2 | Rootfinding

Bisection Method

Bisection Method

The bisection method is to find a root x in the interval $[a, b]$ of a function f, i.e. $f(x) = 0$. You start with the function f, the endpoints $a < b$, a tolerance ε, and a max number of iteration $ITMAX$.

1. Calculate $c = \frac{a+b}{2}$, the midpoint of the interval
2. Calculate the function value at the midpoint, $f(c)$.
3. If convergence is satisfactory, i.e. $\frac{b-a}{2} < \varepsilon$ or iteration is $ITMAX$, return c and stop iterating.
4. Examine the sign of $f(c)$ and replace either a or b by c so that there is a zero crossing within the new interval, which is $[a, c]$ $(b = c)$ or $[c, b]$ $(a = c)$.

Exercise 1: Compute by hand (and a calculator) the first 5 steps of the bisection method to find the smallest root of $x = e^{-x}$. Start with the interval $[0, 2]$

Exercise 2: Write a function that take into argument a function, an interval, a tolerance, and return zero of the function using the bisection method.
To test your function: compute the root of $f(x) = x^6 - x - 1.0$ on the domain $[1, 2]$ using the bisection method with tolerance $5 \cdot 10^{-4}$. It should take 11 iterations and the root is 1.1342773437

Exercise 3: Use bisection method and graph of $f(x)$ to find all the roots of

$$f(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

within 10^{-10} accuracy.

Exercise 4: Let α be the largest root of

$$f(x) = e^x - x - 2$$

Find an interval $[a, b]$ containing α and for which the bisection method will converge to α. Then estimate the number of iterates needed to find α within an accuracy of $5 \cdot 10^{-8}$. Then compute the root using your bisection function and see how many iterations was needed.

Newton's Method

Newton's Method

The Newton's method is to find a root x close to your initial guess x_0 a function f, i.e. $f(x) = 0$. You start with the function f, the intial guess x_0, a tolerance ε, and a max number of iteration $ITMAX$.

1. Calculate the next iteration $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, this is equivalent of finding the zero of the linear approximation of f and x_n, which is $f'(x_n)(x - x_n) + f(x_n) = 0$.
2. If convergence is satisfactory, i.e. $|f(x_{n+1})| < \varepsilon$, or $|x_{n+1} - x_n| < \varepsilon$, or iteration is $ITMAX$, return x_{n+1} and stop iterating.

Exercise 5: Write a function that take into argument a function, its derivative, an initial guess, a tolerance, a maximum number of iteration and return the zero of the function using Newton's method.
To test your function: compute the root of $f(x) = x^6 - x - 1.0$ using Newton's method with tolerance 10^{-10} and with starting point 1.0. It should take 6 iterations and the root is 1.1347

Exercise 6: The equation

$$x + e^{-Bx^2}\cos(x)$$

has a unique root in the interval $[-1, 1]$. Use Newton's method to find it as accurately as possible. Use the values of $B = 1, 5, 10, 25, 50$. What should be a good choice for $x0$? Explain the behavior observed in the iterates for the larger values of B.

Exercise 7: Solve the equation

$$x^3 - 3x^2 + 3x - 1 = 0$$

using Newton's method. (as accurately as possible)

Exercise 8: Give Newton's method for finding $\sqrt[m]{a}$, with $a > 0$ and m a positive integer. Apply it to finding $\sqrt[m]{2}$ for $m = 3, 4, 5, 6, 7, 8$, say to six significant digits.

Exercise 9: For $x^6 - x - 1.0 = 0$, compute the convergence of the Newton's method with initial guess 2.0 and comparing it to the solution 1.13472. Plot in the x axis the iteration number and in the y axis $\log|\alpha - x_n|$, what slope do you expect? (look mostly at the first 10 iterations)

Secant's Method

Secant's Method

It is the same as Newton's method, except you replace $f'(x_n)$ by its approximation $f'(x_n) \simeq \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$.
You start with the function f, the intial guess x_0, a tolerance ε, and a max number of iteration $ITMAX$.

1. Calculate the next iteration $x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$, this is equivalent of finding the zero of the linear approximation of f and x_n, which is $f'(x_n)(x - x_n) + f(x_n) = 0$.
2. If convergence is satisfactory, i.e. $|f(x_{n+1})| < \varepsilon$, or $|x_{n+1} - x_n| < \varepsilon$, or iteration is $ITMAX$, return x_{n+1} and stop iterating.

Exercise 10: Solve the equation

$$x^3 - 3x^2 + 3x - 1 = 0$$

using the secant's method. (as accurately as possible)

Exercise 11: For $x^6 - x - 1.0 = 0$, compute the convergence of the secant's method with initial guesses 1.9 and 2.0, then comparing it to the solution 1.13472. Plot in the x axis the iteration number and in the y axis $\log|\alpha - x_n|$, what slope do you expect? (look mostly at the first 10 iterations)

Fixed point

Fixed Point

Fixed point iteration is a method to find the fixed point of a function, $g(x) = x$. It can also be used to find the root of a function f by setting up $g(x) = f(x) - x$. You start with the function g, the intial guess x_0, a tolerance ε, and a max number of iteration $ITMAX$.

1. Calculate the next iteration $x_{n+1} = g(x_n)$.
2. If convergence is satisfactory, i.e. $|g(x_{n+1}) - x_{n+1}| < \varepsilon$, or $|x_{n+1} - x_n| < \varepsilon$, or iteration is $ITMAX$, return x_{n+1} and stop iterating.

The fixed point iteration algorithm does not always converges, it can be shown that it does converges when $|g'(x)" < 1$.

Exercise 12: Use fixed point iteration to solve the following equation for $x \in [-2, 2]$

$$x = 1 + 2\sin(x)$$

Exercise 13: Convert the equation $x^2 - 5 = 0$ to the fixed-point problem

$$x = x + c(x^2 - 5)$$

with c a non zero constant. Determine the possible values of c to ensure convergence. (The true solution is $\alpha = \sqrt{5}$.