Quantum Notes

antonio.macaluso

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A Ensemble as simple averaging

A.1 Example $d = 2$

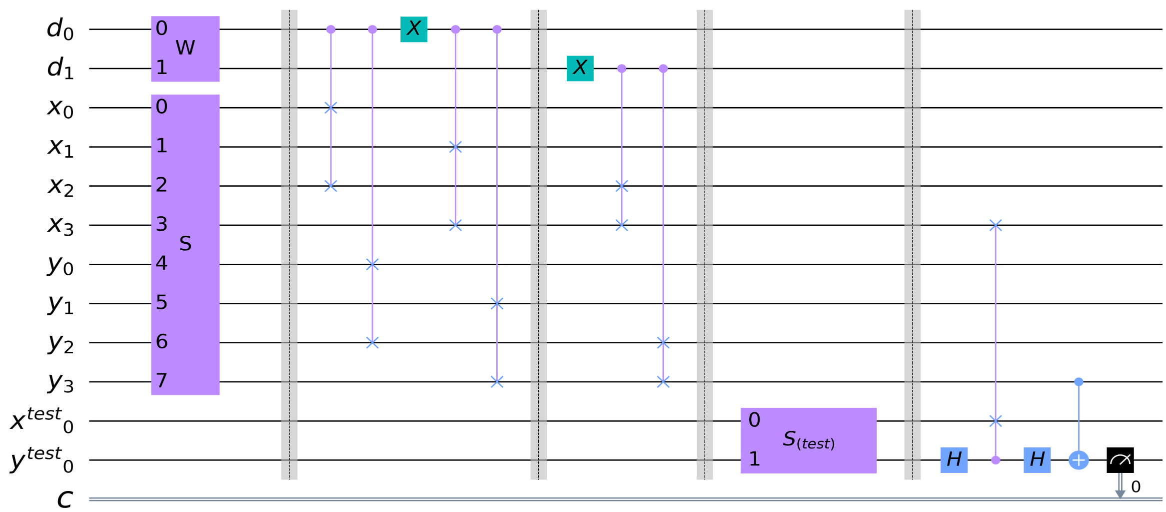

Figure 1

$$
\begin{aligned}
|\Phi_1\rangle =& \big(W \otimes S_{(x,y)}\big) |\Phi_0\rangle = \big(H^{\otimes 2} \otimes S_{(x,y)}\big) |0\rangle \otimes |0\rangle \otimes |0\rangle \\
=& \frac{1}{\sqrt{2}} \big(|0\rangle + |1\rangle \big) \otimes \frac{1}{\sqrt{2}} \big(|0\rangle + |1\rangle \big) \otimes |x_0, x_1, x_2, x_3\rangle |y_0, y_1, y_2, y_3\rangle \\
=& \frac{1}{\sqrt{2}} \big(|0\rangle + |1\rangle \big) \otimes \frac{1}{\sqrt{2}} \big(|0\rangle + |1\rangle \big) \otimes |x\rangle |y\rangle \\
=& |c_1\rangle \otimes |c_2\rangle \bigotimes_{i=1}^{4} |x_b\rangle \bigotimes_{i=1}^{4} |y_b\rangle
\end{aligned}
$$

$$
\begin{aligned}
U_{(1,1)} =& SWAP(x_0, x_2) \times SWAP(y_0, y_2) \\
U_{(1,2)} =& SWAP(x_1, x_3) \times SWAP(y_1, y_3) \\
U_{(2,1)} =& \mathbb{1} \\
U_{(2,2)} =& SWAP(x_2, x_3) \times SWAP(y_2, y_3)
\end{aligned}
$$

This architecture allows to access to all the different points in table (**??**) in superposition and apply the quantum cosine classifier in parallel [1]. Thus, measuring the last qubit we obtain state $|0\rangle$ or $|1\rangle$ with probability equals to the average of the probabilities provided by the quantum cosine classifier with four different training points. The results of implementation are shown in Figure (**??**).

$$
\begin{aligned}
|\Phi_{1.1}\rangle =& \Big(\mathbb{1} \otimes CU_{(1,1)}\Big) |\Phi_1\rangle \\
=& \Big(\mathbb{1} \otimes CU_{(1,1)}\Big) |c_1\rangle \otimes \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle \big) \otimes |x\rangle |y\rangle \\
=& |c_1\rangle \otimes \frac{1}{\sqrt{2}}\big(|0\rangle |x\rangle |y\rangle + |1\rangle U_{(1,1)} |x\rangle |y\rangle \big) \\
=& |c_1\rangle \otimes \frac{1}{\sqrt{2}}\big(|0\rangle |x\rangle |y\rangle + |1\rangle |x_2, x_1, x_0, x_3\rangle \otimes |y_2, y_1, y_0, y_3\rangle \big)
\end{aligned}
$$

$$
\begin{aligned}
|\Phi_{1.2}\rangle =& (\mathbb{1} \otimes X \otimes \mathbb{1}) |\Phi_{1.1}\rangle \\
=& |c_1\rangle \otimes \frac{1}{\sqrt{2}}\big(|1\rangle |x\rangle |y\rangle + |0\rangle |x_2, x_1, x_0, x_3\rangle \otimes |y_2, y_1, y_0, y_3\rangle \big)
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
|\Phi_2\rangle =& (\mathbb{1} \otimes CU_{1,2}) |\Phi_{1.2}\rangle \\
=& |c_1\rangle \otimes \frac{1}{\sqrt{2}}\big(|1\rangle U_{(1,2)} |x_0, x_1, x_2, x_3\rangle |y_0, y_1, y_2, y_3\rangle + |0\rangle |x_2, x_1, x_0, x_3\rangle \otimes |y_2, y_1, y_0, y_3\rangle \big) \\
=& |c_1\rangle \otimes \frac{1}{\sqrt{2}}\big(|1\rangle |x_0, x_3, x_2, x_1\rangle |y_0, y_3, y_2, y_1\rangle + |0\rangle |x_2, x_1, x_0, x_3\rangle \otimes |y_2, y_1, y_0, y_3\rangle \big)
\end{aligned} \tag{2}
$$

Finally, two different transformations ($U_{(1,1)}$ and $U_{(1,2)}$) of the initial state $|x, y\rangle$ are generated in superposition and are entangled with the quantum states of the d–th control qubit.

We repeat these last three steps by considering the two unitaries $U_{(2,1)}$ and $U_{(2,2)}$ and the $(d-1)$–th qubit of the control register. The first controlled-unitary is applied to entangle the transformation of $|x, y\rangle$ with a specific state

[1] For more details about the implementation see the github

of the control qubit:

$$
\begin{aligned}
|\Phi_{2.1}\rangle =& (C \otimes \mathbb{1} \otimes U_{(2,1)}) |\Phi_2\rangle \\
=& \frac{1}{2}\Big[|0\rangle \Big(|1\rangle |x_0, x_3, x_2, x_1\rangle |y_0, y_3, y_2, y_1\rangle + |0\rangle |x_2, x_1, x_0, x_3\rangle |y_2, y_1, y_0, y_3\rangle \Big) + \\
& + |1\rangle \Big(|1\rangle |x_0, x_3, x_2, x_1\rangle |y_0, y_3, y_2, y_1\rangle + |0\rangle |x_2, x_1, x_0, x_3\rangle |y_2, y_1, y_0, y_3\rangle \Big) \Big]
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
|\Phi_{2.2}\rangle =& (X \otimes \mathbb{1} \otimes \mathbb{1}) |\Phi_{2.1}\rangle \\
=& \frac{1}{2}\Big[|1\rangle \Big(|1\rangle |x_0, x_3, x_2, x_1\rangle |y_0, y_3, y_2, y_1\rangle + |0\rangle |x_2, x_1, x_0, x_3\rangle |y_2, y_1, y_0, y_3\rangle \Big) + \\
& + |0\rangle \Big(|1\rangle |x_0, x_3, x_2, x_1\rangle |y_0, y_3, y_2, y_1\rangle + |0\rangle |x_2, x_1, x_0, x_3\rangle |y_2, y_1, y_0, y_3\rangle \Big) \Big]
\end{aligned} \tag{4}
$$

Then, a second controlled-unitary is executed where the position of C gate indicates the control qubit used to apply $U_{(2,2)}$:

$$
\begin{aligned}
|\Phi_3\rangle =& (C \otimes \mathbb{1} \otimes U_{(2,2)}) |\Phi_{2.2}\rangle \\
=& \frac{1}{2}\Big[|1\rangle \Big(|1\rangle U_{(2,2)} |x_0, x_3, x_2, x_1\rangle |y_0, y_3, y_2, y_1\rangle + |0\rangle U_{(2,2)} |x_2, x_1, x_0, x_3\rangle |y_2, y_1, y_0, y_3\rangle \Big) + \\
& + |0\rangle \Big(|1\rangle |x_0, x_3, x_2, x_1\rangle |y_0, y_3, y_2, y_1\rangle + |0\rangle |x_2, x_1, x_0, x_3\rangle |y_2, y_1, y_0, y_3\rangle \Big) \Big] \\
=& \frac{1}{2}\Big[|1\rangle \Big(|1\rangle |x_0, x_3, x_1, x_2\rangle |y_0, y_3, y_1, y_2\rangle + |0\rangle |x_2, x_1, x_3, x_0\rangle |y_2, y_1, y_3, y_0\rangle \Big) + \\
& + |0\rangle \Big(|1\rangle |x_0, x_3, x_2, x_1\rangle |y_0, y_3, y_2, y_1\rangle + |0\rangle |x_2, x_1, x_0, x_3\rangle |y_2, y_1, y_0, y_3\rangle \Big) \Big]
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
|\Phi_3\rangle =& \frac{1}{2}\Big[|11\rangle |x_0\rangle |x_3\rangle |x_1\rangle |x_2\rangle |y_0\rangle |y_3\rangle |y_1\rangle |y_2\rangle + |10\rangle |x_2\rangle |x_1\rangle |x_3\rangle |x_0\rangle |y_2\rangle |y_1\rangle |y_3\rangle |y_0\rangle + \\
& + |01\rangle |x_0\rangle |x_3\rangle |x_2\rangle |x_1\rangle |y_0\rangle |y_3\rangle |y_2\rangle |y_1\rangle + |00\rangle |x_2\rangle |x_1\rangle |x_0\rangle |x_3\rangle |y_2\rangle |y_1\rangle |y_0\rangle |y_3\rangle \Big]
\end{aligned} \tag{6}
$$

$$
|\Phi_3\rangle = \frac{1}{2}\Big[|11\rangle |x_2\rangle |y_2\rangle + |10\rangle |x_0\rangle |y_0\rangle + |01\rangle |x_1\rangle |y_1\rangle + |00\rangle |x_3\rangle |y_3\rangle \Big] = \frac{1}{2}\sum_{i=0}^{3} |i\rangle |x_i\rangle |y_i\rangle \tag{7}
$$

$$
|\Phi_f\rangle = \Big(\mathbb{1}^{\otimes 2} \otimes F\Big) |\Phi_d\rangle = (\mathbb{1}^{\otimes 2} \otimes F)\Big[\frac{1}{\sqrt{2^d}} \sum_{i=1}^{2^d} |i\rangle |x_i\rangle |y_i\rangle |\tilde{x}\rangle |0\rangle \Big] = \frac{1}{\sqrt{2^2}} \sum_{i=0}^{2^2-1} |i\rangle |x_i, y_i\rangle |\tilde{x}\rangle |f_i\rangle \tag{8}
$$

First, the controlled-unitary $CU_{(1,1)}$ is executed to entangle the transformation $U_{(1,1)}\left|x,y\right\rangle$ with the excited state of the d-th control qubit:

$$\begin{aligned}\left|\Phi_{1.1}\right\rangle =&\Big(\mathbb{1}^{\otimes d-1}\otimes CU_{(1,1)}\Big)\left|\Phi_1\right\rangle = \Big(\mathbb{1}^{\otimes d-1}\otimes CU_{(1,1)}\Big)\bigotimes_{j=1}^{d-1}\left|c_j\right\rangle\otimes\frac{1}{\sqrt{2}}\Big(\left|0\right\rangle+\left|1\right\rangle\Big)\otimes\left|x,y\right\rangle\\ =&\bigotimes_{j=1}^{d-1}\left|c_j\right\rangle\otimes\frac{1}{\sqrt{2}}\Big(\left|0\right\rangle\left|x,y\right\rangle+\left|1\right\rangle U_{(1,1)}\left|x,y\right\rangle\Big)\end{aligned} \tag{9}$$

Second, the d–th control qubit is transformed based on Pauli–X gate, so that the two basis states are swapped:

$$\left|\Phi_{1.2}\right\rangle =(\mathbb{1}^{\otimes d-1}\otimes X\otimes\mathbb{1})\left|\Phi_{1.1}\right\rangle = \bigotimes_{j=1}^{d-1}\left|c_j\right\rangle\otimes\frac{1}{\sqrt{2}}\Big(\left|1\right\rangle\left|x,y\right\rangle+\left|0\right\rangle U_{(1,1)}\left|x,y\right\rangle\Big) \tag{10}$$

Third, a second controlled-unitary $CU_{(1,2)}$ is executed:

$$\begin{aligned}\left|\Phi_2\right\rangle =&(\mathbb{1}^{\otimes d-1}\otimes CU_{(1,2)})\bigotimes_{j=1}^{d-1}\left|c_j\right\rangle\otimes\frac{1}{\sqrt{2}}\Big(\left|1\right\rangle\left|x,y\right\rangle+\left|0\right\rangle U_{(1,1)}\left|x,y\right\rangle\Big)\\ =&\bigotimes_{j=1}^{d-1}\left|c_j\right\rangle\otimes\frac{1}{\sqrt{2}}\Big(\left|1\right\rangle U_{(1,2)}\left|x,y\right\rangle+\left|0\right\rangle U_{(1,1)}\left|x,y\right\rangle\Big)\end{aligned} \tag{11}$$

At this point, two different transformations ($U_{(1,1)}$ and $U_{(1,2)}$) of the initial state $\left|x,y\right\rangle$ are generated in superposition and they are entangled with the two basis states of the *d-th* control qubit.

The second step of the algorithm consists in the entanglement between the two basis states of the *(d-1)-th* qubit of the control register with the two unitaries $U_{(2,1)}$ and $U_{(2,2)}$. First, the controlled-unitary $U_{(2,1)}$ is applied to entangle a transformation of $\left|x,y\right\rangle$ with a specific state of the control qubit:

$$\begin{aligned}\left|\Phi_{2.1}\right\rangle =&(\mathbb{1}^{\otimes d-2}\otimes C\otimes\mathbb{1}\otimes U_{(2,1)})\left|\Phi_2\right\rangle\\ =&\bigotimes_{j=1}^{d-2}\left|c_j\right\rangle\otimes\frac{1}{2}\Big[\left|0\right\rangle\Big(\left|1\right\rangle U_{(1,2)}\left|x,y\right\rangle+\left|0\right\rangle U_{(1,1)}\left|x,y\right\rangle\Big)+\\ &+\left|1\right\rangle\Big(\left|1\right\rangle U_{(2,1)}U_{(1,2)}\left|x,y\right\rangle+\left|0\right\rangle U_{(2,1)}U_{(1,1)}\left|x,y\right\rangle\Big)\Big]\end{aligned} \tag{12}$$

where the position of C gate indicates the control qubit used to apply $U_{(2,1)}$.

Second, the current control qubit is transformed based on Pauli–X gate:

$$\begin{aligned}\left|\Phi_{2.2}\right\rangle =&(\mathbb{1}^{\otimes d-2}\otimes\mathbb{1}\otimes X\otimes\mathbb{1})\left|\Phi_{2.1}\right\rangle\\ =&\bigotimes_{j=1}^{d-2}\left|c_j\right\rangle\otimes\frac{1}{2}\Big[\left|1\right\rangle\Big(\left|1\right\rangle U_{(1,2)}\left|x,y\right\rangle+\left|0\right\rangle U_{(1,1)}\left|x,y\right\rangle\Big)+\\ &+\left|0\right\rangle\Big(\left|1\right\rangle U_{(2,1)}U_{(1,2)}\left|x,y\right\rangle+\left|0\right\rangle U_{(2,1)}U_{(1,1)}\left|x,y\right\rangle\Big)\Big]\end{aligned} \tag{13}$$

Third, a second controlled-unitary $CU_{(2,2)}$ is executed:

$$\begin{aligned}|\Phi_3\rangle =&(\mathbb{1}^{\otimes d-2} \otimes C \otimes \mathbb{1} \otimes U_{(2,2)}) |\Phi_{2.2}\rangle \\ =&\bigotimes_{j=1}^{d-2} |c_j\rangle \otimes \frac{1}{2}\Big[|1\rangle \Big(|1\rangle U_{(2,2)}U_{(1,2)} |x, y\rangle + |0\rangle U_{(2,2)}U_{(1,1)} |x, y\rangle \Big)+ \\ &+ |0\rangle \Big(|1\rangle U_{(2,1)}U_{(1,2)} |x, y\rangle + |0\rangle U_{(2,1)}U_{(1,1)} |x, y\rangle \Big)\Big] \end{aligned} \quad (14)$$

Gathering the states of the two control qubits and indicating with natural number the basis states:

$$\begin{aligned}|\Phi_3\rangle =& \bigotimes_{j=1}^{d-2} |c_j\rangle \otimes \frac{1}{2}\Big[|4\rangle U_{(2,2)}U_{(1,2)} |x, y\rangle + |3\rangle U_{(2,2)}U_{(1,1)} |x, y\rangle + \\ & |2\rangle U_{(2,1)}U_{(1,2)} |x, y\rangle + |1\rangle U_{(2,1)}U_{(1,1)} |x, y\rangle \Big] \\ =& \bigotimes_{j=1}^{d-2} |c_j\rangle \otimes \frac{1}{\sqrt{2^2}} \sum_{b=1}^{2^2} |b\rangle V_b |x, y\rangle \end{aligned} \quad (15)$$

where V_b is the product of $d = 2$ unitaries $U_{(i,j)}$ for $i, j = 1, 2$. We can see that using 2 control qubits we generated 4 different quantum trajectories that correspond to 4 different transformations of data $|x, y\rangle$. Repeating this procedure d times with different control qubits result in the following quantum state:

$$|\Phi_d\rangle = \frac{1}{\sqrt{2^d}} \sum_{b=1}^{2^d} |b\rangle V_b |x, y\rangle = \frac{1}{\sqrt{2^d}} \sum_{b=1}^{2^d} |b\rangle |x_b, y_b\rangle \quad (16)$$

where each V_b is the product of d unitaries $U_{(i,j)}$ for $i = 1, \cdots, d$ and $j = 1, 2$.
[**AM**: Qui finiscono i due step (ognnuno composto da 3 sottostep. Scriverne due mi serve a definire V_b]

References