4 Integration and differentiation

Notes

Trapezoidal Rule

Let $a = x_0 < x_1 < \cdot < x_n = b$, with

$$x_i = a + ih \text{ for } i = 0, \ldots, n \text{ and } h = \frac{b-a}{n}.$$

Then

$$\int_A^b f(x)dx = h\left[\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]$$

Simpson's Rule

Let $a = x_0 < x_1 < \cdot < x_n = b$, with

$$x_i = a + ih \text{ for } i = 0, \ldots, n \text{ and } h = \frac{b-a}{n}.$$

Then

$$\int_A^b f(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Monte Carlo integration

In 1D

$$\int_a^b f(x)dx = (b-a)\frac{1}{n}\sum_{i=1}^n f(x_i)$$

In 2D

$$\iint_S f(x)dx = |S|\frac{1}{n}\sum_{i=1}^n f(x_i)$$

First derivative

Forward formula:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \equiv D_h(x)$$

Centered formula:

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \equiv D_h(x)$$

Backward formula:

$$f'(x) \approx \frac{f(x) - f(x-h)}{h} \equiv D_h(x)$$

Centered formula second derivative

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h}{h^2} \equiv D_h^{(2)}(x)$$

Exercises

Exercise 1: Write a function that compute the integral using the Trapezoidal Rule. The function should take in arguments f, a, b, and n (the number of points).

Exercise 2: Write a function that compute the integral using the Simpson's Rule. The function should take in arguments f, a, b, and n (the number of points).

Exercise 3: Use the Trapezoidal Rule and Simpson's Rule with $n = 4, 8, 16, .., 512$ to find approximate values of the following integrals. Then compare it the the exact value of the integral using Sympy and plot the error

1. $\int_0^{10} e^{-x^2} dx$,
2. $\int_0^2 tan^{-1}(1 + x^2) dx$,
3. $\int_0^1 \sqrt{x} e^x dx$,

Exercise 4: Use the Trapezoidal Rule and Simpson's Rule with $n = 4, 8, 16, .., 512$ to find the lenghth of the curves

1. $f(x) = \sin(\pi x)$, $0 \le x \le 1$
2. $f(x) = e^x$, $0 \le x \le 1$
3. $f(x) = e^{x^2}$, $0 \le x \le 1$

Remember the length of the curve is

$$\int_a^b \sqrt{1 + [f'(x)'^2} dx$$

Exercise 5: The degree of precision of a numerical integration formula is defined as follows: if the formula is exact (has zero error) when integration any polynomial of degree $\le r$, and if there is an error for polynomials of degree $> r$, then we say the formula has degree of precision r.

1. Find the degree of precision of the Trapezoidal Rule when $n = 1$.

2. Find the degree of precision of the Simpson's Rule when $n = 2$.

Exercise 6: Using the Monte Carlo integration, compute the value of π.

Exercise 7: In the following instances, find the numerical derivative at the indicated point, usin the backward, forward, and centered formula. Use $h = 0.1, 0.05, 0.025, 0.0125, 0.00625$, then each case plot the error with respect to h.

1. e^x at $x = 0$.
2. $\tan^{-1}(x^2 - x + 1)$ at $x = 1$.
3. $\tan^{-1}(100x^2 - 199x + 100)$ at $x = 1$.

Exercise 8: Using the error formulas, compute how many point you need in order to approximate the ingegral $\int_1^3 x\ln(x)dx$ with an error less than 10^{-6} using the Trapeziodal Rule and the Simpson's Rule. Then compute this integral.