8 Solving Ordinary Differential Equations

Methods

The goal is to solve

$$y'(t) = f(t, y(t))$$

Euler Method Explicit

You replace $y'(t)$ by $\frac{y(t+h)-y(t)}{h}$ and you define

$$y^n = y(t_n), \quad t_n = t_0 + nh$$

which gives you

$$\frac{y^{n+1} - y^n}{h} = f(t_n, y^n) \implies y^{n+1} = y^n + hf(t_n, y^n)$$

Euler Method Implicit

You replace $y'(t)$ by $\frac{y(t+h)-y(t)}{h}$ and you define

$$y^n = y(t_n), \quad t_n = t_0 + nh$$

which gives you

$$\frac{y^{n+1} - y^n}{h} = f(t_{n+1}, y^{n+1}) \implies y^{n+1} = y^n + hf(t_{n+1}, y^{n+1})$$

Runge-Kutta 2 (midpoit method)

The algorithm is

$$y^{n+1} = y^n + hf\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}f(t_n, y_n)\right)$$

Runge-Kutta 4

The algorithm is

$$y^{n+1} = y^n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned}
k_1 &= hf(t_n, y^n) \\
k_2 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right) \\
k_3 &= hf\left(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right) \\
k_4 &= hf\left(t_n + h, y_n + k_3\right)
\end{aligned}$$

Exercise

Exercise 1: Write a function for each methods that solve a general system of ODE.

Exercise 2: Lokta-Volterra Equations:

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta x y \\ \dfrac{dy}{dt} = \delta x y - \gamma y \end{cases}$$

where

- x is the number of prey
- y is the number of some predator
- α, β, γ, $delta$ are positive real parameters describing the interaction of the two species.
- The prey are assumed to have an unlimited food supply and to reproduce exponentially, unless subject to predation; this exponential growth is represented in the equation above by the term αx.
- The rate of predation upon the prey is represented above by $\beta x y$. If either x or y is zero, then there can be no predation.
- $\delta x y$ represents the growth of the predator population.
- γy represents the loss rate of the predators due to either natural death or emigration, it leads to an exponential decay in the absence of prey

1. Solve the Lokta-Volterra equations with the 4 different methods, with $\alpha = 2/3$, $\beta = 4/3$, $\gamma = 1 = \delta$
2. Try another cool set of value and solve it.

Exercise 3: Competitive Lotka–Volterra equations. N species competing for the same resource:

$$\frac{dx_i}{dt} = r_i x_i \left(1 - \sum_{i=1}^{N} \alpha_{ij} x_j\right)$$

- r_i is the rate at which the species i reproduce.
- α_{ij} representes the effect species j has on the population of species i.

1. Solve the Competitive Lokta-Volterrra equations with

$$r = \begin{bmatrix} 1 \\ 0.72 \\ 1.53 \\ 1.27 \end{bmatrix}, \quad \alpha = \begin{bmatrix} 1 & 1.09 & 1.52 & 0 \\ 0 & 1 & 0.44 & 1.36 \\ 2.33 & 0 & 1 & 0.47 \\ 1.21 & 0.51 & 0.35 & \end{bmatrix}$$

2. Solve another cool one.