4 Integration and differentiation

Notes

Trapezoidal Rule

Let $a = x_0 < x_1 < \cdot < x_n = b$, with

$$x_i = a + ih \text{ for } i = 0, \ldots, n \text{ and } h = \frac{b-a}{n}.$$

Then,

$$\int_a^b f(x)dx \simeq T_h = h\left[\frac{1}{2}f(x_0) + f(x_1) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right].$$

The error formula between the exact value of the integral the Trapezoidal Rule is

$$\Big|\int_a^b f(x)dx - T_h\Big| \leq h^2 \frac{b-a}{12}|f''(c)|.$$

To compute the error replace $|f''(c)|$ by its maximum value on $[a, b]$ and replace h by $\frac{b-a}{n}$.

Simpson's Rule

Let $a = x_0 < x_1 < \cdot < x_n = b$, with

$$x_i = a + ih \text{ for } i = 0, \ldots, n \text{ and } h = \frac{b-a}{n}.$$

Then,

$$\int_a^b f(x)dx \simeq S_h = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right].$$

The error formula between the exact value of the integral the Simpson's Rule is

$$\Big|\int_a^b f(x)dx - S_h\Big| \leq h^4 \frac{b-a}{180}|f^{(4)}(c)|.$$

To compute the error replace $|f^{(4)}(c)|$ by its maximum value on $[a, b]$ and replace h by $\frac{b-a}{n}$.

Monte Carlo integration

In 1D, pick n random points x_i between a and b, then

$$\int_a^b f(x)dx \simeq (b-a)\frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

In 2D, pick n random points (x_i, y_i) in S, then

$$\iint_S f(x)dx \simeq |S|\frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

First derivative

Forward formula:

$$f'(x) \approx \frac{f(x+h)-f(x)}{h} \equiv D_h(x)$$

Centered formula:

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h} \equiv D_h(x)$$

Backward formula:

$$f'(x) \approx \frac{f(x)-f(x-h)}{h} \equiv D_h(x)$$

Centered formula second derivative

$$f''(x) \approx \frac{f(x+h)-2f(x)+f(x-h)}{h^2} \equiv D_h^{(2)}(x)$$

Exercises

Exercise 1: Write a function that compute the integral using the Trapezoidal Rule. The function should take in arguments f, a, b, and n (the number of points).

Exercise 2: Write a function that compute the integral using the Simpson's Rule. The function should take in arguments f, a, b, and n (the number of points).

Exercise 3: Use the Trapezoidal Rule and Simpson's Rule with $n = 4, 8, 16, .., 512$ to find approximate values of the following integrals. Then compare it the the exact value of the integral using Sympy and plot the error

1. $\int_0^{10} e^{-x^2} dx$,
2. $\int_0^2 tan^{-1}(1+x^2)dx$,
3. $\int_0^1 \sqrt{x}e^x dx$,

Exercise 4: Use the Trapezoidal Rule and Simpson's Rule with $n = 4, 8, 16, .., 512$ to find the length of the curves

1. $f(x) = \sin(\pi x), 0 \leq x \leq 1$
2. $f(x) = e^x, 0 \leq x \leq 1$
3. $f(x) = e^{x^2}, 0 \leq x \leq 1$

Remember the length of the curve is

$$\int_a^b \sqrt{1 + [f'(x)'^2} dx$$

Exercise 5: In the following instances, find the numerical derivative at the indicated point, using the backward, forward, and centered formula. Use $h = 0.1, 0.05, 0.025, 0.0125, 0.00625$, then each case compute the error.

1. e^x at $x = 0$.
2. $\tan^{-1}(x^2 - x + 1)$ at $x = 1$.
3. $\tan^{-1}(100x^2 - 199x + 100)$ at $x = 1$.

Exercise 6: Using the error formulas, compute how many point you need in order to approximate the integral $\int_1^3 x \ln(x) dx$ with an error less than 10^{-6} using the Trapezoidal Rule and the Simpson's Rule. Then compute this integral.

Exercise 7: *BONUS*. The degree of precision of a numerical integration formula is defined as follows: if the formula is exact (has zero error) when integration any polynomial of degree $\leq r$, and if there is an error for polynomials of degree $> r$, then we say the formula has degree of precision r.

1. Find the degree of precision of the Trapezoidal Rule when $n = 1$.
2. Find the degree of precision of the Simpson's Rule when $n = 2$.

Exercise 8: *BONUS*. Using the Monte Carlo integration, compute the value of π.