3 Polynomial of Interpolations

Notes

Divided Difference

The divided difference formula is

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

and

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The divided difference approximate:

$$\frac{f^{(n)}(c)}{n!} = f[x_0, x_1, \ldots, x_n]$$

Usually we pick c as the middle point.

Polynomial of Interpolation

You have a series of $n+1$ points (x_i, y_i) (where $y_i = f(x_i)$, and you want P_n of degree n with

$$P_n(x_i) = y_i, \quad \text{for} i = 0, \ldots, n.$$

Then

$$P_n(x) = \sum_{i=0}^{n} y_i L_i(x),$$

where the L are the Lagrange polynomial

$$L_i(x) = \prod_{k=0 k\neq i}^{n} \frac{(x - x_k)}{(x_i - x_k)}.$$

Or you can write P_n as

$$P_n(x) = f(x_0) + (x - x_0)f[x_0, x_1] + \cdots + (x - x_0)\cdots(x - x_{n-1})f[x_0, \ldots, x_n]$$

Exercises

Exercise 1:

1. Write a function that compute the divided differences. (You should write this function with recursion).
2. Write a test for your function. (Pick a function you know and see if it gives you the right answer, do not forget to explain your test in your report)

Exercise 2: By using your divided differences function.

1. Approximate the third derivative of $\cos(2x) + e^x + x$ at $x = 0$, using points between -1 and 1.
2. Approximate the second derivative of $\ln(x) + x$ at $x = 1$, using $x = 0.1, 1, 2$. Compute the error (the error is the difference between your approximation and the exact value of the second derivative at $x = 1$)
3. Approximate the second derivative of $\ln(x) + x$ at $x = 1$, using $x = 0.5, 1, 1.5$. Compute the error. How does it compare to the previous part?

Exercise 3: Plot the polynomial of interpolation at $x = -1, -0.5, 0.5, 1$ of

1. $f(x) = e^x$
2. $f(x) = \sin(\pi x)$
3. $f(x) = tan^{-1}(x)$
4. $f(x) = \log(1 + x^2)$

Exercise 4: The following data are taking from a polynomial $p(x)$ of degree ≤ 5. What is its degree? Plot it. And what is this polynomial?

x	-2	-1	0	1	2	3
$p(x)$	-5	1	1	1	7	25

Exercise 5: *Using a polynomial of interpolation for prediction.* Assume we have the data set, that represents the weight of a car and its MPG.

weight	13	21	27	30
MPG	1695	1330	1248	1233

What do you expect the MPG to be if the car weight 1150? Is it a good estimate? What is the issue?

Exercise 6: *BONUS.* Create a Class of polynomials that interpolate the points (x_i, y_i) for $i = 0, \ldots, n$ such that:

1. The polynomial is of degree n,
2. and $P_n(x_i) = y_i$ for $i = 0, \ldots, n$.

The arguments should be a array of points (x_i, y_i), then you have method to evaluate the polynomial at a given point and at a series of points (a vector). You can use either Lagrange or Newton divided difference to contruct such polynomial.
Look at Complex number class in Python.

Exercise 7: *BONUS.* Find the solution to the interpolation problem of finding a polynomial $q(x)$ with $deg(q) \leq 2$ and such that

$$q(x_0) = y_0, \quad q(x_1) = y_1, \quad q'(x_1) = y'_1$$

Hint: Write $q(x) = y_0 M_0(x) + y_1 M_1(x) + y'_1 M_2(x)$.