18847 Final Project - Finite Element Method

Authors

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1. Integrate Triangle into FEGrid (Yao)

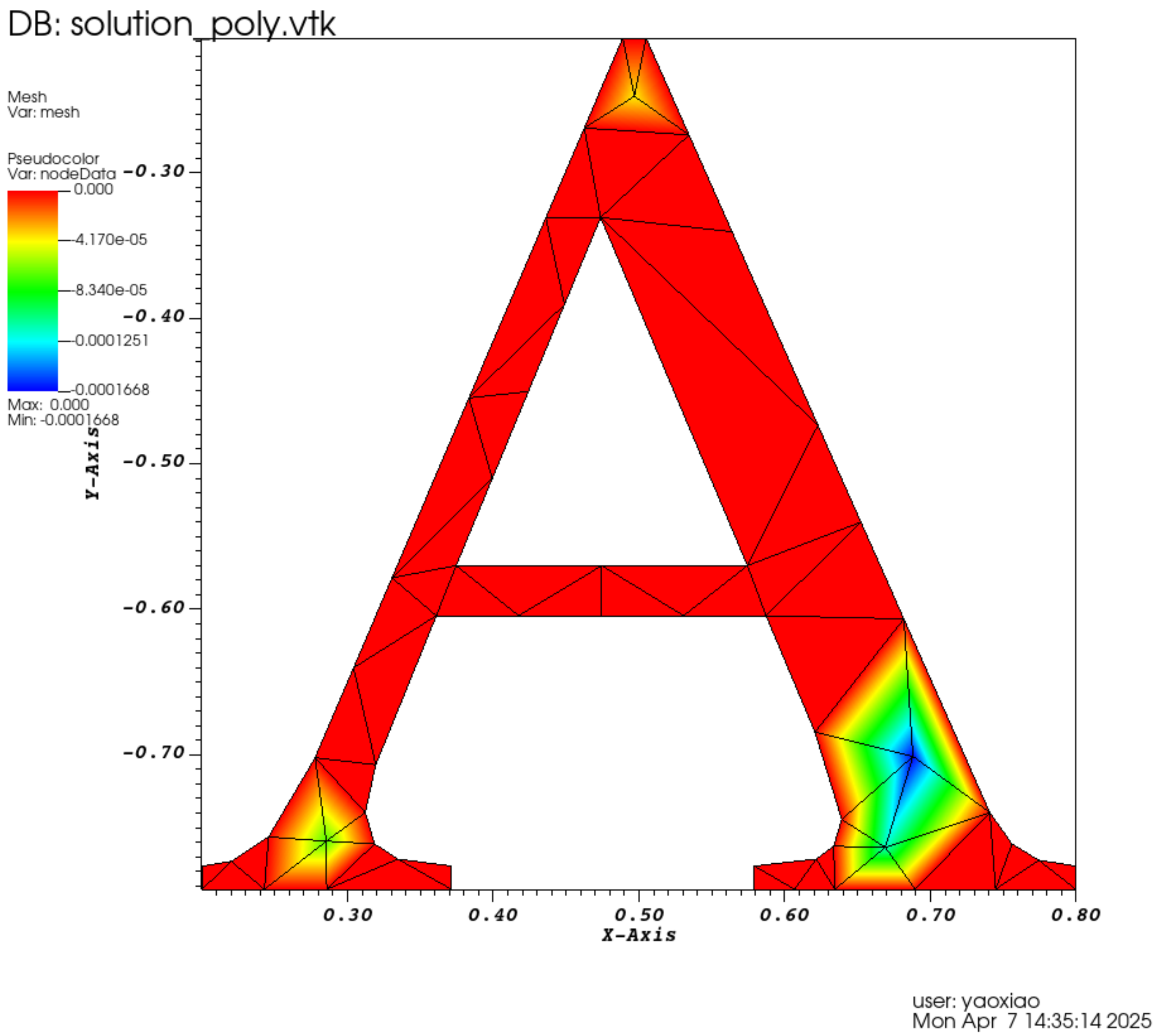
- Scan .poly file
- Call `triangulate()` from Triangle library (written in **C**) to refine triangulation with specified area constraint

  ```
  extern “C”
  ```

- Load nodes and elements from output
  - same as original constructor

Sample VisIt output of .poly file

```
cd exec
make poly
```



2. Template FEPoissonOperator by data type (Myles)

Key changes required

- Converted classes to template <typename T>
- Modified member variables and method signatures to use type T
- Added explicit template instantiations for each supported type
- Updated constructors and operators to handle templated types

Modified core classes

- `FEPoissonOperator`
- `SparseMatrix`
- `JacobiSolver`

Implementation approach

- Separated declarations (.H) from implementations (.hpp)
- Added necessary header includes (e.g., <complex>)

Testing

- Created test cases for each data type
- Verified solution correctness for annulus mesh
- Ensured consistent behavior across all supported types

```
cd exec

make run_float

make run_double

make run_complex_float

make run_complex_double
```

3. Non-zero Dirichlet boundary conditions (Myles)

Goal

Set solution directly at boundaries to known values:

- **Before:** Boundary assumed to be zero.
- **Now:** Allows custom boundary values ϕ_Φ.

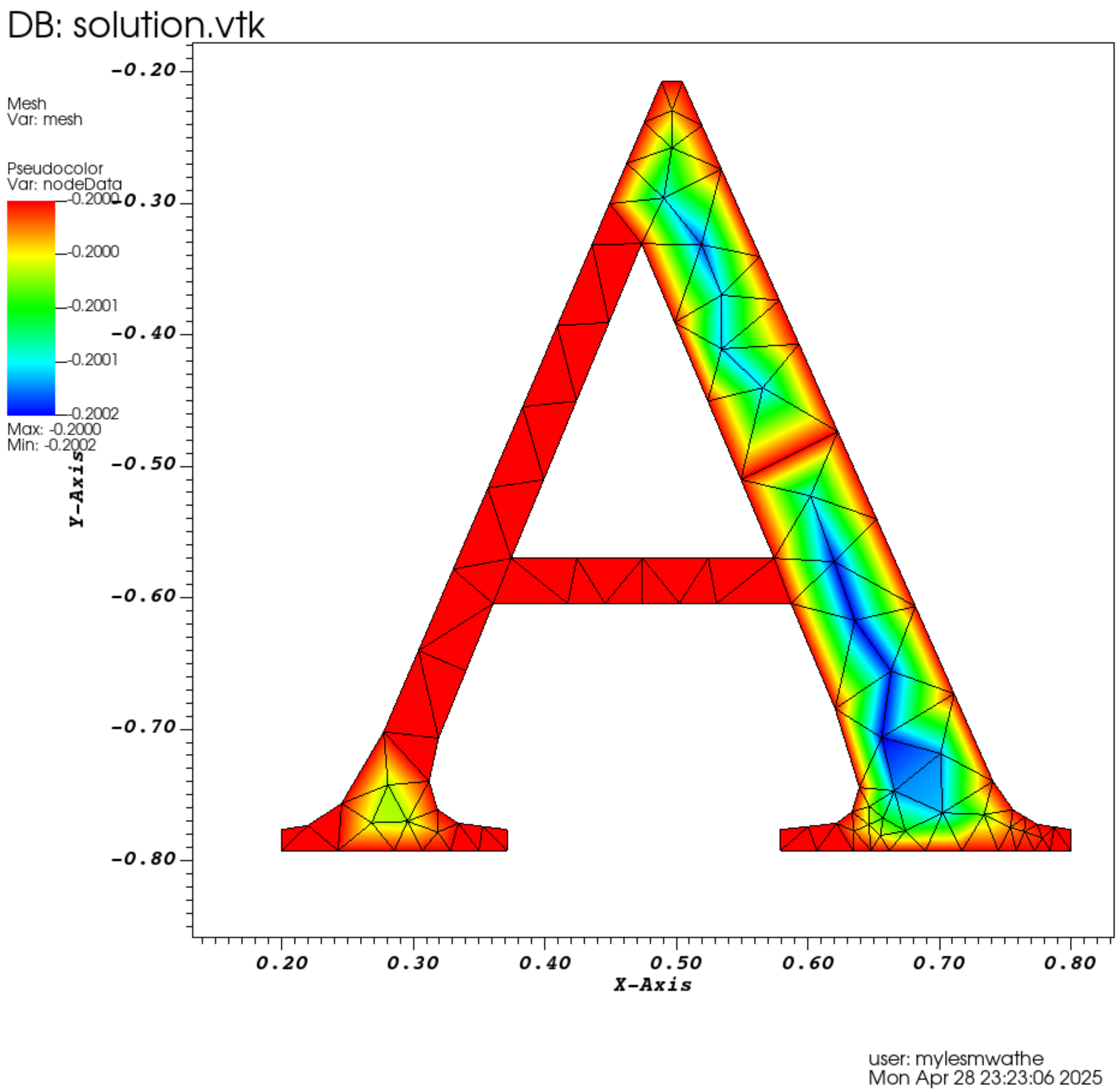
Implementation

- Solve on *all* nodes (interior + boundary).
- Adjust matrix after assembly for boundary nodes:
    - Set diagonal = 1, other entries = 0.
    - Set RHS = known boundary values.

Effect

- System remains solvable (positive definite, diagonally dominant, not symmetric).

Sample .poly file with boundary conditions



4. Demonstrate Piecewise Linear elements converge at 2nd order accuracy

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Setup

- Pick smooth Φ, build RHS from $-\Delta\Phi$.
- Enforce Φ on all boundary nodes.

Exact Solution

```
static auto Phi_exact = [](const array<double, DIM> &X) -> double
{
    // smooth function that vanishes on the unit square boundary
    return X[0] * X[0] + X[1] * X[1];
};

static auto source2D = [](const array<double, DIM> &X) -> double
{
    return -4.0;
};
```

Notes: Why still quadratic? 2D Linear function converges too fast to reveal the order.

Workflow

Refine mesh → smaller max element area.

1. Assemble stiffness matrix & load vector.
2. Stamp Dirichlet rows.
3. Solve for Φ_h.
4. Measure nodal max-error.

$$\Phi\big|_{\partial\Omega} = \Phi, \quad f = -\Delta\Phi, \quad h = 2\sqrt[3]{V}, \quad p \approx \frac{\ln(E_{L-1}/E_L)}{\ln(h_{L-1}/h_L)}.$$

Run the convergence study

```
cd exec
make run2d
```

Results

```
Estimated order p:
between lvl 0->1 : p ≈ 2
between lvl 1->2 : p ≈ 2
between lvl 2->3 : p ≈ 0.265492
between lvl 3->4 : p ≈ 2.68471
```

5. Implement a time-dependent FEM solver (Alan)

Solve the differential equation

$$\frac{\partial \phi_h}{\partial t} = f - L_h \phi_h$$

Via backwards euler, so we get the recurrence

$$\left(\frac{1}{\Delta t} I + L\right)(\phi_h(t + \Delta t)) = \frac{1}{\Delta t}\phi_h(t) + f_h(t)$$

L is the finite element approximation of the laplacian (with boundary conditions)

6. Verification of Time Dependent FEM Solver

General idea

- Pick a Φ
- Let

$$f = \frac{\partial \Phi}{\partial t} + \Delta \Phi$$

The theoretical solution to the differential equation should converge to the Φ we picked regardless of the initial conditions (if we ignore boundary conditions).

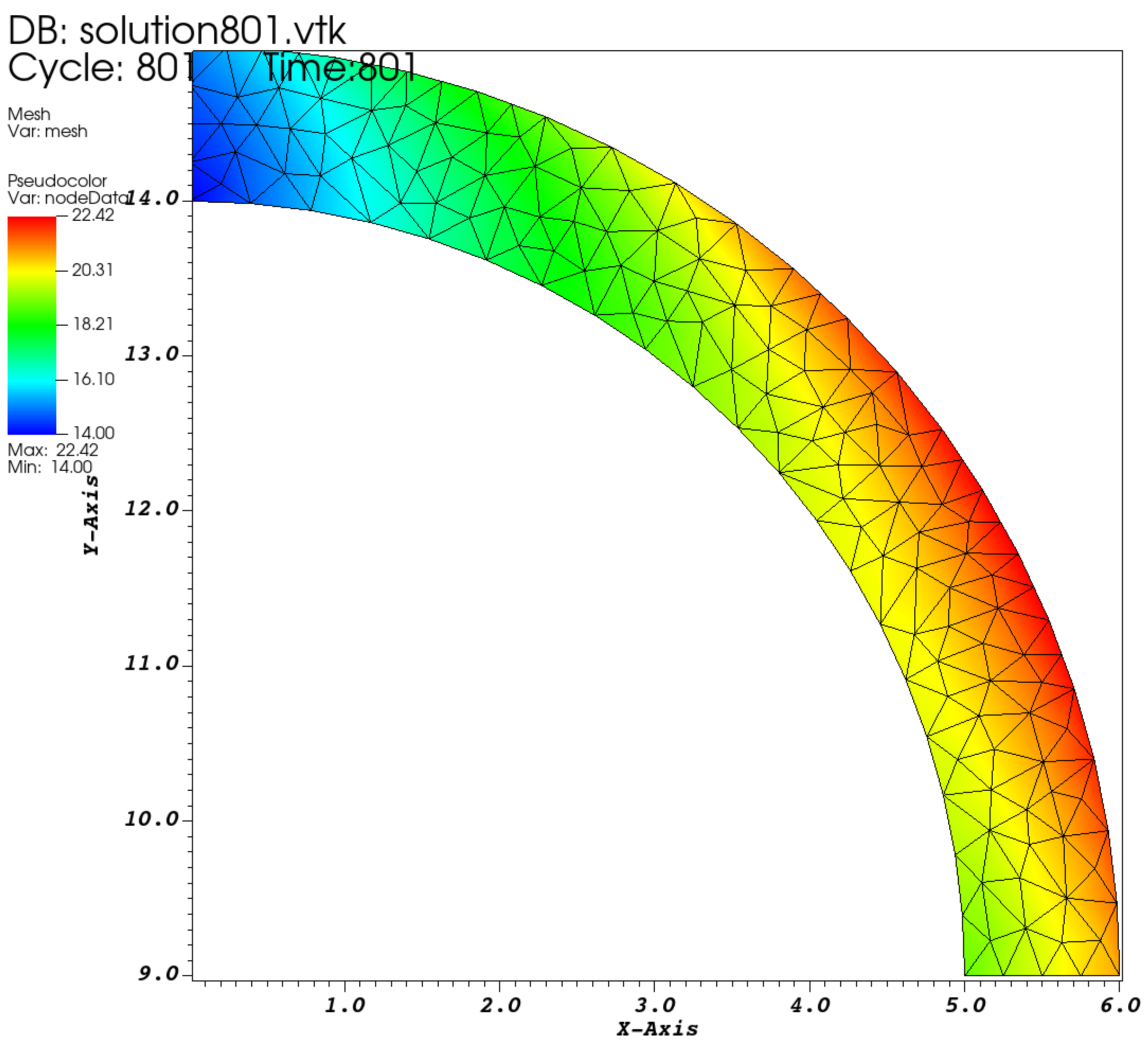
Notes: There is inherent error with a numerical time integrator chasing after the theoretical solution.

Example1: Time independent Φ

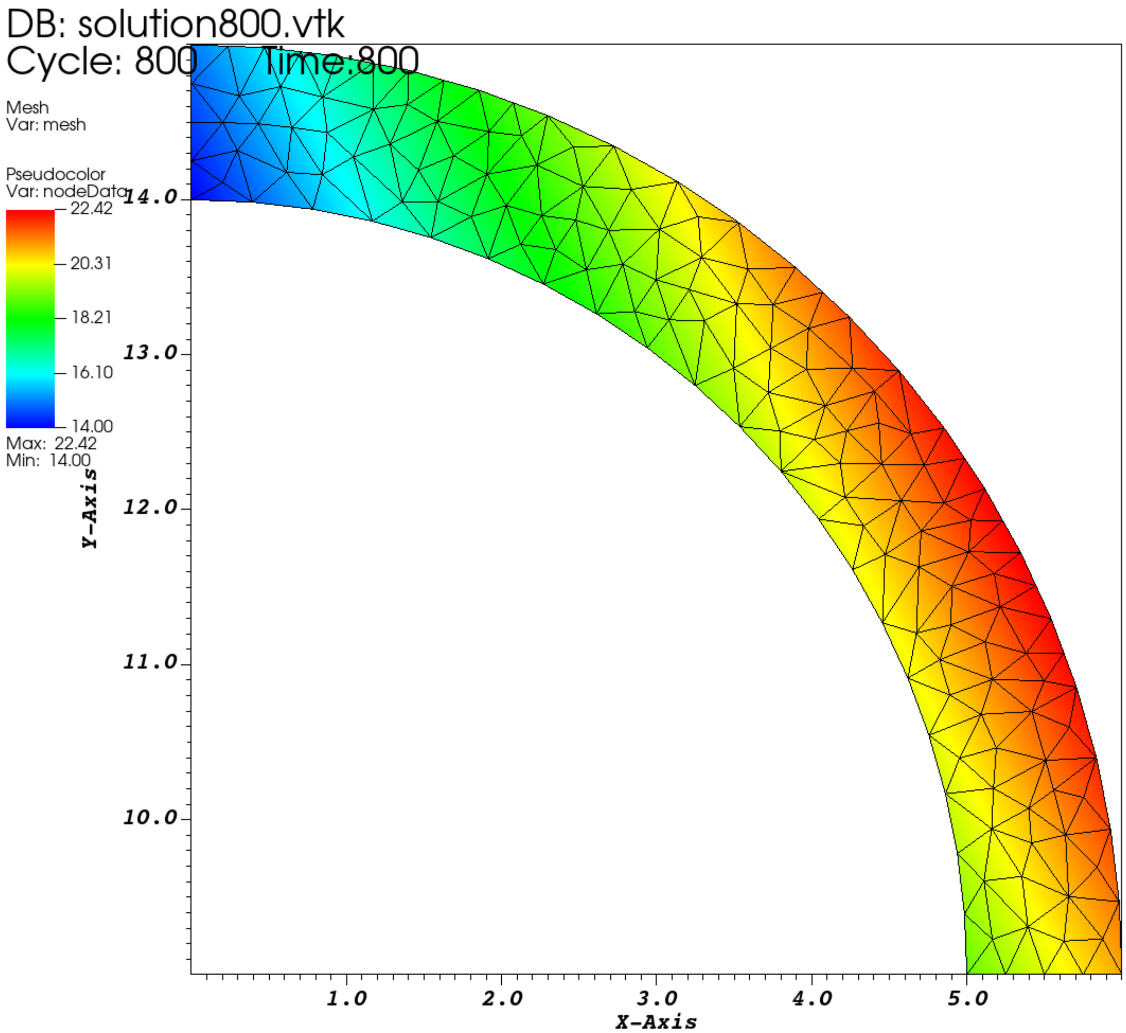
Final relative error **0.009**

```
//our reference phi
double sourcePhi(double time, array<double, DIM> x)
{
    return 2*x[0]+x[1];

}

double derivedf(double time, array<double, DIM> x)
{
    //want to return d^2phi/dx^2 + d^2phi/dy^2 + dphi/dt
    return 0;
}
```

Computed solution



Reference Ψ

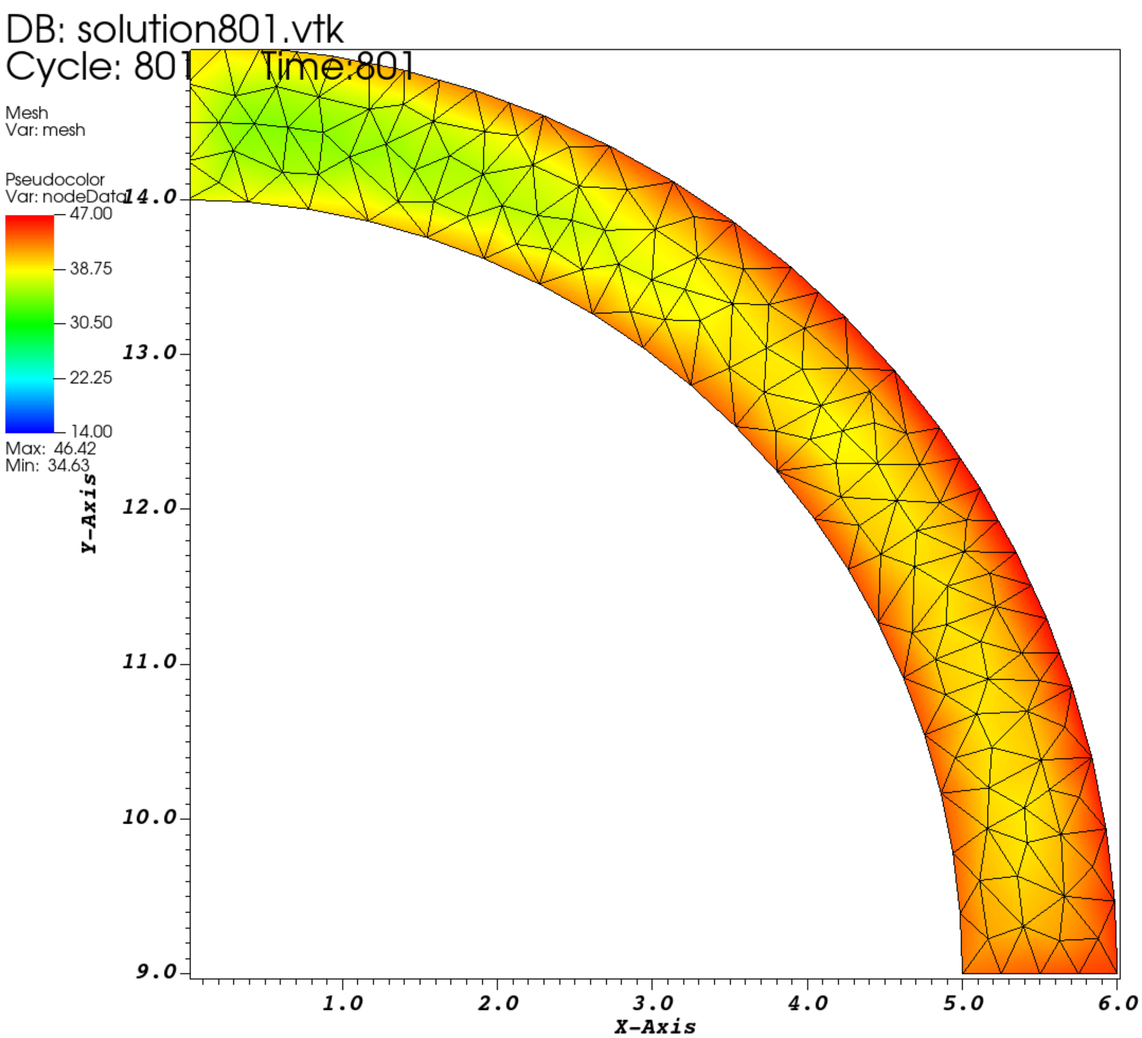


Example2: Simple time dependent Φ

Final relative error **0.1385**

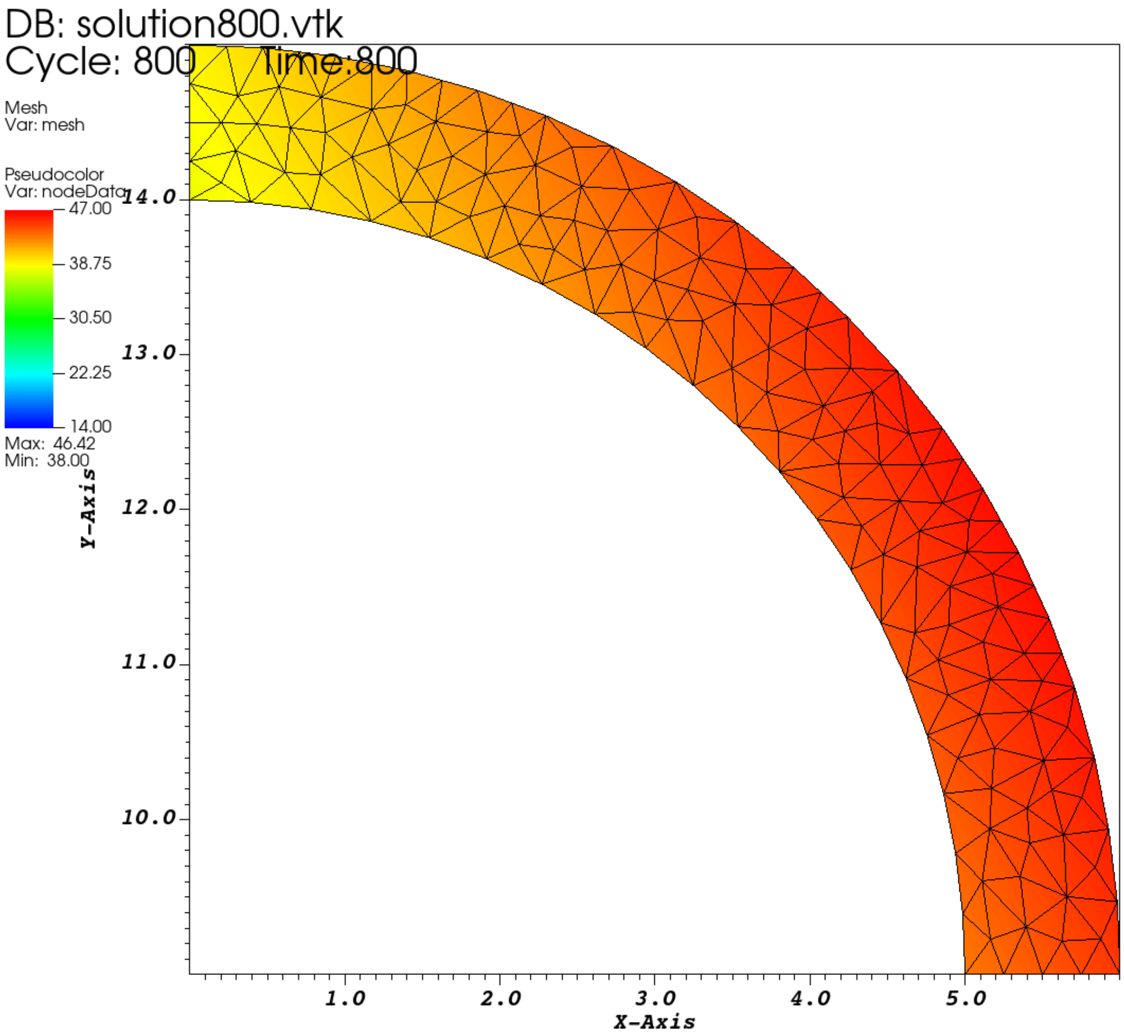
```
//our reference phi
double sourcePhi(double time, array<double, DIM> x)
{
    return 2*x[0]+x[1]+3.0*time;

}

double derivedf(double time, array<double, DIM> x)
{
    //want to return d^2phi/dx^2 + d^2phi/dy^2 + dphi/dt
    return 3.0;
}
```

Computed solution



Reference Φ

Reference Ψ



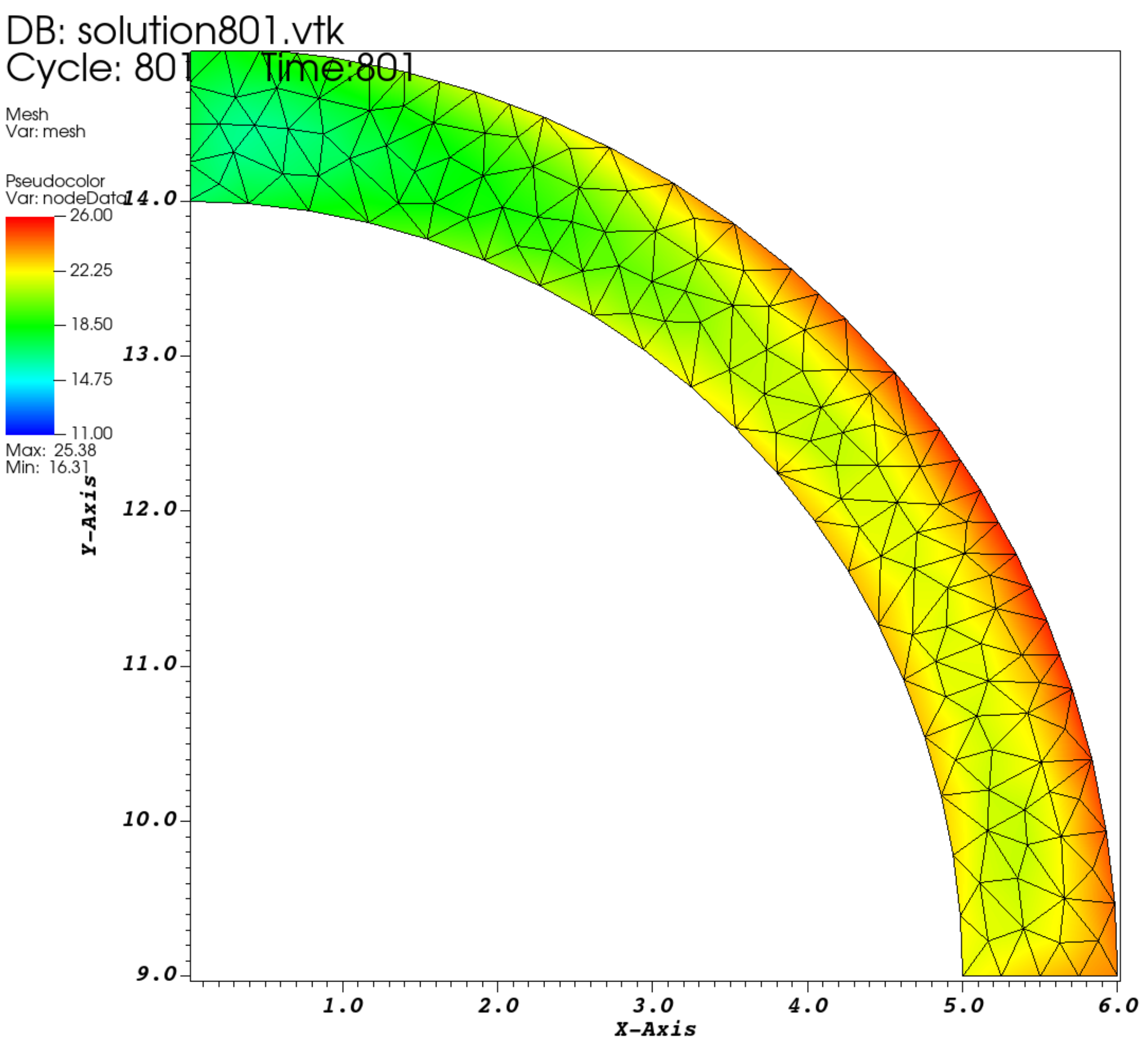
Example3: More complex time dependent Φ

Final relative error **0.1057**

```
//run reference phi
```

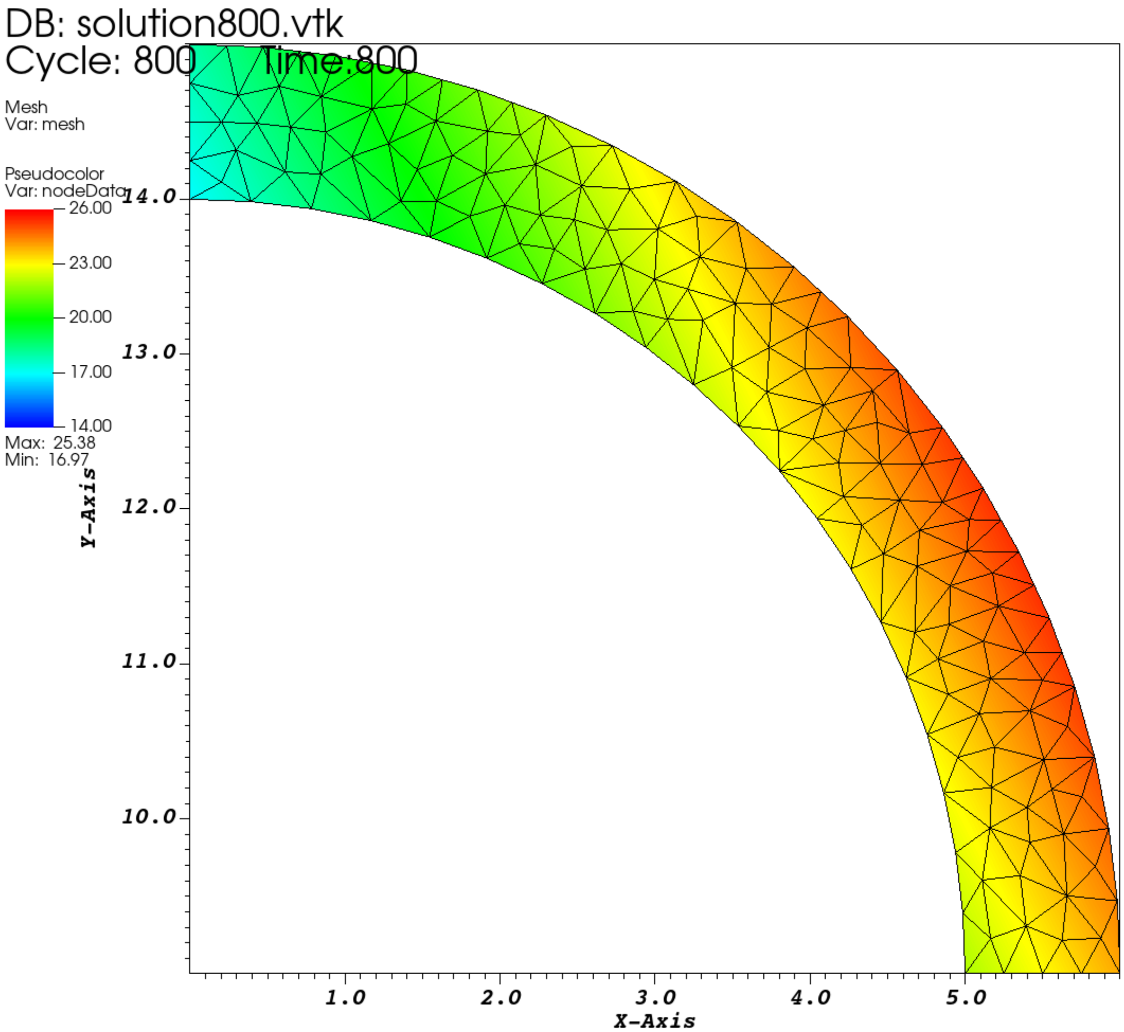
```
//our reference phi
double sourcePhi(double time, array<double, DIM> x)
{
  return 2*x[0]+x[1]+3.0*sin(time);

}

double derivedf(double time, array<double, DIM> x)
{
  //want to return d^2phi/dx^2 + d^2phi/dy^2 + dphi/dt
  return 3.0*cos(time);
}
```

Computed solution



Reference Φ

Reference ψ

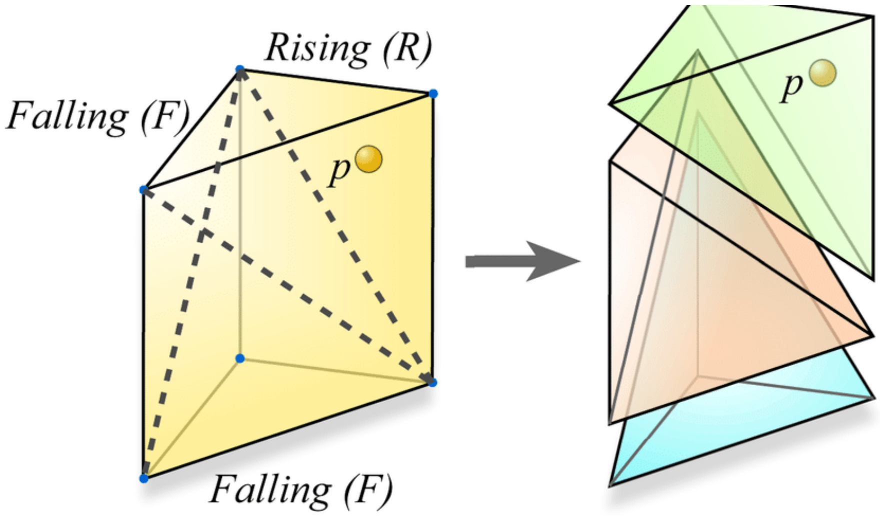


7. Create time-dependent animations of interesting source terms and boundary conditions

TODO

8. Extrude the 2D elements into an extrusion into 3D (Yao)

- Extrude each triangle vertically in a prism
- Split prism into 3 tetrahedrons



9. Solve Poisson Equation in 3D (Yao)

2 key differences of another dimension

FEGrid::gradient()

Solve linear system in 3D (inverse matrix: compute determinant, cofactors)

We have

$$dx \times \Delta\phi = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \tag{1}$$

Then solve

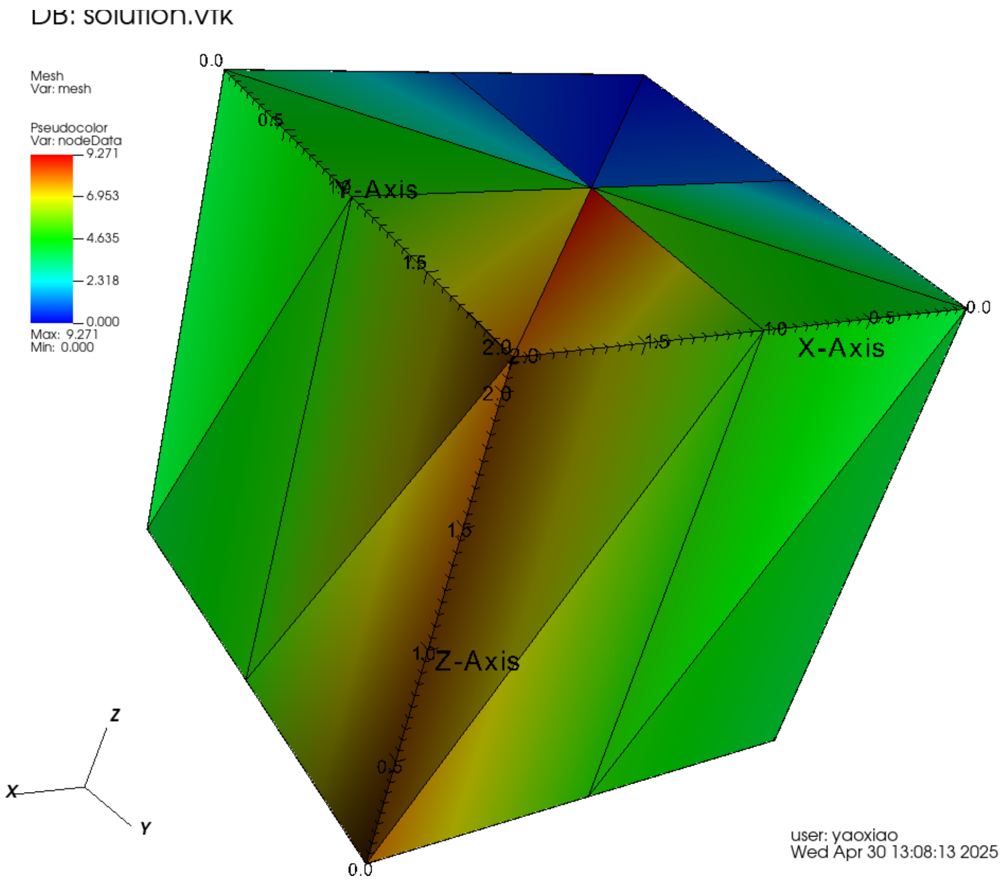
$$\Delta\phi = (dx)^{-1} \times \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \tag{2}$$

FEGrid::elementArea()

Compute tetrahedron volume instead (value: determinant / 6).

Example extrusion plot of square mesh

```
cd exec
make extrude DIM=3
```



10. Verify 3D steady solutions

The process closely resembles the 2D version.

Exact Solution

```
static auto Phi_exact = [](const array<double, DIM> &X) -> double
{
    // smooth function that vanishes on the unit square boundary
    return X[0] + X[1] + X[2];
};

static auto source3D = [](const array<double, DIM> &X) -> double
{
    return 0.0;
};
```

Run the convergence study

Run the convergence study

```
cd exec
make run3d DIM=3
```

Results

```
Estimated order p:
between lvl 0→1 : p ≈ -0.0939246
between lvl 1→2 : p ≈ 1.39756
between lvl 2→3 : p ≈ 0.307041
between lvl 3→4 : p ≈ 0.413396
```

References

Dompierre, Julien & Labbé, Paul & Vallet, Marie-Gabrielle & Camarero, Ricardo. (1999). How to Subdivide Pyramids, Prisms, and Hexahedra into Tetrahedra.. 195-204.