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Solving Linear System

direct method

Quick Review

Matrices

A 2×2 matrix is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A vector is

$$v = \begin{bmatrix} x \\ y \end{bmatrix}$$

The identity is

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For any matrix A we have

$$AI = IA = A$$

For any vector v we have

$$Iv = v$$

Matrix Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bu \\ cx + dz & cy + du \end{bmatrix}$$

Vector Multiplication (usually $AB \neq BA$)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Determinant

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Inverse of A exists if and only if $\det(A) \neq 0$

$$A^{-1}A = AA^{-1} = I$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Eigenvalues and Eigenvectors

Definition. λ is an eigenvalue of A and $v \neq 0$ is an eigenvector associated to λ if

$$Av = \lambda v$$

Theorem. The eigenvalues of A are the roots of

$$\det(A - \lambda I) = 0$$

The eigenvector v associated to λ solves

$$(A - \lambda I)v = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linear System

We have

$$\begin{cases} ax_1 + bx_2 = u \\ cx_1 + dx_2 = v \end{cases} \iff Ax = b \quad \text{where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad b = \begin{bmatrix} u \\ v \end{bmatrix}$$

LU and Cholesky Factorization

LU Factorization

$$A = LU$$

where L is lower triangular

$$L = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ l_{n,1} & \cdots & l_{n,n-1} & 1 \end{bmatrix}$$

where U is upper triangular

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & & 0 & \ddots & u_{n-1,n} \\ 0 & \cdots & 0 & u_{nn} \end{bmatrix}$$

Cholesky Factorization

If A is symmetric definite positive

$$A = LL^T$$

where L is lower triangular

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ l_{n,1} & \cdots & l_{n,n-1} & l_{nn} \end{bmatrix}$$

Exercise

Exercise 1: Write a function that compute the LU factorization of a square matrix. Then test your functions with a test case, and explain why it works.

Exercise 2: BONUS. Write a function that compute the Cholesky factorization of a square matrix. Then test your functions with a test case, and explain why it works.

Exercise 3: Write a function that solve a system $LUx = b$. Then test your functions with a test case, and explain why it works.

Exercise 4: BONUS. Write a function that solve a system $LL^T x = b$. Then test your functions with a test case, and explain why it works.

Exercise 5: Solve the system

$$\begin{cases} 2x_1 + x_2 & = 3 \\ -x_1 + 2x_2 + x_3 & = 2 \\ -x_2 + 2x_3 + x_4 & = 2 \\ -x_3 + 2x_4 + x_5 & = 2 \\ -x_4 + 2x_5 & = 1 \end{cases}$$

Exercise 6: Solve the system

$$\begin{cases} 2x_1 + x_2 + x_3 & = 3 \\ x_1 - x_2 + 2x_3 & = 2 \\ 3x_1 + 2x_2 + 10x_3 + 2x_4 & = 2 \\ -x_3 + 2x_4 + x_5 & = 2 \\ 2x_3 + 3x_4 & = 1 \end{cases}$$

Exercise 7: Solve the tridiagonal system $Ax = f$ with

$$A_{ii} = 4, \quad A_{i,i-1} = A_{i,i+1} = 1$$

for all i . Let the order of the system be $n = 100$, and let

$$f = [1, 1, \dots, 1]^T$$