3 Integration and differentiation

Notes

Trapezoidal Rule

Let $a = x_0 < x_1 < \cdot < x_n = b$, with

$$x_i = a + ih \text{ for } i = 0, \ldots, n \text{ and } h = \frac{b-a}{n}.$$

Then,

$$\int_a^b f(x)dx \simeq T_h = h\left[\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right].$$

The error formula between the exact value of the integral the Trapezoidal Rule is

$$\Big|\int_a^b f(x)dx - T_h\Big| \leq h^2\frac{b-a}{12}|f''(c)|.$$

To compute the error replace $|f''(c)|$ by its maximum value on $[a, b]$ and replace h by $\frac{b-a}{n}$.

Simpson's Rule

Let $a = x_0 < x_1 < \cdot < x_n = b$, with

$$x_i = a + ih \text{ for } i = 0, \ldots, n \text{ and } h = \frac{b-a}{n}.$$

Then,

$$\int_a^b f(x)dx \simeq S_h = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

The error formula between the exact value of the integral the Simpson's Rule is

$$\Big|\int_a^b f(x)dx - S_h\Big| \leq h^4\frac{b-a}{180}|f^{(4)}(c)|.$$

To compute the error replace $|f^{(4)}(c)|$ by its maximum value on $[a, b]$ and replace h by $\frac{b-a}{n}$.

Monte Carlo integration

In 1D, pick n random points x_i between a and b, then

$$\int_a^b f(x)dx \simeq (b-a)\frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

In 2D, pick n random points (x_i, y_i) in S, then

$$\iint_S f(x)dx \simeq |S|\frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

First derivative

Forward formula:

$$f'(x) \approx \frac{f(x+h)-f(x)}{h} \equiv D_h(x)$$

Centered formula:

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h} \equiv D_h(x)$$

Backward formula:

$$f'(x) \approx \frac{f(x)-f(x-h)}{h} \equiv D_h(x)$$

Centered formula second derivative

$$f''(x) \approx \frac{f(x+h)-2f(x)+f(x-h)}{h^2} \equiv D_h^{(2)}(x)$$