5 | Solving Linear System

direct method

Quick Review

Matrices

A 2×2 matrix is

$$\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A vector is

$$\mathsf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The identity is

$$\mathsf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For any matrix A we have

$$\mathsf{AI} = \mathsf{IA} = \mathsf{A}$$

For any vector v we have

$$\mathsf{Iv} = \mathsf{v}$$

Matrix Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bu \\ cx + dz & cy + du \end{bmatrix}$$

Vector Multiplication (usually $\mathsf{AB} \neq \mathsf{BA}$)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + cy \end{bmatrix}$$

Determinant

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cd$$

Inverse of A exists if and only if $\det(A) \neq 0$

$$\mathsf{A}^{-1}\mathsf{A} = \mathsf{AA}^{-1} = \mathsf{I}$$

$$\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow \mathsf{A}^{-1} = \frac{1}{\det(\mathsf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Eigenvalues and Eigenvectors

Definition. λ is an eigenvalue of A and $\mathsf{v} \neq 0$ is an eigenvector associated to λ if

$$\mathsf{Av} = \lambda \mathsf{v}$$

Theorem. The eigenvalues of A are the roots of

$$\det(\mathsf{A} - \lambda \mathsf{I}) = \mathbf{0}$$

The eigenvector v associated to λ solves

$$(\mathsf{A} - \lambda \mathsf{I})\mathsf{v} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linear System

We have

$$\begin{cases} ax_1 + bx_2 = u \\ cx_1 + dx_2 = v \end{cases} \iff \mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{where } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} u \\ v \end{bmatrix}$$

LU and Cholesky Factorization

LU Factorization

$$A = LU$$

where L is lower triangular

$$L = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ l_{n,1} & & \cdots & l_{n,n-1} & 1 \end{bmatrix}$$

where U is upper triangular

$$L = \begin{bmatrix} u_{11} & u_{12} & \cdots & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & & 0 & \ddots & u_{n-1,n} \\ 0 & & \cdots & 0 & u_{nn} \end{bmatrix}$$

Cholesky Factorization

If A is symmetric definite positive

$$A = LL^T$$

where L is lower triangular

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ l_{n,1} & & \cdots & l_{n,n-1} & l_{nn} \end{bmatrix}$$