3 Polynomial of Interpolations

Notes

Divided Difference

The divided difference formula is

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

The divided difference approximate:

$$\frac{f^{(n)}(c)}{n!} = f[x_0, x_1, \ldots, x_n]$$

Usually we pick c as the middle point.

Polynomial of Interpolation

You have a series of $n+1$ points (x_i, y_i) (where $y_i = f(x_i)$, and you want P_n of degree n with

$$P_n(x_i) = y_i, \quad \text{for} i = 0, \ldots, n.$$

Then

$$P_n(x) = \sum_{i=0}^{n} y_i L_i(x),$$

where the L are the Lagrange polynomial

$$L_i(x) = \prod_{k=0 k\neq i}^{n} \frac{(x - x_k)}{(x_i - x_k)}.$$

Or you can write P_n as

$$P_n(x) = f(x_0) + (x - x_0)f[x_0, x_1] + \cdots + (x - x_0)\cdots(x - x_{n-1})f[x_0, \ldots, x_n]$$

Exercises

Exercise 1: Write a function that compute the divided differences. (You should write this function with recursion). Then test your function with a polynomial that you know.

Exercise 2: By using your divided differences function.

1. Approximate the third derivative of $\cos(2x) + e^x + x$ at $x = 1.5$, using $x = 0, 1, 2, 3$.
2. Approximate the second derivative of $\ln(x) + x$ at $x = 1$, using $x = 0.1, 1, 2$. Compute the error.
3. Approximate the second derivative of $\ln(x) + x$ at $x = 1$, using $x = 0.5, 1, 1.5$. Compute the error. How does it compare to the previous part?

Exercise 3:

1. By using your divided differences function, calculate $D_0 = f(x_0)$, $D_1 = f(x_0, x_1]$,...,$D_5 = f[x_0, x_1, x_3, x_4, x_5]$, for $f(x) = e^x$. Use $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $x_4 = 0.8$, $x_5 = 1.0$.
2. Using the previous results calculate $P_j(x)$ of $j = 1, 2, 3, 4, 5$ at $x = 0.1, 0.3, 0.5, 0$. Compare these results to the true value of e^x.

Exercise 4: Create a Class of polynomials that interpolate the points (x_i, y_i) for $i = 0, \ldots, n$ such that:

1. The polynomial is of degree n,
2. and $P_n(x_i) = y_i$ for $i = 0, \ldots, n$.

The arguments should be a array of points (x_i, y_i), then you have method to evaluate the polynomial at a given point and at a series of points (a vector). You can use either Lagrange or Newton divided difference to contruct such polynomial.

Exercise 5: Plot the polynomial of interpolation at $x = -1, -0.5, 0.5, 1$ of

1. $f(x) = \sin(\pi x)$
2. $f(x) = tan^{-1}(x)$
3. $f(x) = \log(1 + x^2)$

Exercise 6: The following data are taking from a polynomial $p(x)$ of degree ≤ 5. What is its degree? Plot it. And what is this polynomial?

x	-2	-1	0	1	2	3
$p(x)$	-5	1	1	1	7	25

Exercise 7: Find the solution to the interpolation problem of finding a polynomial $q(x)$ with $deg(q) \leq 2$ and such that

$$q(x_0) = y_0, \quad q(x_1) = y_1, \quad q'(x_1) = y'_1$$

Hint: Write $q(x) = y_0 M_0(x) + y_1 M_1(x) + y'_1 M_2(x)$.

Exercise 8: Prove that there is only one polynomial $P_3(x)$ among all polynomials of degree ≤ 3 that satisfy the interpolating conditions

$$P_3(x_i) = y_i, \quad i = 0, 1, 2, 3.$$

What can you generalize to the degree n?

Exercise 9: Find the linear and quadratic least square approximation to $f(x) = \sin(x)$ on the interval $[0, \pi]$ and plot it. (use Legendre and sympy)

Exercise 10: BONUS. Create a Class for the least square polynomial that uses the points (x_i, y_i) instead of integral. The class should have a method to evaluate the polynomial at a given point or at a vector of points.

To construct such a class you should use use sympy, compute the Legrendre polynomial and then compute the real lease square approximation

Exercise 11: BONUS. Plot the least square apporixmation of degree 4, using the 10 points in $[-1, 1]$, of

1. $f(x) = \sin(\pi x)$
2. $f(x) = tan^{-1}(x)$
3. $f(x) = \log(1 + x^2)$