10 | Solving equations

1 One Dimensional Heat Equation

Solve for $u(x, t)$,

$$\begin{cases} \frac{du}{dt} - au_{xx} = f \;\; x \in (0, L], \;\; t > 0 \\ u(x, 0) = u_0(x), \;\; x \in [0, L] \\ u(0, t) = g_0(t), \;\; u(L, t) = g_L(t), \;\; t > 0 \end{cases}$$

Remember the stability condition for Euler Explicit is

$$\frac{a\Delta t}{\Delta x^2} \leq \frac{1}{2}$$

1. Write down a program that solve the heat equation using **Euler Implicit**, **Euler Explicit**, and **Runge Kutta 4**.
2. Solve for the case: $L = 1$, $f = 0$, $u_0(x) = \sin(\pi x)$, $g_0(t) = g_L(t) = 0$
3. Solve an interesting problem that you choose.

2 One Dimensional Wave Equation

Solve for $u(x, t)$,

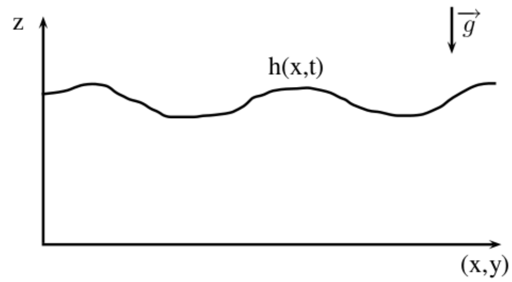
$$\begin{cases} u_{tt} - au_{xx} = f \;\; x \in (0, L], \;\; t > 0 \\ u(x, 0) = u_0(x), \;\; x \in [0, L] \\ u_t(x, 0) = v_0(x), \;\; x \in [0, L] \\ u(0, t) = g_0(t), \;\; u(L, t) = g_L(t), \;\; t > 0 \end{cases}$$

Remember the stability condition for Euler Explicit is

$$\frac{\sqrt{a}\Delta t}{\Delta x} \leq 1$$

1. Write down a program that solve the heat equation using **Euler explicit**.
2. Solve for the case: $L = \pi$, $f = 2e^{-t}\sin(x)$, $u_0(x) = \sin(x)$, $v_0(x) = -\sin(x)$, $g_0(t) = g_L(t) = 0$
3. Solve an interesting problem that you choose.

3 Saint Venant Equations



On $\Omega = [x_0, x_f] \times \{0 < t < T\}$, with Dirichlet boundary conditions.

$$\left\{\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} = 0, \\ \frac{\partial uh}{\partial t} + \frac{\partial hu^2}{\partial x} + \frac{g}{2}\frac{\partial h^2}{\partial x} = 0, \\ h(x_0, f) = j_0(t), \ \ h(x_f, f) = j_f(t), \ \ t > 0, \\ u(x_0, f) = g_0(t), \ \ u(x_f, f) = g_f(t), \ \ t > 0, \\ u(x, t = 0) = u_0(x), \ \ h(x, t = 0) = h_0(x), \end{aligned}\right. \tag{1}$$

Here :

- $h(x, t)$ is the fluid depth above the bottom which is supposed flat
- $u(x, t)$ is the velocity
- g denotes the gravity

Solve the Saint Venant equations using any scheme an explicit and implicit schemes.

1. Write down a program that solve Saint Venant equations using **Euler Implicit**, **Euler Explicit**, and **Runge Kutta 4**.
2. Solve for the case: $x0 = -50$, $xf = 50$, $j_0(t) = j_f(t) = 1$, $g_0(t) = g_f(t) = 1$, and

$$\phi(x) := 0.771B^2\text{sech}(0.395x)^2 \text{ with } B = 0.395$$

$$h_0(x) = \phi(x) \cdot \frac{3}{4} + 1.0$$

$$u_0(x) = \phi(x) \cdot \frac{-9}{4}$$

   This case is the Rossby solition, look at https://marine.rutgers.edu/po/tests/rossby/index.html
3. Solve a Dam-break problem

4 Two Dimensional Heat Equation in 2D

Solve for $u(x, y, t)$

$$\left\{\begin{array}{l} \frac{du}{dt} - au_{xx} - au_{yy} = f \ \ 0 \le x \le L_x, enspace0 \le y \le L_y, \ \ t > 0 \\ u(x, y, 0) = u_0(x, y), \quad 0 \le x \le L_x, \ \ 0 \le y \le L_y \\ u(0, y, t) = g_0(t), \ \ u(L_x, y, t) = g_L(t), \ \ 0 \le y \le L_y, \ \ t > 0 \\ u(x, 0, t) = h_0(t), \ \ u(x, L_y, t) = h_L(t), \ \ 0 \le x \le L_x, \ \ t > 0 \end{array}\right.$$

1. Write down a program that solve 2D heat equations using **Euler Implicit**. **Euler Explicit**, and **Runge Kutta 4**.
2. Solve an interesting problem that you choose.