5 Solving Linear System

direct method

Quick Review

Matrices

A 2×2 matrix is

$$\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A vector is

$$\mathsf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The identity is

$$\mathsf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For any matrix A we have

$$\mathsf{AI} = \mathsf{IA} = \mathsf{A}$$

For any vector v we have

$$\mathsf{Iv} = \mathsf{v}$$

Matrix Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bu \\ cx + dz & cy + du \end{bmatrix}$$

Vector Multiplication (usually $\mathsf{AB} \neq \mathsf{BA}$)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + cy \end{bmatrix}$$

Determinant

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cd$$

Inverse of A exists if and only if $\det(A) \neq 0$

$$\mathsf{A}^{-1}\mathsf{A} = \mathsf{A}\mathsf{A}^{-1} = \mathsf{I}$$

$$\mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow \mathsf{A}^{-1} = \frac{1}{\det(\mathsf{A})}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Eigenvalues and Eigenvectors

Definition. λ is an eigenvalue of A and $\mathsf{v} \neq 0$ is an eigenvector associated to λ if

$$\mathsf{Av} = \lambda\mathsf{v}$$

Theorem. The eigenvalues of A are the roots of

$$\det(\mathsf{A} - \lambda\mathsf{I}) = \mathbf{0}$$

The eigenvector v associated to λ solves

$$(\mathsf{A} - \lambda\mathsf{I})\mathsf{v} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linear System

We have
$$\begin{cases} ax_1 + bx_2 = u \\ cx_1 + dx_2 = v \end{cases} \iff \quad \mathsf{Ax} = \mathbf{b} \quad \text{where } \mathsf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} u \\ v \end{bmatrix}$$

LU and Cholesky Factorization

LU Factorization

$$A = LU$$

where L is lower triangular
$$L = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ l_{n,1} & & \cdots & l_{n,n-1} & 1 \end{bmatrix}$$

where U is upper triangular
$$L = \begin{bmatrix} u_{11} & u_{12} & \cdots & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & & 0 & \ddots & u_{n-1,n} \\ 0 & & \cdots & 0 & u_{nn} \end{bmatrix}$$

Cholesky Factorization

If A is symmetric definite positive
$$A = LL^T$$

where L is lower triangular
$$L = \begin{bmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ l_{n,1} & & \cdots & l_{n,n-1} & l_{nn} \end{bmatrix}$$

Exercise

Exercise 1: Write a function that compute the LU factorization of a square matrix.Then test your functions with a test case, and explain why it works.

Exercise 2: Write a function that compute the Cholesky factorization of a square matrix. Then test your functions with a test case, and explain why it works.

Exercise 3: Write a function that solve a system $LUx = b$. Then test your functions with a test case, and explain why it works.

Exercise 4: Write a function that solve a system $LL^T x = b$. Then test your functions with a test case, and explain why it works.

Exercise 5: Solve the system

$$\begin{cases} 2x_1 + x_2 & = 3 \\ -x_1 + 2x_2 + x_3 & = 2 \\ -x_2 + 2x_3 + x_4 & = 2 \\ -x_3 + 2x_4 + x_5 & = 2 \\ -x_4 + 2x_5 & = 1 \end{cases}$$

Exercise 6: Solve the system

$$\begin{cases} 2x_1 + x_2 + x_3 & = 3 \\ x_1 - x_2 + 2x_3 & = 2 \\ 3x_1 + 2x_2 + 10x_3 + 2x_4 & = 2 \\ -x_3 + 2x_4 + x_5 & = 2 \\ 2x_3 + 3x_4 & = 1 \end{cases}$$

Exercise 7: Solve the tridiagonal system $Ax = f$ with

$$A_{ii} = 4, \ \ A_{i,i-1} = A_{i,i+1} = 1$$

for all i. Let the order of the system be $n = 100$, and let

$$f = [1, 1, \dots, 1]^T$$