*Enumeration of locally-restricted digraphs and pattern-avoiding compositions*

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**Abstract**

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# Contents

# 1 From integer compositions to locally-restricted semigroup-weighted digraphs

convert to local only and homomorphic not isomorphic matching. incorporate literature review of compositions and local restrictions.

We say a directed graph (*digraph*) is a pair $(V, E)$, where $V$ is a finite set (the "vertices"), and $E \subseteq V \times V$ is a binary relation (the "arcs"). If $\Gamma$ is a digraph, we write $V(\Gamma)$ and $E(\Gamma)$ for the sets of vertices and arcs of $\Gamma$. A *weighted digraph* is a digraph $(V, E)$ together with a vertex weight function $w : V \to S$, where $S$ is a fixed semigroup.

Given digraphs $\Gamma_1, \Gamma_2$, a digraph *homomorphism* is a function $h : V(\Gamma_1) \to V(\Gamma_2)$ such that for any two vertices $u, v \in V(\Gamma_1)$, we have

$$(u, v) \in E(\Gamma_1) \implies (h(u), h(v)) \in E(\Gamma_2).$$

If $\Gamma_1, \Gamma_2$ have weight functions $w_1, w_2$, a *weighted homomorphism* is a homomorphism $h$ such that $w_1(v) = w_2(h(v))$ for all $v \in V(\Gamma_1)$. A (weighted) digraph *isomorphism* is a bijective homomorphism.

A "one-step" subdivision of a digraph $(V, E)$ is a new digraph $(V', E')$, where $V' = V \dot\cup \{v\}$ and for some $(v_1, v_2) \in E$, we have

$$E' = \{(v_1, v), (v, v_2)\} \cup E \setminus \{(v_1, v_2)\}.$$

A weighted one-step subdivision is one where the weight function is not modified except for the new vertex $v$, which may take any weight. In general, a *(weighted) subdivision* of a digraph is a graph obtained by any number of one-step subdivisions.

**Example 1.1.** Figure 1 shows digraphs $\Gamma, \Gamma'$. The digraph $\Gamma'$ is a subdivision of $\Gamma$ obtained by adding the vertices $u_5, u_6$ which are shown in bold. $\triangle$

Given weighted digraphs $\Gamma$, $P$, a weighted subdivision $P'$ of $P$, and a subgraph $\Gamma_1$ of $\Gamma$, if we have a weighted isomorphism $h : V(P') \to V(\Gamma_1)$, we say the *match of $h$ with respect to $P$* is $h_{|V(P)}$. [So matches do not include information on arcs, only vertices.] A *local occurrence of $P$ in $\Gamma$* is the match with respect to $P$ of some weighted isomorphism from $P$ to a subgraph of $\Gamma$. A *global occurrence of $P$* is the match with respect to $P$ of some weighted isomorphism from a subdivision $P'$ of $P$ to a subgraph of $\Gamma$.

**Example 1.2.** The digraphs $\Gamma, P$ in Figure 2 shows weighted digraphs $\Gamma, P$ and there exist no local occurrences of $P$ in $\Gamma$ but many global occurrences. One global occurrence is given by matching the vertices

$$u_1 \mapsto v_1, u_2 \mapsto v_2, u_3 \mapsto v_3.$$

Another is given by

$$u_1 \mapsto v_1, u_2 \mapsto v_5, u_3 \mapsto v_6. \qquad \triangle$$

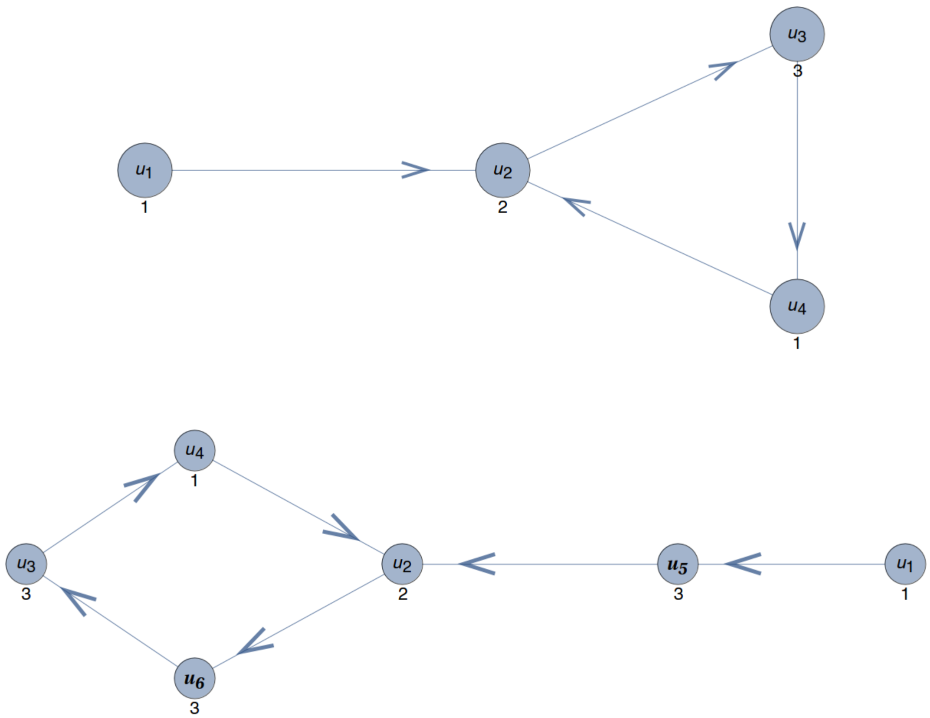


Figure 1: Weighted digraphs $\Gamma$ (above) and $\Gamma'$ (below). Vertex weights are shown below the corresponding vertices.

A *pattern set* is a set $\mathcal{P}$ of weighted digraphs such that the set of diameters of $\mathcal{P}$ is bounded.

Take a set of unweighted digraphs $\mathcal{G}$ and a pattern set $\mathcal{P}$. We define the locally restricted class $L_r(\mathcal{G}, \mathcal{P})$ as all weighted versions of digraphs in $\mathcal{G}$ which contain a total of $r$ local occurrences of patterns in $\mathcal{P}$. The globally restricted class $G_r(\mathcal{G}, \mathcal{P})$ is defined analogously. We define $\tilde{L}_r(\mathcal{G}, \mathcal{P}), \tilde{G}_r(\mathcal{G}, \mathcal{P})$ as the sets of unlabeled versions of the weighted digraphs in $L_r(\mathcal{G}, \mathcal{P}), G_r(\mathcal{G}, \mathcal{P})$. More generally, $L(\mathcal{G}; \mathcal{P}_1, \mathcal{P}_2, \ldots; r_1, r_2, \ldots)$, etc. are the sets of objects with $r_j$ occurrences of patterns in $\mathcal{P}_j$.

**Example 1.3.** Define the set of paths

$$\mathcal{G}_p = \Big\{\{(j, j+1) : 1 \leq j < n\} : n \geq 1\Big\},$$

and define the set of cycles

$$\mathcal{G}_c = \Big\{\{(j, j+1) : 1 \leq j < n\} \cup \{(n, 1)\} : n \geq 1\Big\}.$$

Then weighted digraphs in $\mathcal{G}_p$ are words/compositions, weighted versions of digraphs in $\mathcal{G}_c$ are cyclic words/compositions, and weighted, unlabeled versions of digraphs in $\mathcal{G}_c$ are circular words/compositions. $\triangle$

Take a directed path pattern $P$ with $n$ vertices, with weight function $w$. In the context of global matching, we use a short notation for $P$ using hyphens

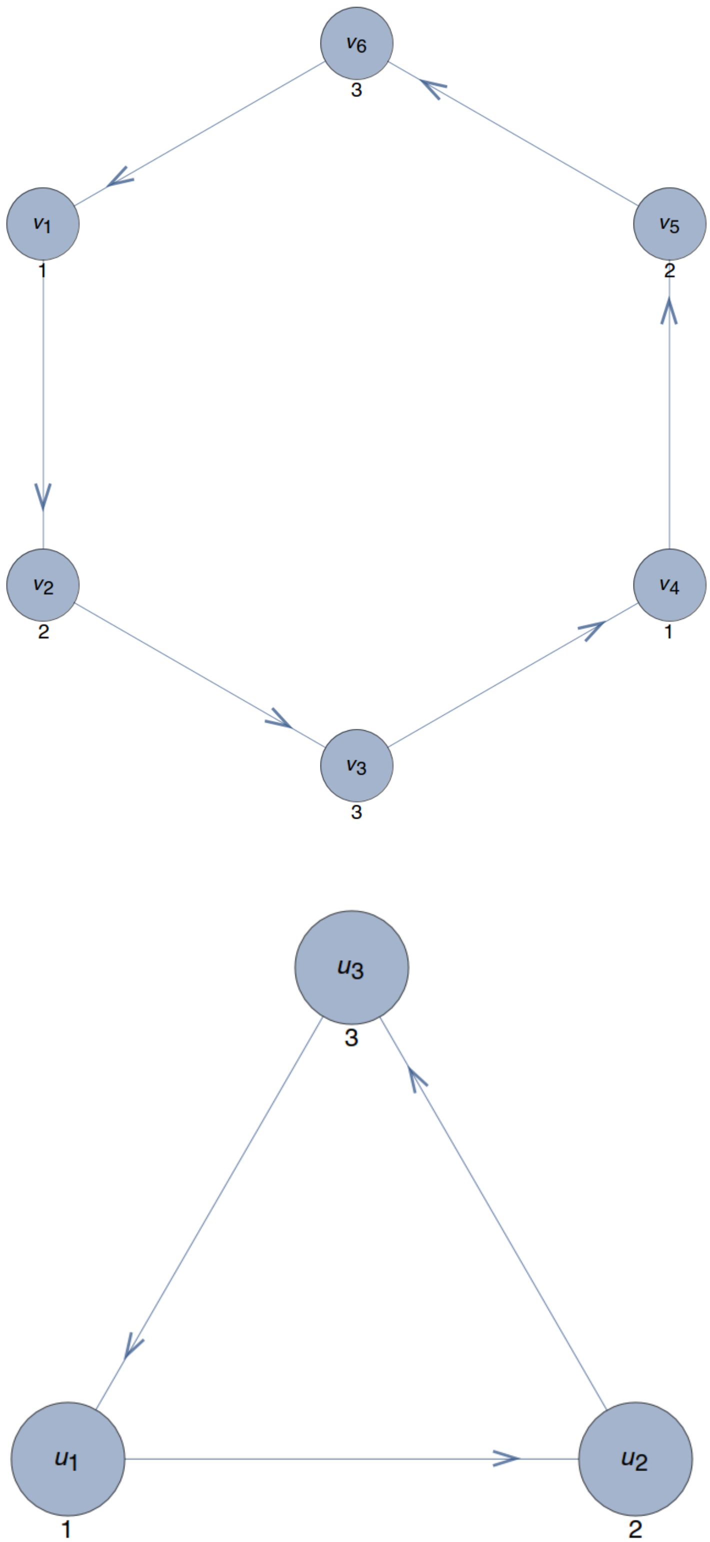


Figure 2: Weighted digraphs $\Gamma$ (above) and $P$ (below).

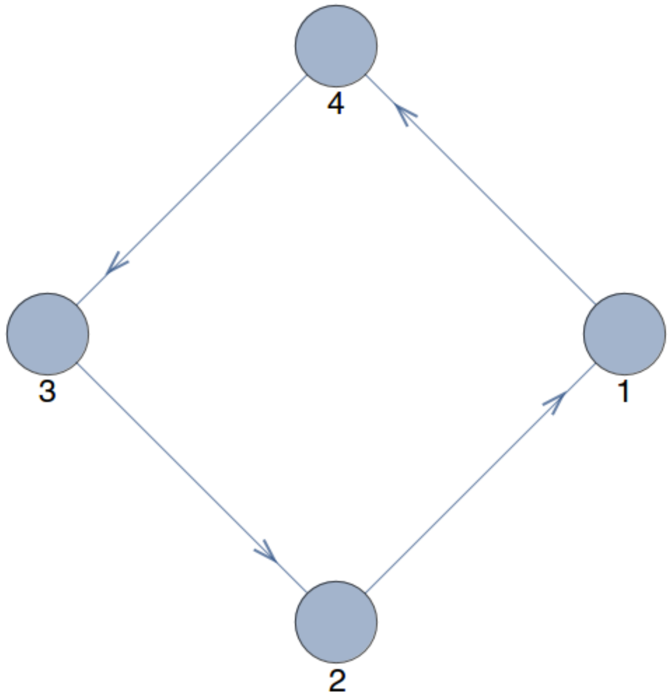


Figure 3: A graph $\Gamma$ which avoids 1-2-3-4.

$w(1)$-$w(2)$-...-$w(n)$. In the context of local matching, where unambiguous we write $P$ with the shorthand $w(1)w(2)\ldots w(n)$.

**Remark 1.1.** Defining pattern matching using homomorphisms rather than isomorphisms simplifies some calculations but has some undesirable properties. The graph $\Gamma$ in Figure 3 is a directed cycle with weights $4, 3, 2, 1$. So $\Gamma$ avoids the pattern 1-2-3-4. Using homomorphisms in the definition of matches, $\Gamma$ contains 1-2-3-4 since it has 4 different weights and is strongly connected.

Avoiding homomorphic matches is a stronger condition which allows us to conclude, for example, that splicing two identical directed cycles that avoid a pattern will result in a cycle that avoids the pattern. But due to the above issue we do not use this definition. $\triangle$

[a definition beyond local and global should be defined if it is necessary to discuss “generalized patterns”]

### Some notation

For power series we use capital Roman letters, such as $A(z)$. For power series coefficients we use lower case Greek letters e.g. $a_n$. For combinatorial classes we use calligraphic capital Roman letters, such as $\mathcal{A}$. These symbols take a tilde to refer to unlabeled graphs.

Directed paths: $p, P, \mathcal{P}$, cycles: $c, C, \mathcal{C}$, trees: $t, T, \mathcal{T}$

Stirling set numbers: ${m \atop k}$

Finite cyclic groups: $\mathbb{Z}_k = \{0, \ldots, k-1\}$

general unstructured sets $\Xi, \Psi, \Phi$

number of occurrences: $r$

span: $\sigma$

digraph: $D$

edge relation: $E(D)$. old: $\bullet\!\!\rightarrow$

transitive closure of edge relation: $\longrightarrow$ (long arrow)

open neighborhood (doesn't include $v$ (even if loop?)): $N(v)$, closed neighborhood: $N[v]$

in-neighborhoods: $N^-(v), N^-[v]$ out-neighborhoods: $N^+(v), N^+[v]$ [BM76]

matrix: $M$

matrix/vector literal: parens or brackets?

matrix entry: $[M]_{i,j}$

derived digraph: $D_\times$

arc (directed edge): $e$

real numbers: $x, y$

pattern: $\tau$

complex number/indeterminate: $z, u, v, w$

disjoint union: $\dot\cup$

vertices: $u, v, w$

eigenvectors: $u_\lambda, v_\lambda$

total: $n$, length: $m$

concatenation of finite sequences: $a^\frown b$

group: $G$

group elements: $a, b, c$

semigroup: $S$

semigroup elements: $s, t$

random variables: $X, Y, Z$

integers: $n, j, k$

named $m$-tuples/vectors/sequences/words/compositions: lowercase Roman or Greek e.g. $d$ with domain $[n]$, $d = (d(1), \ldots, d(n))$

sequence terms/set elements: $d_i, d_j, \ldots, d_k$ (unnamed)

secondary sequence terms/set elements: $d^{\langle i\rangle}, d^{\langle j\rangle}, \ldots, d^{\langle k\rangle}$ (unnamed)

sum of finite sequence $a$: $\Sigma a$ (note it's a capital $\sigma$)

all $n$-tuples: $\textsc{Seq}_n(set)$

subset: $\subseteq$, strict subset: $\subset$

don't use: $I, l, \ell, \iota$

unused so far: bold caligraphic, bold normal, hat above, arrow above

We use the Iverson bracket $[\phi]$ which is equal to 1 if the statement $\phi$ is true and 0 otherwise [Knu92].

If $k$ is an integer, we use $[k]$ to represent the set $\{n : 1 \leq n \leq k, n \in \mathbb{Z}\}$.

Nonnegative integers $\mathbb{Z}_{\geq 0}$

Convergence in distribution/weak convergence: $\Rightarrow$

Big o: $f(n) = O(g(n))$ iff there is $c > 0$ s.t. eventually $f(n) \leq c|g(n)|$

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intro to local restriction literature

*Directed paths* give the familiar notion of locally restricted compositions. Over the integers, these objects have been studied successfully in a number of papers by Bender et al. [BC09; BCG12; BG14]. Those works include somewhat more general restrictions, where a subword may or may not be allowed based on the residue of its position in a composition, and special rules can apply to initial and final parts.

Counting:

**Theorem 1.1** (Theorem 3 in [BC09]). *Let $\mathcal{L}$ be a regular, locally restricted class of compositions, and let $a_n$ be the number of compositions of $n$ in the class $\mathcal{L}$. Then $a_n \sim A \cdot B^n$ for some $A > 0$ and $B > 1$.*

Counting compositions by pattern occurrences in the sense of subword patterns is covered by the following result.

**Theorem 1.2** (Theorem 4 in [BC09]). *Let $\mathcal{C}_n$ be the compositions of $n$ in $\mathcal{C}$ made into a probability space with the uniform distribution. Let the random variables $Y_i(n), 1 \leq i \leq I$ count occurrences of recurrent local events. Then $E(Y_i(n)) = nm_i + o(n)$ where $m_i > 0$. Let $\vec{Z}(n) = n^{-1/2}\left(\vec{Y}(n) - E(\vec{Y}(n))\right)$. If the $Y_i(n)$ are unrelated, then $\vec{Z}(n)$ converges in distribution to an $I$-dimensional normal.*

e.g. from [BC09]:

**Theorem 1.3.** *Say $p$ is a fixed composition such that $p \cdots p$ is always allowed in a restricted class $\mathcal{C}$. Then the length of the longest run of $p$ within random restricted composition of $n$ is $\Theta(\log n)$ whp. [add detail]* runs, etc.

Mansour and others in [HM10, Ch. 4] count compositions by number of occurrences of specific subword patterns such as 123 and 112. These results can be seen as giving more explicit information than the general results obtained by Bender. The umbral technique in [Zei00] is also used to explicitly count locally-restricted objects.

[state summary of main theorems from above literature]

# 2 Trees over a finite group

## 2.1 Paths (compositions)

**Remark 2.1** (Undirected paths)**.** Here "undirected" means that for any arc $(u, v)$, the arc $(v, u)$ is added if not already present. We assume that patterns are still directed paths. Take a weighted *directed* path $\Gamma_p = (V_p, E_p)$ and a pattern $P$. We define $s(\Gamma_p)$ as the underlying undirected graph of $\Gamma_p$, i.e. where the arc set is the symmetric closure of $E_p$. Let $P^{-1}$ be the graph $P$ with all arcs reversed. If $P \simeq P^{-1}$, then the number of occurrences of $P$ in $s(\Gamma_p)$ is the number of occurrences of $P$ in $\Gamma_p$. Otherwise, we add the number of occurrences of $P^{-1}$ in $\Gamma_p$. △

Locally restricted compositions over finite fields and even finite abelian groups were counted in [GMW18] under some conditions, and in less generality in the preceding papers mentioned therein. The method used in that paper involves obtaining the relevant generating function $F(z)$ for compositions over the *(non-negative) integers*, and working with $\sum_{j \equiv s \pmod k} [z^j]F(z)$. Below we give an alternative counting method that expands the range of applicability beyond abelian groups, addressing a problem posed in [GMW18].

Let $(S, +)$ be a finite semigroup with the operation of addition. A composition over $S$ is a finite sequence $x$ with terms in $S$; typically the terms are called *parts*. If the length of $x$ is $m$ and $\sum_j^m x(j) = s$, we say $x$ is an $m$-composition of $s$.

**Definition 2.1.** *Let $\Xi$ be a finite set, and let $n$ be a positive integer. The $n$-dimensional* de Bruijn graph *(actually a digraph) on $\Xi$ has vertex set $V = \textsc{Seq}_n(\Xi)$ and includes the arc from $(u_1, \ldots, u_n)$ to $(v_1, \ldots, v_n)$ iff*

$$(u_2, \ldots, u_n) = (v_1, \ldots, v_{n-1}).$$

Let $\sigma$ be a positive integer which we call the *span*, and let $D$ be a subgraph of the $\sigma$-dimensional de Bruijn graph on $S$. The digraph $D$ is associated with a set of locally restricted compositions as follows. An $m$-composition over $S$ is legal according to $D$ iff it takes the form

$$(w_1(1), \ldots, w_1(\sigma), w_2(\sigma), \ldots, w_{m-\sigma+1}(\sigma)),$$

where $w_1, \ldots, w_{m-\sigma+1}$ is a walk in $D$. In other words, we build compositions from $D$ by starting at any vertex, and taking a walk in which we append the last element of each vertex we visit after the first. Additionally, we may designate sets of start and end vertices which are the allowed vertices for walks to start and end at.

We write the set of all $m$-compositions of $s$ that are legal according to $D$ with start set $\Psi$ and finish set $\Phi$ as $\mathcal{P}_s(m; D, \Psi, \Phi)$. Also, $p_s(m; D, \Psi, \Phi) = |\mathcal{P}_s(m; D, \Psi, \Phi)|$, $P_s(z; D, \Psi, \Phi) = \sum_{m \geq 0} p_s(m; D, \Psi, \Phi) z^m$. We let $\mathcal{P}(m; D, \Psi, \Phi), p(m; D, \Psi, \Phi), P(z; D, \Psi, \Phi)$ be the corresponding objects where there is no stipulated value for the total. If the start and finish sets are omitted, they are taken to be maximal.

Define a new digraph $D_\times$ with vertex set $V(D) \times S$ such that $((u,s),(v,t)) \in E(D_\times)$ iff $(u,v) \in E(D)$ and $s + v(\sigma) = t$. We call $D_\times$ the *derived digraph* (of $D$). We define the start set $\Psi_\times \subseteq V(D_\times)$ to contain all $(v,s)$ such that $\sum v = s$ and $v \in \Psi$. For each $s \in S$ the finish set $\Phi_s \subseteq V(D)$ for $s$ contains all vertices $(v,t)$ with $t = s$ and where $v \in \Phi$.

Fix an ordering on $V(D_\times)$ so we can define an adjacency matrix $M_\times$ of $D_\times$. Let $\psi_\times \in \mathbb{R}^{|V(D_\times)|}$ be the indicator vector for $\Psi_\times$, and let $\phi_s \in \mathbb{R}^{|V(D_\times)|}$ be the indicator vector for $\Phi_s$

**Proposition 2.1.** *For $m \geq \sigma$, we have*

$$p_s(m; D, \Psi, \Phi) = \psi_\times^\top M_\times^{m-\sigma} \phi_s.$$

*Proof.* Let $W_s$ be a walk in $D_\times$ starting in $\Psi_\times$ and ending in $\Phi_s$, and say the vertices in $D$ corresponding to $W_s$ are $w(1), \ldots, w(m-\sigma+1)$. We say the $m$-composition of $s$ defined by $W_s$ is

$$w(1)w(2)_\sigma w(3)_\sigma \cdots w(m-\sigma+1)_\sigma.$$

That is, the compositions defined by $D_\times$ and $D$ are the same, but $D_\times$ also directly keeps track of the total.

Recurrence

$$[M_\times^q]_{i,j} = \sum_{k=1}^{\ell} [M_\times]_{i,k} [M_\times^{q-1}]_{k,j}$$

implies result. □

**Example 2.1.** Figure 4 shows an example of $D_\times$ for Carlitz compositions over $\mathbb{Z}_3$.

Under a particular vertex ordering we get

$$M_\times = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix},$$

$$\psi_\times = [1,1,0,0,0,1,0,0,0]^\top,$$
$$\phi_0 = [1,0,1,0,1,0,0,0,0]^\top,$$
$$\psi_\times^\top M^{3-1} \phi_0 = 6.$$

So the number of 3-compositions of 0 in $\mathbb{Z}_3$ is 6. △

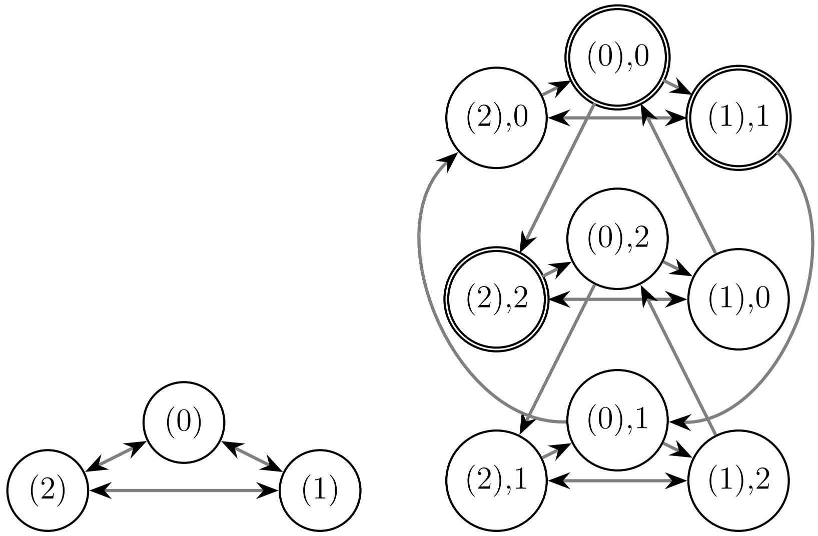



Figure 4: A base digraph $D$ (left) and derived digraph $D_\times$ (right) representing Carlitz compositions over $\mathbb{Z}_3$. Here all vertices of $D$ are allowed start and finish vertices.

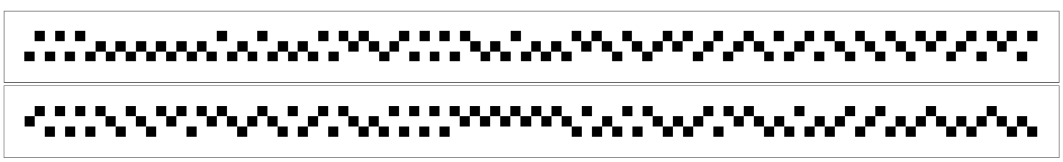


Figure 5: Uniform-randomly generated Carlitz 100-compositions of 0 (above) and 1 (below) over $\mathbb{Z}_3$. (The vertical axis represents the value of a part.)

**Remark 2.2.** The following procedure generates a random walk in $D_\times$ of length $m$, where all such walks are equally likely.

First pick a start vertex $v_1$ weighted by the number of $m$-walks from that vertex to a finish vertex. Given the current vertex $v_i$, select an out-neighbor where such neighbors are weighted by the number of $m-i+1$-walks from that vertex to a finish vertex.

Figure 5 shows examples.

The computer code used is available in `random_comp.nb` in §A. $\triangle$

If $D_\times$ is strongly connected and aperiodic, then we can obtain a highly-precise asymptotic expression for $p_s(m; D, \Psi, \Phi)$, $m \to \infty$, via Proposition 2.1 and the Perron-Frobenius theorem. (A digraph is *aperiodic* iff the set of all cycle lengths has no common divisor besides 1.) We now give some general facts about the strong connectedness of $D_\times$.

If $D$ is not strongly connected then certainly $D_\times$ is not strongly connected either. However, if $D$ decomposes into disconnected strong components, then naturally we are able to simply count with each component separately and add. In the following, *we assume $D$ is strongly connected.*

Unfortunately, if $D$ *is* strongly connected, $D_\times$ is not necessarily strongly con-

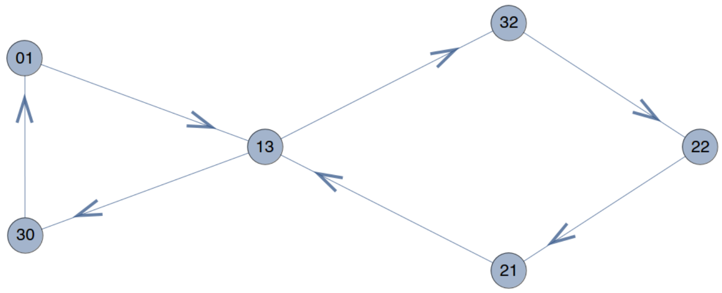



Figure 6: A digraph $D$ with vertices in $\mathbb{Z}_4^2$.

nected. Say $D$ is the digraph given in Figure 6, over $\mathbb{Z}_4$ with span $\sigma = 2$. In $D_\times$, there is a path from $((3,0),3)$ to $((1,3),3)$, but there is no path from $((1,3),0)$ to $((1,3),3)$.

If the entirety of $D_\times$ is not strongly connected then we would hope it is simply a disjoint union of strong components. This is not true for general finite semigroups $S$. For example, if there is $s^* \in S$ satisfying

$$\forall s \in S : s^* + s = s + s^* = s^*,$$

then the digraph $D_\times$ will get "stuck" at $s^*$ and some connected vertices will not be strongly connected. We do obtain this desideratum, however, if $S$ is a group, as we show eventually below. *In the following we assume that $S = G$ is a group.*

**Definition 2.2.** *Let $D$ be an arbitrary digraph, referred to as the* base digraph*. Let $G$ be a finite group, and let $\alpha : E(D) \to G$ map arcs of $D$ to group elements. Together, $D$ and $\alpha$ are known as a* voltage graph*. We define the* derived digraph *$D_\alpha$ such that $V(D_\alpha) = V(D) \times G$ and $((u,a),(v,b))$ is an arc iff $(u,v) \in E(D)$ and $a + \alpha(u,v) = b$.*

The digraphs $D_\times$ directly give derived digraphs in the sense of voltage graphs, specifically "right derived ordinary voltage graphs", if we associate the group element $u(\sigma)$ to all incoming arcs to $u$ in the base digraph $D$.

**Remark 2.3.** Let $(V_1, E_1)$ and $(V_2, E_2)$ be graphs. Then $(V_2, E_2)$ is a *covering graph* of $(V_1, E_1)$ iff there is a surjection $f : V_2 \to V_1$ such that for each $v \in V_2$, the restriction $f_{|N[v]}$ is a bijection. In that case, $f$ is called a *covering map*. We note that derived graphs can be seen as a covering graphs of the base graph, but directed. The book [GT87] provides a basic introduction to covering graphs in Chapter 2. Covering graphs are more generally known as covering spaces in topology. △

**Lemma 2.1.** *The derived digraph $D_\times$ is a disjoint union of strong components.*

*Proof.* Select a vertex $(u,a)$, and take another vertex $(v,b)$ such that there is a path $(u,a) \longrightarrow (v,b)$ in $D_\times$. Since $D$ is strongly connected, there is a path

$(v,b) \longrightarrow (u,c)$ in $D_\times$ for some $c \in G$. This implies that $(u,a) \longrightarrow (u,c)$. We are done if we can show that $(u,c) \longrightarrow (u,a)$.

Since there is a path $(u,a) \longrightarrow (u,c)$, we know that for any positive integer $j$, there is a path $(u,a) \longrightarrow (u, a + j(-a+c))$, which is found by repeating the path in $D$. In a finite digraph we will eventually get $g > j > 0$ with $a+j(-a+c) = a+g(-a+c)$, thus $j(-a+c) = g(-a+c)$ and $(g-j)(-a+c) = 0$. We conclude that

$$\begin{aligned}(u,a) &\longrightarrow (u, a + (-a+c)) \\ &\longrightarrow \cdots \\ &\longrightarrow (u, a + (g-j)(-a+c)) = (u,a).\end{aligned}$$

□

**Lemma 2.2.** *For each $v \in V(D)$ and $a, b \in G$, there is a digraph automorphism $f$ on $D_\times$ with $f(v,a) = (v,b)$. In particular, the strong components of $D_\times$ are isomorphic.*

*Proof.* Let $f : V(D) \times G \to V(D) \times G$ be defined $f(v,c) = (v, b-a+c)$. We have $f(v,a) = (v,b)$, and clearly $f$ is a bijection. Take an arc from $(u,c)$ to $(w,d)$. Then $c + w_\sigma = d$, so $b - a + c + w_\sigma = b - a + d$, so there is also an arc from $f(u,c)$ to $f(w,d)$. This automorphism is mentioned in [GT87, §2.2.1].

The second claim follows since every strong component contains a vertex $(v,c)$ for some $c \in G$, which follows from the strong connectedness of $D$. □

Aperiodicity of $D_\times$ does not follow from aperiodicity of $D$, as shown in Example 2.2. However, aperiodicity can be checked for individual digraphs or digraph families and periodic digraphs can transformed to equivalent aperiodic ones.

**Example 2.2.** The condition of aperiodicity of $D_\times$ cannot be transfered from $D$. Figure 7 shows a counterexample digraph $D$. △

**Proposition 2.2.** *Assume $D_\times$ is aperiodic. Say $p(m; D, \Psi, \Phi) \sim A \cdot B^m$. Then either $p_s(m; D, \Psi, \Phi) = 0$ or*

$$p_s(m; D, \Psi, \Phi) = C_s \cdot B^m(1 + O(\theta^m)),$$

*where $C_s > 0$ can be computed from $D_\times$ and $0 \le \theta < 1$.*

*Proof.* Given Lemma 2.1 and Lemma 2.2, standard results on asymptotics for the transfer matrix method found in [FS09] imply that since the only difference between a composition of $s$ and an arbitrary word as represented by walks in $D_\times$ is the set of allowed finish vertices, the only possible difference in the dominant asymptotic term is the positive leading coefficient, unless $p_s(m; D, \Psi, \Phi) = 0$. □

**Definition 2.3.** *If $A$ is an $m \times n$ matrix and $B$ is a $p \times q$ matrix, then the* Kronecker product *$A \otimes B$ is the $mp \times nq$ matrix $C$ such that $[C]_{p(r-1)+v, q(s-1)+w} =$*

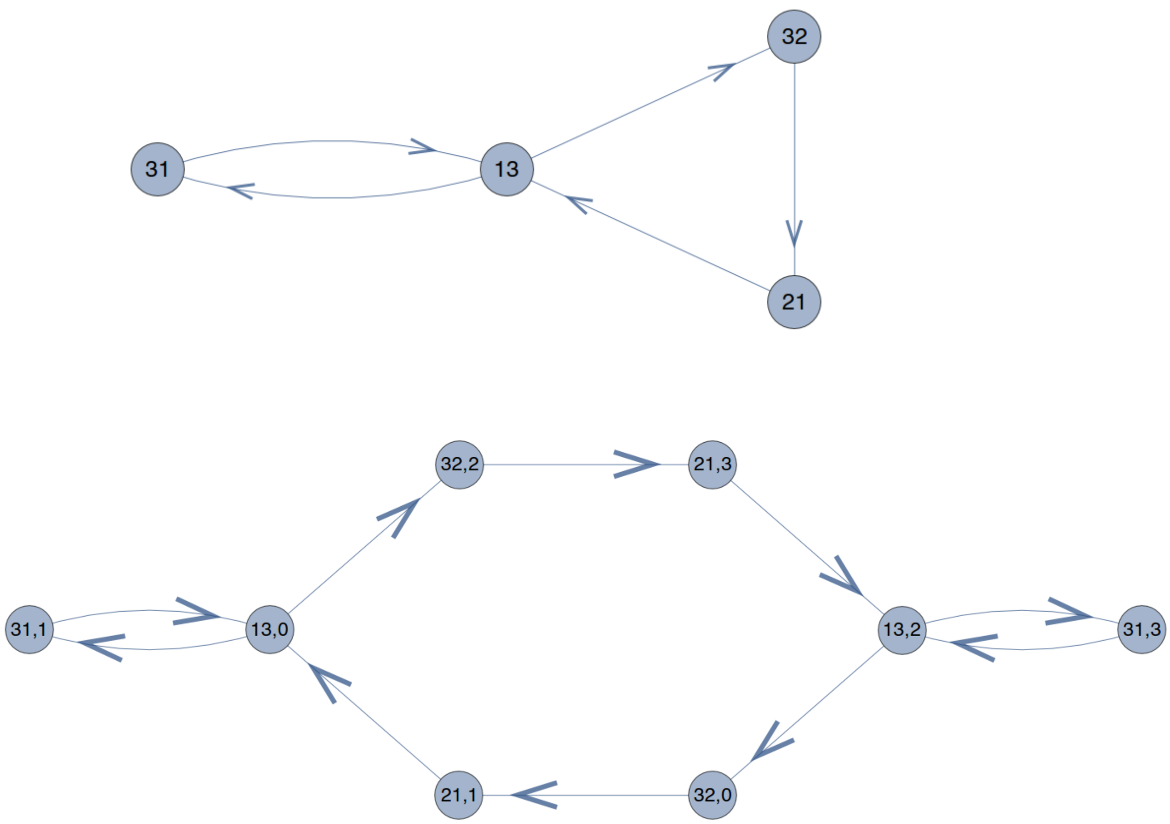



Figure 7: An aperiodic strongly connected digraph $D$ (above) with vertices in $\mathbb{Z}_4^2$ such that $D_\times$ (one component shown below) has period 2. Examples with connected $D_\times$ exist as well, such as the above $D$ over $\mathbb{Z}_8$ with 7 replacing 3 and 3 replacing 2.

$[A]_{r,s}[B]_{v,w}$. *Visually,*

$$A \otimes B = \begin{bmatrix} [A]_{1,1}B & \cdots & [A]_{1,n}B \\ \vdots & \ddots & \vdots \\ [A]_{m,1}B & \cdots & [A]_{m,n}B \end{bmatrix}.$$

**Theorem 2.1.** *Assume*

- *for some $v \in V(D)$ we have that for all $a \in G$ there is a legal composition starting and ending with $v$ with total $a$, and*
- *for some $u \in V(D)$ the set*

$$\left\{ m : \exists \text{ a walk } x = (u, v, \cdots, w, u) \text{ of length } m, \sum x = \sum u \right\},$$

*has a GCD of 1, where $\sum x$ is the total of the composition corresponding to $x$.*

*Let $A, B > 0$ be the values such that $p(m; D, \Psi, \Phi) \sim A \cdot B^m$. Then*

$$p_a(m; D, \Psi, \Phi) \sim \frac{A}{|G|} \cdot B^m(1 + O(\theta^m)), \qquad m \to \infty, 0 \leq \theta < 1.$$

*Proof.* From the first condition we know there is a single strong component i.e. $D_\times$ is strongly connected. The second condition ensures that this component is aperiodic. This allows us to conclude that the Perron-Frobenius Theorem applies directly to $D_\times$.

Say $|V(D)| = \alpha$ and fix a vertex ordering $v_1, \ldots, v_\alpha$. Let $M$ be the adjacency matrix of $D$ with respect to this ordering. Also fix an ordering on $G = \{a_1, a_2, \ldots, a_\beta\}$ where $a_1 = 0$. Finally, define a vertex ordering on $D_\times$ as

$$(v_1, a_1), \ldots, (v_\alpha, a_1), (v_1, a_2), \ldots, (v_\alpha, a_2), \ldots, (v_1, a_\beta), \ldots, (v_\alpha, a_\beta).$$

Let $M_\times$ be the adjacency matrix of $D_\times$ with respect to this ordering.

For each $a \in G$, define the $\alpha \times \alpha$ matrix $M_a$ and $\beta \times \beta$ matrix $P_a$ such that

$$\begin{aligned} [M_a]_{i,j} &= [v_j(\sigma) = a, (v_i, v_j) \in E(D)], \\ [P_a]_{i,j} &= [a_i + a = a_j]. \end{aligned}$$

By (the proof of) Theorem 1 in [MS95] we have $M_\times = \sum_{a \in G} P_a \otimes M_a$ and $M = \sum_{a \in G} M_a$.

Let $\lambda > 0$ be the dominant eigenvalue of $M$, let $v_\lambda$ be an associated positive eigenvector, and let $u_\lambda$ be an associated positive left eigenvector. Let $\xi \in \mathbb{R}^\beta$ be the all-1 vector $[1, 1, \ldots, 1]$.

We claim that $\xi \otimes v_\lambda$ is an eigenvector of $M_\times$ with eigenvalue $\lambda$. First, by Lemma 4.2.10 in [HJ94], $(P_a \otimes M_a)(\xi \otimes v_\lambda) = P_a\xi \otimes M_a v_\lambda$. Since $P_a$ is a permutation matrix, we have $P_a\xi = \xi$. Thus

$$\begin{aligned} M_\times(\xi \otimes v_\lambda) &= \left(\sum_{a\in G} P_a \otimes M_a\right)(\xi \otimes v_\lambda) \\ &= \sum_{a\in G}(P_a \otimes M_a)(\xi \otimes v_\lambda) \\ &= \sum_{a\in G} P_a\xi \otimes M_a v_\lambda \\ &= \sum_{a\in G} \xi \otimes M_a v_\lambda. \end{aligned}$$

By Equation 4.2.8 in [HJ94], $\sum_{a\in G} \xi \otimes M_a v_\lambda = \xi \otimes \sum_{a\in G} M_a v_\lambda$. We conclude

$$\begin{aligned} M_\times(\xi \otimes v_\lambda) &= \xi \otimes \sum_{a\in G} M_a v_\lambda \\ &= \xi \otimes \left(\sum_{a\in G} M_a\right) v_\lambda \\ &= \xi \otimes M v_\lambda \\ &= \xi \otimes \lambda v_\lambda \\ &= \lambda\xi \otimes v_\lambda. \end{aligned}$$

Similarly we have that $\xi \otimes v_\lambda$ is a left eigenvector for $M_\times$ with eigenvalue $\lambda$.

By the Perron-Frobenius Theorem (and a Jordan decomposition of $M_\times$), we know that

it's start vector dot right eigenvector

$$p_a(m; D, \Psi, \Phi) = \psi_\times^\top M_\times^{m-\sigma} \phi_a = C_a \lambda^m (1 + O(\theta^m)),$$

where $C_s = c \cdot (\psi_\times \cdot (\xi \otimes u_\lambda))((\xi \otimes v_\lambda) \cdot \phi_s)$ for some fixed $c > 0$.

Suppose $a = a_i$; then $\phi_a = e_i \otimes \phi$. Thus

$$(\xi \otimes v_\lambda) \cdot \phi_s = (\xi \otimes v_\lambda) \cdot (e_i \otimes \phi) = \xi e_i \otimes v_\lambda \phi = v_\lambda \phi.$$

Since $v_\lambda \phi$ does not depend on $a$, the proof is now complete. □

**Corollary 2.1.** *Assume the conditions of Theorem 2.1. Construct a probability space from $\mathcal{P}(m; D, \Psi, \Phi)$ and the uniform probability measure. Then for $a \in G$, let $\mathbb{P}_m(a)$ be the probability that an element drawn randomly from $\mathcal{P}(m; D, \Psi, \Phi)$ has total $a$. We have*

$$\mathbb{P}_m(a) \to \frac{1}{|G|}, \qquad m \to \infty,$$

*or in other words, $\mathbb{P}_m$ converges strongly to the uniform measure on $G$.*

*Proof.* Direct from Theorem 2.1. □

**Example 2.3.** We show a case where the connectedness condition in Theorem 2.1 is required. Let $D$ be the digraph given in Figure 8, where $G = \mathbb{Z}_2$. Assume $\Psi = \Phi = V(D)$.

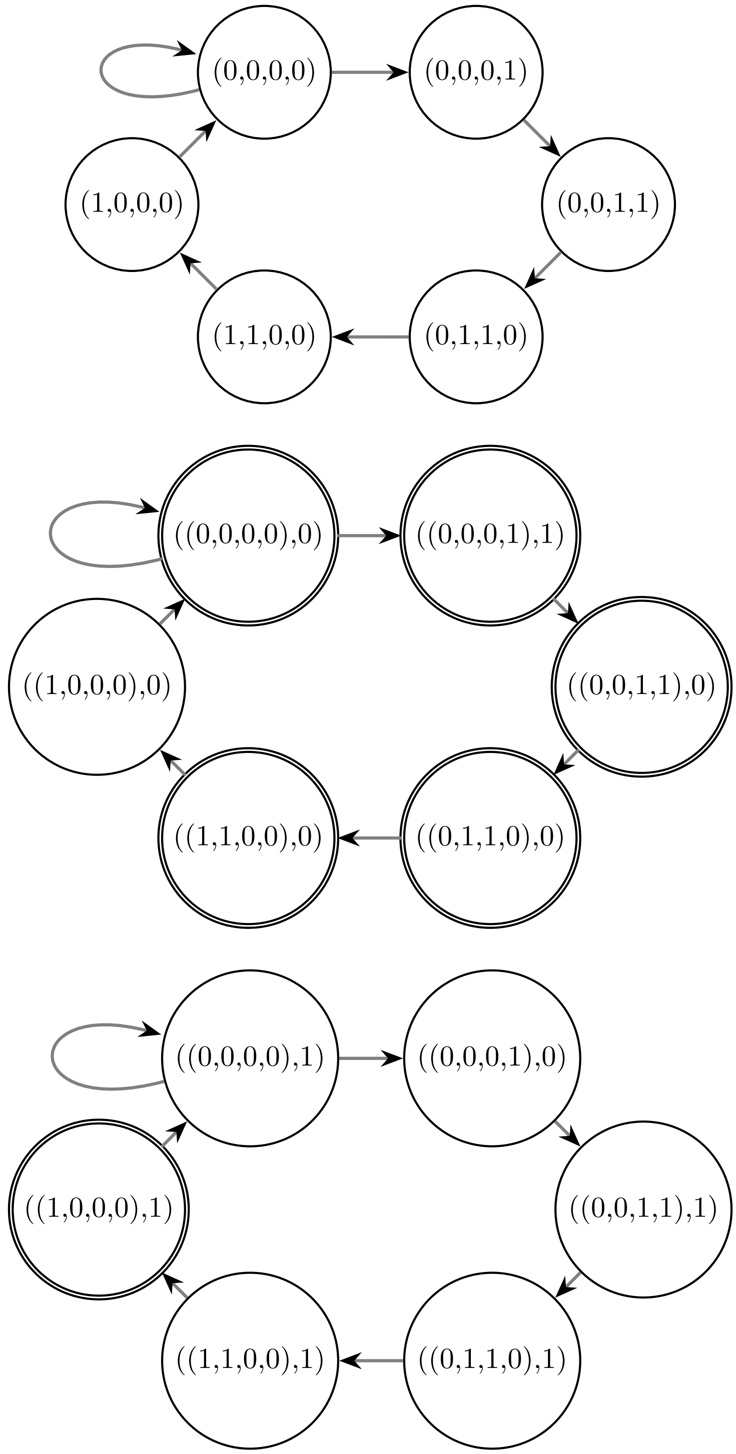



Figure 8: A base digraph $D$ (above) and derived digraph $D_\times$ with 2 strong components (below). The vertices of $D$ are 4-tuples over $\mathbb{Z}_2$.

Let $M^{\langle 1\rangle}, M^{\langle 2\rangle}$ be adjacency matrices of the two strong components of $D_\times$, under a particular vertex ordering. We have

$$M^{\langle 1\rangle} = M^{\langle 2\rangle} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

From a Jordan decompositions, we get a left eigenvalue

$$u_\lambda \doteq [0.368841, 0.286991, 0.223305, 0.173751, 0.135194, 0.105193],$$

and a right eigenvector

$$v_\lambda \doteq [1.2852, 0.366538, 0.471074, 0.605423, 0.77809, 1.0].$$

Also

$$\psi_\times^{\langle 1\rangle} = [1,1,1,1,1,0]^\top, \phi_0^{\langle 1\rangle} = [1,0,1,1,1,1]^\top, \phi_1^{\langle 1\rangle} = [0,1,0,0,0,0]^\top,$$

and

$$\psi_\times^{\langle 2\rangle} = [0,0,0,0,0,1]^\top, \phi_0^{\langle 2\rangle} = [0,1,0,0,0,0]^\top, \phi_1^{\langle 2\rangle} = [1,0,1,1,1,1]^\top.$$

We compute

$$\begin{aligned} p_0(m;D) &= \psi_\times^{\langle 1\rangle}(M^{\langle 1\rangle})^{m-4}\phi_0^{\langle 1\rangle} + \psi_\times^{\langle 2\rangle}(M^{\langle 2\rangle})^{m-4}\phi_0^{\langle 2\rangle} \\ &\sim \lambda^{m-4}((\psi_\times^{\langle 1\rangle}v_\lambda)(u_\lambda\phi_0^{\langle 1\rangle}) + (\psi_\times^{\langle 2\rangle}v_\lambda)(u_\lambda\phi_0^{\langle 2\rangle})) \\ &\doteq \lambda^{m-4}(3.50632 \cdot 1.00628 + 1.0 \cdot 0.286991) \\ &= 3.81533 \cdot \lambda^{m-4} \end{aligned}$$

and

$$\begin{aligned} p_1(m;D) &= \psi_\times^{\langle 1\rangle}(M^{\langle 1\rangle})^{m-4}\phi_1^{\langle 1\rangle} + \psi_\times^{\langle 2\rangle}(M^{\langle 2\rangle})^{m-4}\phi_1^{\langle 2\rangle} \\ &\sim \lambda^{m-4}((\psi_\times^{\langle 1\rangle}v_\lambda)(u_\lambda\phi_1^{\langle 1\rangle}) + (\psi_\times^{\langle 2\rangle}v_\lambda)(u_\lambda\phi_1^{\langle 2\rangle})) \\ &\doteq \lambda^{m-4}(3.50632 \cdot 0.286991 + 1.0 \cdot 1.00628) \\ &= 2.01256 \cdot \lambda^{m-4}. \end{aligned}$$

So indeed Theorem 2.1 does not hold. $\triangle$

**Lemma 2.3.** *Assume $G$ is abelian. Define a $D_\times$-automorphism $f$ by $f(v,b) = (v, a+b)$ for some $a \in G$. If $f$ maps any vertex to its own strong component, then $f$ maps all vertices to their own strong component.*

*Proof.* Let $(u,c), (v,b)$ be arbitrary vertices and say $(v,b) \longrightarrow (v, a+b)$ in $D_\times$. We seek to show that $(u,c) \longrightarrow (u, a+c)$.

There is some $(u,d)$ in the same strong component as $(v,b)$ and so $(u,d) \longrightarrow (u, a+d)$.

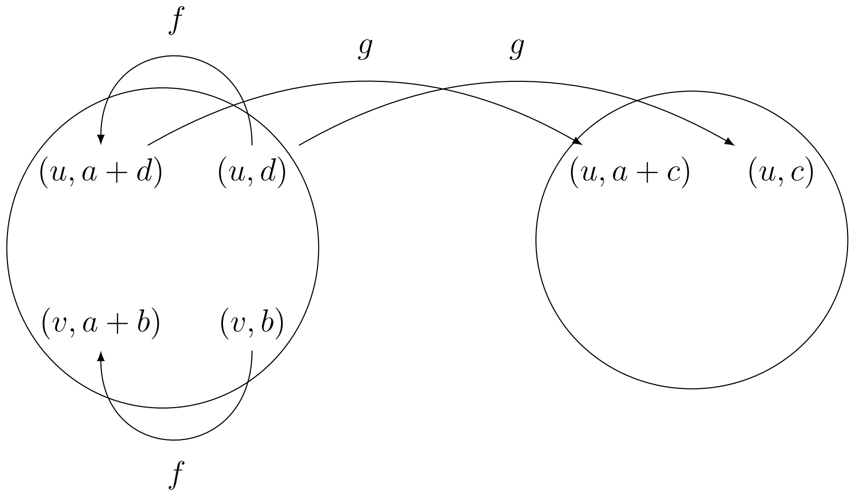



Figure 9: Strong components in the proof of Lemma 2.3.

The automorphism $g(w, r) = (w, c - d + r)$ maps $(u, d)$ to $(u, c)$ and $(u, a + d)$ to $(u, c - d + a + d) = (u, a + c)$.

Thus $(u, d) \longrightarrow (u, a + d)$ implies $g(u, d) = (u, c) \longrightarrow g(u, a + d) = (u, a + c)$. This is illustrated in Figure 9. $\square$

The following is a useful characterization of strong connectedness of $D_\times$ for abelian $G$.

**Lemma 2.4.** *Assume $G$ is abelian. Let $A$ be a generating set for $G$, i.e. $\langle A \rangle = G$. If for all $a_i \in A$ there is a vertex $(v, 0) \in V(D_\times)$ such that $(v, 0) \longrightarrow (v, a_i)$ in $D_\times$, then $D_\times$ is strongly connected.*

*Proof.* We show that for any $u \in V(D)$ and $r \in G$, the vertices $(u, 0)$ and $(u, r)$ are in the same strong component of $D_\times$.

Say $r = j(1)a_1 + \cdots + j(p)a_p$ for $a_i \in A, j(i) \in \mathbb{Z}_{\geq 0}$. We know from Lemma 2.3 that the $D_\times$-automorphism $f_j(v, s) = (v, a_j + s)$ maps strong components to themselves. Thus the composition $f_r = f_1^{j(1)} \circ \cdots \circ f_p^{j(p)}$ also maps strong components to themselves. We conclude that $(u, 0)$ and $f_r(u, 0) = (u, j(1)a_1 + \cdots + j(p)a_p) = (u, r)$ belong to the same strong component. $\square$

We now consider some examples of $D$.

We generalize Carlitz compositions as follows. A sequence $x \in \mathrm{S{\small EQ}}_m(G)$ is a *$d$-Carlitz composition* iff every subsequence $x(i), \ldots, x(i + d)$ contains no repeated part. Thus Carlitz compositions are 1-Carlitz. We note that unlike for integer compositions, we generally allow the identity element as a part. We also note that this definition is consistent with [GMW18] but different from Definition 4.33 in [HM10, p. 115]. Words with no equal adjacent letters are also called Smirnov words as in [FS09, Example III.24].

**Proposition 2.3.** *The number of $d$-Carlitz $m$-compositions of $s \in G$ over a finite group $G$ is asymptotic to*

$$\frac{1}{|G|}|G|^{\underline{d}}(|G|-d)^{m-d}, \qquad m \to \infty,$$

*provided $|G| \geq d+2$.*

*Proof.* Take as vertex set for $D$ all $(d+1)$-tuples of distinct elements of $G$. The allowed start and finish vertices are all of $V(D)$. The strong connectedness of $D$ is established in [GMW18].

To show strong connectedness of $D_\times$, we fix a vertex $(a_1, \ldots, a_{d+1}) \in V(D)$ and for any $s \in G$ exhibit a path from $((a_1, \ldots, a_{d+1}), 0)$ to $((a_1, \ldots, a_{d+1}), s)$. Let $n$ be the order of $\Sigma a = a_1 + \cdots + a_{d+1}$. We consider two cases.

Case 1: $s \notin \{a_1, \ldots, a_{d+1}\}$. The first step is to $(a_2, \ldots, a_{d+1}, s)$. Follow this by the $(d+1)$-step path to $(a_1, \ldots, a_{d+1})$ Take the $(d+1)$-step path back to $(a_1, \ldots, a_{d+1})$ exactly $n-1$ times. The total of this path is

$$s + \Sigma a + (n-1)\Sigma a = s,$$

thus $((a_1, \ldots, a_{d+1}), 0) \longrightarrow ((a_1, \ldots, a_{d+1}), s)$ in $D_\times$.

Case 2: $s = a_j$ for some $1 \leq j \leq d+1$. Let $b$ represent some element of $G$ not in $\{a_1, \ldots, a_{d+1}\}$. Begin with the $(d+1)$-step path to $(b, a_1, \ldots, a_{j-1}, a_{j+1}, a_{d+1})$. Let $n_1$ be the order of $b + a_1 + \cdots + a_{j-1} + a_{j+1} + a_{d+1}$. Follow the $(d+1)$-step path back to $(b, a_1, \ldots, a_{j-1}, a_{j+1}, a_{d+1})$ exactly $n_1 - 1$ times. Traverse one arc to $(a_1, \ldots, a_{j-1}, a_{j+1}, a_{d+1}, s)$, then follow the $(d+1)$-step path to $(a_1, \ldots, a_{j-1}, a_{j+1}, a_{d+1}, b)$. Let $n_2$ be the order of $a_1 + \ldots + a_{j-1} + a_{j+1} + a_{d+1} + b$. Take the $(d+1)$-step path back to $(a_1, \ldots, a_{j-1}, a_{j+1}, a_{d+1}, b)$ exactly $n_2 - 1$ times. Finally take the $(d+1)$-step path to $(a_1, \ldots, a_{d+1})$ and cycle $(a_1, \ldots, a_{d+1})$ the suitable number of times. The total of this walk is $0 + s + 0 + 0 = s$ so $((a_1, \ldots, a_{d+1}), 0) \longrightarrow ((a_1, \ldots, a_{d+1}), s)$ in $D_\times$.

We now turn to aperiodicity. Take a $D$-vertex $u = (a_1, \ldots, a_{d+1})$ that does not contain the part 0. We give two closed walks starting from $u$ with total 0 and lengths that differ by 1.

Let $n$ be the order of $a_1 + \cdots + a_{d+1}$. The first walk repeats the $(d+1)$-step cycle back to $u$ exactly $n$ times. The second walk first takes a step to $(u_2, \ldots, u_{d+1}, 0)$ followed by the $(d+1)$-step path to $u$. Then we cycle back to $u$ exactly $n-1$ times.

We conclude that the conditions of Theorem 2.1 are satisfied.

In $D$ each vertex has an out-degree of $|G| - d$. This allows us to count walks in $D$ directly. We have $V(D) = |G|^{\underline{d+1}}$. Thus the number of $m$-compositions represented by $D$ is $|G|^{\underline{d+1}}(|G|-d)^{m-d-1} = |G|^{\underline{d}}(|G|-d)^{m-d}$. We conclude by applying Theorem 2.1. $\square$

Figure 10 shows randomly generated 2-Carlitz 100-compositions over $\mathbb{Z}_5$. Table 1 gives counts for Carlitz $m$-compositions of $a$ over $S_3$.

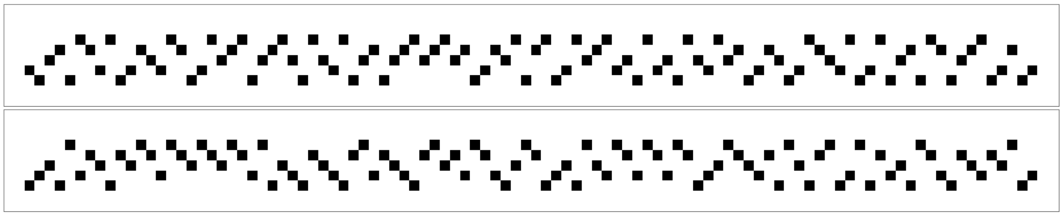


Figure 10: Uniform-randomly generated 2-Carlitz 100-compositions of 0 (above) and 1 (below) over $\mathbb{Z}_5$. (The vertical axis represents the value of a part.)

| $m$ \ $a$ | $id$ | (123) | (12) |
|---|---:|---:|---:|
| 3 | 27 | 24 | 25 |
| 4 | 134 | 128 | 120 |
| 5 | 613 | 631 | 625 |
| 6 | 3096 | 3102 | 3150 |
| 7 | 15667 | 15604 | 15625 |
| 8 | 78224 | 78263 | 78000 |
| 9 | 390513 | 390681 | 390625 |
| 10 | 1952696 | 1952402 | 1953750 |
| 11 | 9765817 | 9765529 | 9765625 |
| 12 | 48830424 | 48831663 | 48825000 |
| 13 | 244140763 | 244140556 | 244140625 |
| 14 | 1220690096 | 1220686202 | 1220718750 |
| 15 | 6103512717 | 6103517079 | 6103515625 |
| 16 | 30517650374 | 30517659188 | 30517500000 |

Table 1: Exact counts of Carlitz $m$-compositions of $a$ over $S_3$.

**Proposition 2.4.** *The number of $m$-compositions of $s \in G$ over a finite group $G$ such that the sum of any $d+1$ consecutive parts is not $0$ is asymptotic to*

$$|G|^{d-1}(|G|-1)^{m-d}, \qquad m \to \infty$$

*provided $d \leq |G|-2$.*

*Proof.* Define the appropriate $D$ so that $V(D)$ contains all $(d+1)$-tuples of vertices that do not sum to 0. The strong connectedness of $D$ is established in [GMW18].

Let $u = (a_1, \dots, a_{d+1})$ be an arbitrary vertex in $V(D)$, and let $s$ be an element of $G$. We seek a path (or a walk) from $(u, 0)$ to $(u, s)$ in $D_\times$.

Let $b \in G$ satisfy the system

$$\begin{aligned} a_1 + \cdots + a_d + b &\neq 0 \\ a_2 + \cdots + a_d + b + s &\neq 0. \end{aligned}$$

This gives at least $|G|-2$ possible values for $b$.

Let $b' \in G$ satisfy the system

$$\begin{aligned} a_j + \cdots + a_d + b + s + b' + a_2 + \cdots + a_{j-2} &\neq 0, \qquad 3 \leq j \leq d+1 \\ s + b' + a_2 + \cdots + a_d &\neq 0 \\ b' + a_2 + \cdots + a_{d+1} &\neq 0. \end{aligned}$$

This gives at least $|G|-d-1$ possible values for $b'$.

Starting from $u$, we take a $(d+1)$-step walk to $(a_1, \dots, a_d, b)$. Let $n_1$ be the order of $a_1 + \ldots + a_d + b$. We cycle back to $(a_1, \dots, a_d, b)$ exactly $n_1$ times. Now we take one step by appending $s$. Then we take a $(d+1)$-step walk to $b', a_2, \dots, a_{d+1}$ and cycle that vertex the appropriate number of times. Finally walk to and cycle $a_1, \dots, a_{d+1}$. The total of this walk is $0 + s + 0 + 0 = s$. We conclude that $D_\times$ is strongly connected.

To establish aperiodicity, let $u = (a_1, \dots, a_{d+1})$ be a $D$-vertex satisfying the following. Set $a_d$ so that $a_1 + \cdots + a_d \neq 0$. Set $a_{d+1}$ so that for $i = 1, \dots, d$ we have $\Sigma a - a_i \neq 0$. Thus for $i = 1, \dots, d+1$ we have $\Sigma a - a_i \neq 0$. There are at least $|G|-d$ possible values for $a_{d+1}$. Then we may take the same approach as in the proof of Proposition 2.3 where we consider two cycles from $u$, one with an extra 0 inserted.

We have $|V(D)| = |G|^d(|G|-1)$ and each vertex has out-degree $|G|-1$. Thus there are $|G|^d(|G|-1)(|G|-1)^{m-d-1} = |G|^d(|G|-1)^{m-d}$ walks in $D$ defining an $m$-composition. Applying Theorem 2.1 gives the result. □

Figure 11 shows uniformly-randomly generated 100-compositions over $\mathbb{Z}_5$ also such that no part may be followed by its (additive) inverse.

**Proposition 2.5.** *Let $p_a(m)$ be the number of $m$-compositions of $a \in G$ such that the sum of any $d+1$ consecutive parts is not $0$. Then for $a \neq 0, b \neq 0$, we have $p_a(m) = p_b(m)$. If $m$ is not a multiple of $d+1$, then $p_0(m) = p_a(m)$.*

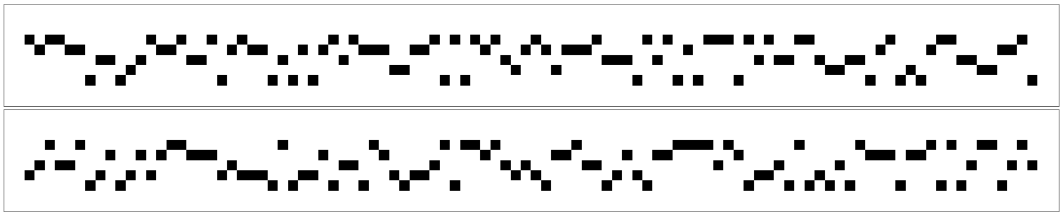


Figure 11: Uniform-randomly generated 100-compositions of 0 (above) and 1 (below) over $\mathbb{Z}_5$ with $x(i) \neq -x(i+1)$. (The vertical axis represents the value of a part.)

| $m$ \ $a$ | $id$ | (1854)(2763) | (1256)(3478) |
|---|---|---|---|
| 2 | 0 | 8 | 8 |
| 3 | 49 | 49 | 49 |
| 4 | 392 | 336 | 336 |
| 5 | 2401 | 2401 | 2401 |
| 6 | 16464 | 16856 | 16856 |
| 7 | 117649 | 117649 | 117649 |
| 8 | 825944 | 823200 | 823200 |
| 9 | 5764801 | 5764801 | 5764801 |
| 10 | 40336800 | 40356008 | 40356008 |
| 11 | 282475249 | 282475249 | 282475249 |
| 12 | 1977444392 | 1977309936 | 1977309936 |
| 13 | 13841287201 | 13841287201 | 13841287201 |
| 14 | 96888186864 | 96889128056 | 96889128056 |
| 15 | 678223072849 | 678223072849 | 678223072849 |
| 16 | 4747567274744 | 4747560686400 | 4747560686400 |

Table 2: Exact counts of $m$-compositions of $a$ with no part followed by its inverse, over $Q_8$ (written as a subgroup of $S_8$).

*Proof.* Let $x = (x(1), \dots, x(m))$ be an $m$-composition. Let $y(i) = \sum_{n=1}^{i} x(n)$. Clearly $x$ uniquely determines $y$ and vice versa. Also, $x$ has total $a$ iff $y(m) = a$.

Let $y^{\langle j \rangle}(i) = y((i-1)(d+1)+j-1)$ for $j \in [d+1]$. Then $x$ satisfies the condition iff each $y^{\langle j \rangle}$ is Carlitz and $y^{\langle d+1 \rangle}(1) \neq 0$.

First assume $m$ is not a multiple of $d+1$, so $y(m)$ is the last part of some $y^{\langle j \rangle}, j \neq d+1$. Let $\pi : G \to G$ be defined $\pi(b) = b$ for all $b \notin \{0, a\}$, and $\pi(a) = 0, \pi(0) = a$. Then if we apply $\pi$ to $y^{\langle j \rangle}$ within $y$ and take differences, we get a new $x'$ which satisfies the condition and has total 0. Thus $p_0(m) = p_b(m)$.

Second, if $m$ is a multiple of $d+1$, the previous $\pi$ does not work since it may change whether $y^{\langle d+1 \rangle}(1) \neq 0$. However, if we take some bijective $\pi' : G \to G$ which fixes 0 and swaps two nonzero elements $a$ and $b$, and apply it to $y^{\langle d+1 \rangle}$ in $y$ we conclude $p_a(m) = p_b(m)$. $\square$

Table 2 gives counts for $m$-compositions of $a$ over the quaternion group $Q_3$ such that no part may be followed by its inverse.

**Example 2.4.** We examine restrictions where all parts are simply required to lie in a fixed set $\Xi$. We assume without loss of generality that the subset $\Xi$ generates $G$.

If $\Xi = G$ then the number of compositions of $a$ is always $|G|^{m-1}$ since the first $m-1$ parts are arbitrary and the last part is uniquely determined. However if $\Xi \subset G$ this is no longer the case.

The digraph $D$ with vertex set $\Xi$ is clearly strongly connected, and it is straightforward to see that $D_\times$ is strongly connected as well.

For any cycle with final edge labeled $a$ in the Cayley graph constructed from $\Xi$, there is a cycle of equal length at $((a), 0)$ in $D_\times$. This implies that $D_\times$ is aperiodic iff the Cayley graph is aperiodic.

One way to ensure an aperiodic Cayley graph is to include $0 \in \Xi$. In general Cayley graphs are not aperiodic e.g. only an even number of transposition permutations can equal the identity since the identity is an even permutation. $\triangle$

**Theorem 2.2.** *Let $\bar{D}$ be a de Bruijn graph. Let $U \subset V(\bar{D})$ and $\Psi, \Phi \subseteq V(\bar{D})$ be nonempty and let $D = \bar{D} - U$. Let $\mu = [\Psi \subseteq U] + [\Phi \subseteq U]$.*

*Assume that $D_\times$ is aperiodic. Also assume that for all sufficiently large values of $m$ there exist $m$-compositions with exactly 1 occurrence of $U$. If $r \geq \mu$ then the number of unrestricted $m$-compositions of $a \in G$ starting in $\Psi$ and finishing in $\Phi$ with exactly $r$ occurrences of $U$ is*

$$p_a(m, r; D, \Psi, \Phi) = m^{r-\mu} A_{a,r,\mu} \cdot B^m (1 + O(m^{-1})), \qquad m \to \infty.$$

*If $D$ satisfies the requirements of Theorem 2.1 then $A_{a,r,\mu}$ does not depend on $a$.*

*Proof.* Let $x$ be an $m$-composition of $a$ containing $r$ occurrences of $U$. Fix elements of $\Psi$ and $\Phi$ as start and finish segments, and let $\nu$ be the number of these segments that lie in $U$. Fix an $(r-\nu)$-tuple over $U$ to be the interior occurring elements, in order. Also fix how much each pair of adjacent occurrences overlap, for all $r-1$ such pairs. This determines the number $m'$ of parts of $x$ not in an occurrence, and $m - r\sigma \leq m' \leq m - \sigma - r + 1$. These $m'$ parts are located within the $j \leq r+1$ maximal subcompositions which are between occurrences. (If adjacent occurrences do not overlap but are contiguous, we say there is an empty subcomposition between them). The value of $j$ attains a maximum of $r + 1 - \nu$. Fix a total $a_i$ of each subcomposition $x_i$ so that the total of $x$ is $a$.

Then each $x_i$ is constrained to avoid $U$ and is allowed certain sets of start and finish segments. By Proposition 2.2, the number of combinations of these subcompositions is then $n_1 = C_a \cdot B^{m'}(1 + O(\theta^{m'}))$ for some $C > 0$. The number of ways of selecting the positions of the $r - \nu$ interior occurrences within $x$ is $n_2 = Dm^{j-1}(1 + O(m^{-1})), D > 0$. Taking $n_1 n_2$ and adding over the (fixed and finite) set of ways to fix the objects above, we conclude the result. $\square$

**Theorem 2.3.** *Let $Y = (Y_{i,j}(z,u))_{i,j=1}^n$ be a matrix of bivariate complex functions analytic at $(0,0)$ with non-negative coefficients satisfying*

$$Y = TY + V,$$

*where $V = (V_{i,j}(z,u))_{i,j=1}^n$, $T = (T_{i,j}(z,u))_{i,j=1}^n$ and each entry $V_{i,j}, T_{i,j}$ is a polynomial in $z, u$ with non-negative coefficients. Assume also that $T(z,1)$ is transitive and primitive, and that $T(0,1)$ is nilpotent. Let $\rho(u)$ be the unique solution to*

$$|I - T(\rho(u), u)| = 0,$$

*assumed to be analytic at $1$, such that $\rho(1) = \rho$. Let $A \subseteq [n]^2$ be nonempty and define $F(z,u) = \sum_{(i,j)\in A} Y_{i,j}(z,u)$. Then, provided the variability condition,*

$$\frac{\rho''(1)}{\rho(1)} + \frac{\rho'(1)}{\rho(1)} - \left(\frac{\rho'(1)}{\rho(1)}\right)^2 \neq 0$$

*is satisfied, a Gaussian Limit Law holds for the coefficients of $F(z,u)$ with mean and variance that are $\Theta(n)$ and speed of convergence that is $O(n^{-1/2})$. The mean and variance are asymptotic to sequences that depend only on $T$.*

*Proof.* See [FS09, p. 665]. $\square$

**Proposition 2.6.** *Let $M$ be the adjacency matrix of a strongly connected aperiodic digraph. Let $(\Psi_i, \Phi_i)$ be a finite sequence of nonempty sets of start and finish vertices such that we do not have*

$$u \in \Psi_i, v \in \Phi_i,\ u \in \Psi_j, v \in \Phi_j$$

*for some $u, v$ and $i \neq j$.*

*If we consider all possible length-$m$ walks from $\Psi_i$ to $\Phi_i$, over all $i$, to be equally likely, then the number of times that a designated set of vertices $\Xi$ is reached in such a walk is asymptotically normal with mean and variance asymptotic to sequences that do not depend on $(\Psi_i, \Phi_i)$.*

*Proof.* The result follows from Theorem 2.3 as in [FS09, Example IX.17]. Further discussion of the variability condition is found in e.g. [Ber+03, Theorem 3]. $\square$

**Theorem 2.4.** *Distinguish a set of words $\Xi$ over $G$, all having common length $\sigma$. Then the number of occurrences of $\Xi$ in a uniform random $m$-composition of $a \in G$ is asymptotically normal with mean and variance asymptotic to those of the number of occurrences of $\Xi$ in a uniform random word over $G$.*

*Proof.* Let $D$ be the $\sigma$-dimensional de Bruijn graph over $G$ and let $D_\times$ be the associated derived graph. Then $D_\times$ is clearly strongly connected and is aperiodic since $((0,\ldots,0),0)$ has a loop. We set $\Psi_1 = \Psi_\times$ to contain all valid start vertices and set $\Phi_1 = \Phi_a$ to contain those vertices of the form $(u,a)$, $u \in V(D)$. Proposition 2.6 immediately allows us to conclude. $\square$

Joint distributions and local limit phenomena are derivable, under conditions, from adaptations of [BRW83] and/or [Ber+03].

**Lemma 2.5.** *All subword patterns $\tau$ other than $1^p2$ and its symmetries ($12^p, 2^p1$, and $21^p$) are such that any digraph $D$ whose walks represent words over $[k]$ that avoid $\tau$ is strongly connected.*

*Proof.* The patterns $1^p2$ do not satisfy this because $1^{p-1}$ and $2^{p-1}$ are both allowed but there is no allowed word of the form $1^{p-1}\cdots 2^{p-1}$.

Let $\tau = (\tau(1), \cdots, \tau(\sigma+1))$, and let the maximum letter in $\tau$ be $j^*$. Let $D$ be the digraph of span $\sigma$ representing $k$-ary words avoiding $\tau$. Let $x = (x(1), \ldots, x(\sigma))$ and $y = (y(1), \ldots, y(\sigma))$ be vertices of $D$. We proceed by cases, establishing either that $x \longrightarrow y$ and $y \longrightarrow x$ or $1^\sigma \longrightarrow x$ and $x \longrightarrow 1^\sigma$.

Case 1: $j^* = 1$. If $x(\sigma) \neq y(1)$, then the concatenation $x^\frown y$ is allowed. Otherwise, take $c \neq x(\sigma) = y(1)$ and then $x^\frown(c)^\frown y$ is allowed.

Case 2: $j^* \geq 3$. Assume WLOG $\tau(1) > 1$. Then $1^{\sigma\frown}x$ is always allowed. If $\tau(\sigma) > 1$ then $x^\frown 1^\sigma$ is allowed too. Otherwise $\tau(\sigma) = 1$ and $x^\frown 2^\sigma 1^\sigma$ is allowed.

Case 3: $j^* = 2$. Assume WLOG $\tau(1) = 2$. Again $1^{\sigma\frown}x$ is allowed. If $\tau(\sigma) = 2$ then $x^\frown 1^\sigma$ is allowed too. If $\tau(\sigma) = 1$ and $\tau$ is not monotonic then $x^\frown 2^\sigma 1^\sigma$ is allowed. Finally, if $\tau = 2^p1^q$ with $p, q > 1$ then $x^\frown(21)^p1^{p-1}$ is allowed.

This also implies that any digraph $D$ representing the same set of compositions but possibly with a greater span $\sigma$ is also strongly connected. $\square$

**Lemma 2.6.** *Let $G$ be a totally ordered finite group and let $\tau$ be a subword pattern other than $1^p2$ and its symmetries. There is a digraph $D$ for compositions over $G$ avoiding $\tau$ such that $D_\times$ is strongly connected and aperiodic.*

*Proof.* Let $\sigma + 1$ be the length of the pattern, and take $D$ to have span $\sigma$.

We show strong connectedness of $D_\times$. Let $a \in G$ be nonzero. If the pattern is not $i^pji^q$ where $p, q \geq 1$ and $i \neq j$, then $0^\sigma a 0^\sigma$ is allowed and therefore $(0^\sigma, 0) \longrightarrow (0^\sigma, a)$ in $D_\times$. For patterns in the form $i^pji^q$, let $n$ be the order of $\sigma a$. Then $0^\sigma a^{\sigma n} a 0^\sigma$ suffices.

We show aperiodicity of $D_\times$. The vertex $(0^\sigma, 0)$ exists in $D_\times$ and has a loop iff the pattern is not $1^{\sigma+1}$. For the pattern $1^{\sigma+1}$, let $b \in G$ be nonzero and let $a = -\sigma b$. Then the two sequences $b^\sigma 0a0b^\sigma$ and $b^\sigma 0a00b^\sigma$ are allowed and correspond to paths $(b^\sigma, 0) \longrightarrow (b^\sigma, 0)$. $\square$

**Theorem 2.5.** *Let $G$ be a finite group with a total order and let $\tau$ be a subword pattern not $1^p2$ or its symmetries. The number of $m$-compositions of $a \in G$ containing $r$ occurrences of $\tau$ is*

$$A_r m^r B^m (1 + O(m^{-1})), \qquad m \to \infty.$$

*If $X_m^{\langle a \rangle}$ is the number of occurrences of $\tau$ in a uniform random $m$-composition*

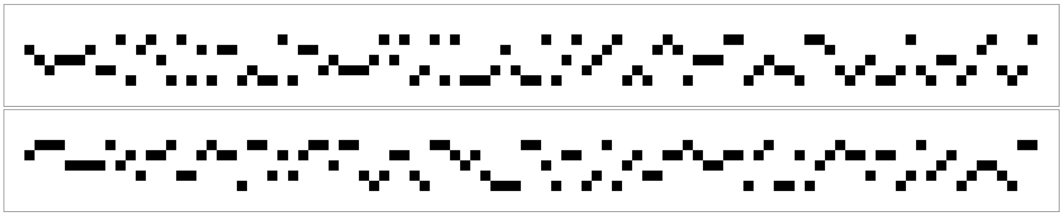


Figure 12: Uniform-randomly generated 100-compositions of 0 (above) and 1 (below) over $\mathbb{Z}_5$ which avoid 132. (The vertical axis represents the value of a part.)

*of $a \in G$ then*

$$\frac{X_m^{\langle a \rangle} - \mathbb{E}(X_m^{\langle a \rangle})}{\sqrt{\mathrm{Var}(X_m^{\langle a \rangle})}} \Rightarrow N(0,1).$$

*Proof.* Direct from Lemma 2.6, Theorem 2.2, and Theorem 2.4. ☐

We are immediately able to modify results where they are available for words containing subwords patterns.

**Proposition 2.7.** *Let $G$ be a totally ordered group with $|G| = k$, e.g. $\mathbb{Z}_k$ where $0 < 1 < \cdots < k-1$. Define*

$$C(y) = [q^r]\frac{1}{1 - y - \sum_{p=3}^{d} \sum_{j=0}^{p-3} \binom{p-3}{j}\binom{k}{p+j} y^{p+j}(q-1)^{p-2}},$$

*as in [HM10, p. 112]. Let $\rho > 0$ be the radius of convergence of $C(y)$, and let $A_r = \lim_{y \to \rho}((1 - y/\rho)^{r+1} C(y))$.*

*The number of $m$-compositions of $a \in G$ containing $r$ occurrences of the subword pattern 123 is*

$$\frac{1}{k} A_r m^r \left(\frac{1}{\rho}\right)^m (1 + O(m^{-1})), \qquad m \to \infty.$$

*Proof.* Theorem 4.30 in [HM10] states that $[y^m]C(y)$ is the number of $m$-compositions with any total containing $r$ occurrences of 123. The result then follows from Theorem 2.5. ☐

Table 3 shows counts of $m$-compositions of $a$ avoiding $\tau = 132$ over $\mathbb{Z}_5$. Figure 12 gives randomly selected compositions avoiding 132 over $\mathbb{Z}_5$. Figure 13 gives the same for compositions avoiding 121.

## 2.2 Simplification of transfer matrices

What's the simplest recurrence relation we can get for the number of words in a regular language? Can we get results that depend "explicitly" on the alphabet size $k$ and parameters of the language?

| $m$ \ $a$ | 0 | 1 | 2 |
|---|---|---|---|
| 2 | 5 | 5 | 5 |
| 3 | 23 | 23 | 23 |
| 4 | 105 | 105 | 105 |
| 5 | 478 | 477 | 477 |
| 6 | 2171 | 2171 | 2171 |
| 7 | 9869 | 9868 | 9868 |
| 8 | 44861 | 44861 | 44861 |
| 9 | 203930 | 203930 | 203930 |
| 10 | 927032 | 927032 | 927033 |
| 11 | 4214147 | 4214147 | 4214147 |
| 12 | 19156861 | 19156861 | 19156865 |
| 13 | 87084158 | 87084158 | 87084158 |
| 14 | 395871195 | 395871195 | 395871198 |
| 15 | 1799569607 | 1799569609 | 1799569610 |
| 16 | 8180566793 | 8180566793 | 8180566793 |

Table 3: Exact counts of $m$-compositions of $a$ avoiding 132 over $\mathbb{Z}_5$.

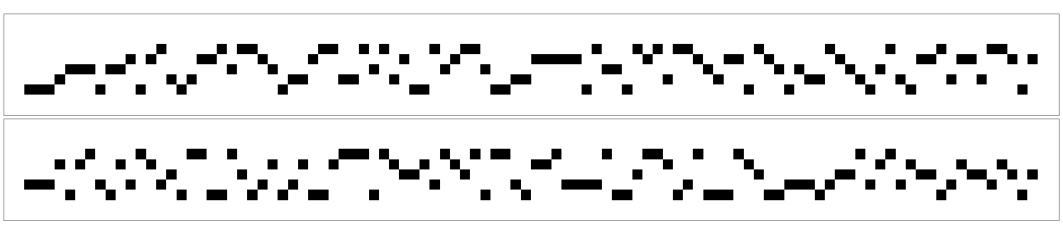


Figure 13: Uniform-randomly generated 100-compositions of 0 (above) and 1 (below) over $\mathbb{Z}_5$ which avoid 121. (The vertical axis represents the value of a part.)

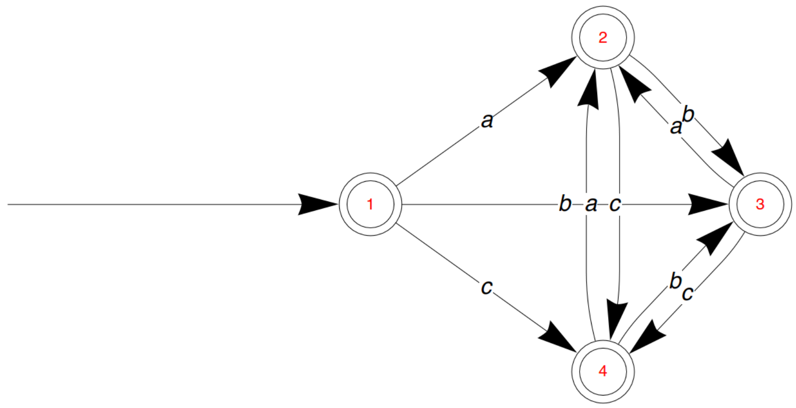



Figure 14: blah

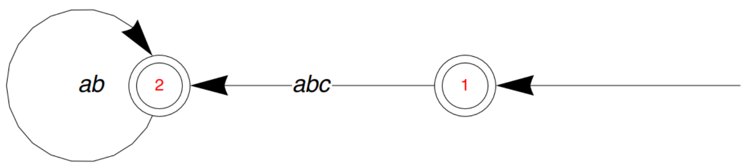



Figure 15: blah

"Weak equivalence" of DFAs: DFAs accept same number of words of each length. approximate weak minimization algorithm (RavEis algorithm) given in [RE04, Sec. 4.2]. The RavEis algorithm works by finding states which are equivalent in the counting sense: there are equal numbers of words of length $m$ starting at each state for each $m$. These states are then merged. However, unlike in the case of traditional minimization this may not yield the absolute minimal possible DFA. The procedure at least cannot increase the number of states and in practice may reduce the number substantially.

get simpler recurrences rather than to get faster computation. apply symbolically as a technique.

**Example 2.5** (Carlitz words)**.** Figure 14 shows the naive automaton for Carlitz words on the alphabet $\{a, b, c\}$. In fact this automaton is minimal. However, there is a length-preserving bijection between Carlitz 3-words and the words accepted by automaton in Figure 15.

For general $k$, we need 2 states rather than the naive $k+1$, and the final recurrences are $f_1(n) = [n = 0]$, $f_2(1) = kf_1(0), f_2(n) = (k-1)f_2(n-1)$, where $f_i$ counts words starting [finishing?] at state $i$. We get the solution

$$f(n) = f_1(n) + f_2(n) = k(k-1)^{n-1}. \qquad \triangle$$

**Grammars, automata, and graphs**

[discuss grammars and recurrence relations, graphs vs. automata]

the relationship between regular languages and recurrence relations for their counting sequences is clearest when considering unambiguous regular grammars ...

Let $\Sigma$ be a finite alphabet. We first define a "regular transfer digraph". This is $(D, w, S, F, \epsilon_a)$, where $D$ is a digraph, $w : V(D) \to \Sigma$ is a weight function, $S, F \subseteq V(D)$ are the start and finish vertices, and $\epsilon_a$ is 1 or 0 depending on whether the empty word $\epsilon$ is legal. A walk in $D$ is legal if it starts in $S$ and finishes in $F$. A regular transfer digraph is ambiguous if more than one legal walk represents the same word. Words defined by unambiguous regular transfer digraphs can be counted in a straightforward manner using powers of the adjacency matrix and start and finish indicator vectors.

It is clear that the digraphs in §2.1 used for the transfer matrix method can be converted to unambiguous regular transfer digraphs.

**Proposition 2.8.** *Let $L$ be the formal language given by all concatenated legal walks in an unambiguous canonical transfer digraph $(D, w, S, F, \epsilon_a)$. Then $L$ is an arbitrary regular language.*

*Proof.* Digraph: $D, w, S, F, \epsilon_a$

NFA: state set $Q$, transition function $\Delta$, initial state $q_0$, accept states $Q^*$

We give constructions in two directions which show equivalence between regular transfer digraphs and other representations of regular languages.

Given the digraph, we define an equivalent NFA as follows. The state set is all arcs, with a start state and finish state added, so $Q = E(D) \,\dot\cup\, \{q_0\} \,\dot\cup\, \{q^*\}$, $Q^* = \{q^*\}$. For arcs $(u, v), (v, w)$, the transition function is defined

$$\Delta((u, v), a) = \{(v, w) : w(v) = a\} \,\dot\cup\, \{q^* : v \in F\},$$

and transitions from the start state are given by

$$\Delta(q_0, a) = \{(u, v) : u \in S, w(u) = a\} \,\dot\cup\, \{q^* : \epsilon_a = 1, a = \epsilon\}.$$

Thus walks from start vertex to finish vertex correspond to a sequence of transitions. The digraph is not required to be ambiguous.

DFA to digraph

$V(D) = \{(q, q', a) : q' = \Delta(q, a)$

$((q_1, q_1', a_1), (q_2, q_2', a_2)) \in E(D)$ iff $q_1' = q_2$

$w(q, q', a) = a$

$S = \{(q_0, q', a) : (q_0, q', a) \in V(D)\}$

$F = \{(q, q', a) : q' \in Q^*, (q, q', a) \in V(D)\}$

$\epsilon_a = 1$ iff $q_0 \in Q^*$

It can be verified that the resulting digraph is unambiguous. □

### Multivariate version

now states are equivalent iff the MGFs for the languages accepted starting at that state are the same. The MGF may be defined by replacing letters by commutative indeterminates.

Obviously you can always extract the multivariate rational counting GFs and compare by subtracting and testing for 0. This gives a brute-force algorithm for exact minimization.

The RavEis algorithm terminates after it has found all possible equivalent states. This follows from a bound on the maximum word length required to check in order to know whether two states are equivalent for all word lengths. The following lemmas are used.

**Lemma 2.7** (Lemma A1 in [SH85])**.** *Let $S$ be a finite nonempty set. For all $s \in S$ let $A_s(n)$ be a real sequence such that*

$$A_s(n+1) = \sum_{t \in S} c_{s,t} A_t(n), \qquad n \geq 0$$

*where $c_{s,t} \in \mathbb{R}$. Then each $A_s$ satisfies a linear difference equation of degree $|S|$ or less with real coefficients.*

**Lemma 2.8** (Lemma A2 in [SH85])**.** *Let $A(n)$ and $B(n)$ be real sequences satisfying linear difference equations of degrees $a$ and $b$ with real coefficients. If for $0 \leq n \leq a+b$ we have $A(n) = B(n)$ then the sequences $A(n)$ and $B(n)$ are identical for all $n$.*

These lemmas allow us to conclude that if two states have equal counts for words of length $\leq 2|Q| - 1$, where $Q$ is the state set, then they are equivalent (simply define new automata for each state with that state as start state). In the multivariate case, the proofs may be simply modified to use linear algebra over the field of multivariate rational functions rather than the field of rational numbers. We conclude that the bounds employed by the RavEis algorithm still hold. The rest of the RavEis algorithm can be directly adapted.

example?

### Subword patterns in words

If we generalize Carlitz words to avoidance of the pattern $1^p$, we can generalize the weak-minimized automaton where the states keep track of how long the run is at the end of the current window. However, there exist subword patterns which do not have a significantly smaller weak-minimized automaton.

Define the pattern $\tau_p$ of length $2^p - 1$ as follows. First, $\tau_1 = 1$. Assume $\tau_p = a_1 \dots a_{2^p-1}$. Then

$$\tau_{p+1} = 1, 2a_1, 3, 2a_2, 5, \dots, 2^{p+1} - 3, 2a_{2^p-1}, 2^{p+1} - 1,$$

where commas are used to separate letters.

Claim: the weak-minimized automaton for $\tau_p$ is not much smaller than the naive one. [actually it is not the weak-minimized one, it's just the one returned by the RavEis algorithm. so it's the minimal one using the partition technique but perhaps not overall.]

### Locally Mullen compositions

We give one case of local restriction which is not based on a subword pattern.

[define locally Mullen]

The following observation establishes a connection between Carlitz compositions and locally Mullen compositions.

**Proposition 2.9.** *(quoting from [GMW18]) For each $m$-composition $\boldsymbol{u} = u_1, u_2, \dots, u_m$ over $G$, let $\boldsymbol{v} = \phi(\boldsymbol{u})$ be an $m$-composition defined by $v_j = u_1 + \cdots + u_j, 1 \leq j \leq m$. Then $\phi$ is a bijection between locally $d$-Mullen $m$-compositions and $d$-Carlitz weak $m$-compositions over $G$ such that the first $d$ parts are nonzero.*

**Proposition 2.10.** *(quoting from [GMW18])*

1. *Let $\mathcal{F}$ be a family of compositions over a finite (additive) abelian group $G$. Suppose $\mathcal{F}$ is closed under multiplication, that is, $\mathbf{c} := (c_1, \dots, c_m)$ belongs to $\mathcal{F}$ implies that $a\mathbf{c} := (ac_1, \dots, ac_m)$ also belongs to $\mathcal{F}$ for every $a \in G^*$. Let $c_m(\mathcal{F}; s)$ be the number of $m$-compositions in $\mathcal{F}$. If $s \in G^*$ has a multiplicative inverse $s^{-1}$ in $G^*$, then $c_m(\mathcal{F}; s) = c_m(\mathcal{F}; \mathbf{1})$.*

2 *Let $c_m(s)$ be the number of locally $d$-Mullen $m$-compositions of $s \in G$ then $c_m(s) = c_m(\mathbf{1})$ for every $s \in G^*$.*

**Remark:** The above proposition implies that the number of Carlitz $m$-compositions of $s$ over a finite field is equal to that of 1 when $s \neq 0$. And the same is true for Carlitz weak compositions.

**Corollary 2.2.** *Let $(h_1, \dots, h_{d-1})$ be a locally $(d-1)$-Mullen composition over $\mathbb{Z}_k$. Let $c_m(s; h_1, \dots, h_{d-1})$ be the number of locally $d$-Mullen compositions of $s$ over $\mathbb{Z}_k$ with prefix $h_1, \dots, h_{d-1}$. Let $b_i = \left[\sum_{j=1}^{i} h_j = s\right]$ for $i \in \{1, \dots, d-1\}$. Then the value of $c_m(s; h_1, \dots, h_{d-1})$ only depends on the $b_i$ values and whether $s = 0$.*

probably could use any prefix length $\leq d$.

*Proof.* Similar to previous. $\square$

more details needed here

**Lemma 2.9.** *The number of locally $d$-Mullen $d$-compositions $(x_1, \dots, x_d)$ over $\mathbb{Z}_k$ where $x_1 + \cdots + x_j = t$ is $(k-2)^{\underline{d-1}}$ (if $t > 0$). In particular, this is the number of locally $d$-Mullen $d$-compositions of $s$ if $s > 0$.*

*Proof.* Employ bijection. Single parts in the $d$-Carlitz (nonweak) $d$-compositions may be freely chosen among $\mathbb{Z}_k \setminus \{0\}$. For the second statement, take $j = m$. ☐

Let $c_m(s; h_1, \dots, h_{d-1})$ and $b_i$ be defined as in Corollary 2.2. We write $\boldsymbol{h} = (h_1, \dots, h_{d-1})$. Note that $\sum_i b_i \leq 1$. Let $i^*$ be the index such that $\sum_{j=1}^{i^*} h_j = s$, or if no such index exists we take $i^* = \infty$. We know that $c_m(s; \boldsymbol{h})$ only depends on whether $s = 0$ and the value of $i^*$. We write this quantity as $c_m(s, i^*)$.

state properly with initial conditions

**Proposition 2.11.**

$$c_m(0, \infty) = (k-d)c_{m-1}(1, \infty)$$

$$c_m(1, 1) = (k-d)c_{m-1}(0, \infty)$$

$$c_m(1, i) = (k-d)c_{m-1}(1, i-1), \qquad 1 < i < \infty$$

this line can be simplified, trivial recurrence in $i$

$$c_m(1, \infty) = (k-d-1)c_{m-1}(1, \infty) + c_{m-1}(1, d-1)$$

might be cleaner using 0 instead of $\infty$

are there obvious simplifications to this proof? known Carlitz results?

*Proof.* Let $(h_1, \dots, h_{d-1})$ be a locally $(d-1)$-Mullen composition over $\mathbb{Z}_k$.

Clearly for $m < d-1$ we have $c_m(s; \boldsymbol{h}) = 0$ so we consider $m \geq d-1$. We proceed by induction on $m$. Base case: $m = d-1$. Here $c_{d-1}(s; \boldsymbol{h}) = [s > 0][i^* = d-1]$.

Inductive case: Let $R$ be the relation such that $R(x_1, \dots, x_n)$ holds iff $(x_1, \dots, x_n)$ is an locally $\infty$-Mullen composition ($d$-Mullen for all $d$). We have the following recurrence relation:

$$c_m(s; \boldsymbol{h}) = \sum_{j : R(h_1, \dots, h_{d-1}, j)} c_{m-1}(s - h_1; h_2, \dots, h_{d-1}, j). \tag{1}$$

The condition $R(h_1, \dots, h_{d-1}, j)$ here is equivalent to the inequations

$$\begin{aligned} j &\neq 0, \\ j + h_{d-1} &\neq 0, \\ j + h_{d-1} + h_{d-2} &\neq 0, \\ &\vdots \\ j + h_{d-1} + h_{d-2} + \cdots + h_1 &\neq 0. \end{aligned}$$

why the horizontal gap

Note that the sum always contains $k-d$ terms because each of the $d$ inequations excludes a different value of $j$.

Using Equation 1 we may now apply the induction hypothesis on $c_{m-1}$. We break into cases.

(Case 1.) $s = 0, i^* < \infty$. Impossible since this implies $\boldsymbol{h}$ is not a locally $(d-1)$-Mullen composition.

(Case 2.) $s = 0, i^* = \infty$. Let $i^*_s(x_1, \ldots, x_n) = \min\left\{i : \sum_{j=1}^{i} x_j = s\right\}$. Considering the general term $c_{m-1}(-h_1; h_2, \ldots, h_{d-1}, j)$ from the RHS of Equation 1, we have $-h_1 \neq 0$ and we must have $i^*_{-h_1}(h_2, \ldots, h_{d-1}, j) = \infty$ since $R(h_1, \ldots, h_{d-1}, j)$ holds. Thus

$$c_m(s; \boldsymbol{h}) = (k-d)c_{m-1}(1, \infty).$$

(Case 3.) $s \neq 0, i^* = 1$. We have $s = h_1$ so the general term is $c_{m-1}(0; h_2, \ldots, h_{d-1}, j)$. Hence

$$c_m(s; \boldsymbol{h}) = (k-d)c_{m-1}(0, \infty).$$

(Case 4.) $s \neq 0, 1 < i^* < \infty$. Here the general term is $c_{m-1}(s - h_1; h_2, \ldots, h_{d-1}, j)$, where $s - h_1 \neq 0$. We have

$$i^* - 1 = i^*_s(h_1, \ldots, h_{d-1}) - 1 = i^*_{s-h_1}(h_2, \ldots, h_{d-1}) = i^*_{s-h_1}(h_2, \ldots, h_{d-1}, j),$$

thus

$$c_m(s; \boldsymbol{h}) = (k-d)c_{m-1}(1, i^* - 1).$$

(Case 5.) $s \neq 0, i^* = \infty$. Again the general term is $c_{m-1}(s - h_1; h_2, \ldots, h_{d-1}, j)$, with $s - h_1 \neq 0$, and we must have $i^*_{s-h_1}(h_2, \ldots, h_{d-1}, j) \geq d-1$. We can only have $i^*_{s-h_1}(h_2, \ldots, h_{d-1}, j) = d-1$ if $j$ takes on the value $s - h_1 - h_2 - \cdots - h_{d-1}$ in the sum. Assume this value of $j$ is excluded by some inequation $j \neq -h_{d-1} - \cdots - h_{d-n}$. Then we have $s - h_1 - h_2 - \cdots - h_{d-1} = -h_{d-1} - \cdots - h_{d-n}$, so $s = h_1 + \cdots + h_{d-n-1}$ which contradicts $i^* = \infty$. So we have

$$c_m(s; h_1, \ldots, h_{d-1}) = (k-d-1)c_{m-1}(1, \infty) + c_{m-1}(1, d-1). \qquad \square$$

**Corollary 2.3.** *If $c_m(s)$ is the number of locally $d$-Mullen $m$-compositions of $s \in \mathbb{Z}_k$ then $c_m(s) =$?.*

complete

*Proof.* Assume $s \neq 0$. If $m \leq d$, then Lemma 2.9 implies $c_m(s) = c_m(1) =$?.

implement and compute values

If $m \geq d-1$, let $\boldsymbol{h}$ represent a composition $(h_1, \ldots, h_{d-1}) \in \mathrm{SEQ}_{d-1}(\mathbb{Z}_k)$. Let $i^*_s(x_1, \ldots, x_n) = \min\left\{i : \sum_{j=1}^{i} x_j = s\right\}$. Let $R$ be the relation such that $R(x_1, \ldots, x_n)$ holds iff $(x_1, \ldots, x_n)$ is an locally $\infty$-Mullen composition. By Proposition 2.11 we have

do other case too

$$\begin{aligned} c_m(s) &= \sum_{\boldsymbol{h}} c_m(s; \boldsymbol{h}) \\ &= \sum_{\boldsymbol{h}: R(\boldsymbol{h})} c_m(1, i^*_s(\boldsymbol{h})) \\ &= \sum_{i=0}^{k-1} \#\{\boldsymbol{h} : R(\boldsymbol{h}), i^*_s(\boldsymbol{h}) = i\} c_m(1, i). \end{aligned}$$

By Lemma 2.9 the quantity $\#\{\boldsymbol{h} : R(\boldsymbol{h}), i^*_s(\boldsymbol{h}) = i^*\}$ does not depend on $s$. $\square$

## 2.3 Binary trees

unweighted binary plane trees, tree patterns: see [Row10]. Sec. 5 gives algorithm for system of algebraic equations in GF. See also ternary and $m$-ary tree extension in [Gab+12, Sec. 3].

remark: non-contiguous but still unweighted avoidance in [Dai+12]

counting unweighted pattern occurrences in [Chy+08]

A weighted version seems pretty straightforward. extension: *functional graphs, DAGs, cactus graphs.* Finite group $G$. Sum order given by preorder traversal.

We consider trees on the vertices $[n]$. Parents must be less than children, and vertices in each row must form a contiguous interval. Thus we essentially have plane (embedded) trees. Binary trees give us somewhat simpler expressions. binary = 0, 1, or 2 children. When talking of isomorphisms we mean graph isomorphisms that are also order isomorphisms on vertex labels. Height is defined as number of vertices along path from root to leaf. Let $G$ be a finite group; in the following, all trees are $G$-weighted binary trees.

analogy to §2.1 as context-free languages to regular languages. does this generalize paths? no, slightly different premises.

**Definition 2.4.** *Let $T$ be a binary tree. Then the $h$-prefix of $T$ is the subgraph $T_{|h}$ induced by vertices of height at most $h$.*

Let $\mathcal{H}$ be a set of $G$-weighted binary trees of height $h$. For each tree $H_i \in \mathcal{H}$, say there are $\ell_i$ leaves at height $h$. Let $S_i$ be a set of $\ell_i$-tuples $s_{i,j}$ where each component of $s_{i,j}$ is either $\{\}$, or $T$, or $(T, R)$, where $T$ and $R$ are weighted binary trees with height in $[h]$. We say $\{\}$ matches the empty tree, $T$ matches single trees with $h$-prefix $T$, and $(T, R)$ matches pairs of trees with $h$-prefixes $T$ and $R$. We handle empty trees separately to avoid having both an empty tree and a pair of empty trees as possibilities. A binary tree $T$ is considered legal according to $\mathcal{H}$ and the $S_i$ iff

1. $T_{|h}$ is isomorphic to $H_i$ for some $i$,
2. there is $j$ such that for each $1 \le k \le \ell_i$ the children of the $k^{\text{th}}$-largest height-$h$ vertex match $s_{i,j}(k)$, and
3. the subtrees of $T$ with roots at height $h+1$ which themselves have height at least $h$ are all legal.

Let $z$ mark the number of nodes. Let $\Phi_i(z)$ be the OGF for locally restricted binary trees with $h$-prefix $H_i$.

**Proposition 2.12.** *We have an algebraic system of equations, one for each i of the form*

$$\Phi_i(z) = z^{|H_i|} \left( \sum_{s_{i,j} \in S_i} \prod_{k=1}^{\ell_i} \Xi_{i,j,k}(z) \right), \qquad \forall i \tag{2}$$

*where*

$$\Xi_{i,j,k}(z) = \begin{cases} 1, & s_{i,j}(k) = \{\} \\ \Phi_t(z), & s_{i,j}(k) = H_t \\ \Phi_r(z)\Phi_t(z), & s_{i,j}(k) = (H_r, H_t) \\ z^{|T|}\Phi_t(z), & s_{i,j}(k) = (T, H_t), T \notin \mathcal{H} \\ \Phi_t(z)z^{|R|}, & s_{i,j}(k) = (H_t, R), R \notin \mathcal{H} \\ z^{|T|+|R|}, & s_{i,j}(k) = (T, R), T, R \notin \mathcal{H} \\ z^{|T|}, & s_{i,j}(k) = T, T \notin \mathcal{H} \end{cases}.$$

*Proof.* follows from combinatorial specification □

**Example 2.6.** example with 2-Carlitz binary trees over $\mathbb{Z}_2$. Let $H$ be the following tree.

```
  0
 / \
0   0
```

To "grow" $H$, each of the height-2 leaves of $H$ can have children chosen independently. Specifically, there may be 0, 1, or 2 children. Each child can chosen from the following set of binary trees of height at most 2. If the child subtree has height less than 2, no vertices can be added to that tree.

```
  1       1       1       1     1     1     1
 / \     / \     / \     / \    |     |
0   0   0   1   1   0   1   1   0     1
```

Each of these trees can be found as child subtrees of $H$. △

**Proposition 2.13.** *If the solution to the system (3) has no rational components, all components are aperiodic, and the dependency graph is strongly connected, then the number of locally restricted binary trees with $n$ nodes is given by …*

*Proof.* [FS09, Sec. VII.6.1] gives asymptotics … □

We now define a set $\mathcal{H}^G$ containing all pairs $H_i^g = (H_i, g)$ where $g \in G$. We define $S_i^g$ from $S_i$ as follows. The set $S_i^g$ contains tuples of length $\ell_i$ where components are either $\{\}$, or $(T, a)$, or $(T, a, R, b)$ where $a, b \in G$ and $T$ and $R$ are trees with height in $[h]$. If the height of $T$ is less than $h$, then $a = \sum T$, similarly if the height of $R$ is less than $h$, then $b = \sum R$. For each tuple $s_{i,j} \in S_i$, we include in $S_i^g$ all such tuples $s_{i,j}^g$ so that $H_i + s_{i,j}^g = g$.

Let $\Phi_i^g(z)$ be the OGF for locally restricted binary trees with $h$-prefix $H_i$ and total $g$. We have an algebraic system of equations, one for each $i$ of the form

$$\Phi_i^g(z) = z^{|H_i|}\left(\sum_{s_{i,j}^g \in S_i^g} \prod_{k=1}^{\ell_i} \Xi_{i,j,k}(z)\right), \qquad \forall i, g \tag{3}$$

where $\Xi_{i,j,k}(z)$ is as above.

**Example 2.7.** leaves must be 0 and cannot have adjacent 0 nodes in a row, over $\mathbb{Z}_2$. $A_i^g$ stands for trees with sum $g$ and root weighted $i$.

$$A_1^0(z) = z(A_0^1(z) + 2A_0^0(z)A_0^1(z) + 2A_1^0(z)A_0^1(z) + A_1^1(z) + 2A_0^0(z)A_1^1(z) + 2A_1^0(z)A_1^1(z)),$$
$$A_0^0(z) = z(A_1^0(z) + A_1^0(z)^2 + A_1^1(z)^2),$$
$$A_1^1(z) = z(1 + A_0^0(z) + A_0^0(z)^2 + A_1^0(z) + 2A_0^0(z)A_1^0(z) + A_1^0(z)^2 + A_0^1(z)^2 + 2A_0^1(z)A_1^1(z) + A_1^1(z)^2),$$
$$A_0^1(z) = z(A_1^1(z) + 2A_1^0(z)A_1^1(z))$$

△

TOWRITE: following proposition. generalization to $r$ occurrences, obtained by taking derivatives of system wrt $u$ and setting $u = 0$ and checking strong connectedness, aperiodicity, irrationality which I think are direct. limiting distribution.

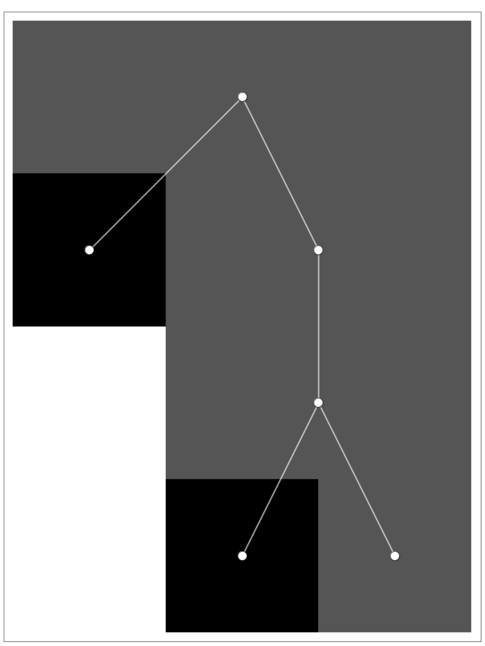
**Proposition 2.14.** *If for all binary trees $H$ and $s \in S$ we have that $H$ is the truncation of some valid tree with sum $s$, the dependency graph of $\{\Phi_i(z)\}$ is strongly connected and all components $\Phi_i(z)$ are irrational, then the dependency graph of $\{\Phi_{i,s}(z)\}$ is strongly connected and all components $\Phi_{i,s}(z)$ are irrational.*

*Proof.* show strong connectedness of new system, then use that to show irrationality □

**Corollary 2.4.** *asymptotics of $\Phi_i^g(z)$ ...*

*Proof.* ... □

random generation



```
(2, ((3), (2, (2, ((3), (2))))))
```

Number of occurrences. See Drmota's book [Drm09]; attempt to emulate strategy used for the kind of patterns discussed there which is based on earlier paper [Chy+08].

the proof works exactly the same. doesn't depend on $s$, i.e. each component of the system and any sum of components will have same limiting dist by Cor 2.34 in the book (choose $G$ to be any desired sum of components) where $x_0$ depends only on the system.

Note on computation: Computation of series terms given the system can be done by eliminating all but one component, then using Newton iteration (as in Genfunlib) on one variable which includes the other variables in the series coefficients, e.g.

```
In[1]:= CoefsByNewton[
  f[z, u] == u + z (f[z, u]^2 + f[z, u]),
  f[z, u], {z, 0, 5}]

Out[1]= u + (u+u^2)z + (u+3 u^2+2 u^3)z^2
  + (u+6 u^2+10 u^3+5 u^4)z^3 +(u+10 u^2+30 u^3+35 u^4+14 u^5)z^4
  + (u+15 u^2+70 u^3+140 u^4+126 u^5+42 u^6)z^5 + O[z]^6
```

# 3 Cycles (cyclically restricted compositions)

*Undirected* cycles: Same idea as in Remark 2.1 (page 9).

## 3.1 Compositions over a finite group

Let $D$ and $D_\times$ be digraphs as in §2.1. Let $a \in G$ and consider walks $w_1, \ldots, w_{m+\sigma-2}$ in $D_\times$ where $w_1 = (v, \sum v)$ and $(w_{m+\sigma-2}, (v, a+\sum v)) \in E(D_\times)$. Let $(x(1), \ldots, x(m+\sigma-1))$ be the composition represented by the walk. Then $(x(1), \ldots, x(m))$ is an $m$-composition of $a$ which is *cyclically restricted* according to $D$. Let $\mathcal{C}_a(m; D)$ be the set of all $m$-compositions of $a$ that are cyclically restricted according to $D$, and define

$$c_a(m; D) = |\mathcal{C}_a(m; D)|, \qquad C_a(z; D) = \sum_{m \geq 0} c_a(m; D) z^m.$$

**Lemma 3.1.** *For $v \in V(D)$, let $\Sigma' v = v(1) + \cdots + v(\sigma - 1)$. We have*

$$c_a(m; D) = \sum_{\substack{v \in V(D) \\ u \in N^-(v)}} p_{a+\Sigma' v}(m + \sigma - 1; D, \{v\}, \{u\}).$$

*Proof.* straightforward from definition ☐

**Proposition 3.1.** *Fix an ordering on $V(D_\times)$ and let $M_\times$ be the adjacency matrix of $D_\times$. For $(v, a) \in V(D_\times)$, let $\xi_{v,a} \in \mathbb{R}^{|V(D_\times)|}$ be the indicator vector for vertex $(v, a)$. Then*

$$c_a(m; D) = \sum_{\substack{v \in V(D) \\ u \in N^-(v)}} (\xi_{v, \sum v})^\top M_\times^{m-1} \xi_{u, a+\Sigma' v}.$$

*Proof.* follows from Lemma 3.1 and Proposition 2.1. ☐

**Proposition 3.2.** *Assume $D_\times$ is aperiodic. We have*

$$c_a(m; D) = A_a \cdot B^m (1 + O(\theta^m)), \qquad m \to \infty$$

*for $A, B > 0, 0 \leq \theta < 1$.*

*Proof.* via Lemma 3.1 and Proposition 2.2 ☐

**Theorem 3.1.** *Assume $D_\times$ is strongly connected and aperiodic. Suppose we have $c_a(m; D) = A_a \cdot B^m (1 + O(\theta^m))$ for $a \in G$. Then $A_a$ does not depend on $a$.*

*Proof.* via Lemma 3.1 and Theorem 2.1 ☐

Let $x = (x(1), \dots, x(m))$ and $y = (y(1, \dots, y(\sigma)))$ be compositions. A *cyclic occurrence* of $y$ in $x$ is an occurrence of $y$ in $(x(1), \dots, x(m), x(1), \dots, x(\sigma - 1))$.

**Theorem 3.2.** *Let $\bar{D}$ be a de Bruijn graph. Let $U \subset V(\bar{D})$ be nonempty and let $D = \bar{D} - U$. Assume that $D_\times$ is aperiodic. Also assume that for all sufficiently large values of $m$ there exist $m$-compositions with exactly 1 occurrence of $U$.*

*Let $\mu = \min_{v \in V(D)} \Big([\{v\} \subseteq U] + [N^-(v) \subseteq U]\Big)$. Then if $r \geq \mu$, then the number of unrestricted $m$-compositions of $a \in G$ with exactly $r$ cyclic occurrences of $U$ is*

$$c_a(m, r) = m^{r-\mu} A_a \cdot B^m (1 + O(m^{-1})), \qquad m \to \infty.$$

*If $D$ satisfies the requirements of Theorem 2.1 then $A_a$ does not depend on $a$.*

*Proof.* The result follows from Lemma 3.1 and Theorem 2.2. □

**Theorem 3.3.** *Distinguish a nonempty set of words $\Xi \subset \mathrm{S}\textsc{eq}_\sigma(G)$. Then the number of cyclic occurrences of $\Xi$ in a uniform random $m$-composition of $a \in G$ is asymptotically normal with mean and variance asymptotic to those of the number of occurrences of $\Xi$ in a uniform random word over $G$.*

*Proof.* replay of that of Theorem 2.4 □

A *cyclic Carlitz composition* of $n$ with $m$ parts is a Carlitz composition $x_1 \cdots x_m$ of $n$ such that $x_1 \neq x_m$. Research Direction 3.3 in [HM10, p. 87] asks for explicit generating functions for the number of cyclic Carlitz compositions/words.

**Proposition 3.3.** *The number of cyclic Carlitz $m$-compositions of $a \in G$ over a finite group $G$ is asymptotic to*

$$\frac{(|G| - 1)^m}{|G|}, \qquad m \to \infty,$$

*provided $|G| \geq 3$.*

*Proof.* First let us consider cyclic Carlitz $m$-words over $[k]$. Assume the first letter is $k$. A cyclic Carlitz word is then a sequence of a single letter $k$ followed by a non-empty Carlitz word on $[k-1]$. Let $\bar{G}_k = kz/(1-(k-1)z)$ be the ordinary generating function for non-empty Carlitz words on $[k]$. Thus if $F_k(z)$ is the ordinary generating function for cyclic Carlitz words on $[k]$, we have

$$\begin{aligned} F_k(z) =& k \frac{z \bar{G}_{k-1}(z)}{1 - z\bar{G}(z)} \\ =& k \frac{(k-1)z^2}{(z+1)(1-(k-1)z)} \end{aligned}$$

and

$$[z^m] F_k(z) = (k-1)^m + k(-1)^m + (-1)^{m+1}, \qquad m > 1.$$

Figure 16: Uniform-randomly generated 100-compositions of 0 (above) and 1 (below) over $\mathbb{Z}_5$ which cyclically avoid 132. (The vertical axis represents the value of a part.)

It remains to recall from Proposition 2.3 that there is a digraph $D$ representing Carlitz compositions such that $D_\times$ is aperiodic and strongly connected; Theorem 3.1 applies. $\square$

A word $w = w(1) \cdots w(n)$ is *p-smooth* iff $|w(i) - w(i+1)| \leq p$ for all $i = 1, 2, \ldots, n-1$. Additionally, $w$ is *p-smooth cyclic* iff $(w(1), \ldots w(n), w(1))$ is an $p$-smooth word. The definition generalizes to totally ordered finite groups in the obvious way. Research Direction 6.5 in [HM10, p. 239] asks for an explicit formula for the number of $p$-smooth cyclic $k$-ary words of length $n$. We use *smooth* as a synomym for 1-smooth.

**Proposition 3.4.** *Let $G$ be a totally ordered finite group, with $|G| = k$. Let*

$$C(z) = 1 + \frac{kz(1+3z)}{(1+z)(1-3z)} - \frac{2(k+1)z}{(1+z)(1-3z)} \frac{U_{k-1}(\frac{1-z}{2z})}{U_k(\frac{1-z}{2z})}$$

*be the ordinary generating function for $k$-ary smooth cyclic words as in [HM10, Exercise 6.10] where $U_k$ is the $k^{th}$ Chebyshev polynomial of the second kind.*

*Let $\rho > 0$ be the radius of convergence of $C(z)$, and let $A = \lim_{z\to\rho}(1-z/\rho)C(z)$. Then the number of smooth cyclic $m$-compositions of $a \in G$ is asymptotic to*

$$\frac{1}{|G|} A \cdot \left(\frac{1}{\rho}\right)^m, \qquad m \to \infty.$$

*Proof.* It is easy to see that there is a corresponding aperiodic and strongly connected $D_\times$, so we apply Theorem 3.1. $\square$

**Example 3.1.** 2018.Jul.13 table and random generation for subword pattern

Table 4 shows counts of $m$-compositions of $a$ cyclicaly avoiding $\tau = 132$ over $\mathbb{Z}_5$. Figure 16 contains uniform randomly generated compositions over $\mathbb{Z}_5$ that cyclically avoid 132.

$\triangle$

variation: *wheel graph.* center vertex can now be the first vertex in a pattern occurrence. count cycles where in any $\pi_2 \ldots \pi_l$ pattern the range of possible values for center vertex to create occurrence is at most $j$.

| $m$ \ $a$ | 0 | 1 |
|---|---|---|
| 3 | 19 | 19 |
| 4 | 85 | 85 |
| 5 | 390 | 385 |
| 6 | 1763 | 1763 |
| 7 | 8023 | 8016 |
| 8 | 36469 | 36469 |
| 9 | 165790 | 165790 |
| 10 | 753660 | 753660 |
| 11 | 3426039 | 3426039 |
| 12 | 15574231 | 15574231 |
| 13 | 70798118 | 70798118 |
| 14 | 321837325 | 321837325 |
| 15 | 1463023035 | 1463023045 |
| 16 | 6650677797 | 6650677797 |

Table 4: Exact counts of $m$-compositions of $a$ cyclically avoiding 132 over $\mathbb{Z}_5$.

### 3.2 Integer compositions

The definition of cyclic restrictions according to isomorphic matching is as follows. A subword $w = w_1 \cdots w_p$ cyclically occurs in a word $x = x_1 \cdots x_m$ iff for some $i$ we have $x_i \cdots x_{i+p-1} = w$ or $x_i \cdots x_m x_1 \cdots x_{p-m+i-1}$ where $i > p - m + i - 1$. Words shorter than the "span" require special attention. For example, is 1 a cyclically-Carlitz word? One definition is that it isn't; this allows the general property that $u \in S \implies uu \in S$. However, our use of isomorphisms requires that 1 avoids the word pattern 11. to $\mathcal{T}$.

Take a pattern set $\mathcal{P}$ consisting of directed paths, and let $\rho < \infty$ be the maximum length of any of the paths, which we name the *span* of $P$. Then the class $L(\mathcal{G}_c, \mathcal{P})$ consists of all words (over $\mathbb{Z}_+$) that cyclically avoid local occurrences of $\mathcal{P}$.

define vertices $v(j)$.

Let $\ell^2$ be the set of all complex sequences $(a_n)$ such that $\sum |a_n|^2 < \infty$.

**Theorem 3.4** (Theorem 1 in [BC09]). *Let $\rho > 0$ and let $T_n$ be a sequence of infinite matrices. Suppose that the power series $T(z) = zT_1 + z^2T_2 + \cdots$ satisfies*

1. $\sum_n |z|^n \|T_n\|_2 < \infty$ *for* $|z| < \rho$ *[must be* $< 1$ *not just convergent?]*
2. $T_n \geq 0$.
3. $T(z_0)$ *is recurrent for all* $z \in (0, \rho)$.

*Then for each $z_0 \in (0, \rho)$ the matrix $T(z_0)$ has an eigenvalue $\lambda(z_0) > 0$ which is simple and strictly larger in absolute value than the other eigenvalues of $T(z_0)$. On the interval $(0, \rho)$ the function $\lambda(z)$ is analytic and $\lambda'(z) > 0$.*

*Assume further that we have $r \in (0,\rho)$, an integer $k_0$, and functions $\boldsymbol{s}(z), \boldsymbol{f}(z) \in \ell^2(|z| < \rho)$ such that*

4. $\lambda(r) = 1$,
5. $\boldsymbol{s}(r), \boldsymbol{f}(r) > 0$,
6. $|T(z)^{k_0}| < T(|z|)^{k_0}$ *for* $z \neq \pm|z|, 0 < |z| < \rho$.

*Then the function*

$$\phi(z) = \boldsymbol{s}(z)^\top \left(\sum_{k \geq 0} T(z)^k\right) \boldsymbol{f}(z)$$

*is analytic for $|z| < r$, has a simple pole at $z = r$, and has at most one additional singularity on the circle of convergence at $z = -r$.*

**Theorem 3.5** (Theorem 2 in [BC09])**.** *Let $\mathcal{L}$ be a regular, locally restricted class of compositions, and let $\Phi(z)$ be the OGF for $\mathcal{L}$ where $z$ marks the total. There is a power series $T(z) = zT_1 + z^2T_2 + \cdots$ satisfying hypotheses (a)-(c) of Theorem 3.4 above with $\rho = 1$, as well as $k_0, r, \boldsymbol{s}(z), \boldsymbol{f}(z)$ satisfying (d)-(f), such that $\Phi(z^2) = \Xi(z) + \Phi_{NR}(z^2)$, where $\Xi(z) = \boldsymbol{s}(z)^\top \left(\sum_{k \geq 0} T(z)^k\right) \boldsymbol{f}(z)$ and $F_{NR}(z)$ has radius of convergence at least 1.*

define $T$ and $S$.

**Definition 3.1** (Definition 9 in [BC09])**.** *Let $\mathcal{C}$ be a locally restricted class of compositions with span $\sigma$ associated with a digraph $D$. We say that $D$ is* regular *provided:*

*(R1) The directed graph within $D$ spanned by the recurrent vertices contains at least 2 vertices and is strongly connected.*

*(R2) Given any 2 recurrent vertices $v_1, v_2$ there is a third recurrent vertex $v_3$ and an integer $k$ such that there are paths $v_3 \to v_1$ and $v_3 \to v_2$ both of length $k$.*

*(R3) There is an integer $k > 0$ and (possibly equal) recurrent vertices $v_1, v_2$ such that*

$$\gcd\{m - n : m, n \in S\} = 1,$$

*where*

$$S = \left\{n : n = \sum c_1 + \cdots + \sum c_{k-1} \text{ for some path } v_1, c_1, \ldots, c_{k-1}, v_2\right\}.$$

*(R4) There exists $K \in \mathbb{Z}$ such that any path $\pi$ in $D$ of length more than $K$ contains at least one recurrent vertex.*

**Lemma 3.2.** *There is a regular digraph associated with compositions avoiding a subword pattern $\tau$, as long as $\tau$ is not $12^\sigma$ or its symmetries.*

*Proof.* We verify the conditions in Definition 3.1 for the digraph of span $\sigma$, where the length of $\tau$ is $\sigma + 1$.

R1. Lemma 2.5 suffices.

R2. If $\tau_1 = 1$, $v_3 = M^\sigma$ where $M$ is the max in $v_1, v_2$. If $\tau_1 > 1$, then $v_3 = 1^\sigma$. In both cases $v_3$ has an arc to both $v_1$ and $v_2$.

R3. For $\tau = 1^{\sigma+1}$, take $v_1 = v_2 = 1^\sigma$. Two paths with sum differing by 1 are given by $1^\sigma 2^{\sigma-1} 3 1^\sigma$ and $1^\sigma 2^\sigma 1^\sigma$. For $\tau = 1^p 2^q$ with $p, q \leq 2$, we take $1^\sigma 2 \cdots (\sigma+1) 1^\sigma$ and $1^\sigma 2 \cdots \sigma(\sigma+2) 1^\sigma$. For other $\tau$, again take $v_1 = v_2 = 1^\sigma$. Two paths are given by $1^\sigma 1^{\sigma-1} 2 2^\sigma 1^\sigma$ and $1^\sigma 1^{\sigma-2} 2 2 2^\sigma 1^\sigma$.

R4. Trivial. □

**Theorem 3.6.** *The number of cyclically locally restricted compositions of $n$ is asymptotic to $A \cdot B^n$, where $A > 0, B > 1$.*

*Proof.* Let $\vec{s}(z;i)$ be $\vec{s}(z)$ with all entries set to 0 except the $i^{\text{th}}$.

Let $\vec{f}(z;i)$ be the vector such that $\vec{f}_j(z;i) = \sum_{x'} z^{\sum v(j) + 2\sum x'}$ where $x'$ is any composition with length in $[\sigma..2\sigma-1]$ such that $v(j)x'$ is legal and $v(i)$ is a suffix of $x'$.

Let $[z^n]C(z)$ be the number of cyclically locally restricted compositions of $n$. [we exclude some short compositions.] We have

$$C(z^2) = \sum_{i \geq 1} z^{-2\sum v(i)} \vec{s}(z;i)^\top S(z) \vec{f}(z;i)$$

Want to show that any subseries of $sSf$ where $s$ and $f$ are full is holomorphic where $S$ is. We know $s, S, f$ are holomorphic and uniformly $\ell^2$ on compact subsets. Any subseries converges by Hölder's inequality. Need to show convergence is uniform on compact sets $K$ by [Rud87, p. 214]. Montel's theorem [Rud87, p. 282] applies. The pointwise limit and any subsequence having further subsequence with uniform convergence suffice.

...

The following fact is stated in [BC09, p. 27]:

Let $\Phi(z)$ be a power series and $r > 0$. Suppose that $\Phi(z^2)$ has radius of convergence $r^{1/2}$ and that the only singularity of $\Phi(z^2)$ on $\{|z| = r^{1/2}\} \cap \{Re(z) \geq 0\}$ is a simple pole at $z = r^{1/2}$. Then the radius of convergence of $\Phi(z)$ is $r$, and the only singularity of $\Phi(z)$ on its circle of convergence is a simple pole at $z = r$.

Applying this fact to $C(z)$ gives the result. □

random generation plots ...

**Example ??** (Continued)**.** ... △

Research Direction 4.4 in [HM10] begins as follows. "We say that a sequence (composition, word, partition) $s_1 \cdots s_m$ cyclically avoids a subword $\tau = \tau_1 \cdots \tau_\ell$ if $s_1 \cdots s_m s_1 \cdots s_{\ell-1}$ avoids $\tau$. For example, the composition 33412 avoids the subword 123, but does not cyclically avoid 123 (since 3341233 contains 123)." The problem is to find the generating function for the number of compositions of $n$ that cyclically avoid a subword pattern of length $k$. Lemma 3.2 implies that we always get regular classes of locally restricted compositions. Presumably we cannot get a GF that is both explicit and incorporates a large set of patterns at once. So we consider 112 and 321.

**Example 3.2.** For 122, no letters $k$ can be adjacent unless no other letters are in the word. Also, the word cannot start and end with $k$. Inside the word, the subwords in between letters $k$ are $(k-1)$-ary words that avoid 122. The concatenation of the subword after the last $k$ and the subword before the first $k$ is also a $(k-1)$-ary word that avoids 122.

... △

**Example 3.3.** For $\tau = 321$, we compute two sequences. Let ^21 be the pattern 21 except that it only counts if it appears at the beginning of a word. We count compositions over $[k]$ that avoid both 321 and ^21. Case 1: the word has no letter $k$. This is counted by word over $[k-1]$. Case 2: the word has at least 1 letter $k$. Say the word can be written $\sigma_1 k \sigma'$, where $\sigma_1$ is a word on $[k-1]$ and avoids $\{321, \hat{}21\}$, and $\sigma'$ is a word on $[k]$ avoiding $\{321, \hat{}21\}$. The word $\sigma_1$ must have length at least 1; if not we must have $k\sigma' = kk\sigma''$ where $\sigma''$ is a word on $[k]$ avoiding $\{321, \hat{}21\}$. This method proceeds similarly to the proof of Lemma 4.29 in [HM10].

Now we go back to words cyclically avoiding just 321. Case 1: The word has no letter $k$ (familiar procedure). Case 2: There are at least 2 letters $k$. Such a word can be written $\sigma_1 k \sigma' k \sigma_2$, where $\sigma'$ and $\sigma_2\sigma_1$ are words on $[k-1]$ avoiding $\{321, \hat{}21\}$ Case 3: There is 1 letter $k$. Then the word is $\sigma_1 k \sigma_2$ where $\sigma_2\sigma_1$ is a word over $[k-1]$ avoiding $\{321, \hat{}21\}$.

... △

define unrelated local events. note that local events can be infinite sets.

The proof of Theorem 1.2 given in [BC09] needs no modification to prove the same result for cyclically restricted compositions. The authors further state, "With further restrictions on the $Y_i(n)$, it would be possible to extend this central limit theorem to a local limit theorem, but we have not worked out the details".

TOWRITE: include length as a parameter but don't do asymptotics with it here

# 4 Unlabeled graphs

from $L$ to $\tilde{L}$.

abelian semigroups only!

*aperiodic* generalizes to *core*

**Lemma 4.1** (Burnside)**.** *The number of orbits of a permutation group $S$ on a set $X$ is*

$$|X/S| = \frac{1}{|S|} \sum_{s \in S} \mathrm{fix}(s),$$

*where* $\mathrm{fix}(s)$ *is the number of fixed points of $s$.*

## 4.1 Undirected paths (counting palindromic compositions)

also avoid reverses of any disallowed subwords.

even $m$: special finish vertices $v$ such that $vv$ is legal

odd $m$: special finish vertices $v$ such that $v\overline{v}$ is legal, where $\overline{v}$ is $v$ with first part deleted

asymptotically does not matter because palindromes are not very numerous

## 4.2 Cycles (circular compositions)

occurrences and sum are additive for concatenating

### 4.2.1 Finite groups

A circular composition is an equivalence class of cyclically-restricted compositions where the equivalence is under circular shift. Let $\tilde{\mathcal{C}}_a(m; D)$ be the set of all circular $m$-compositions of $a$ that are cyclically-restricted according to $D$, and define

$$\tilde{c}_a(m; D) = |\tilde{\mathcal{C}}_a(m; D)|, \qquad \tilde{C}_a(z; D) = \sum_{m \geq 0} \tilde{c}_a(m; D) z^m.$$

Let $P = \mathbb{Z}_{>0} \times G$ be a poset where $(j, a) \preceq (k, b)$ iff $j|k$ and $(k/j)a = b$. The Moebius function $\mu_P$ of $P$ is defined recursively by $\mu_P(s, s) = 1$ for $s \in P$ and $\mu_P(s, u) = -\sum_{s \preceq t \prec u} \mu_P(s, t)$ for $s \prec u$ in $P$.

**Proposition 4.1.** *For $m \geq 0$, we have*

$$\tilde{c}_a(m; D) = \sum_{(d,b) \preceq (m,a)} \frac{1}{d} \sum_{(d',b') \preceq (d,b)} c_{b'}(d'; D) \mu_P((d', b'), (d, b)).$$

*Proof.* Let $\mathrm{acyc}(m, a)$ be the number of aperiodic cyclically-restricted $m$-compositions of $a$. We have

$$c_a(m; D) = \sum_{(d,b)\preceq(m,a)} \mathrm{acyc}(d, b).$$

By Proposition 3.7.1 in [Sta12],

$$\mathrm{acyc}(m, a) = \sum_{(d,b)\preceq(m,a)} c_b(d; D)\mu_P((d, b), (m, a)).$$

Now,

$$\tilde{c}_a(m; D) = \sum_{(d,b)\preceq(m,a)} \frac{1}{d}\,\mathrm{acyc}(d, b),$$

which gives the result. □

**Theorem 4.1.** *Assume $D_\times$ is aperiodic.*

$$\tilde{c}_a(m; D) = \frac{1}{m} A_a \cdot B^m(1 + O(\theta^m))$$

*If $D$ satisfies the assumptions of Theorem 2.1, then $A_a$ does not depend on $a$.*

*Proof.* We have

$$\begin{aligned} c_a(m) - \sum_{(d,b)\prec(m,a)} c_a(m) &\le c_a(m) - \sum_{(d,b)\prec(m,a)} \mathrm{acyc}(m, a) \\ &= \mathrm{acyc}(m, a) \\ &\le c_a(m). \end{aligned}$$

Together with Proposition 3.2, this implies

$$\sum_{(d,b)\prec(m,a)} \frac{1}{d}\,\mathrm{acyc}(d, b) \le (m/2)A' \cdot B^{m/2}(1 + O(\theta^{m/2}))$$

and similarly

$$A_a \cdot B^m(1 + O(\theta^m)) \le \mathrm{acyc}(m, a) \le A_a \cdot B^m(1 + O(\theta^m)).$$

The result then follows from Proposition 4.1. □

**Theorem 4.2.** *Let $\bar{D}$ be a de Bruijn graph. Let $U \subset V(\bar{D})$ be nonempty and let $D = \bar{D} - U$. Assume that $D_\times$ is aperiodic. Also assume that for all sufficiently large values of $m$ there exist $m$-compositions with exactly 1 occurrence of $U$.*

*Let $\nu = \min_{v\in V(D)} \big([\{v\} \subseteq U] + [N^-(v) \subseteq U]\big)$. Then if $r \ge \nu$, then the number of circular $m$-compositions of $a \in G$ with exactly $r$ cyclic occurrences of $U$ is*

$$\tilde{c}_a(m, r) = m^{r-\nu-1} A_a \cdot B^m(1 + O(m^{-1})), \qquad m \to \infty.$$

*If $D$ satisfies the requirements of Theorem 2.1 then $A_a$ does not depend on $a$.*

*Proof.* Let $Q = \mathbb{Z}_{>0} \times G \times \mathbb{Z}_{>0}$ be a poset where $(j_1, a, j_2) \preceq (k_1, b, k_2)$ iff $j_1 | k_1$, $(k_1/j_1)a = b$, and $(k_1/j_1)j_2 = k_2$. The Moebius function $\mu_Q$ of $Q$ is defined recursively by $\mu_Q(s, s) = 1$ for $s \in Q$ and $\mu_Q(s, u) = -\sum_{s \preceq t \prec u} \mu_Q(s, t)$ for $s \prec u$ in $Q$. By analogy to Proposition 4.1 we have

$$
\begin{aligned}
&\tilde{c}_a(m, r; D) \\
&= \sum_{(d_1,b,d_2) \preceq (m,a,r)} \frac{1}{d} \sum_{(d'_1,b',d'_2) \preceq (d_1,b,d_2)} c_{b'}(d'_1, d'_2; D) \mu_P((d'_1, b', d'_2), (d_1, b, d_2)).
\end{aligned}
$$

Following the proof of Theorem 4.1, the dominant term is $m^{-1} c_a(m, r; D)$, so we conclude with reference to Theorem 3.2. □

**Lemma 4.2.** *Let $X_n$ and $Y_n$ be non-degenerate $L^2$ random variables for $n \geq 1$. Let $Z_n$ have a mixture distribution of $X_n$ and $Y_n$ with weights $p_n$ and $1 - p_n$. Assume that $p_n \to 1$ and $(1 - p_n)E(Y_n^2) = o(\mathrm{Var}(X_n))$, $p_n E(X_n)E(Y_n) = o(\mathrm{Var}(X_n))$. If $(X_n - E(X_n))/\sqrt{\mathrm{Var}(X_n)} \Rightarrow F$ then $(Z_n - E(Z_n))/\sqrt{\mathrm{Var}(Z_n)} \Rightarrow F$.*

*Proof.* From the assumptions, is straightforward to verify that

$$E(Z_n) \sim E(X_n) \text{ and } \mathrm{Var}(Z_n) \sim \mathrm{Var}(X_n).$$

By Slutsky's theorem, $F_{X_n}(\sqrt{\mathrm{Var}(X_n)}x + E(X_n)) \sim F_{X_n}(\sqrt{\mathrm{Var}(Z_n)}x + E(Z_n))$. And for $x$ where $F$ is continuous,

$$
\begin{aligned}
F_{Z_n}(\sqrt{\mathrm{Var}(Z_n)}x + E(Z_n)) =& p_n F_{X_n}(\sqrt{\mathrm{Var}(Z_n)}x + E(Z_n)) \\
&+ (1 - p_n) F_{Y_n}(\sqrt{\mathrm{Var}(Z_n)}x + E(Z_n)) \\
\to& F_{X_n}(\sqrt{\mathrm{Var}(Z_n)}x + E(Z_n)) \\
\to& F_{X_n}(\sqrt{\mathrm{Var}(X_n)}x + E(X_n)) \\
\to& F(x).
\end{aligned}
$$

□

**Theorem 4.3.** *Distinguish a nonempty set of words $\Xi \subset \mathrm{SEQ}_\sigma(G)$. Then the number of cyclic occurrences of $\Xi$ in a uniform random circular $m$-composition of $a \in G$ is asymptotically normal with mean and variance asymptotic to those of the number of occurrences of $\Xi$ in a uniform random word over $G$.*

*Proof.* With exponentially high probability a uniform random $m$-composition is aperiodic. And with exponentially high probability a uniform random circular $m$-composition is an equivalence class of $m$ aperiodic compositions with a common number of cyclic occurrences of $U$. Since the expected number of occurrences is bounded by $m$ and the variance is bounded by $m^2$, we may apply Lemma 4.2 along with Theorem 3.3. □

**Example 4.1.** For a composition $x = x_1 \cdots x_m$, we define

$$\mathrm{gap}(x) = \max_i x_i - \min_i x_i + 1 - |\{x_i : i = 1, 2, \ldots, m\}|,$$

which is the number of parts missing between the minimum and maximum parts of $x$. If $\mathrm{gap}(x) = 0$ we say $x$ is *gap-free*. Research Direction 3.1 parts (3) and (4) in [HM10, p. 86] ask for an explicit generating function for the number of circular compositions/words $x$ such that $\mathrm{gap}(x) = \ell$.

Let $c(m)$ be the number of gap-free $k$-ary words. The number of $m$-words with exactly $k$ distinct letters is $k!\{{m \atop k}\}$ and the number of gap-free $k$-ary words with $j$ distinct letters is $k - j + 1$. Thus

$$c(m) = \sum_{j=1}^{k} (k - j + 1) j! \left\{ {m \atop j} \right\} \sim \sum_{j=1}^{k} (k - j + 1) k^m \sim k^m.$$

Using the familiar Moebius function $\mu$, as in [BG75], we have

$$\begin{aligned} \tilde{c}(m) =& \sum_{d|m} \frac{1}{d} \sum_{d'|d} \mu(d/d') c(d') \\ \sim& \frac{1}{m} k^m. \end{aligned}$$

It does not follow from Theorem 4.1 that

$$c_a(m) \sim \frac{1}{m} k^{m-1};$$

however, this is easy to establish since the last part is arbitrary for almost all such compositions. △

**Example 4.2.** Considering avoidance of the subword pattern 132, for any total $a$ there will be 1 composition with 1 part, namely $(a)$. For $m \geq 2$, some compositions are grouped into non-trivial equivalence classes. For $m = 1, \ldots, 5$, the numbers of 132-avoiding circular $m$-compositions of 0 over $\mathbb{Z}_5$ are $1, 3, 7, 23, 82$, and the counts for $m$-compositions of 1 are $1, 3, 7, 23, 77$. △

#### 4.2.2 other

Fact: Two cyclically-restricted words $u$ and $v$ are circularly-equivalent iff they have the same period $p$ and $u_{1:p}$ is a cyclic shift of $v_{1:p}$.

A cyclically-restricted word $w$ is determined by its period $d$ and its aperiodic prefix $w_{1:d}$. Any cyclically-restricted word may be an aperiodic prefix since if $u$ is a cyclically-restricted word, so is $uu$. A circular restricted word with period $d$ is an equivalence class of $d$ cyclically-restricted words, corresponding to possible shifts of the $d$-letter aperiodic prefix. Thus we have

$$\begin{aligned} && \mathrm{circ}(n) &= \sum_{d|n} \mathrm{acyc}(d)/d \\ && \mathrm{cyc}(n) &= \sum_{d|n} \mathrm{acyc}(d), \\ \Longrightarrow && \mathrm{acyc}(n) &= \sum_{d|n} \mu(n/d)\, \mathrm{cyc}(d), \\ \Longrightarrow && \mathrm{circ}(n) &= \sum_{d|n} \frac{1}{d} \sum_{c|d} \mu(d/c)\, \mathrm{cyc}(c). \end{aligned}$$

Thus we can express circ($n$) in terms of cyc.

special case of Carlitz compositions done in [Had17]

For integer compositions, if we sum over all lengths, we get

$$\sum_m \sum_{d|m} \frac{1}{d} \sum_{c|d} \mu(d/c)\delta(nc/m, c),$$

where $\delta(n, m)$ is count for sum $n$ and length $m$. We break this up into terms with sum $n$ and those with lower sums.

$$\sum_{m=1}^{n} \frac{\delta(n,m)}{m} + \sum_m \sum_{d|m, d<m} \frac{1}{d} \sum_{c|d} \mu(d/c)\delta(nc/m, c)$$

and

$$\sum_m \sum_{d|m, d<m} \frac{1}{d} \sum_{c|d} \mu(d/c)\delta(nc/m, c) \leq (n/2)(n/2) \sum_m \delta(n/2, m)$$

The term $\sum_m \delta(n/2, m)$ has a simple bound, basically $A \cdot B^{n/2}$ with uniform error.

If we look at $\sum_{m=1}^{n} \frac{\delta(n,m)}{m}$, we see that we have to take the GF $S(z, u) = 1/(1 - uT(z))$, integrate wrt $u$, then set $u = 1$. This gives something like $\log 1/(1 - T(z))$ which I think is not to hard to derive asymptotics from given the details in [BC09]. This gives a term of $A \cdot B^n/n$

# 5 Subsequence pattern avoidance in compositions and words

One could define a general "global restriction" function but typical examples such as requiring distinctness or disallowing a certain sum no longer make sense. We focus on subsequence patterns.

I guess no general "implicit" results are available, e.g. D-finiteness? rationality?

usual notion of subsequence patterns in compositions/words

Reduction of word: $red(22454) = 11232$. Subsequence pattern occurrence is subsequence w/ same reduction as pattern, e.g. pattern 122 in $\underline{4}\underline{7}34\underline{7}2$.

Avoiding permutation patterns in words: [Bur98]. Somewhat simpler proofs given in [HM10, Sec. 6.5]. Enumeration schemes given in [Pud08] give counting algorithms but are fairly opaque.

With $r$ occurrences of $1^p$:

$$\sum_{m,r\geq 0} \gamma_r(m,k)\frac{z^m}{m!}u^r = \Gamma_k(z,u) = \left(\sum_{i\geq 0} u^{\binom{i}{p}}\frac{z^i}{i!}\right)^k$$

amenable to representation using integral transforms as discussed in [Sch17]

[further literature survey. words, compositions avoiding subsequence patterns. origins with permutations. symmetries. generalized patterns, partialy-ordered patterns. main reference is [HM10]. ]

**Remark 5.1** (Cycles and undirected paths)**.** Here we assume patterns are directed paths. Given a directed path $\Gamma_p$, and pattern $P$, an occurrence of $P$ in $s(\Gamma_p)$ is either an occurrence of $P$ in $\Gamma_p$ or an occurrence of $P^{-1}$ in $\Gamma_p$.

Let $c(\Gamma_p)$ be the directed cycle formed by adding an arc to $\Gamma_p$. Then an occurrence of $P$ in $c(\Gamma_p)$ is the occurrence of some cyclic shift of $P$ in $\Gamma_p$.

Occurrences of $P$ in $s(c(\Gamma_p))$ are occurrences of cyclic shifts and/or reversals of $P$ in $\Gamma_p$.

Enumeration schemes are available for avoidance of all combinations of 3 patterns with 3 letters each [Pud08] for words. △

## 5.1 Words and integer compositions

compositions avoiding 3-letter permutation patterns GF given in [SW06] based on alphabet vector GF in [ALW95]; recurrence also given in [Alb+01]. Compositions avoiding 3-letter word patterns, and pairs of 3-letter patterns: [HM06]. That paper also looks at the subsequence pattern $1^p$-2-$1^q$. avoiding PoPs: [HKM06].

In this section we fill some gaps in the literature. For simplicity we only consider avoidance of patterns.

general techniques for recurrences: strip off all copies of the smallest letter, refine based on patterns found among smallest few letters. this is a bottom-up approach rather than the left-right approach of enumeration schemes.

**Remark 5.2.** Note on random generation.

Presumably can adapt simple MCMC algorithm from [ML10] to generate words and create `ArrayPlot`s for various patterns.

and/or Boltzmann sampling? △

### 5.1.1 The pair $\{1\text{-}1\text{-}2, 1\text{-}2\text{-}2\}$

This pair $\{1\text{-}1\text{-}2, 1\text{-}2\text{-}2\}$ is left open in [HM06] (even for words in [BM02, Sec. 4]).

We count words for simplicity.

$f_k(m, m')$: here $m'$ is indicates that words have no occurrence of 1-1 among the first $m'$ letters. So $f_k(m) = \sum_{m'} f_k(m, 1)$. Given a word, let $k'$ be maximal such that the word has at least to letters $k'$. Let $M_1$ be the position of the second $k'$, and let $M_2$ be the position of the second-last $k'$ Let $b$ be the number of letters $k'$.

Case 1: $M_1 \le M_2$. The word from position 1 to position $M_1$, excluding letters $k'$, is restricted to the letters between $k'+1$ and $k$, inclusive, in order to avoid occurrences of 1-2-2, and thus it also has no repeated letters. There are $\binom{k-k'}{M_1-2}(M_1-2)!$ such words, and $M_1 - 1$ locations for the first letter $k'$. The word from position $M_1$ to $M_2$ is only the letter $k'$ repeated some number of times. From $M_2$ to the end, with letters $k'$ removed there is a word on the alphabet $[k'-1]$ with $m - M_2 - 2$ letters. This word must avoid the patterns, and it must have no occurrence of 1-1 among the first $\overline{M} - M_2 - 1$ letters.

Case 2: $M_1 > M_2$, i.e. $b = 2$. Same as above except letters between $M_2$ and $M_1$ could be part of either of the smaller words.

Putting these together …

### 5.1.2 Pairs of generalized patterns of length 3

singles: compositions in [HM10, Sec. 5.3], words in [HM10, Sec. 6.6]. No work done for pairs, even for words. Here we count compositions to illustrate the involved techniques.

The techniques for counting each possible pair get fairly repetitive, so we merely give a handful of representative examples.

TODO: continue the following

**The pair** $\{11\text{-}2, 12\text{-}3\}$

Words avoiding 12-3 counted in Theorem 5.21 from [HM10, p. 147].

We use $F_k(w; z, u)$ to count compositions over $[k]$ starting with the subword $w$ which avoid this pair of patterns, where $z$ marks the total and $u$ marks the length.

$$\begin{aligned} F_k(i; z, u) &= z^i u + \sum_{j=1}^{i-1} F_k(ij; z, u) + \sum_{j=i}^{d} F_k(ij; z, u) \\ &= z^i u + z^i u \left( \sum_{j=1}^{i-1} F_k(j; z, u) + \sum_{j=i}^{d} z^j u F_j(z, u) \right) \end{aligned}$$

**The pair** $\{23\text{-}1, 11\text{-}2\}$

This can be done, by combining the proof of each. We now have a decomposition where the letters 1 must not be adjacent, and the subwords $\sigma$ avoid 12 and 11-2, which involves a modification of Theorem 5.41 in [HM10, p. 159] where the first iterated summation goes up to $i - 1$ but not to greater values.

**The pair** $\{11\text{-}2, 12\text{-}1\}$

$$\begin{aligned} F_k(i; z, u) &= z^i u + \sum_{j=1}^{i-1} F_k(ij; z, u) + F_k(ii; z, u) + \sum_{j=i+1}^{k} F_k(ij; z, u) \\ &= z^i u + \sum_{j=1}^{i-1} z^i u F_k(j; z, u) + z^{2i} u^2 F_i(z, u) + \sum_{j=i+1}^{k} z^i u F_{k \setminus i}(j; z, u) \end{aligned}$$

where $k \setminus i$ means $\{1, \ldots, i-1, i+1, \ldots, k\}$. For words, we can replace $k \setminus i$ with $k - 1$.

**The pair** $\{12\text{-}1, 1\text{-}21\}$

Assume the lowest letter present in the word is 1. All copies of 1 must be contiguous in order to avoid the patterns. If we delete these copies, the remaining word has the same structure. Thus

$$\phi_k(m) = \phi_{k-1}(m) + \sum_{j=1}^{m} (m - j + 1)\phi_{k-1}(m - j), \qquad k, m \geq 1,$$

and $\phi_0(m) = [m = 0], \phi_k(0) = 1$. Passing to OGFs gives

$$\Phi_k(z) = \Phi_{k-1}(z) + \frac{z}{1 - z} D_z(z\Phi_{k-1}(z)), \Phi_0(z) = 1.$$

Note that it is not true that letters $k$ must be found only in contiguous blocks at the very beginning and/or very end. For example, 132 avoids the patterns and has the greatest letter in the middle.

Look at how recurrences are solved in [HM10] and see if the technique applies. For compositions, look at the vertically reflected patterns and remove letters $k$ rather than 1.

**The pair $\{12\text{-}3, 3\text{-}21\}$**

$k$-decomposition like 1 decomposition for 23-1 pattern

### 5.1.3 Some partially-ordered patterns with 2 distinct letters

TODO: continue the following

**The pattern $2\text{-}\cdots\text{-}2\text{-}1'\text{-}1''\text{-}\cdots\text{-}1^{(q)}\text{-}2\text{-}\cdots\text{-}2$.**

We consider the pattern $2^p\text{-}1'\text{-}\cdots\text{-}1^{(q)}\text{-}2^r$. For general $q$, neither words nor compositions avoiding this pattern have been counted. We count words for simplicity.

Let $b$ be the number of letters $k$ in a word. If $b \le p+r-1$, their positions do not matter, so there are $\binom{n}{b} h_r(n-b, k-1)$ such words $\sigma$. If $b \ge p+r$, then between the $p$th $k$ from the left and the $r$th $k$ from the right there must be at most $q-1$ letters that are not $k$. Let $w$ be the number of all letters between the $p$th $k$ from the left and the $r$th $k$ from the right, and let $M = b-(p-1)-(r-1)$ be the number of letters $k$ among those letters. Then there are

$$\sum_{w=M}^{M+q-1} \binom{w-2}{M-2} \binom{m-w+1}{(p-1)+(r-1)+1}$$

possible ways of placing the letters $k$ in the word.

So for $m \ge p+r$, this gives

$$\begin{aligned} h_k(n,m) = &\sum_{b=0}^{p+r-1} \binom{m}{b} h_{k-1}(n-bk, m-b) \\ &+ \sum_{b=p+r}^{m} \sum_{w=M}^{M+q-1} \binom{w-2}{M-2} \binom{m-w+1}{(p-1)+(r-1)+1} h_{k-1}(n-bk, m-b). \end{aligned}$$

**The pattern $1'\text{-}1''\text{-}\cdots\text{-}1^{(p)}\text{-}2^q\text{-}1^{(p+1)}\text{-}1^{(p+2)}\text{-}\cdots\text{-}1^{(p+s)}$.**

Generalization of previously studied “peak” pattern.

Let $\tau = 1'\text{-}1''\text{-}\cdots\text{-}1^{(p)}\text{-}2^q\text{-}1^{(p+1)}\text{-}1^{(p+2)}\text{-}\cdots\text{-}1^{(p+r)}$. We count words for simplicity. Simple initial conditions: $g_0(n,m) = [n = m = 0]$, for $m < p + r$, $g_k(n,m) = [z^n u^m](\sum_{j=1}^k z^j u)^m$.

Case 1: $0 \le b < q$ or $m - p - r < b$. For each $b$,

$$\binom{m}{b} g_{k-1}(n - bk, m - b)$$

Case 2: $q \le b \le m - p - r$. Here we must place the letters $k$ in the following manner. Between the $p^{\text{th}}$ non-$k$ from the beginning and the $r^{\text{th}}$ non-$k$ from the end, there must be at most $q - 1$ letters $k$. Let $P$ be the position of the $p^{\text{th}}$ non-$k$ from the beginning, and let $R$ be the position of the $r^{\text{th}}$ non-$k$ from the end. Let $Q'$ be the number of letters not $k$ strictly between positions $P$ and $R$. Summing the possibilities, we have

$$\sum_{P=p}^{m-r} \binom{P}{p} \sum_{R=P+1}^{m} \binom{m-R-1}{r} \sum_{Q'=R-p-q}^{R-P-1} \binom{R-P-1}{Q'} g_{k-1}(n-bk, m-b),$$

where $b = m - 2p + P - r - Q'$.

**Special case:** $s = 0$

$\tau = 1'\text{-}1''\text{-}\cdots\text{-}1^{(p)}\text{-}2^q$. Now we count compositions rather than words. Words avoiding $\tau$ were counted in [GMP11, Sec. 2]; compositions were left as an open problem. Instead of $r$ we have a new parameter $n$ for the sum, where the old $n$ is now $m$, the number of parts.

First, we deal with the case where the set of parts is $[k]$, then we can go beyond. Let $f_k^\tau(n,m) = f_k(n,m)$ be the number of $m$-compositions of $n$ over $[k]$ that avoid $\tau$.

For the range $m \ge p + q$, we recursively count these compositions $x$ by first counting $x$ such that at least one of the first $p$ letters is $k$. By the principle of inclusion-exclusion, the number of such $x$ is

$$\sum_{j=1}^{p} N_j (-1)^{j+1},$$

where $N_j$ is the sum, over all $j$-subsets of the first $p$ positions, of the number of compositions $x$ with $k$'s in the positions given by the subset. The quantity $N_j$ is given by

$$N_j = \binom{p}{j} f_k(n - jk, m - j),$$

since inserting $j$ copies of $k$ into any of the first $p$ positions of an $(m - j)$-composition is reversible and does not affect the number of occurrences of $1'2_{p,q}$.

Now we count the $x$'s that have no $k$'s in their first $p$ positions. Let $b$ be the number of $k$'s in $x$. If $b \le q-1$, then there are not enough $k$'s to be part of a pattern, so there are

$$\sum_{b=0}^{q-1} \binom{m-p}{b} f_{k-1}(n-bk, m-b),$$

compositions of this kind.

If $b \ge q$ then there will be at least one occurrence of the pattern. Thus we have, for $m \ge p+q, k \ge 1$,

$$\begin{aligned} f_k(n,m) = \sum_{j=1}^{p} \binom{p}{j} f_k(n-jk, m-j)(-1)^{j+1} \qquad (4) \\ + \sum_{b=0}^{q-1} \binom{m-p}{b} f_{k-1}(n-bk, m-b). \qquad (5) \end{aligned}$$

For $m < p+q$, we have $f_k(n,m) = [z^n u^m] \left(\sum_{j=1}^{k} z^j u\right)^m$. If $k = 0$, then $f_k(n,m) = [n = m = 0]$. Thus for $k \ge 1$,

$$\begin{aligned} \sum_{n,m\ge 0} f_k(n,m) z^n u^m = F_k(z,u) = \sum_{n\ge 0, m\ge p+q} \Bigg( \sum_{j=1}^{p} \binom{p}{j} f_k(n-jk, m-j)(-1)^{j+1} \\ + \sum_{b=0}^{q-1} \binom{m-p}{b} f_{k-1}(n-bk, m-b) \Bigg) z^n u^m \\ + \sum_{m=0}^{p+q-1} \left( \sum_{j=1}^{k} z^j u \right)^m \end{aligned}$$

$$\begin{aligned} A &= \sum_{n\ge 0, m\ge p+q} \sum_{j=1}^{p} \binom{p}{j} f_k(n-jk, m-j)(-1)^{j+1} z^n u^m \\ &= \sum_{j=1}^{p} \binom{p}{j} (-1)^{j+1} \sum_{m\ge p+q} u^m \sum_{n\ge jk} f_k(n-jk, m-j) z^n \\ &= \sum_{j=1}^{p} \binom{p}{j} (-1)^{j+1} \sum_{m\ge p+q} u^m \sum_{n\ge 0} f_k(n, m-j) z^{n+jk} \\ &= \sum_{j=1}^{p} \binom{p}{j} (-1)^{j+1} (z^j)^k \sum_{n\ge 0} z^n \sum_{m\ge p+q} f_k(n, m-j) u^m \\ &= \sum_{j=1}^{p} \binom{p}{j} (-1)^{j+1} (z^j)^k \sum_{n\ge 0} z^n \sum_{m\ge p+q-j} f_k(n,m) u^{m+j} \\ &= \sum_{j=1}^{p} \binom{p}{j} (-1)^{j+1} (z^j)^k u^j \left( \sum_{n\ge 0} \sum_{m\ge p+q-j} f_k(n,m) z^n u^m \right) \\ &= \sum_{j=1}^{p} \binom{p}{j} (-1)^{j+1} (z^j)^k u^j \left( F_k(z,u) - \sum_{n\ge 0} \sum_{m=0}^{p+q-j-1} f_k(n,m) z^n u^m \right) \end{aligned}$$

$$\begin{aligned}
A_1 &= \sum_{n\geq 0}\sum_{m=0}^{p+q-j-1} f_k(n,m)z^n u^m\\
&= \sum_{m=0}^{p+q-j-1}\left(\sum_{j=1}^{k} z^j u\right)^m\\
&= \frac{(z-1)\left(\frac{uz\left(z^k-1\right)}{z-1}\right)^{-j+p+q}-z+1}{z\left(u\left(z^k-1\right)-1\right)+1}
\end{aligned}$$

$$\begin{aligned}
B &= \sum_{n\geq 0, m\geq p+q}\sum_{b=0}^{q-1}\binom{m-p}{b} f_{k-1}(n-bk,m-b)z^n u^m\\
&= \sum_{n\geq 0}\sum_{m\geq p+q}\sum_{b=0}^{q-1}\binom{m-p}{b} f_{k-1}(n-bk,m-b)z^n u^m\\
&= \sum_{n\geq 0}\sum_{m\geq p+q}\sum_{b=0}^{q-1}\frac{(m-p)^{\underline{b}}}{b!} f_{k-1}(n-bk,m-b)z^n u^m\\
&= \sum_{n\geq 0}\sum_{m\geq p+q}\sum_{b=0}^{q-1}\frac{1}{b!}\sum_{c=0}^{b}\begin{bmatrix} b\\ c\end{bmatrix}(m-p)^c f_{k-1}(n-bk,m-b)z^n u^m\\
&= \sum_{n\geq 0}\sum_{m\geq p+q}\sum_{b=0}^{q-1}\frac{1}{b!}\sum_{c=0}^{b}\begin{bmatrix} b\\ c\end{bmatrix}\sum_{d=0}^{c}\binom{c}{d} m^d(-p)^{c-d} f_{k-1}(n-bk,m-b)z^n u^m\\
&= \sum_{b=0}^{q-1}\frac{1}{b!}\sum_{c=0}^{b}\begin{bmatrix} b\\ c\end{bmatrix}\sum_{d=0}^{c}\binom{c}{d}(-p)^{c-d}\sum_{n\geq 0}\sum_{m\geq p+q} m^d f_{k-1}(n-bk,m-b)z^n u^m\\
&= \sum_{b=0}^{q-1}\frac{1}{b!}\sum_{c=0}^{b}\begin{bmatrix} b\\ c\end{bmatrix}\sum_{d=0}^{c}\binom{c}{d}(-p)^{c-d}(z^k)^b\sum_{n\geq 0}\sum_{m\geq p+q} m^d f_{k-1}(n,m-b)z^n u^m
\end{aligned}$$

$$\begin{aligned}
B_1 &= \sum_{n\geq 0}\sum_{m\geq p+q} m^d f_{k-1}(n,m-b)z^n u^m\\
&= \sum_{n\geq 0}\sum_{m\geq 0} m^d f_{k-1}(n,m-b)z^n u^m - \sum_{n\geq 0}\sum_{m=0}^{p+q-1} m^d f_{k-1}(n,m-b)z^n u^m\\
&= \sum_{n\geq 0}\sum_{m\geq b} m^d f_{k-1}(n,m-b)z^n u^m - u^b\sum_{n\geq 0}\sum_{m=0}^{p+q-1-b}(m+b)^d f_{k-1}(n,m)z^n u^m\\
&= u^b\sum_{n\geq 0}\sum_{m\geq 0}(m+b)^d f_{k-1}(n,m)z^n u^m - u^b\sum_{m=0}^{p+q-1-b}(m+b)^d\left(\sum_{j=1}^{k} z^j u\right)^m\\
&= u^b(D_u+b)^d F_{k-1}(z,u) - u^b\sum_{m=0}^{p+q-1-b}(m+b)^d\left(\sum_{j=1}^{k} z^j u\right)^m
\end{aligned}$$

$$C = \sum_{m=0}^{p+q-1}\left(\sum_{j=1}^{k} z^j u\right)^m = \frac{(z-1)\left(\left(\frac{uz\left(z^k-1\right)}{z-1}\right)^{p+q}-1\right)}{uz^{k+1}-uz-z+1}$$

And so we get the simplified recurrence relation $F_k(z,u) = A + B + C$. Some things we could do:

- "Solve" GF equations.
- Find asymptotics for fixed $k$.
- Set $k = n$.

We can also attempt the above for special values of $p, q$, e.g. $p = 2, q = 1$ or $p = 2, q = 2$.

**Special case:** $p = 2, q = 1$

We have, for $m \geq 3, k \geq 1$,

$$f_k(n,m) = 2f_k(n-k, m-1) - f_k(n-2k, m-2) + f_{k-1}(n,m) \qquad (6)$$

### 5.1.4 Longest runs

count words avoiding pattern(s) with given longest run

subword patterns covered in [BG16] no previous work for subsequence patterns? Equivalent to counting words avoiding a subsequence pattern $\tau$ and the subword pattern $1^p$. (Subtract words avoiding $1^p$ from words avoiding $1^{p+1}$ to get words with longest run of length $p$.)

use substitution trick

## 5.2 Compositions over $\mathbb{Z}_k$

no previous work on avoiding subsequence patterns. Here we look at subsequence pattern avoidance for some fundamental patterns.

addressing a problem posed in [GMW18]

TODO: Try two possible methods. 1. Restrict parts to $[k] - 1$ and use multisection formula. 2. Modify recurrence relation to keep track of total $\pmod{k}$. This second method is complicated because in recurrence relations on $k$, we want to keep track of the total $\pmod{k}$, but count words on $[k-1]-1, [k-2]-1$, et cetera; i.e. the alphabet size does not match the modulus as soon as we recurse. So really we get a system of recurrence relations on $m$, parametrized by $k$.

Multisection formula:

$$\frac{1}{k}\sum_{j=0}^{k-1} e^{-2\pi i j s/k}\Phi(e^{2\pi i j/k} z) = \sum_{n \equiv s \pmod{k}} \phi_n z^n$$

**The pattern 1-3-2.**

From [SW06] we have that the GF for integer compositions over $[k]$ avoiding a length-3 pattern, with $x$ marking total and $y$ marking length is

$$F_k(x,y) = \sum_{i=1}^{k} \frac{x^{i(k-1)}(1-x^i y)^{k-2}}{\prod_{j=1, j\neq i}^{k}(x^i - x^j)(1-x^i y - x^j y)}.$$

**The pattern 1-1-2.**

The following GF is given in Theorem 5.13 in [HM10, p. 139], where $x$ marks the total and $y$ marks the length, and the set of parts is $[k]$. If $b_i = \frac{x^i y^2}{1-x^i y}$, $c_i = \frac{1}{x^i y}$, then

$$F_k(x,y) = b_1 e^{c_1} D_y b_2 e^{c_2 - c_1} D_y \cdots C_y b_{k-1} e^{c_{k-1}-c_{k-2}} D_y \frac{e^{-c_{k-1}}}{1-x^k y}.$$

**The pattern set $\{1\text{-}3\text{-}2, 2\text{-}3\text{-}1, 1\text{-}2\text{-}1\}$.**

Let $A = \{a_1, \ldots, a_k\}$ be an ordered set of positive integers. Example 5.62 in [HM10] gives the generating function

$$F_A(x,y) = \frac{1}{\prod_{a\in A}(1-x^a y)^2} - \sum_{a\in A} \frac{x^a y}{\prod_{a\leq b\in A}(1-x^b y)^2}.$$

## 5.3 Counting under symmetries

### 5.3.1 Reversal

different for odd and even $m$.

even $m$. say we want to avoid pattern $\tau$. we now need to avoid all possible "folds", meaning the following. Let $\tau$ be some concatenation $\tau_1\tau_2$ where the $\tau_1, \tau_2$ may be empty. A fold is a word $\tau'$ obtained by taking $\tau_2$, reversing it, and combining it with $\tau_1$ by interspersing the letters and possibly merging values equal in $\tau_1$ and $\tau_2$.

odd $m$. similar to above except we break $\tau$ into three parts $\tau = \tau_1\tau_2\tau_3$ where $\tau_2$ has length at most 1, and a fold involves interspersing the reversal of $\tau_3$ with $\tau_1$ and leaving $\tau_2$ at the end.

### 5.3.2 Circular shift

No longer have property that if $u$ avoids pattern so does $uu$. Example: $T = \{123, 312, 231\}$, $u = 321$, then $u$ avoids the pattern but $uu = 321321$ contains

231. For $w = u^a$, where $a$ is less than the length of the pattern, we need to count $u$ such that $u^a$ avoids the pattern.

if we have $w = u^a$ and $a$ is at least the length of the pattern, $u$ is any word with fewer letters than the pattern has.

# 6 Conclusion

We list some other graph families: the complete graphs, which have the structure of multisets; 2-regular graphs, i.e. sets of cycles; star graphs, where there are no paths longer than 3.

...

## 6.1 Open problems

### Finitely generated (infinite) groups

Consider compositions over a finite generating set. The base graph $D$ is finite but the derived graph $D_\times$ is possibly infinite.

example. group $\mathbb{Z}$, generating set $\{-1, 0, 1\}$, total $s$.

count is $\sim \frac{2^{m+\frac{3}{2}}}{\sqrt{\pi m}}$, $m \to \infty$

note no dependence on $s$ but irrational counting sequence

extend to finitely generated groups? Locally-restricted compositions over finitely generated abelian groups with a given total are recognized by pushdown automata so they form a CFL. Recognizing compositions over general finitely-generated groups with a given total involves the word problem.

can they be recognized with pushdown automata?

### Weighted (infinite) groups

The group elements are now themselves weighted so vertex weights are now pairs $(a, k(a))$, where $a \in G$ and $k(a) \in \mathbb{Z}_+$.

Possibly above results along with those in [BC09] can be adapted to count locally restricted compositions over groups which are infinite but weighted by positive integers such that each weight is associated to a finite subset of the group. E.g. where the weight of each element is some kind of positive, additive cost.

### Colored compositions

Much of the above results can likely be generalized to a setting where the group elements or integer parts come in different colors, possibly an infinite set of colors. Some open problems mentioned in [BG18, Sec. 12] are relevant.

### “Explicit” counting for local patterns

One could ask for more “explicit” counts/asymptotics with regards to §2. Mansour has some results for some subword patterns. It’s hard to say precisely what “explicit” means here since the general results give an algorithm that produces the counts/asymptotics, given the specifics of the pattern. See [Wil82; Pak18] for more discussion. this could include e.g. words/compositions with $r$ occurrences of variations on subword patterns, such as sets, POPs, generalized. Three letter patterns and some pattern sets are done in [HM10, Sec. 4.3]. [presumably in the sections above we already have enough recurrence relations for one document]

### Restricted-preimage mappings

Main paper [Mar+17].

Could find limiting distribution of normalization of $\log \boldsymbol{T}$ for arbitrary indegree set. Also, analyze $\boldsymbol{T}$ rather than the log (since even the expectation will be quite different from just an application of log, so at least analyze that).

possible applications? other restrictions? does $k$ have to be a function of $n$ going to infinity?

# A Computer code

include `treeplot.nb`, `random_comp.nb`

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