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## Polynomial of Interpolations

### Notes

#### Divided Difference

The divided difference formula is

$$f[x_0, x_1, \dots, x_n] = \frac{f[x_1, \dots, x_n] - f[x_0, x_1, \dots, x_{n-1}]}{x_n - x_0}$$

The divided difference approximate:

$$\frac{f^{(n)}(c)}{n!} = f[x_0, x_1, \dots, x_n]$$

Usually we pick  $c$  as the middle point.

#### Polynomial of Interpolation

You have a series of  $n + 1$  points  $(x_i, y_i)$  (where  $y_i = f(x_i)$ ), and you want  $P_n$  of degree  $n$  with

$$P_n(x_i) = y_i, \text{ for } i = 0, \dots, n.$$

Then

$$P_n(x) = \sum_{i=0}^n y_i L_i(x),$$

where the  $L$  are the Lagrange polynomial

$$L_i(x) = \prod_{k=0, k \neq i}^n \frac{(x - x_k)}{(x_i - x_k)}.$$

Or you can write  $P_n$  as

$$P_n(x) = f(x_0) + (x - x_0)f[x_0, x_1] + \dots + (x - x_0) \cdots (x - x_{n-1})f[x_0, \dots, x_n]$$

### Exercises

#### Exercise 1:

1. Write a function that compute the divided differences. (You should write this function with recursion).

- Write a test for your function. (Pick a function you know and see if it gives you the right answer, do not forget to explain your test in your report)

**Exercise 2:** By using your divided differences function.

- Approximate the third derivative of  $\cos(2x) + e^x + x$  at  $x = 0$ , using points between  $-1$  and  $1$ .
- Approximate the second derivative of  $\ln(x) + x$  at  $x = 1$ , using  $x = 0.1, 1, 2$ . Compute the error (the error is the difference between your approximation and the exact value of the second derivative at  $x = 1$ )
- Approximate the second derivative of  $\ln(x) + x$  at  $x = 1$ , using  $x = 0.5, 1, 1.5$ . Compute the error. How does it compare to the previous part?

**Exercise 3:** Plot the polynomial of interpolation at  $x = -1, -0.5, 0.5, 1$  of

- $f(x) = e^x$
- $f(x) = \sin(\pi x)$
- $f(x) = \tan^{-1}(x)$
- $f(x) = \log(1 + x^2)$

**Exercise 4:** The following data are taking from a polynomial  $p(x)$  of degree  $\leq 5$ . What is its degree? Plot it. And what is this polynomial?

$x$	-2	-1	0	1	2	3
$p(x)$	-5	1	1	1	7	25

**Exercise 5:** Using a polynomial of interpolation for prediction. Assume we have the data set, that represents the weight of a car and its MPG.

weight	13	21	27	30
MPG	1695	1330	1248	1233

What do you expect the MPG to be if the car weight 1150? Is it a good estimate? What is the issue?

**Exercise 6:** BONUS. Create a Class of polynomials that interpolate the points  $(x_i, y_i)$  for  $i = 0, \dots, n$  such that:

- The polynomial is of degree  $n$ ,
- and  $P_n(x_i) = y_i$  for  $i = 0, \dots, n$ .

The arguments should be a array of points  $(x_i, y_i)$ , then you have method to evaluate the polynomial at a given point and at a series of points (a vector). You can use either Lagrange or Newton divided difference to construct such polynomial.

Look at [Complex number class in Python](#).

**Exercise 7:** BONUS. Find the solution to the interpolation problem of finding a polynomial  $q(x)$  with  $\deg(q) \leq 2$  and such that

$$q(x_0) = y_0, \quad q(x_1) = y_1, \quad q'(x_1) = y'_1$$

Hint: Write  $q(x) = y_0 M_0(x) + y_1 M_1(x) + y'_1 M_2(x)$ .