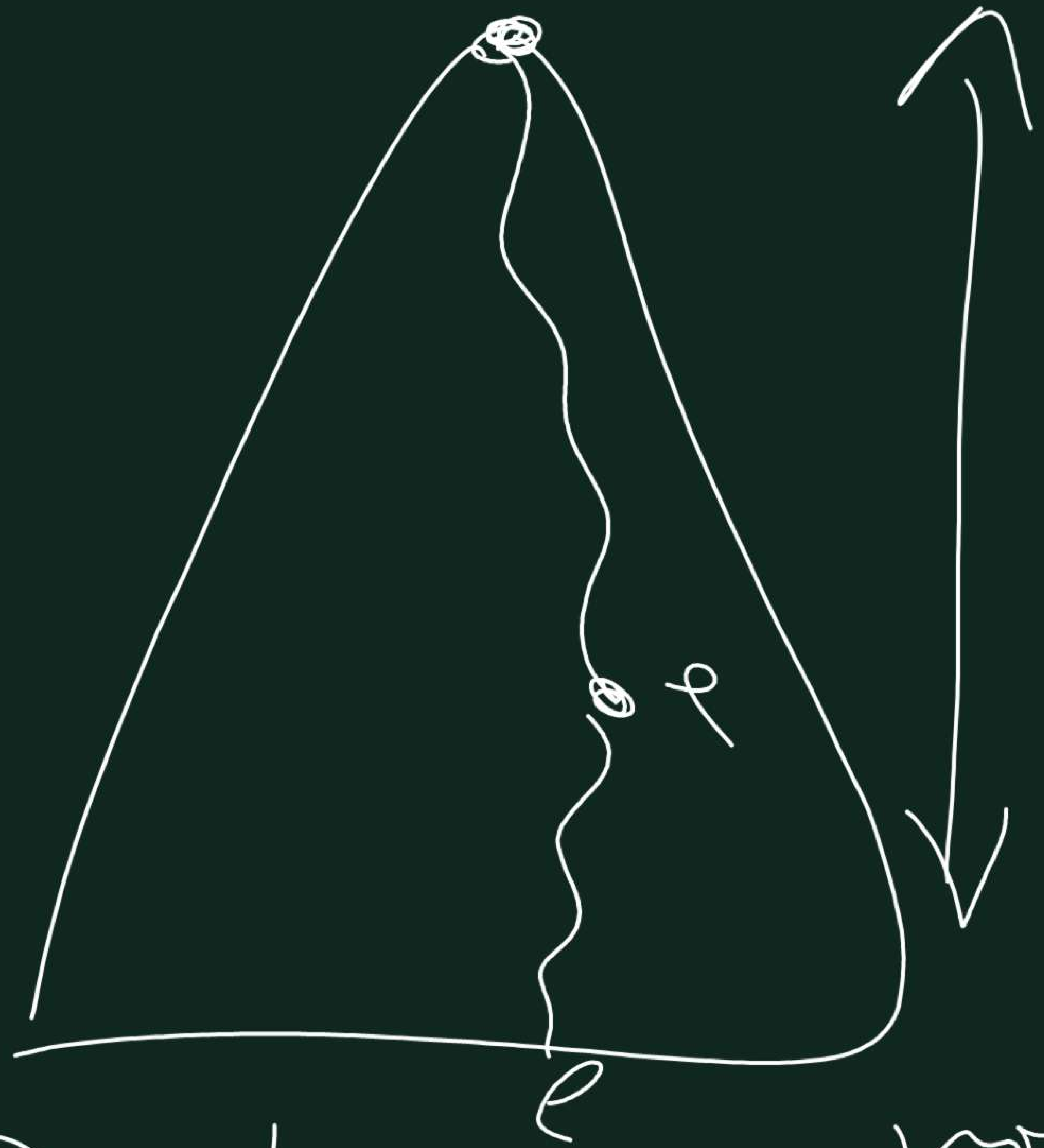


Q: relation between size(t) and height(t)

$$\text{size} = \Theta(2^h)$$

$$\Updownarrow$$

$$h = \Theta(\log(\text{size}))$$

worst case

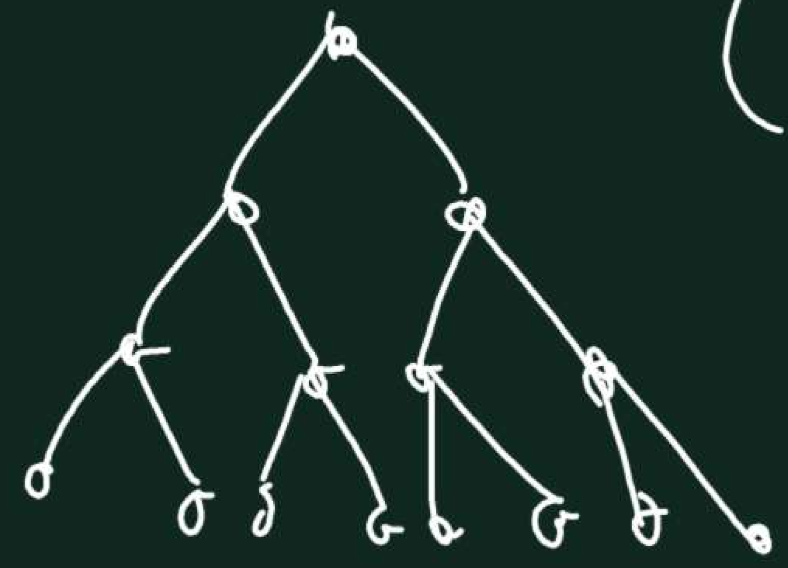
height = size − 1

$= \Theta(\text{height})$

avg?

still log

best case (perfect tree)



h	size
0	1
1	3
2	7
3	15

$h \mid 2^{h+1} - 1$

important quantity: average depth of elements in the tree

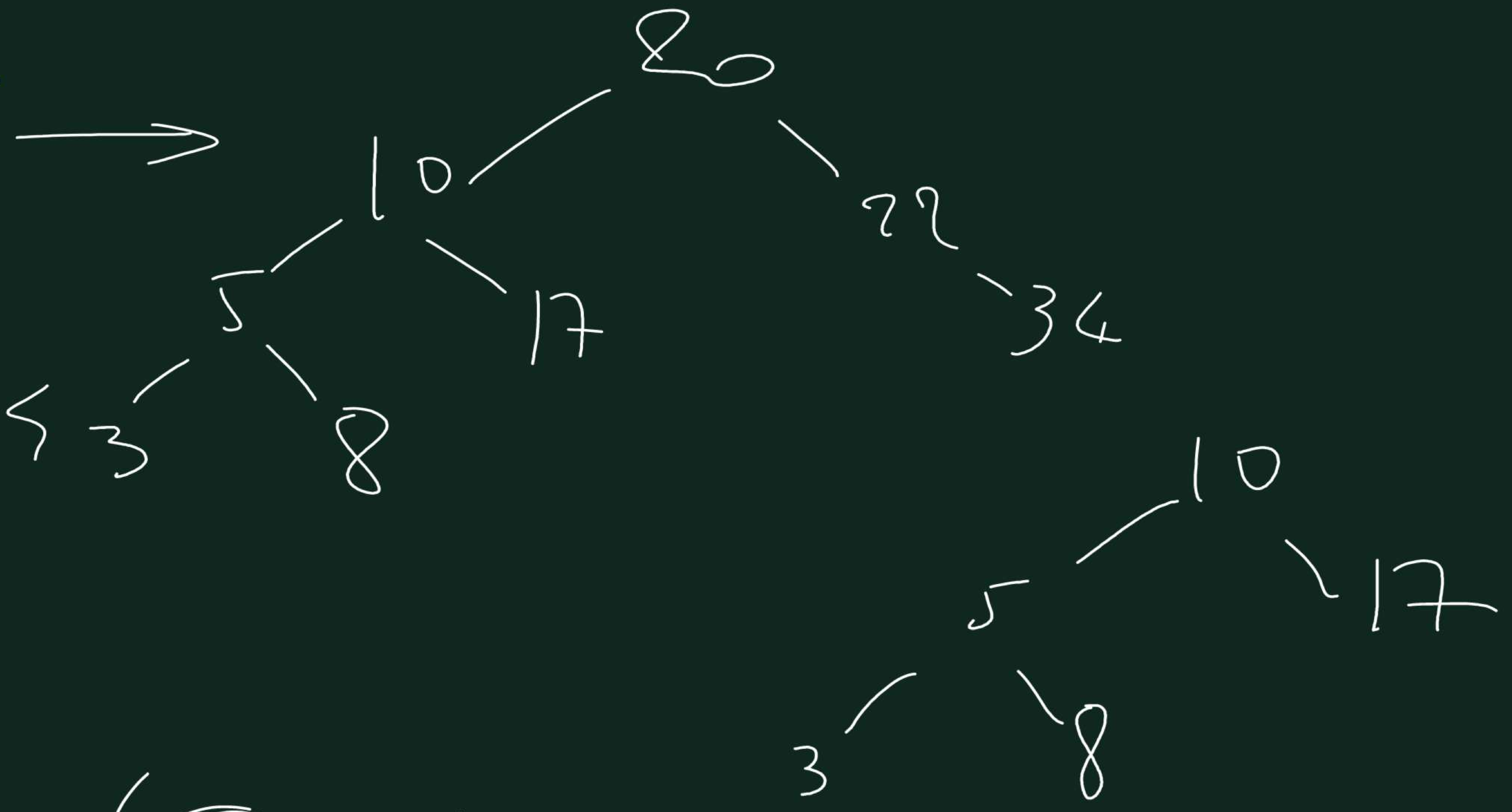
$$\hookrightarrow \frac{\sum_{n \in t} \overset{\text{height}}{\text{depth}}(n)}{\text{size}(t)} = \frac{PL(t)}{\text{size}(t)}$$

"leaf_addition"

18?

root_addition

20 ?



lessThan(10, 20)

A_4 $\langle 10, 5, \text{lessThan}(22, 20)\rangle$

A_3 $\langle 10, 5, \text{lessThan}(17, 20)\rangle$

A_2 $\langle 10, 5, \langle 17, \text{new}, \text{lessThan}(\text{new}, 20)\rangle\rangle$

A_1 $\langle 10, 5, \langle 17, \text{new}, \text{new}\rangle\rangle$

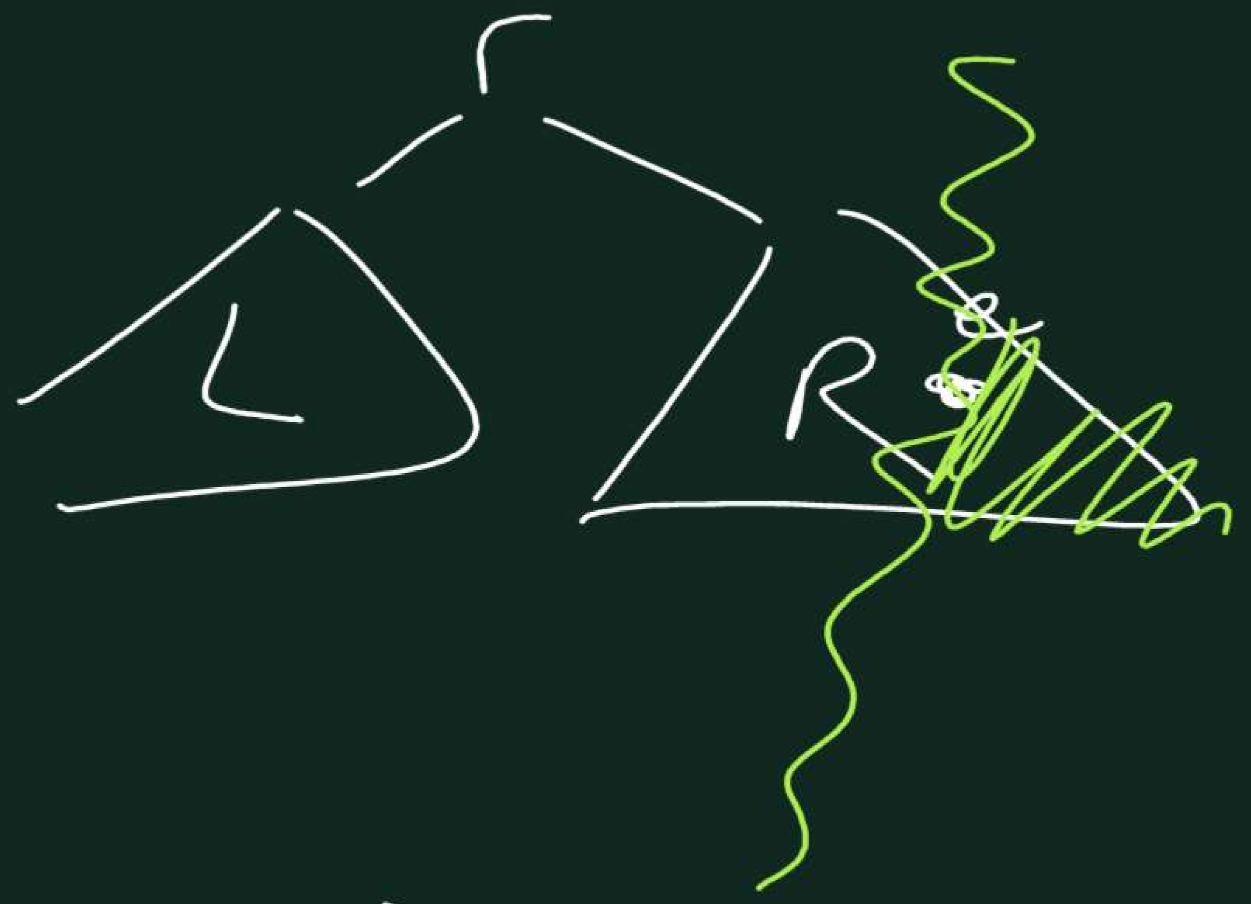
ADT extension BST

Operations

lessThan : BST × Element ⟶ BST

greaterThan : BST × Element ⟶ BST

rootAdd : BST × Element ⟶ BST



Axioms

A_1 $\text{lessThan}(\text{new}, e) \equiv \text{new}$

A_2 $\text{lessThan}(\langle r, L, R\rangle, r) \equiv L$

A_3 $e<r$ $\text{lessThan}(\langle r, L, R\rangle, e) \equiv \text{lessThan}(L, e)$

A_4 $\text{lessThan}(\langle r, L, R\rangle, e) \equiv \langle r, L, \text{lessThan}(R, e)\rangle$

A_5 $\text{greaterThan}(\text{new}, e) \equiv \text{new}$

A_6 $\text{greaterThan}(\langle r, L, R\rangle, r) \equiv R$

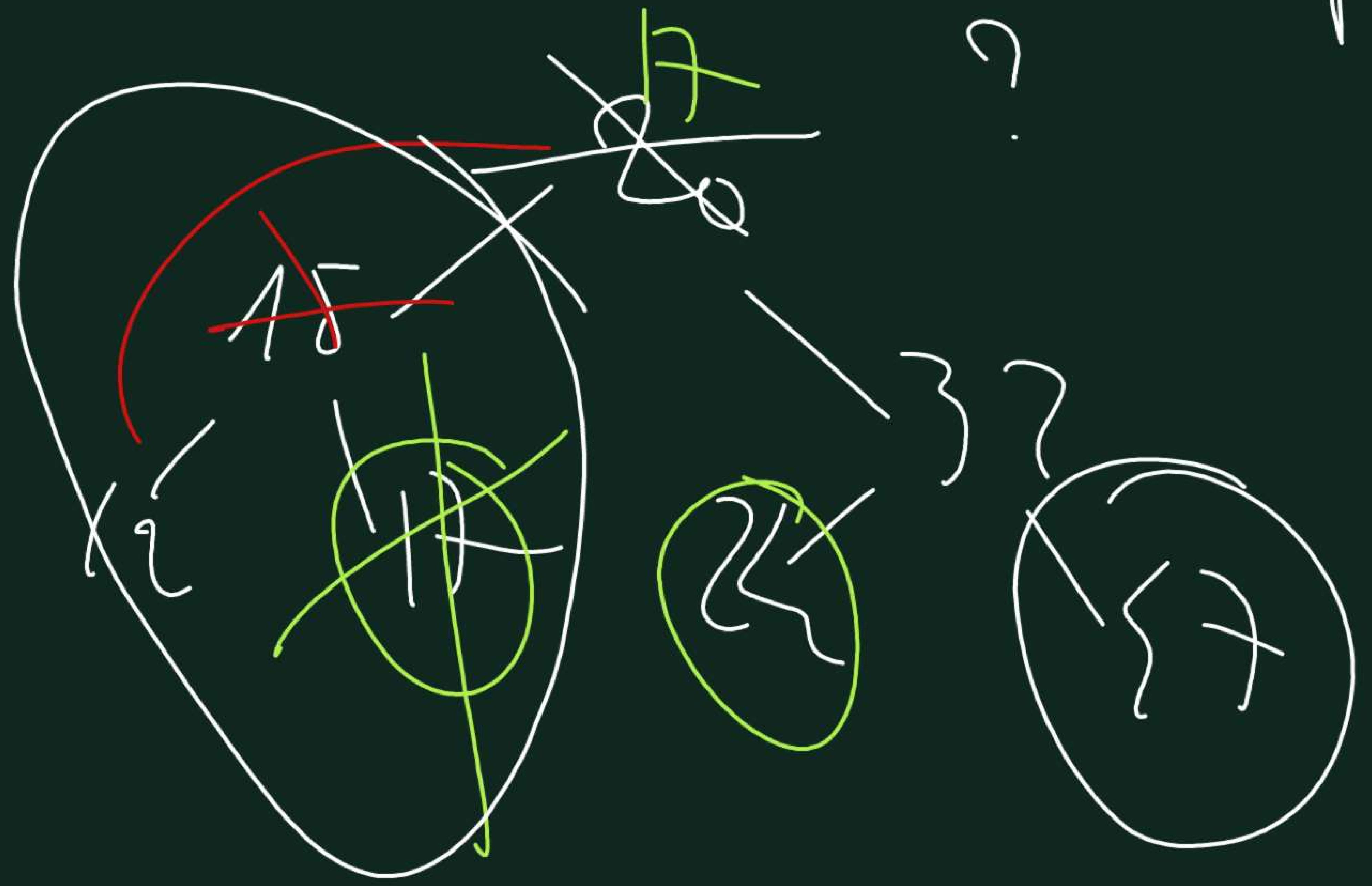
A_7 $e>r \Rightarrow \text{greaterThan}(\langle r, L, R\rangle, e) \equiv \text{greaterThan}(R, e)$

A_8 $\text{greaterThan}(\langle r, L, R\rangle, e) \equiv \langle r, \text{greaterThan}(L, e), R\rangle$

A_9 $\text{rootAdd}(t, e) \equiv \langle e, \text{lessThan}(t, e), \text{greaterThan}(t, e)\rangle$

Removing an element from a BST

- if element $\notin$ BST, don't change anything
- If element is a leaf, just remove it (still BST)
- If element has only 1 child → replace with the child
- 2 children ? : replace root with
  - ↗ either the max elt of left child
  - ↘ or min elt of right child



ADT extension BST

Operations

$max : BST \longrightarrow Element$

$delMax : BST \longrightarrow BST$

$remove : BST \times Element \longrightarrow BST$

precondition

$max(t)$ iff $t \mathrel{!}\equiv new$

Axioms

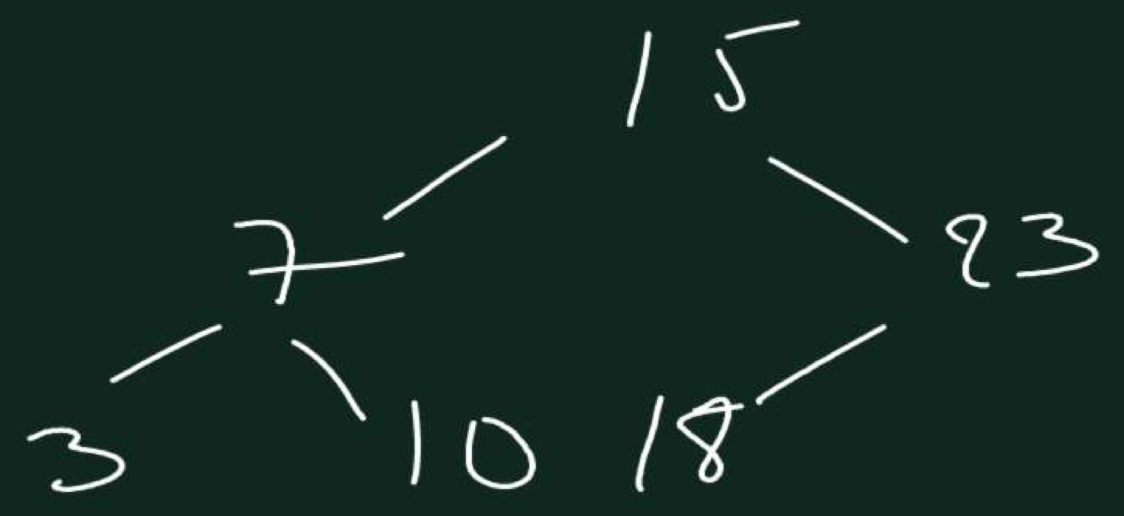
A_1 $max(<r, L, new>) \equiv r$

A_2 $max(<r, L, R>) \equiv max(R)$

A_3 $delMax(new) \equiv new$

A_4 $delMax(<r, L, new>) \equiv L$

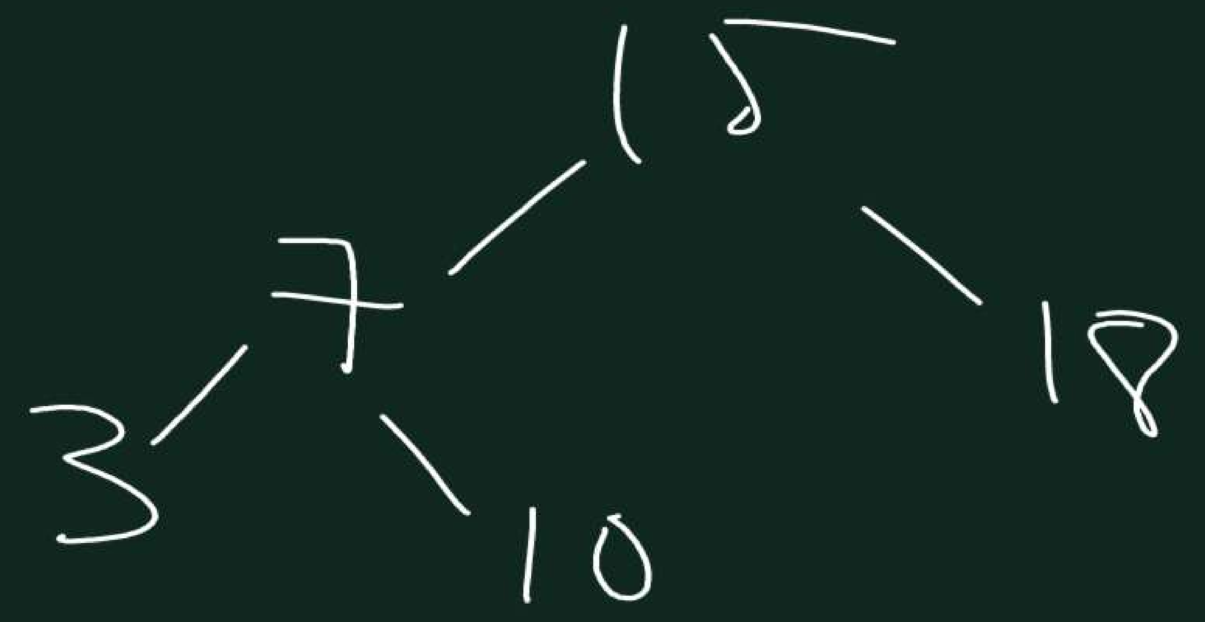
A_5 $delMax(<r, L, R>) \equiv <r, L, delMax(R)>$



$delMax(15)$

A_5 $<15, 7, delMax(23)>$

A_4 $<15, 7, 18>$



$A_6 \quad remove(new, e) \equiv new$

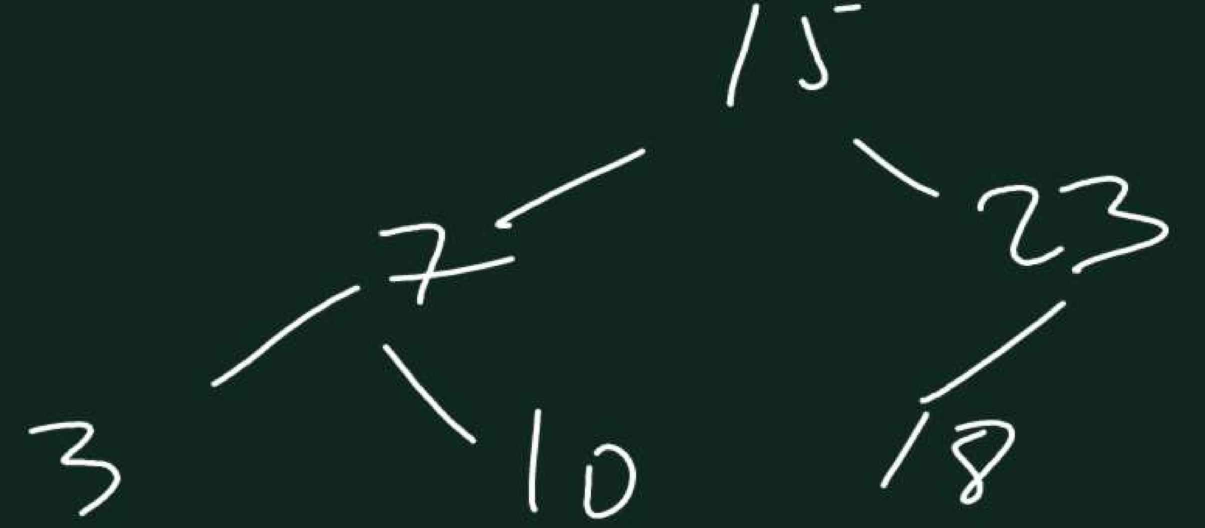
$A_7 \quad remove(\langle r, L, new \rangle, r) \equiv L$

$A_8 \quad remove(\langle r, new, R \rangle, r) \equiv R$

$A_9 \quad e < r \Rightarrow remove(\langle r, L, R \rangle, e) \equiv \langle r, remove(L, e), R \rangle$

$A_{10} \quad r < e \Rightarrow remove(\langle r, L, R \rangle, e) \equiv \langle r, L, remove(R, e) \rangle$

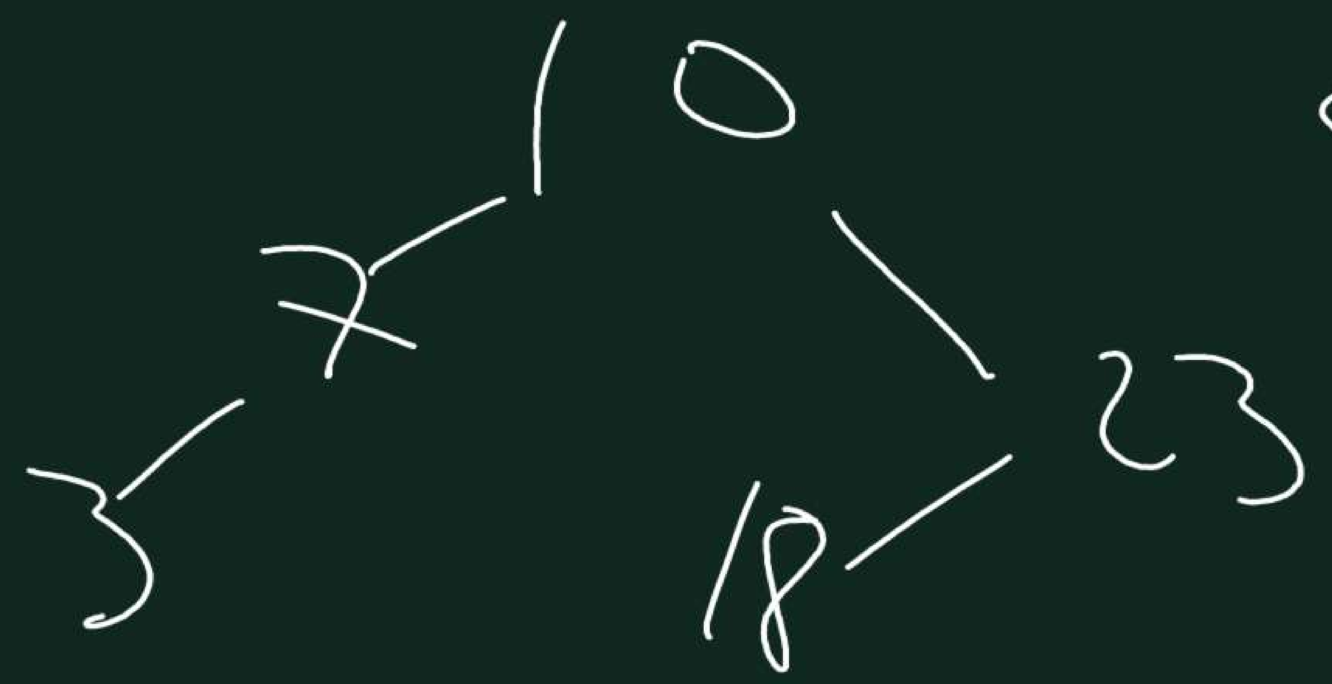
$A_{11} \quad remove(\langle r, L, R \rangle, r) \equiv \langle max(L), delMax(L), R \rangle$



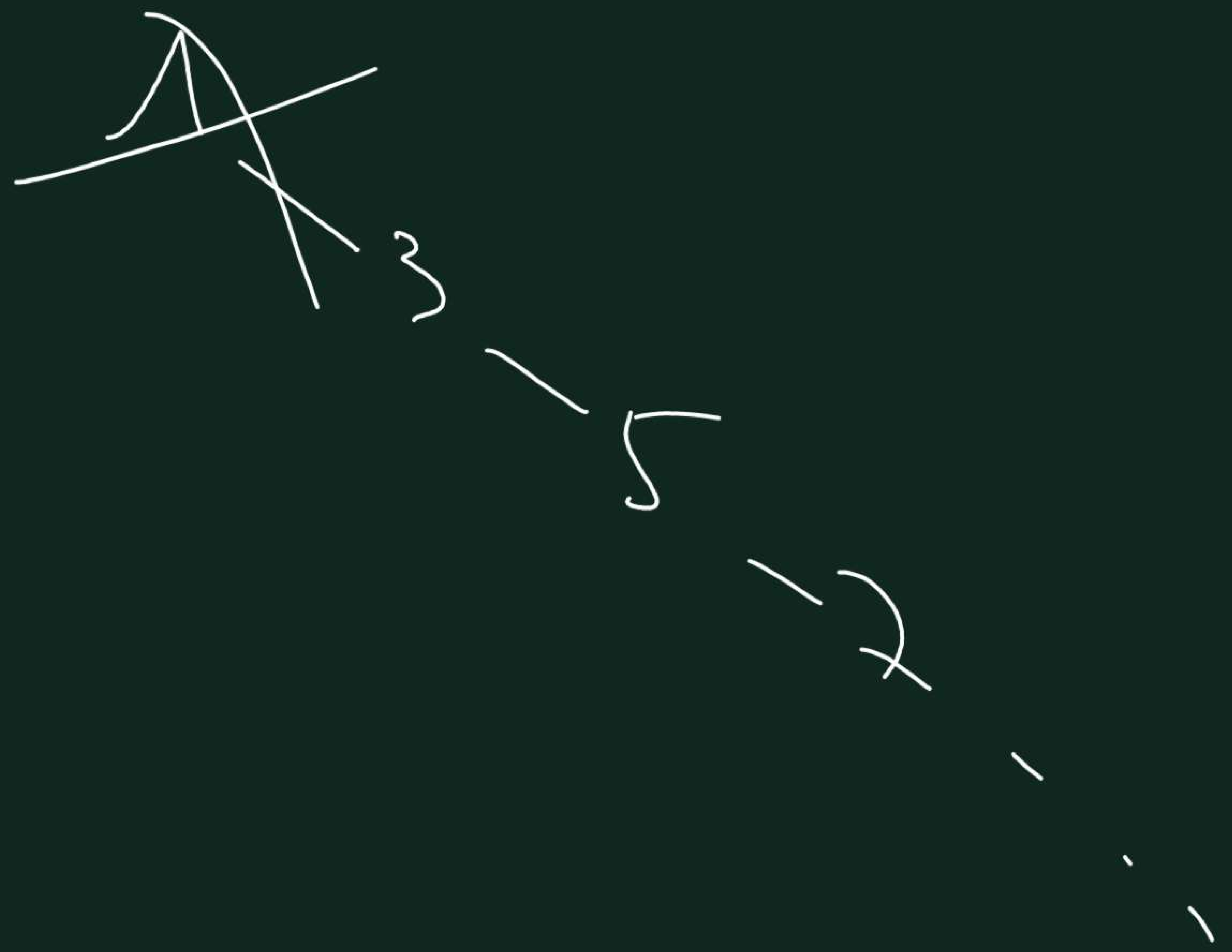
$remove(15, 15)$

$A_{11} \quad \langle max(7), delMax(7), 23 \rangle$

$\langle 10, \langle 7, 3, new \rangle, \langle 23, 18, new \rangle \rangle$



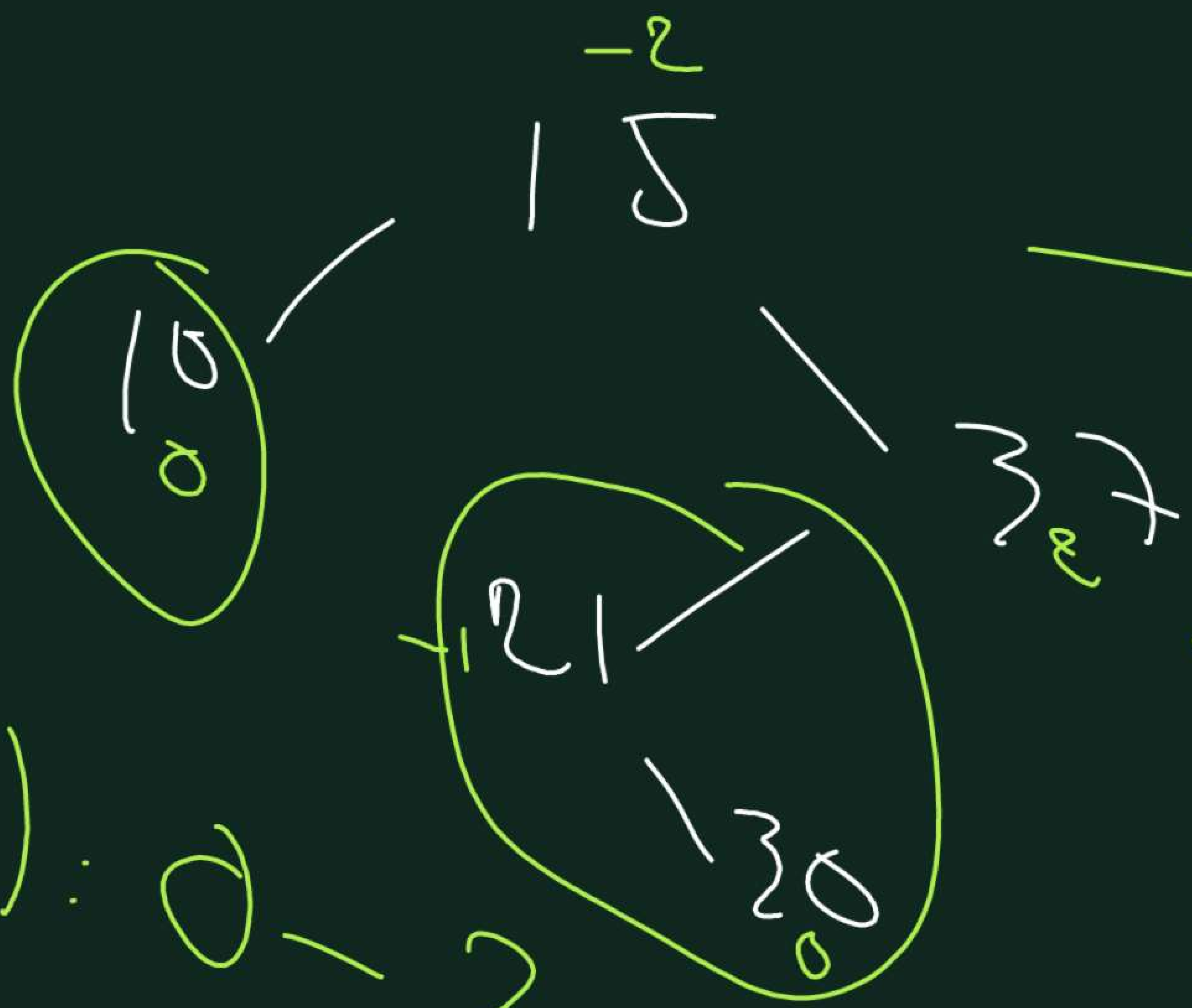
$1 \quad 3 \quad 5 \quad 7 \quad 9 \quad \ldots$



AVL (Adelson-Velskii & Landis)

def: • **degree** of node n in t: $\text{height}(\text{left}(n)) - \text{height}(\text{right}(n))$

• AVL: a BST whose nodes have a degree in $\{-1, 0, +1\}$



→ not an AVL

$h(l): 0 - 2$

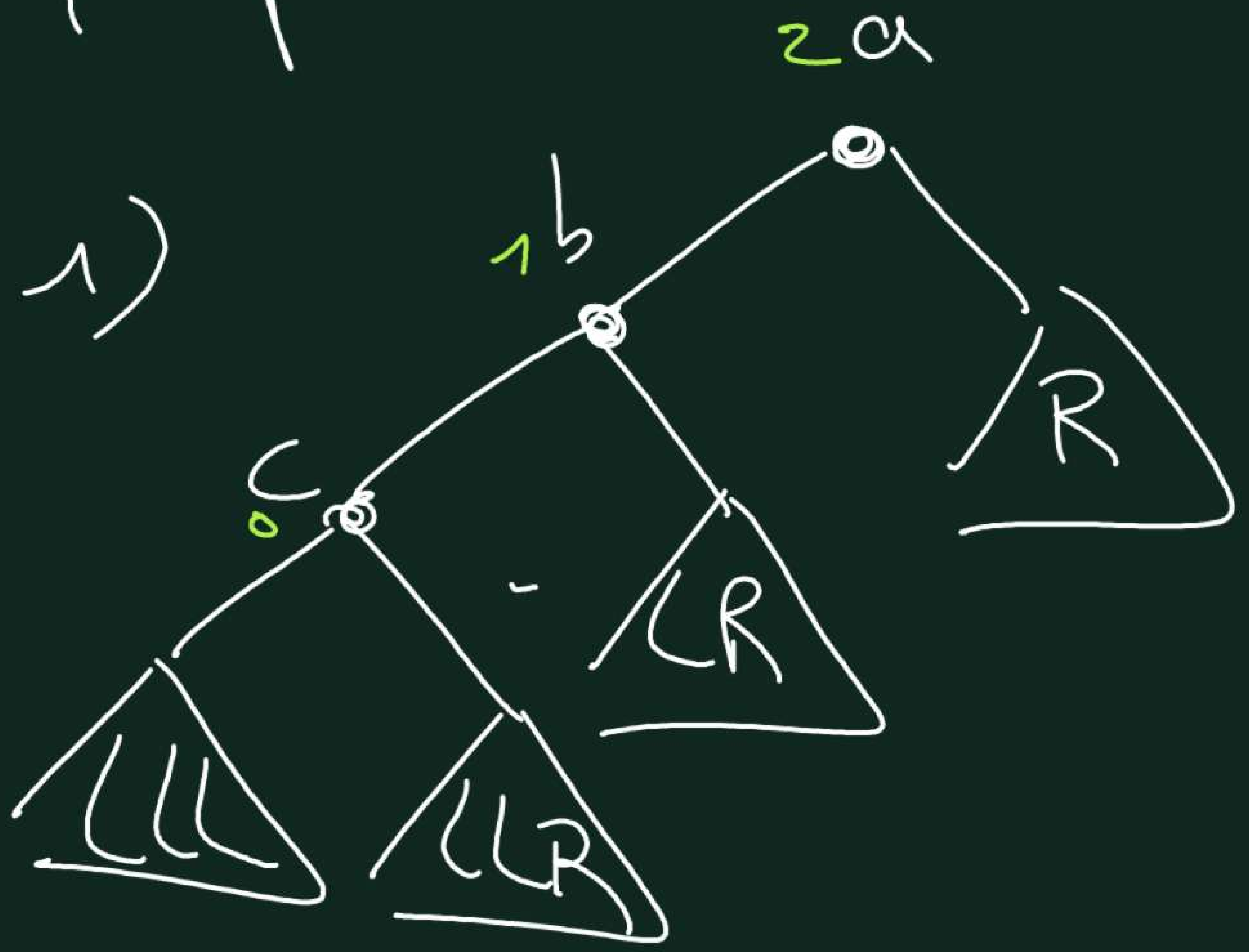
$1 - -1 = 2$

How can we re-balance a non-AVL? → rotations

4 possible cases:

right rotation

1)

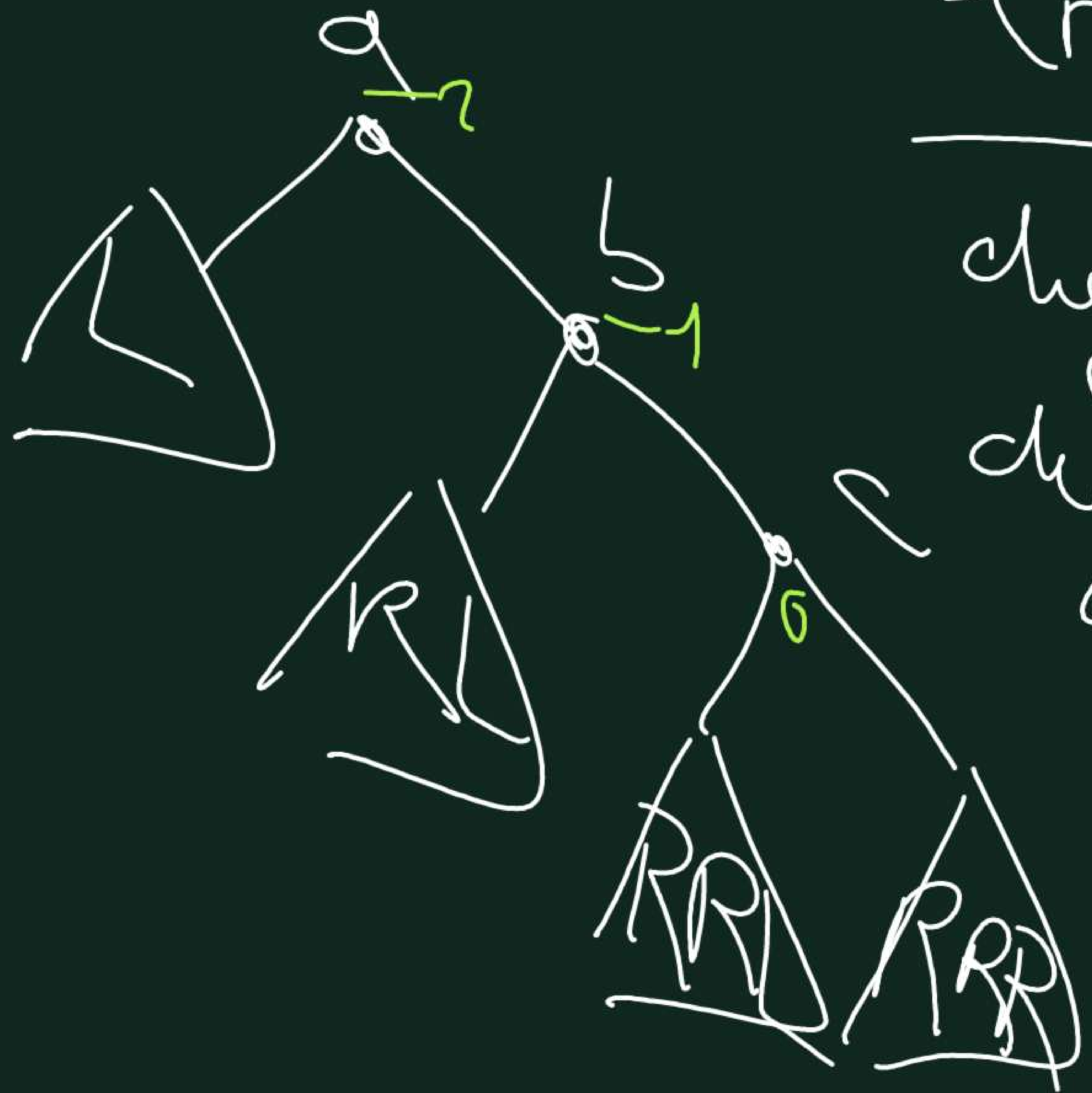


$$\left.\begin{array}{l} deg(t) = 2 \\ deg(left(t)) = 1 \end{array}\right\}$$

$\xrightarrow{rr}$

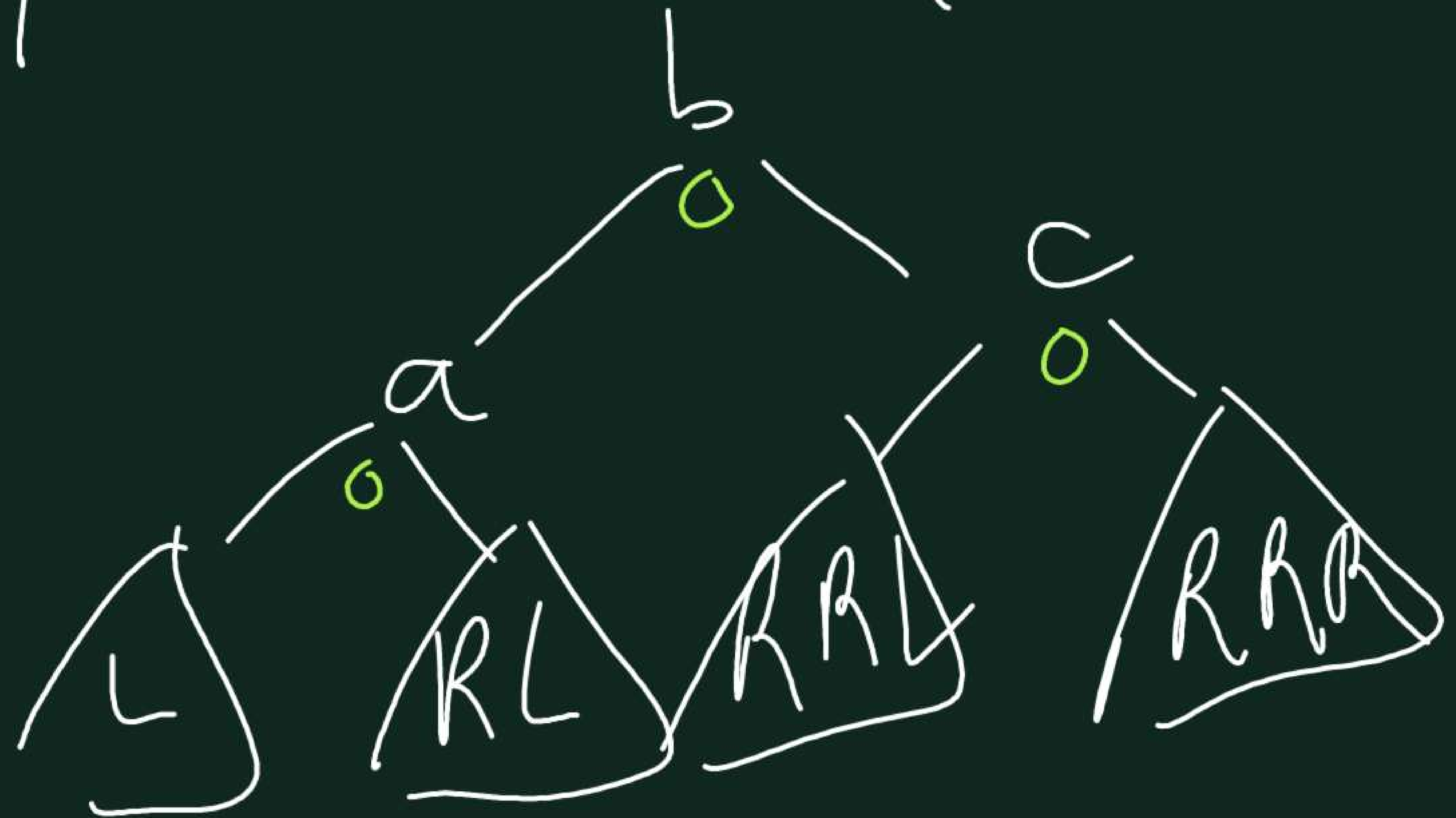
left rotation

2)

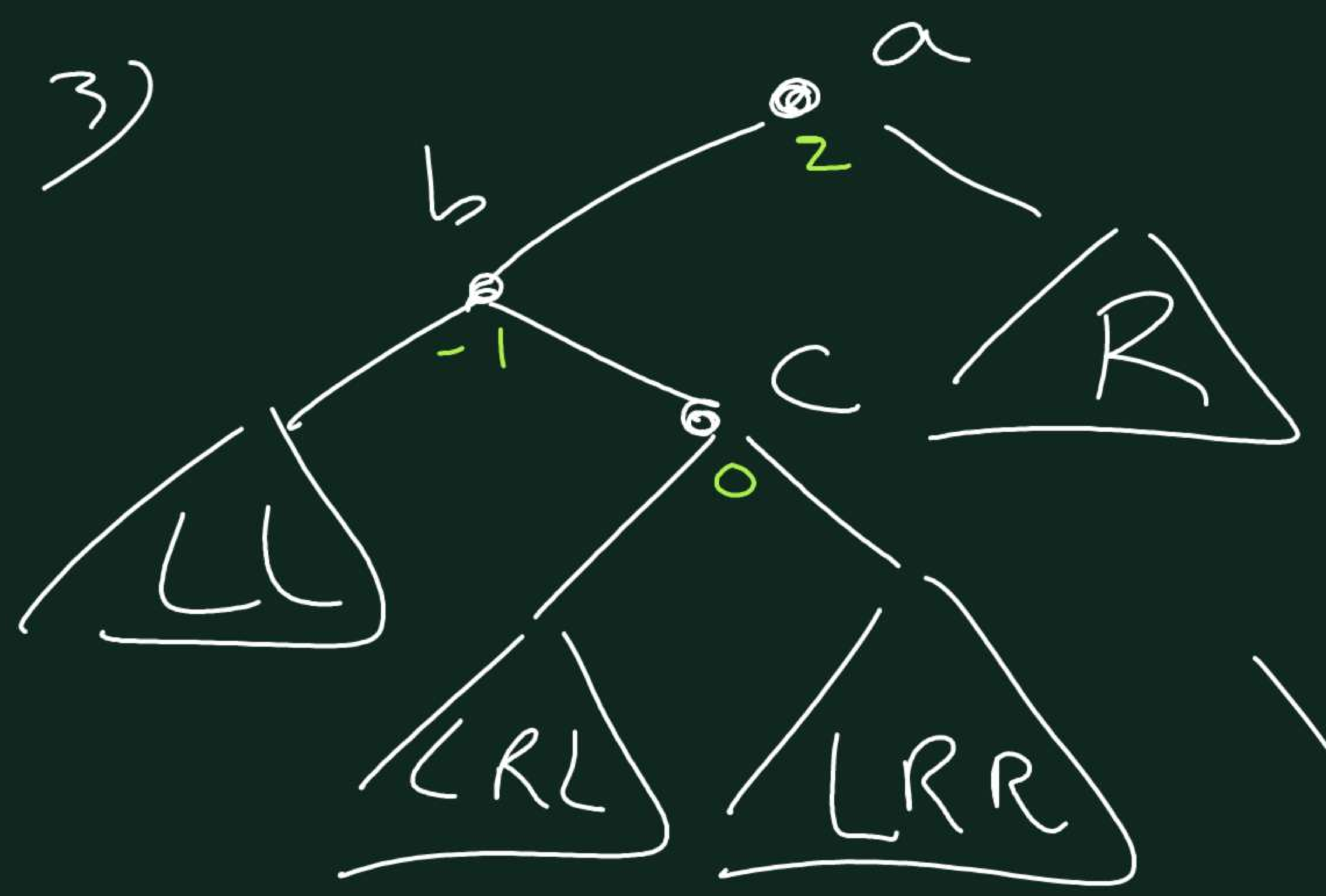


$\xrightarrow{lr}$

$$\left.\begin{array}{l} deg(t) = -2 \\ deg(right(t)) = -1 \end{array}\right)$$



3)



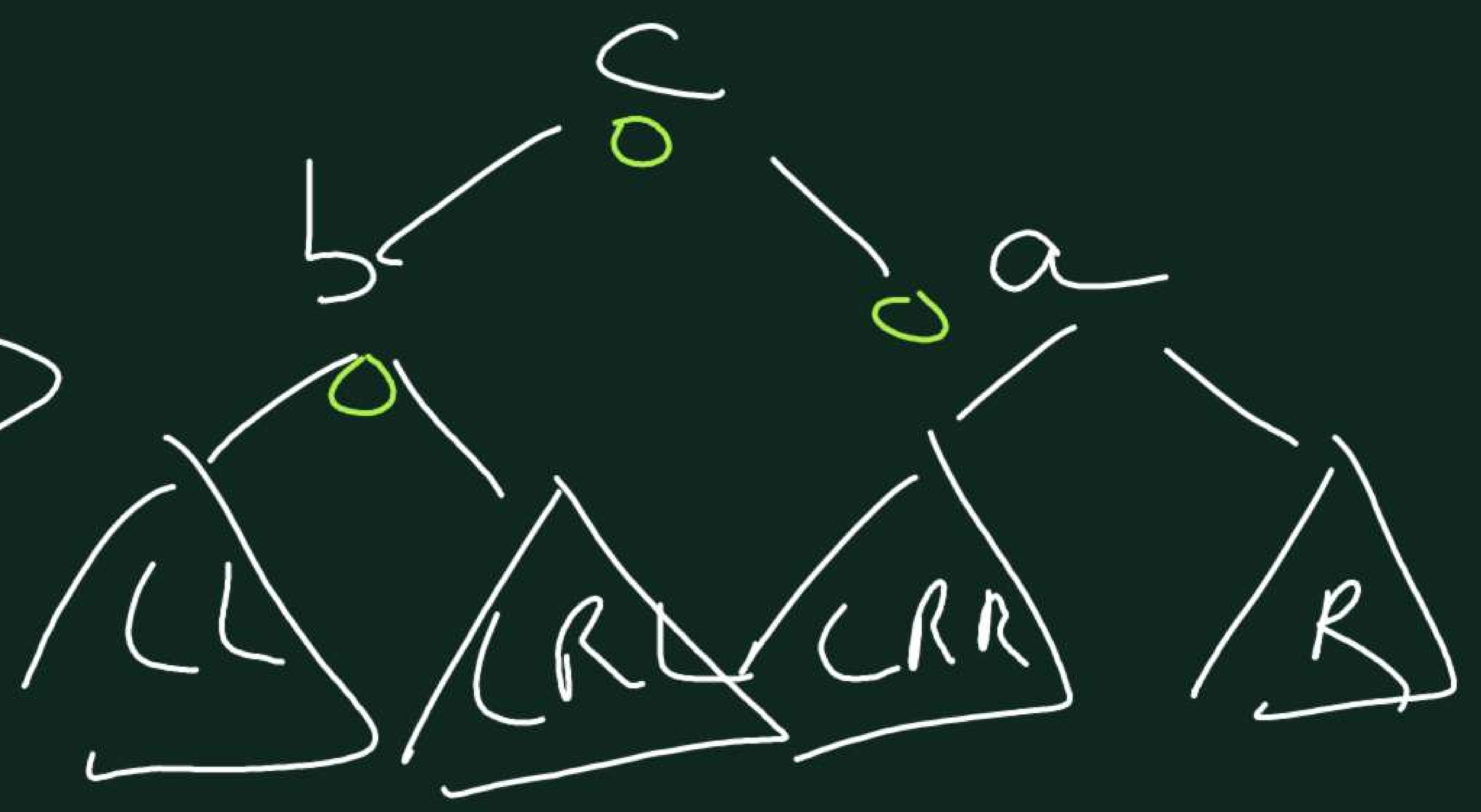
$bal(t) = +2$

$bal(left(t)) = -1$

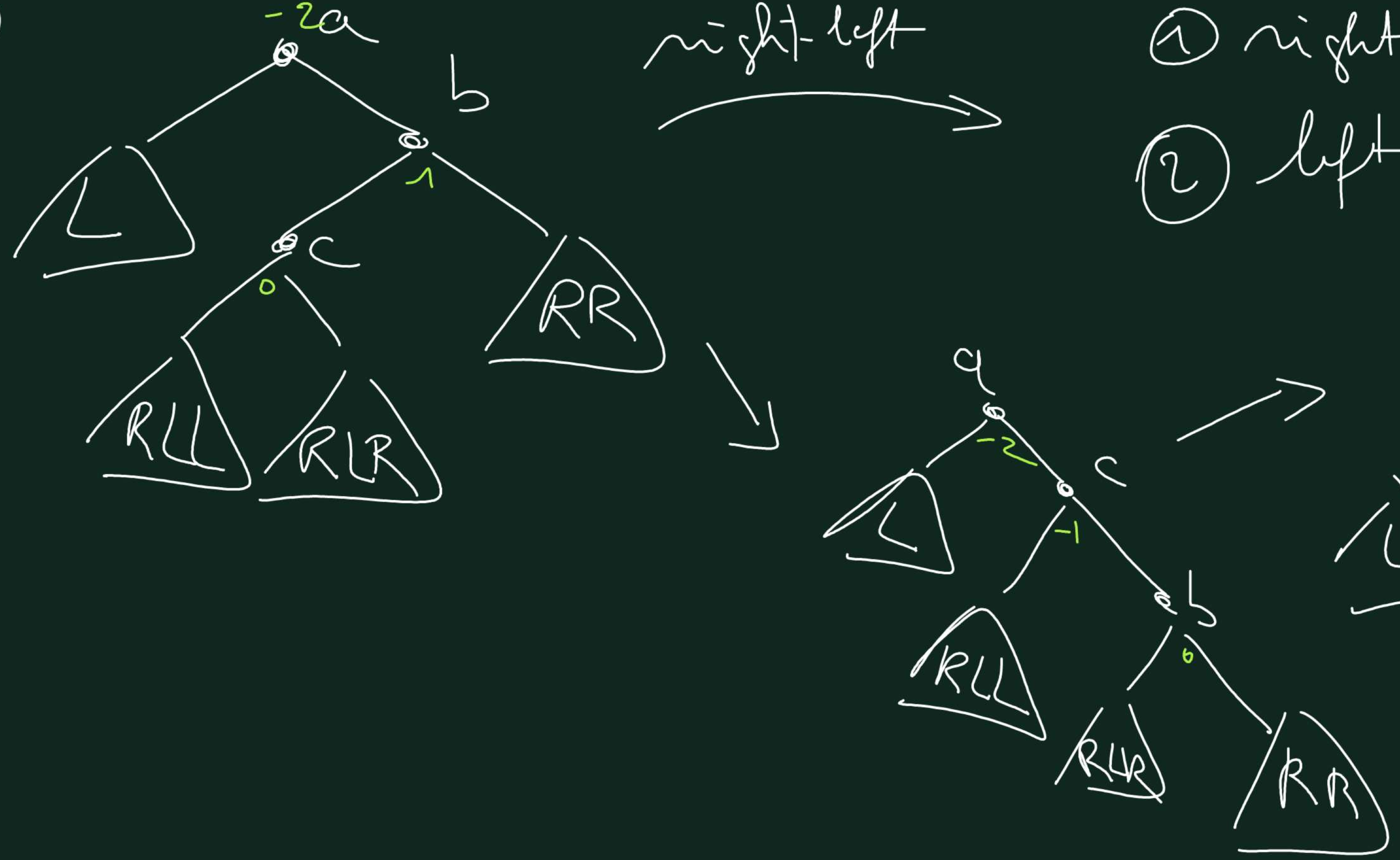
left_right

① left-rotate left child

② right-rotate root



4)



① right-rotate right child

② left_rotate tree

<u>exercise</u>: starting from an empty BST, add the following values:

2, 8, 5, 7, 1, 4, 0, 9

1) using leaf-addition
2) using root_addition
3) using "AVL" addition (leaf add + balance)

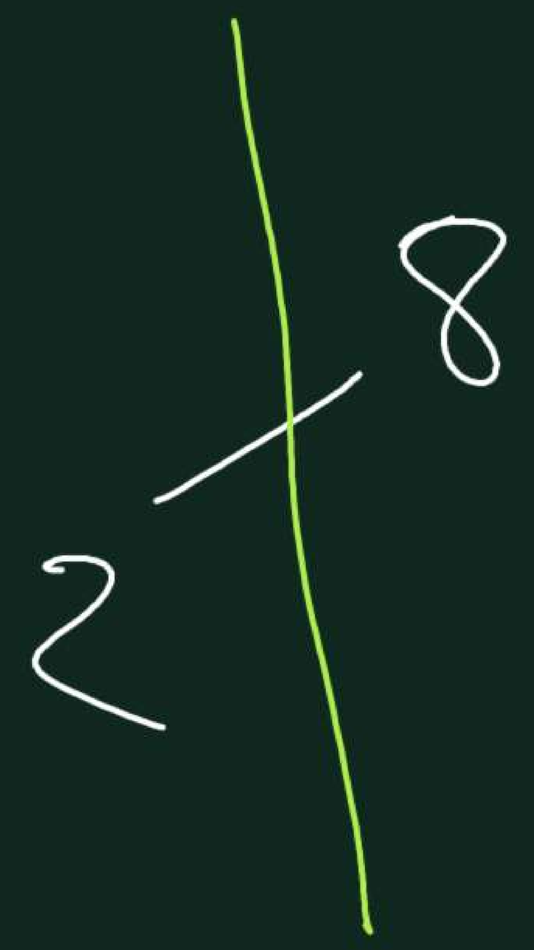
1) leaf addition

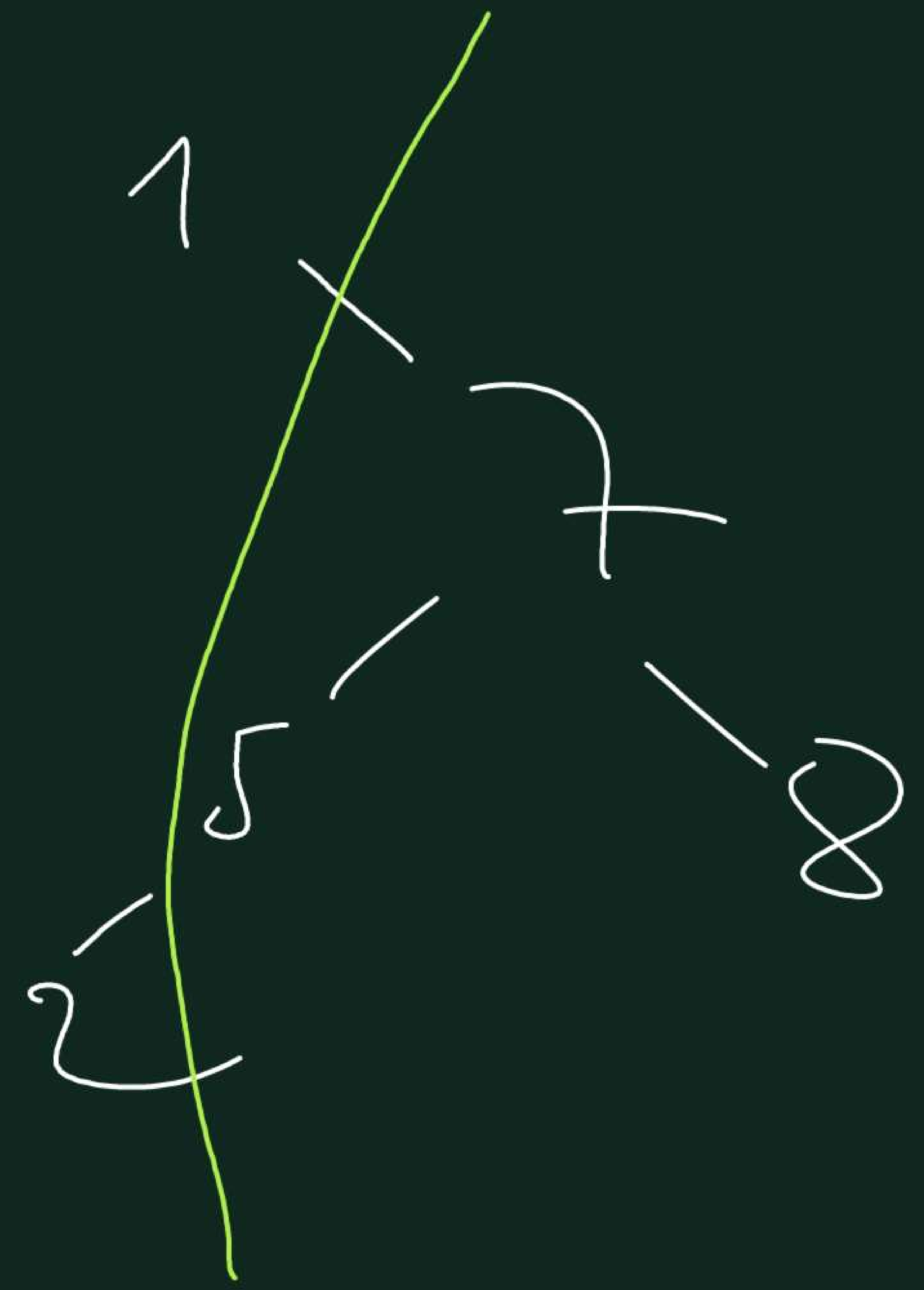
H : 3

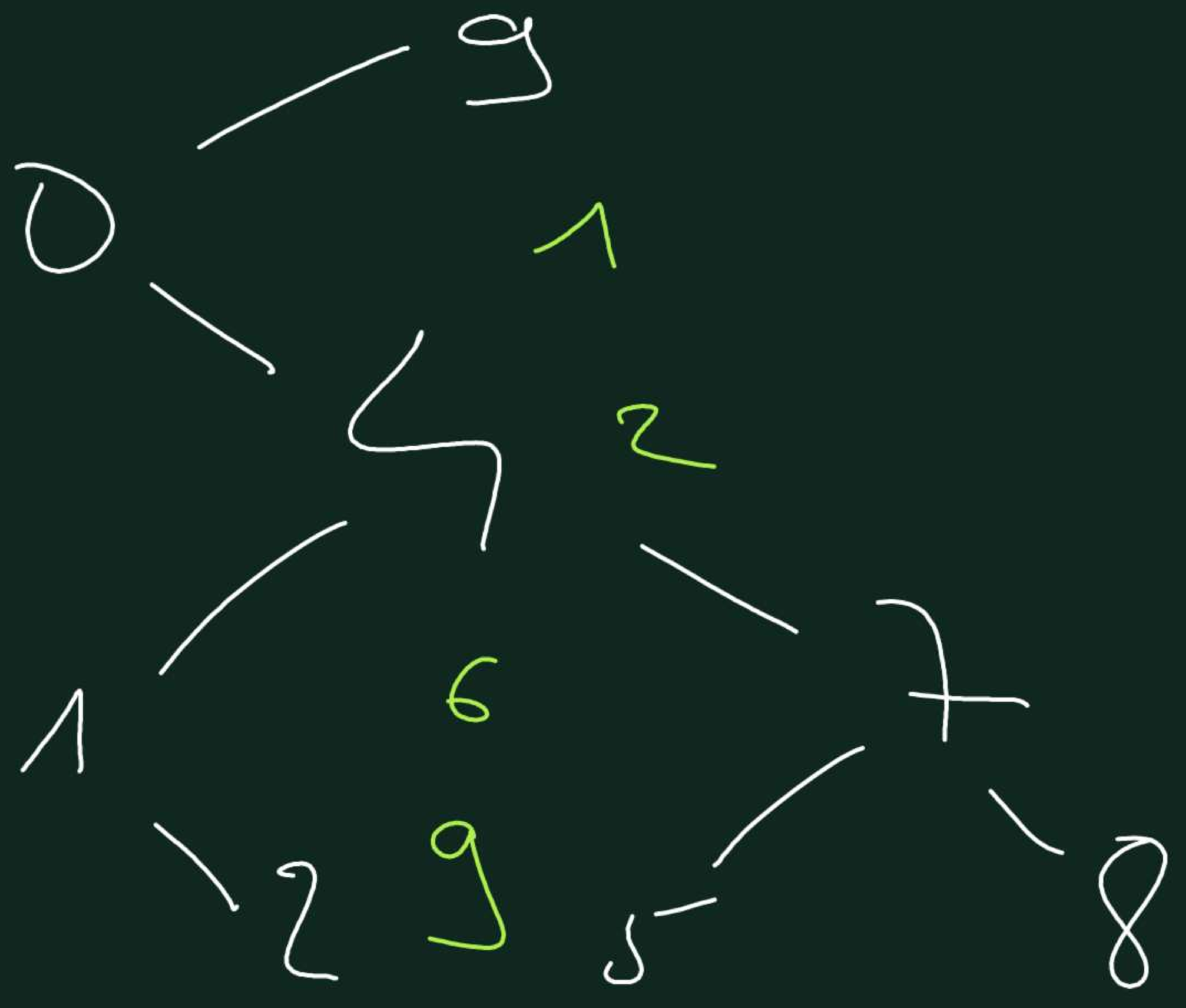
PL : 14

~~depth~~ = $\frac{14}{8}$

root addition





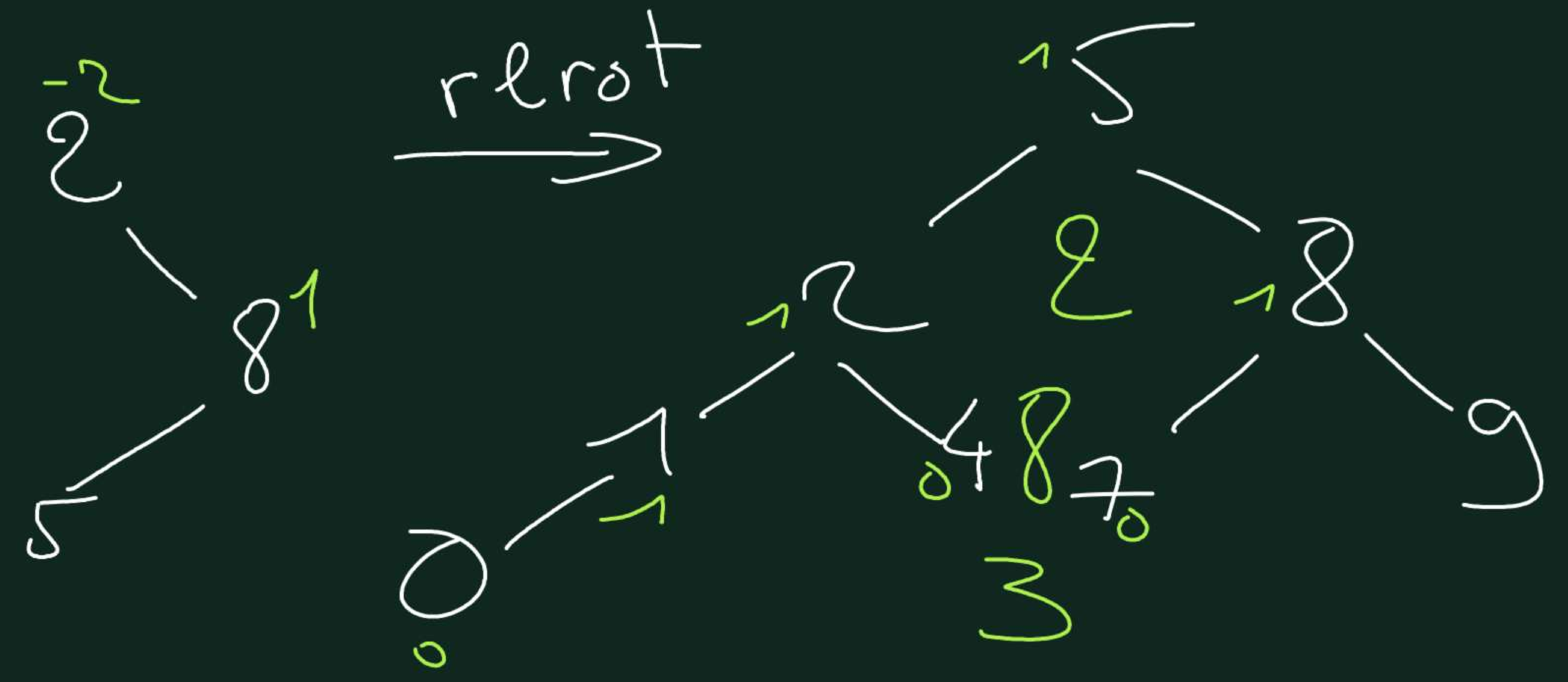


H: 4

RL: 18

depth: $\frac{18}{8}$

"AVL" addition



H : 3

PL : 13

~~depth~~ : $\frac{13}{8}$

ADT AVL extends BST

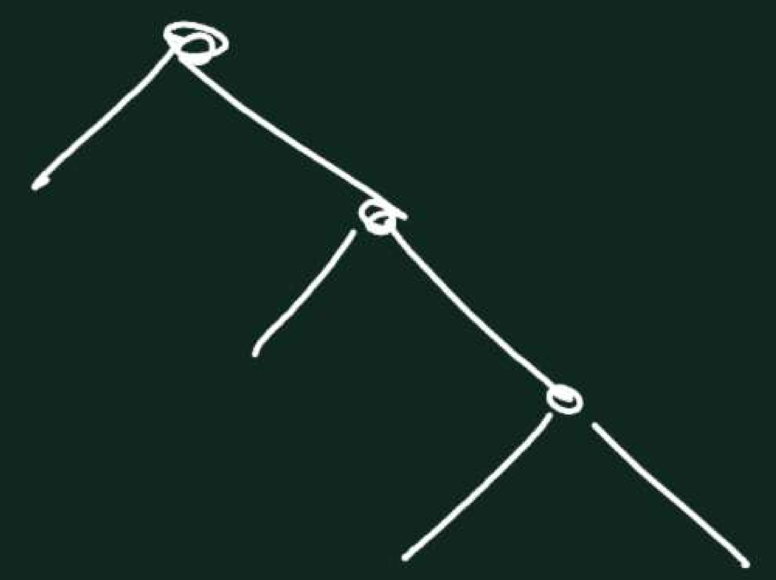
Operations

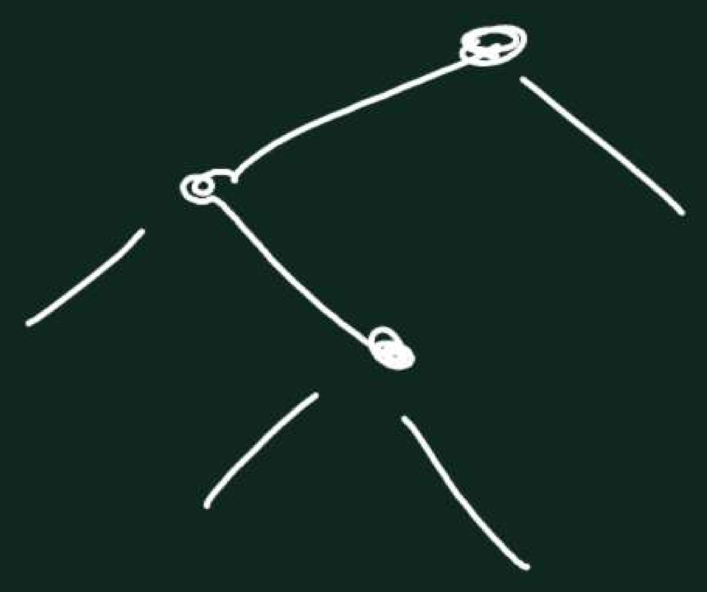
lrot : BST → AVL

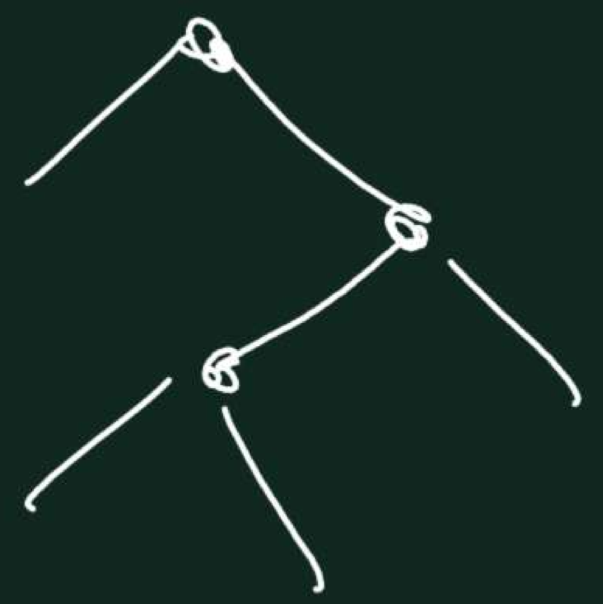
rrot : BST → AVL

lrrot : BST → AVL

rlrot : BST → AVL







preconditions

lrot(t) iff t != new and right(t) != new and right(right(t)) != new

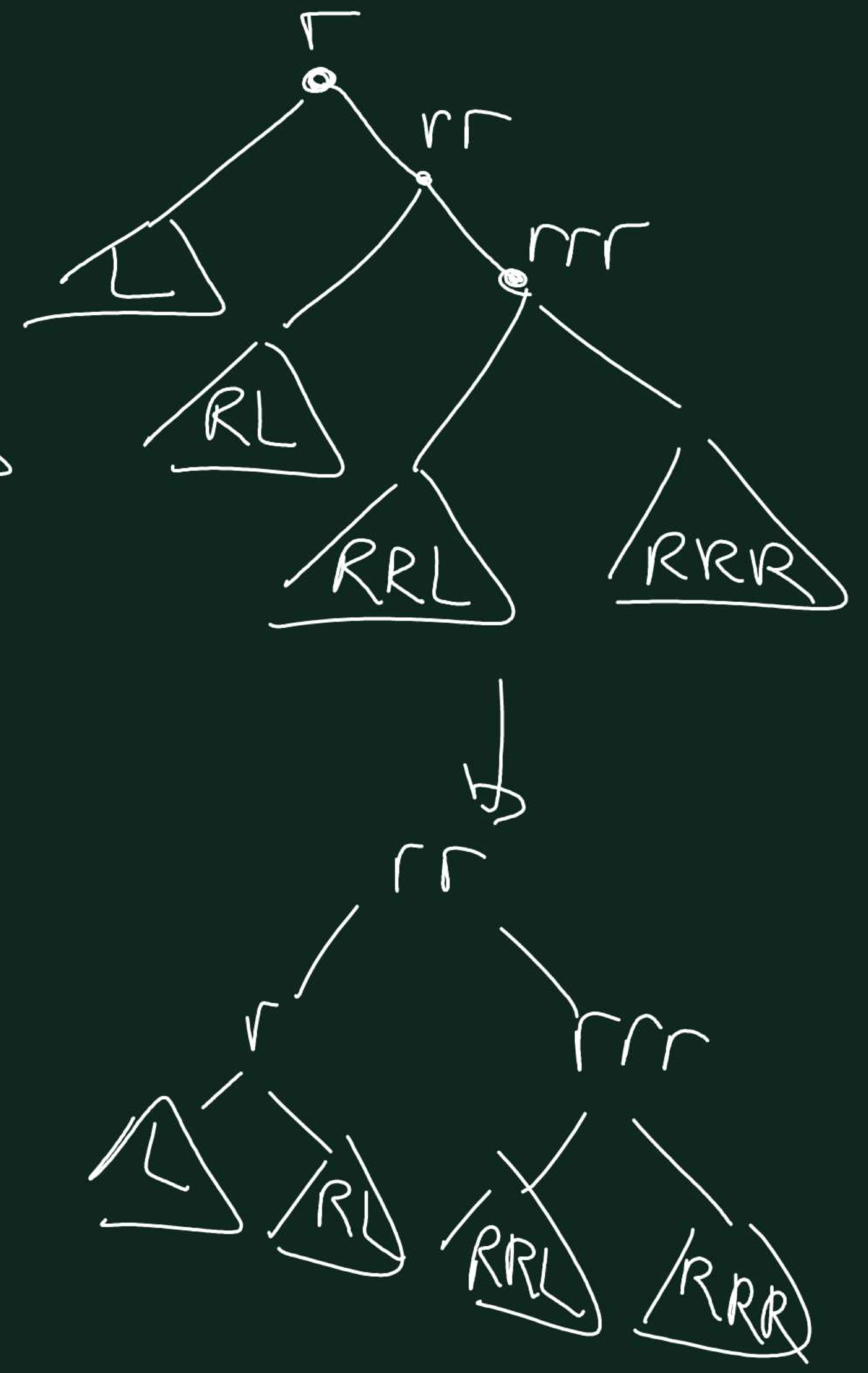
rrot(t) iff ______ left ______ left left ______

lrrot(t) iff ______ ______ right ______

rlrot(t) iff ______ right ______ left right ______ axioms

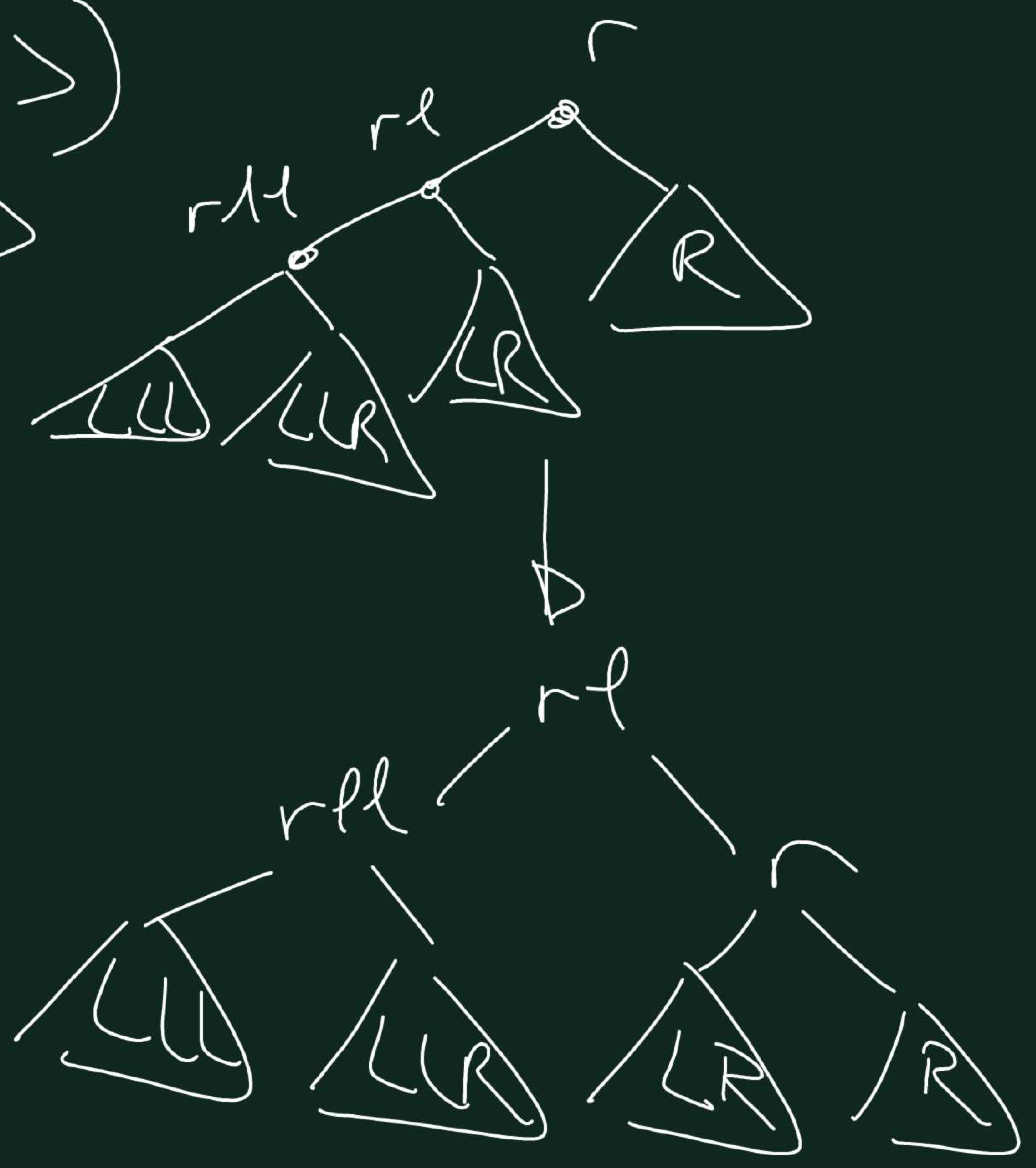
$$A_1: \mathrm{lrot}(\langle r, L, \langle rr, RL, \langle rrr, RRL, RRR\rangle\rangle\rangle)$$

$$\equiv \langle rr, \langle r, L, RL\rangle, \langle rrr, RRL, RRR\rangle\rangle$$



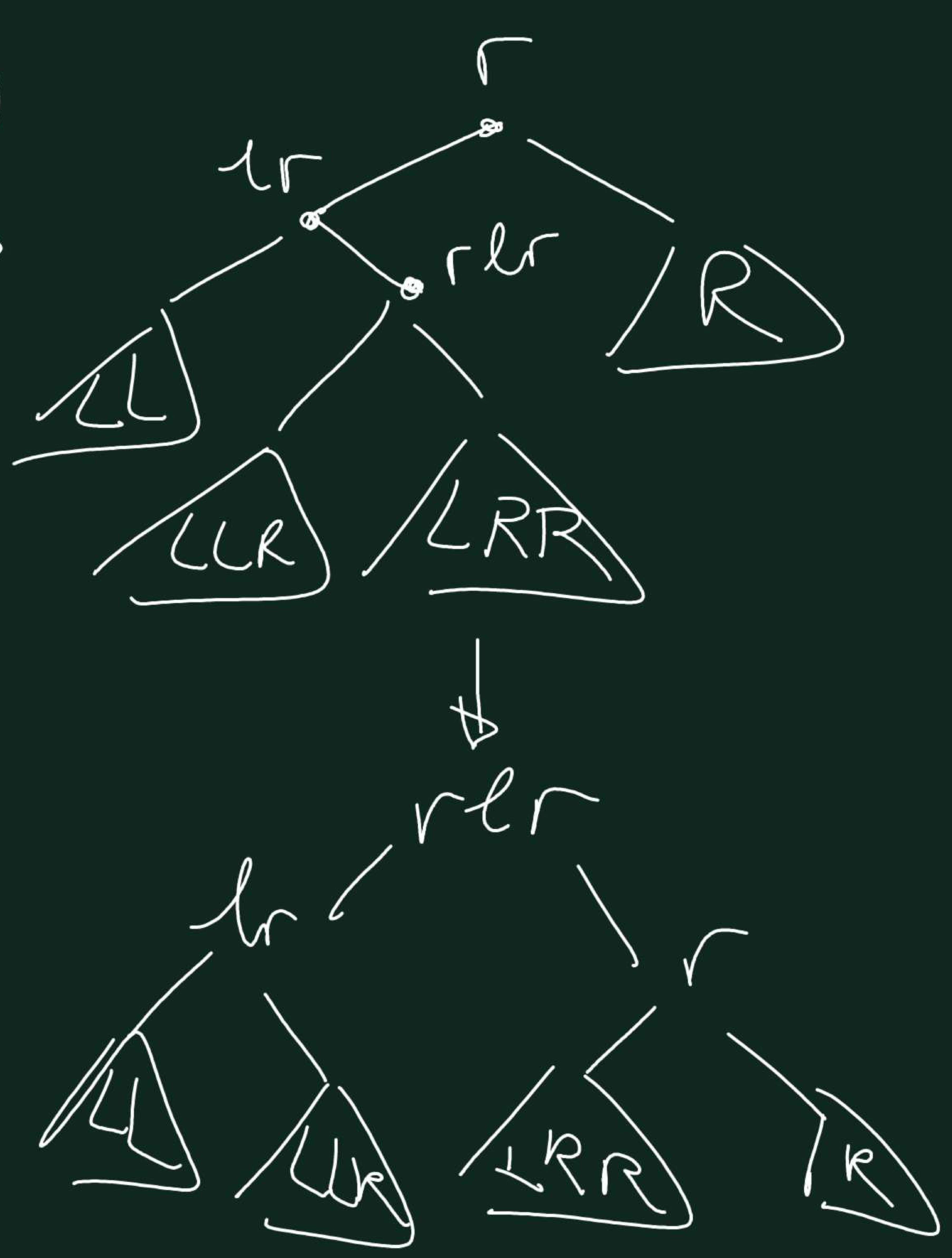
A_2 $\text{rrot}(\langle r, \langle rl, \langle rll, LLL, LLR\rangle, LR\rangle, R\rangle)$

$\equiv \langle rl, \langle rll, LLL, LLR\rangle, \langle r, LR, R\rangle\rangle$



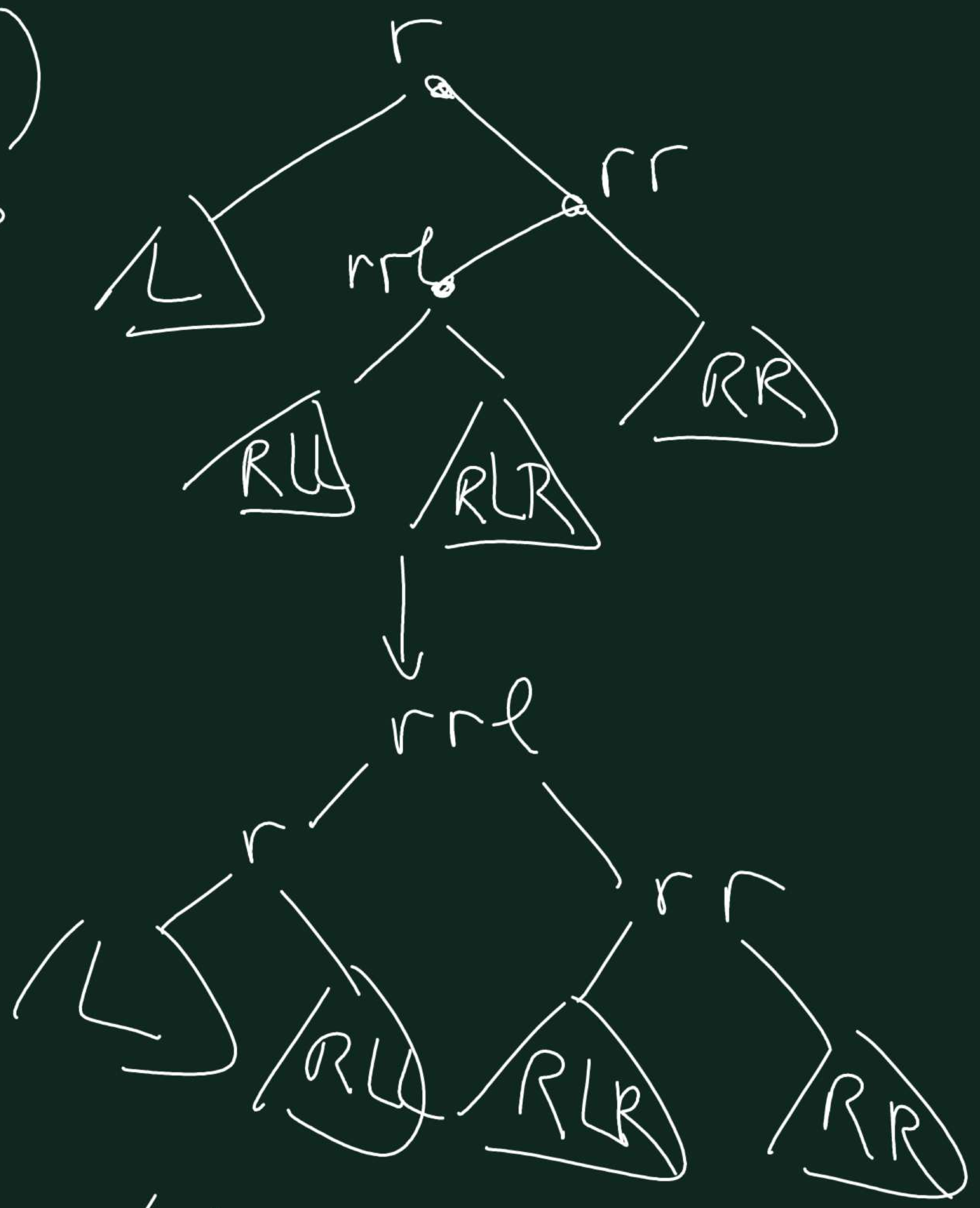
$$\mathrm{lrrot}(\langle r, \langle lr, LL, \langle rlr, LLR, LRR\rangle\rangle, R\rangle)$$

$$\equiv \langle rlr, \langle lr, LL, LLR\rangle, \langle r, LRR, R\rangle\rangle$$



$\mathrm{rlrot}(\langle r, L, \langle rr, \langle rrl, RLL, RLR\rangle, RR\rangle\rangle)$

$\equiv \langle rrl, \langle r, L, RLL\rangle, \langle rr, RLR, RR\rangle\rangle$



rotations are $\Theta(1)$

$\longrightarrow$ addition in an AVL is <u>always</u> $\Theta(\lg(\mathrm{size}))$

https://tiny.one/algo56

Complexity of rotations?

param: N (size of tree)

unit op: $<,>$