CSCI 650 Assignment 1

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Question 1: Asymptotic Growth Comparison (with $f_1, \ldots, f_{12}$ and Θ-Groups)

Objective

Sort the given functions from **smallest to largest** asymptotic growth as $n \to \infty$, producing an order $f_1, \ldots, f_{12}$ such that $f_1 = O(f_2)$, $f_2 = O(f_3)$, ..., and **identify groups** of functions that are Θ-equivalent.

Simplifications Used

$$\begin{aligned}
\ln(19n^4) &= \ln 19 + 4\ln n = \Theta(\ln n),\\
7\ln^2(n^3) &= 7(3\ln n)^2 = \Theta\big((\ln n)^2\big),\\
\frac{n^3}{\sqrt{n}} &= n^{5/2},\\
\log_\pi\Big(\big(\tfrac{2}{23}n\big)^{5n}\Big) &= 5n\log_\pi\big(\tfrac{2}{23}n\big) = \Theta(n\ln n).
\end{aligned}$$

Constants such as 2, 121!, $\ln 19$, and fixed log bases contribute only constant factors.

Final Ordering (Smallest $\to$ Largest)

The list below satisfies $f_1 = O(f_2)$, $f_2 = O(f_3)$, ...

$$\begin{aligned}
f_1 &= 2 && (\Theta(1))\\
f_2 &= 121! && (\Theta(1))\\
f_3 &= \ln(19n^4) && (\Theta(\ln n))\\
f_4 &= 7\ln^2(n^3) && \big(\Theta((\ln n)^2)\big)\\
f_5 &= 42n^{3/5} && (\Theta(n^{3/5}))\\
f_6 &= n && (\Theta(n))\\
f_7 &= n\ln n && (\Theta(n\ln n))\\
f_8 &= \log_\pi\Big(\big(\tfrac{2}{23}n\big)^{5n}\Big) && (\Theta(n\ln n))\\
f_9 &= \frac{n^3}{\sqrt{n}} && (\Theta(n^{5/2}))\\
f_{10} &= \left(\frac{3}{2}\right)^n && (\Theta((1.5)^n))\\
f_{11} &= 2^n && (\Theta(2^n))\\
f_{12} &= (n!)^2 && (\text{super-exponential})
\end{aligned}$$

Θ-Equivalent Groups

$$
\begin{aligned}
&\{2,\ 121!\} = \Theta(1),\\
&\{\ln(19n^4)\} = \Theta(\ln n),\\
&\{7\ln^2(n^3)\} = \Theta((\ln n)^2),\\
&\{42n^{3/5}\} = \Theta(n^{3/5}),\\
&\{n\} = \Theta(n),\\
&\{\, n\ln n,\ \log_\pi(((2/23)n)^{5n})\,\} = \Theta(n\ln n),\\
&\left\{\frac{n^3}{\sqrt{n}}\right\} = \Theta(n^{5/2}),\\
&\left\{\left(\frac{3}{2}\right)^n\right\} = \Theta((1.5)^n),\\
&\{2^n\} = \Theta(2^n),\\
&\{(n!)^2\} \text{ dominates all exponentials (via Stirling).}
\end{aligned}
$$

Why This Ordering Is Correct (One-Liners)

- Constants $= \Theta(1)$; any non-constant eventually outgrows them.
- $\ln n \ll (\ln n)^2 \ll n^\alpha$ for any fixed $\alpha > 0$.
- Among polynomials: $n^{3/5} \ll n \ll n\ln n \ll n^{5/2}$.
- Exponentials beat all polynomials: $(3/2)^n \ll 2^n$.
- By Stirling, $(n!)^2 \approx \mathrm{poly}(n)\left(\frac{n}{e}\right)^{2n}$, which dominates c^n for any constant $c > 1$.

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Question 2: Master Theorem Applications

(a)

$$
T(n) = \begin{cases} 1, & n \le 2,\\ 4T\left(\frac{n}{2}\right) + 7n^2\log(n^2), & \text{otherwise.} \end{cases}
$$

Identify parameters: $a = 4,\ b = 2,\ f(n) = 7n^2\log(n^2) = 14n^2\log n = \Theta(n^2\log n)$.
Critical exponent:

$$
n^{\log_b a} = n^{\log_2 4} = n^2.
$$

Comparison:

$$
f(n) = \Theta\big(n^2\log^1 n\big) = \Theta\big(n^{\log_2 4}\log^1 n\big).
$$

This matches **Master Theorem Case 2** with $k = 1$:

$$
T(n) = \Theta\big(n^{\log_2 4}\log^{k+1} n\big) = \Theta\big(n^2(\log n)^2\big).
$$

$$
\boxed{T(n) = \Theta\big(n^2\log^2 n\big)}
$$

(b)

$$T(n) = \begin{cases} 42, & n \le 2, \\ 5T\left(\frac{n}{7}\right) + 8n^5, & \text{otherwise.} \end{cases}$$

Identify parameters: $a = 5,\ b = 7,\ f(n) = 8n^5 = \Theta(n^5)$.
Critical exponent:

$$n^{\log_b a} = n^{\log_7 5} \approx n^{0.699}.$$

Since $5 > 0.699$, we have:

$$f(n) = \Theta\big(n^5\big) = \Omega\big(n^{\log_7 5+\varepsilon}\big) \quad \text{for some } \varepsilon > 0.$$

Regularity check:

$$a\, f\left(\frac{n}{b}\right) = 5 \cdot 8 \left(\frac{n}{7}\right)^5 = \frac{40}{7^5} n^5 = \left(\frac{5}{7^5}\right) f(n), \quad \frac{5}{7^5} < 1.$$

Condition satisfied.
This is **Master Theorem Case 3**:

$$\boxed{T(n) = \Theta(n^5)}.$$