7 Nonlinear system of equations

Methods

The goal is to solve

$$\mathbf{F}(x_1, x_2, \ldots, x_n) = \begin{bmatrix} F_1(x_1, x_2, \ldots, x_n) \\ F_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ F_n(x_1, x_2, \ldots, x_n) \end{bmatrix} = \mathbf{0}$$

For simplicity

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \partial x_2 \\ \vdots \\ \partial x_n \end{bmatrix} \text{ and } \mathbf{F} = \begin{bmatrix} F_1 \\ \partial F_2 \\ \vdots \\ \partial F_n \end{bmatrix} \Longrightarrow \mathbf{F}(\mathbf{x}(t)) = 0$$

The gradient

$$\nabla F(\mathbf{x}) = \nabla F(x_1, x_2, \ldots, x_n) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & . & \cdots & \vdots \\ \frac{\partial F_n}{\partial x_n} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix}$$

Newton's Method

By using Taylor formula in higher dimension:

$$\mathbf{x}^{k+1} = x^k - \alpha \left[\nabla F(\mathbf{x}^k) \right]^{-1} \mathbf{F}(\mathbf{x}^k)$$

Now if the goal is to minimize

$$F(x_1, x_2, \ldots, x_n)$$

Gradient Descent

$$\mathbf{x}^{k+1} = x^k - \alpha \nabla F(\mathbf{x}^k)$$

Exercise

Exercise 1: Write a function and a test for each methods.

Exercise 2: Solve the system

$$\begin{cases} x^2 + y^2 = 4 \\ x^2 - y^2 = 1 \end{cases}$$

Exercise 3: Find all the solutions to the system

$$\begin{cases} x^2 + xy^3 = 9 \\ 3x^2y - y^3 = 4 \end{cases}$$

Use each initial guess $(x_0, y_0) = (1.2, 2.5)$, $(-2, 2.5)$, $(-1.2, -2.5)$, $(2, -2.5)$. Observe the root to which each of the iterations converges and the number of iterates computed. Comment on your results.

Exercise 4: Bioremediation involves the use of bacteria to consume toxic wastes. At a steady state, the bacterial density x and the nutrient concentration y satisfy the system of nonlinear equations

$$\begin{aligned} \gamma xy - x(1+y) &= 0 \\ -xy + (\delta - y)(1+y) &= 0, \end{aligned}$$

where γ and δ are parameters that depend on various physical features of the system.
For this problem, assume the typical values $\gamma = 5$ and $\delta = 1$, find the 3 solutions.

Exercise 5: Minimize the function below using Newton's method and gradient descent methods.

$$D(x_1, x_2) = x_1^4 + x_1x_2 + (1 + x_2)^2$$

Exercise 6: Minimize the function (use gradient descent)

$$f(x_1, x_2) = (1 - x_1)^2 + 100(x_2 - x_1^2)^2$$

Exercise 7: Minimize the function (use gradient descent)

$$f(x, y) = \sin\left(\frac{1}{2}x^2 - \frac{1}{4}y^2 + 3\right)\cos(2x + 1 - e^y)$$