Maths for DS Assignment

https://s3-us-west-2.amazonaws.com/secure.notion-static.com/e92321bd-dc20-43e6-a362-2fc46ef3a134/StatsProbs.pdf

PART I Linear algebra (6.5 points)

a) Do you think that { $\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}-1\\0\end{pmatrix},\begin{pmatrix}0\\3\end{pmatrix}$ } is a basis of $\mathbb{R}^2$? **[0.5 point]**

b) What are the eigenvalues and eigenvectors of the matrix A = $\begin{pmatrix}3 & -2\\-2 & 3\end{pmatrix}$? **[2 points]**

c) Is the matrix A diagonalizable and why? If it is, what is a related diagonal matrix looks like? (reminder : there are several possible diagonal matrices if A is diagonalizable) **[2 points]**

d) Calculate A^3 ? **[1 point]**

e) Based on d), what is A^2 ? **[1 point]**

Part 1 - Linear Algebra

a)

Since all 3 vectors are linearly independent from each other and also none of them is multiple of one other, the collection of vectors is a basis of R^3.

b)

$Av = \lambda v$ is known as well as $Av - \lambda I v = 0$ and $|A - \lambda I| = 0$
If we start with the absolute equation;

| $\begin{bmatrix}3 & -2\\-2 & 3\end{bmatrix} - \lambda \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ | = 0

$|\begin{bmatrix}3-\lambda & -2\\-2 & 3-\lambda\end{bmatrix}| = 0$

$\lambda^2 - 6\lambda + 5 = 0$

so,

$\lambda = 5$ and $\lambda = 1$ are our possible **eigenvalues**.

1st Eigenvector if $\lambda = 5$;
$Av = \lambda v$ still must be present so,

$$\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$$

$3x - 2y = 5x$
$-2x + 3y = 5y$
$x = -y$ is our eigenvector equality, and it's vectorized phase is:

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ → 1st **Eigenvector** when $\lambda = 5$

2nd Eigenvector if $\lambda = 1$;

$$\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix}$$

$3x - 2y = x$
$-2x + 3y = y$
$x = y$ is our eigenvector equality, and it's vectorized phase is:

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ → 2nd **Eigenvector** when $\lambda = 1$

c)

A matrix is diagonalizable if there is an invertible matrix P and a diagonal matrix D, such that $A = PDP^{-1}$.

Considering our eigenvalues as $\lambda = 5$ and $\lambda = 1$, eigenvectors as $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The diagonal matrix D is composed of eigenvalues:

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

And, the eigenvectors corresponding to the eigenvalues in D compose the columns of:

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

So, our inverted matrix will look like:

$$P^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

To verify that $A = PDP^{-1}$:

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

Considering all, our matrix A is **diagonalizable**. And, the other possible diagonal matrix containing eigenvalues in different order would be $D^* = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$.

d)

The third power of our matrix can be equaled to A multiplied 3 times by itself, such as:

$A^3 = AAA$

$A^3 = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

$A^3 = \begin{bmatrix} 63 & -62 \\ -62 & 63 \end{bmatrix}$

e)

Since we're already having the 3rd power of A, A^2 can be found by multiplying the 3rd power with inverted A which is (A^{-1}).

$A^2 = A^3 A^{-1}$

$A^3 = \begin{bmatrix} 63 & -62 \\ -62 & 63 \end{bmatrix}$

$A^{-1} = inverted(A) = \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix}$

$A^2 = \begin{bmatrix} 63 & -62 \\ -62 & 63 \end{bmatrix} \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 13 & -12 \\ -12 & 13 \end{bmatrix}$

PART II Optimization (7 points)

a) Solve $max\, 3x - y$
s.t. $x^2 + y^2 = 5$ **[3.5 points]**

b) Do you think it is possible to find 2 functions f and g such as, for a specific point $(x_0, y_0) \in \mathbb{R}^2$ we have $\nabla f(x_0, y_0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\nabla g(x_0, y_0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and (x_0, y_0) solution of the optimization problem

max f(x,y)
s.t. g(x,y) = c (with a specific $c \in \mathbb{R}$) ? **[3.5 point]**

Part 2 - Optimization

a)

By the constraint that field of possible solutions lies on a disk of radius $\sqrt{5}$ which is a closed and bounded region → $-\sqrt{5} \leq x, y \leq \sqrt{5}$

The system need to be solved is;

$$3 = 2\lambda x$$
$$-1 = 2\lambda y$$
$$x^2 + y^2 = 5$$

After solving equations within each other, we will have;

$$x = 3/2\lambda$$
$$y = -1/2\lambda$$

if we place these values to $x^2 + y^2 = 5$;

$$9/4\lambda^2 + 1/4\lambda^2 = 10/4\lambda^2 = 5/2\lambda^2 = 5$$

we will have;

$$\lambda^2 = 1/2$$

now it's obvious that λ will be equal to either $1/\sqrt{2}$ or $-1/\sqrt{2}$.

if $\lambda = 1/\sqrt{2}$:

$x = 3\sqrt{2}/2$ and $y = -2/\sqrt{2}$ (by replacing the λ in initial equality)
if $\lambda = -1/\sqrt{2}$:

$x = -3\sqrt{2}/2$ and $y = 2/\sqrt{2}$ (by replacing the λ in initial equality)

Finally, the min value of the function is: $f(-3\sqrt{2}/2, 2/\sqrt{2}) = -5\sqrt{2}$
And, the **max** value of the function is: $f(3\sqrt{2}/2, -2/\sqrt{2}) = 5\sqrt{2}$

b)

The Lagrange-multiplier method is intended to locate points on the curve, surface, etc. where the gradient vector ∇f has the same direction as the gradient for the constraint function, ∇g. (Thus, one is *at least* a scalar multiple λ of the other, giving this technique its name.)

In this case, our derivates, $\binom{1}{2}$ and $\binom{2}{3}$ are always in a different direction from the specific point. The method will not produce a solution. So, it is **not possible to find** 2 functions for a specific point at this point.

PART III Probability and statistics (6.5 points)

A TV channel displays a football game and just after that, a show about mathematics.

We have the following information:
- 56% of the viewers watched the game
- one quarter of the viewers who have seen the game also have seen the math show
- 16.2% of viewers watched the show

We asked a few questions to a random viewer. We define 2 events:

- G: “the viewer watched the game”
- S: “the viewer watched the show”

a) What is the probability $P(G \cap S)$? **(1.5 points)**
b) If x is the probability that the viewer watched the show given the fact that he didn’t watch the game, show that P(S) = 0.44x + 0.14 **(2 points)**
c) Using b), find the value of x. **(1.5 points)**
d) It appears that the viewer hasn’t seen the show: what is the probability (up to 2 decimals) that he watched the game? **(1.5 points)**

Part 3 - Probability and Statistics

a)

$P(G \cap S)$ means that the possibility of the two events happened together (one after the other), mathematically, it's equal to $P(G)P(S)$.

$P(G) = \%56$

$P(S) = \%16.2$

$P(S) = 16.2/100$

$P(G \cap S) = P(G)P(S) = 907.2/10000 = \%9.072$

b)

If we consider $P(nG)$ as the possibility of the viewers who didn't watched the game;

$P(nG)$ will be equal to $1 - P(G)$ which is 0.44

also as mentioned in the second point of given details in the question, $P(G|S)$ is equal to 0.25

so, $P(G \cap G|S)$ is equal to 0.56x$0.25 = 0.14$ which is the probability of the viewers watch the game and also the viewers who have seen the game and also have seen the math show.

After having all these equations we will have the equation of;

$P(S) = P(nG \cap (S|nG) \cup (G \cap G|S))$

since $P(S|nG)$ is equal to x as given in the question. our final probability will look like:

$P(S) = P(nG \cap x \cup (G \cap G|S))$

$P(S) = 0.44x + 0.14$

c)

We already have $P(S) = 0.44x + 14$ and $P(S)$ given in the question, which makes it equal to;

$0.162 = 0.44x + 0.14$

$0.44x = 0.022$

$x = 0.05$ which makes $\%5$

d)

$P(nS)$ is considered as the probability of viewers who didn't watch the game (also known as $P(S')$).

Mathematically, the viewers who have seen the game but hasn't seen the show will be equal to $P(G|nS)$

$P(nS) = 1 - P(S) = 1 - 0.162 = 0.838$

and, $P(G) = 0.56$

so, just need to put the values in the following formula of conditional probability as of;

$P(G|nS) = P(G \cap nS)/P(nS)$

$P(G \cap nS)$ will be equal to $P(G)P(nS)$ because of having independent events in this case

$P(G|nS) = (0.56\text{x}0.838)/0.838 = 0.56 = \%56$