term in the sum vanishes. This follows from a similar argument as above. $\square$

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