Maths for DS Assignment

https://s3-us-west-2.amazonaws.com/secure.notion-static.com/e92321bd-dc20-43e6-a362-2fc46ef3a134/StatsProbs.pdf

PART I Linear algebra (6.5 points)

a) Do you think that { $\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}0\\3\end{pmatrix}$ } is a basis of $\mathbb{R}^2$? **[0.5 point]**

b) What are the eigenvalues and eigenvectors of the matrix A = $\begin{pmatrix}3 & -2\\-2 & 3\end{pmatrix}$? **[2 points]**

c) Is the matrix A diagonalizable and why? If it is, what is a related diagonal matrix looks like? (reminder : there are several possible diagonal matrices if A is diagonalizable) **[2 points]**

d) Calculate A^3 ? **[1 point]**

e) Based on d), what is A^2 ? **[1 point]**

Part 1 - Linear Algebra

a)

Since all 3 vectors are linearly independent from each other and also none of them is multiple of one other, the collection of vectors is a basis of R^3.

b)

$Av = \lambda v$ is known as well as $Av - \lambda Iv = 0$ and $|A - \lambda I| = 0$
If we start with the absolute equation;

| $\begin{bmatrix}3 & -2\\-2 & 3\end{bmatrix} - \lambda \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ | = 0

$|\begin{bmatrix}3-\lambda & -2\\-2 & 3-\lambda\end{bmatrix}| = 0$

$\lambda^2 - 6\lambda + 5 = 0$

so,

$\lambda = 5$ and $\lambda = 1$ are our possible **eigenvalues**.

1st Eigenvector if $\lambda = 5$;
$Av = \lambda v$ still must be present so,

$$\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$$

$3x - 2y = 5x$
$-2x + 3y = 5y$
$x = -y$ is our eigenvector equality, and it's vectorized phase is:

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ → 1st **Eigenvector** when $\lambda = 5$

2nd Eigenvector if $\lambda = 1$;

$$\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix}$$

$3x - 2y = x$
$-2x + 3y = y$
$x = y$ is our eigenvector equality, and it's vectorized phase is:

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ → 2nd **Eigenvector** when $\lambda = 1$

c)

A matrix is diagonalizable if there is an invertible matrix P and a diagonal matrix D, such that $A = PDP^{-1}$.

Considering our eigenvalues as $\lambda = 5$ and $\lambda = 1$**,** eigenvectors as $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The diagonal matrix D is composed of eigenvalues:

$$D = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

And, the eigenvectors corresponding to the eigenvalues in D compose the columns of:

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

So, our inverted matrix will look like:

$$P^{-1} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

To verify that $A = PDP^{-1}$:

$$A = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

Considering all, our matrix A is **diagonalizable**. And, the other possible diagonal matrix containing eigenvalues in different order would be $D^* = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$.

d)

The third power of our matrix can be equaled to A multiplied 3 times by itself, such as:

$A^3 = AAA$

$A^3 = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

$A^3 = \begin{bmatrix} 63 & -62 \\ -62 & 63 \end{bmatrix}$

e)

Since we're already having the 3rd power of A, A^2 can be found by multiplying the 3rd power with inverted A which is (A^{-1}).

$A^2 = A^3 A^{-1}$

$A^3 = \begin{bmatrix} 63 & -62 \\ -62 & 63 \end{bmatrix}$

$A^{-1} = inverted(A) = \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix}$

$A^2 = \begin{bmatrix} 63 & -62 \\ -62 & 63 \end{bmatrix} \begin{bmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 13 & -12 \\ -12 & 13 \end{bmatrix}$

PART II Optimization (7 points)

a) Solve $max\, 3x - y$
s.t. $x^2 + y^2 = 5$ **[3.5 points]**

b) Do you think it is possible to find 2 functions f and g such as, for a specific point $(x_0, y_0) \in \mathbb{R}^2$ we have $\nabla f(x_0, y_0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\nabla g(x_0, y_0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and (x_0, y_0) solution of the optimization problem

max f(x,y)
s.t. g(x,y) = c (with a specific $c \in \mathbb{R}$) ? **[3.5 point]**

Part 2 - Optimization

a)

By the constraint that field of possible solutions lies on a disk of radius $\sqrt{5}$ which is a closed and bounded region → $-\sqrt{5} \leq x, y \leq \sqrt{5}$

The system need to be solved is;

$$3 = 2\lambda x$$
$$-1 = 2\lambda y$$
$$x^2 + y^2 = 5$$

After solving equations within each other, we will have;

$$x = 3/2\lambda$$
$$y = -1/2\lambda$$

if we place these values to $x^2 + y^2 = 5$;

$$9/4\lambda^2 + 1/4\lambda^2 = 10/4\lambda^2 = 5/2\lambda^2 = 5$$

we will have;

$$\lambda^2 = 1/2$$

now it's obvious that λ will be equal to either $1/\sqrt{2}$ or $-1/\sqrt{2}$.

if $\lambda = 1/\sqrt{2}$:

$x = 3\sqrt{2}/2$ and $y = -2/\sqrt{2}$ (by replacing the λ in initial equality)
if $\lambda = -1/\sqrt{2}$:

$x = -3\sqrt{2}/2$ and $y = 2/\sqrt{2}$ (by replacing the λ in initial equality)

Finally, the min value of the function is: $f(-3\sqrt{2}/2, 2/\sqrt{2}) = -5\sqrt{2}$
And, the **max** value of the function is: $f(3\sqrt{2}/2, -2/\sqrt{2}) = 5\sqrt{2}$

b)

The Lagrange-multiplier method is intended to locate points on the curve, surface, etc. where the gradient vector ∇f has the same direction as the gradient for the constraint function, ∇g. (Thus, one is *at least* a scalar multiple λ of the other, giving this technique its name.)

In this case, our derivates, $\binom{1}{2}$ and $\binom{2}{3}$ are always in a different direction from the specific point. The method will not produce a solution. So, it is **not possible to find** 2 functions for a specific point at this point.

PART III Probability and statistics (6.5 points)

A TV channel displays a football game and just after that, a show about mathematics.

We have the following information:
- 56% of the viewers watched the game
- one quarter of the viewers who have seen the game also have seen the math show
- 16.2% of viewers watched the show

We asked a few questions to a random viewer. We define 2 events:

- G: "the viewer watched the game"
- S: "the viewer watched the show"

a) What is the probability $P(G \cap S)$? **(1.5 points)**
b) If x is the probability that the viewer watched the show given the fact that he didn't watch the game, show that P(S) = 0.44x + 0.14 **(2 points)**
c) Using b), find the value of x. **(1.5 points)**
d) It appears that the viewer hasn't seen the show: what is the probability (up to 2 decimals) that he watched the game? **(1.5 points)**

Part 3 - Probability and Statistics

a)

$P(G \cap S)$ means that the possibility of the two conditionally independent events happened one after the other, mathematically, it's equal to $P(S \cap G) = P(S|G)P(G)$.

$P(S|G) = 0.25$ (Which is given in the second point of the informations)

$P(G) = 0.56$

$P(G \cap S) = P(S \cap G) = P(S|G)P(G) = 0.25x0.56 = 0.14 = \%14$

b)

If we consider $P(G')$ as the possibility of the viewers who didn't watched the game;

$P(G')$ will be equal to $1 - P(G)$ which is 0.44

also as mentioned in the second point of given details in the question, $P(S|G)$ is equal to 0.25

$P(S \cap G) = P(G \cap S) = 0.14$

$P(S) = 0.162 \rightarrow P(S') = 0838$

After having all these infos, we can write the equation as following;

$P(S|G') = P(S \cap G')/P(G')$

$P(S|G').P(G') = P(S \cap G')$

$P(S|G').P(G') = P(S) - P(S \cap G)$

since $P(S|nG)$ is equal to x as given in the question. our final probability will look like:

$x(0.44) = P(S) - 0.14$

$P(S) = 0.44x + 0.14$

c)

We already have $P(S) = 0.44x + 14$ and $P(S)$ given in the question, which makes it equal to;

$0.162 = 0.44x + 0.14$

$0.44x = 0.022$

$x = 0.05$ which makes $\%5$

d)

$P(S')$ is considered as the probability of viewers who didn't watch the game.

Mathematically, the viewers who have seen the game but hasn't seen the show will be equal to $P(G|S')$

$P(G|S') = P(G \cap S')/P(S')$

$P(G|S') = (P(G) - P(S \cap G))/P(S')$

and,

$P(G) = 0.56$

$P(S') = 0.838$

$P(S \cap G) = 0.14$

so, just need to put the values in the following formula of conditional probability as of;

$P(G|S') = (0.56 - 0.14)/0.838 \approx 0.50$

**This file can also be found @

https://github.com/ediziks/EPITA-DSA-Notes/blob/main/MathematicsforDataScience%5BMfDS%5D/Assignments/Maths_for_DS_Assignment.pdf