

11.3: The Dot Product

Learning Objectives

- Calculate the dot product of two given vectors.
- Determine whether two given vectors are perpendicular.
- Find the direction cosines of a given vector.
- Explain what is meant by the vector projection of one vector onto another vector, and describe how to compute it.
- Calculate the work done by a given force.

If we apply a force to an object so that the object moves, we say that work is done by the force. Previously, we looked at a constant force and we assumed the force was applied in the direction of motion of the object. Under those conditions, work can be expressed as the product of the force acting on an object and the distance the object moves. In this chapter, however, we have seen that both force and the motion of an object can be represented by vectors.

In this section, we develop an operation called the dot product, which allows us to calculate work in the case when the force vector and the motion vector have different directions. The dot product essentially tells us how much of the force vector is applied in the direction of the motion vector. The dot product can also help us measure the angle formed by a pair of vectors and the position of a vector relative to the coordinate axes. It even provides a simple test to determine whether two vectors meet at a right angle.

The Dot Product and Its Properties

We have already learned how to add and subtract vectors. In this chapter, we investigate two types of vector multiplication. The first type of vector multiplication is called the dot product, based on the notation we use for it, and it is defined as follows:

Definition: dot product

The *dot product* of vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \quad (11.3.1)$$

Note that if u and v are two-dimensional vectors, we calculate the dot product in a similar fashion. Thus, if $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$, then

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2. \quad (11.3.2)$$

When two vectors are combined under addition or subtraction, the result is a vector. When two vectors are combined using the dot product, the result is a scalar. For this reason, the dot product is often called the *scalar product*. It may also be called the *inner product*.

Example 11.3.1: Calculating Dot Products

- Find the dot product of $\vec{u} = \langle 3, 5, 2 \rangle$ and $\vec{v} = \langle -1, 3, 0 \rangle$.
- Find the scalar product of $\vec{p} = 10\hat{i} - 4\hat{j} + 7\hat{k}$ and $\vec{q} = -2\hat{i} + \hat{j} + 6\hat{k}$.

Solution:

- Substitute the vector components into the formula for the dot product:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= 3(-1) + 5(3) + 2(0) \\ &= -3 + 15 + 0 \\ &= 12. \end{aligned}$$

b. The calculation is the same if the vectors are written using standard unit vectors. We still have three components for each vector to substitute into the formula for the dot product:

$$\begin{aligned}\vec{p} \cdot \vec{q} &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= 10(-2) + (-4)(1) + (7)(6) \\ &= -20 - 4 + 42 \\ &= 18.\end{aligned}$$

Exercise 11.3.1

Find $\vec{u} \cdot \vec{v}$, where $\vec{u} = \langle 2, 9, -1 \rangle$ and $\vec{v} = \langle -3, 1, -4 \rangle$.

Like vector addition and subtraction, the dot product has several algebraic properties. We prove three of these properties and leave the rest as exercises.

Properties of the Dot Product

Let \vec{u} , \vec{v} , and \vec{w} be vectors, and let c be a scalar.

i. *Commutative property*

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \quad (11.3.3)$$

ii. *Distributive property*

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \quad (11.3.4)$$

iii. *Associative property*

$$c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v}) \quad (11.3.5)$$

iv. *Property of magnitude*

$$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2 \quad (11.3.6)$$

Proof

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\begin{aligned}\vec{u} \cdot \vec{v} &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= v_1 u_1 + v_2 u_2 + v_3 u_3 \\ &= \langle v_1, v_2, v_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= \vec{v} \cdot \vec{u}.\end{aligned}$$

The associative property looks like the associative property for real-number multiplication, but pay close attention to the difference between scalar and vector objects:

$$\begin{aligned}c(\vec{u} \cdot \vec{v}) &= c(u_1 v_1 + u_2 v_2 + u_3 v_3) \\ &= c(u_1 v_1) + c(u_2 v_2) + c(u_3 v_3) \\ &= (cu_1)v_1 + (cu_2)v_2 + (cu_3)v_3 \\ &= \langle cu_1, cu_2, cu_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= c\langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (c\vec{u}) \cdot \vec{v}.\end{aligned}$$

The proof that $c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$ is similar.

The fourth property shows the relationship between the magnitude of a vector and its dot product with itself:

$$\begin{aligned}\vec{v} \cdot \vec{v} &= \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (v_1)^2 + (v_2)^2 + (v_3)^2 \\ &= \left[\sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2} \right]^2 \\ &= \|\vec{v}\|^2.\end{aligned}$$

□

Note that by property iv. we have $\vec{0} \cdot \vec{v} = 0$. Also by property iv. if $\vec{v} \cdot \vec{v} = 0$, then $\vec{v} = \vec{0}$.

Example 11.3.2: Using Properties of the Dot Product

Let $\vec{a} = \langle 1, 2, -3 \rangle$, $\vec{b} = \langle 0, 2, 4 \rangle$, and $\vec{c} = \langle 5, -1, 3 \rangle$.

Find each of the following products.

- $(\vec{a} \cdot \vec{b})\vec{c}$
- $\vec{a} \cdot (2\vec{c})$
- $\|\vec{b}\|^2$

Solution

a. Note that this expression asks for the scalar multiple of \vec{c} by $\vec{a} \cdot \vec{b}$:

$$\begin{aligned}(\vec{a} \cdot \vec{b})\vec{c} &= (\langle 1, 2, -3 \rangle \cdot \langle 0, 2, 4 \rangle)\langle 5, -1, 3 \rangle \\ &= (1(0) + 2(2) + (-3)(4))\langle 5, -1, 3 \rangle \\ &= -8\langle 5, -1, 3 \rangle \\ &= \langle -40, 8, -24 \rangle.\end{aligned}$$

b. This expression is a dot product of vector \vec{a} and scalar multiple $2\vec{c}$:

$$\begin{aligned}\vec{a} \cdot (2\vec{c}) &= 2(\vec{a} \cdot \vec{c}) \\ &= 2(\langle 1, 2, -3 \rangle \cdot \langle 5, -1, 3 \rangle) \\ &= 2(1(5) + 2(-1) + (-3)(3)) \\ &= 2(-6) = -12.\end{aligned}$$

c. Simplifying this expression is a straightforward application of the dot product:

$$\begin{aligned}\|\vec{b}\|^2 &= \vec{b} \cdot \vec{b} \\ &= \langle 0, 2, 4 \rangle \cdot \langle 0, 2, 4 \rangle \\ &= 0^2 + 2^2 + 4^2 \\ &= 0 + 4 + 16 \\ &= 20.\end{aligned}$$

Exercise 11.3.2

Find the following products for $\vec{p} = \langle 7, 0, 2 \rangle$, $\vec{q} = \langle -2, 2, -2 \rangle$, and $\vec{r} = \langle 0, 2, -3 \rangle$.

- $(\vec{r} \cdot \vec{p})\vec{q}$

b. $\|\vec{p}\|^2$

Using the Dot Product to Find the Angle between Two Vectors

When two nonzero vectors are placed in standard position, whether in two dimensions or three dimensions, they form an angle between them (Figure 11.3.1). The dot product provides a way to find the measure of this angle. This property is a result of the fact that we can express the dot product in terms of the cosine of the angle formed by two vectors.

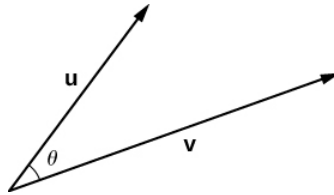


Figure 11.3.1: Let θ be the angle between two nonzero vectors \vec{u} and \vec{v} such that $0 \leq \theta \leq \pi$.

Evaluating a Dot Product

The dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle between them:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta. \quad (11.3.7)$$

Proof

Place vectors \vec{u} and \vec{v} in standard position and consider the vector $\vec{v} - \vec{u}$ (Figure 11.3.2). These three vectors form a triangle with side lengths $\|\vec{u}\|$, $\|\vec{v}\|$, and $\|\vec{v} - \vec{u}\|$.

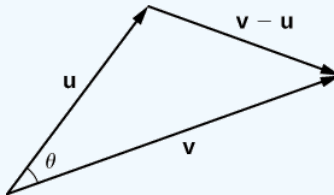


Figure 11.3.2: The lengths of the sides of the triangle are given by the magnitudes of the vectors that form the triangle.

Recall from trigonometry that the law of cosines describes the relationship among the side lengths of the triangle and the angle θ . Applying the law of cosines here gives

$$\|\vec{v} - \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \|\vec{v}\| \cos \theta. \quad (11.3.8)$$

The dot product provides a way to rewrite the left side of Equation 11.3.8

$$\begin{aligned} \|\vec{v} - \vec{u}\|^2 &= (\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u}) \\ &= (\vec{v} - \vec{u}) \cdot \vec{v} - (\vec{v} - \vec{u}) \cdot \vec{u} \\ &= \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{u} \cdot \vec{u} \\ &= \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{u} \\ &= \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{u}\|^2. \end{aligned}$$

Substituting into the law of cosines yields

$$\begin{aligned} \|\vec{v} - \vec{u}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \|\vec{v}\| \cos \theta \\ \|\vec{v}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{u}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\| \|\vec{v}\| \cos \theta \\ -2\vec{u} \cdot \vec{v} &= -2\|\vec{u}\| \|\vec{v}\| \cos \theta \\ \vec{u} \cdot \vec{v} &= \|\vec{u}\| \|\vec{v}\| \cos \theta. \end{aligned}$$

□

We can use the form of the dot product in Equation 11.3.7 to find the measure of the angle between two nonzero vectors by rearranging Equation 11.3.7 to solve for the cosine of the angle:

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}. \quad (11.3.9)$$

Using this equation, we can find the cosine of the angle between two nonzero vectors. Since we are considering the smallest angle between the vectors, we assume $0^\circ \leq \theta \leq 180^\circ$ (or $0 \leq \theta \leq \pi$ if we are working in radians). The inverse cosine is unique over this range, so we are then able to determine the measure of the angle θ .

Example 11.3.3: Finding the Angle between Two Vectors

Find the measure of the angle between each pair of vectors.

- $\hat{i} + \hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} - 3\hat{k}$
- $\langle 2, 5, 6 \rangle$ and $\langle -2, -4, 4 \rangle$

Solution

- To find the cosine of the angle formed by the two vectors, substitute the components of the vectors into Equation 11.3.9:

$$\begin{aligned} \cos \theta &= \frac{(\hat{i} + \hat{j} + \hat{k}) \cdot (2\hat{i} - \hat{j} - 3\hat{k})}{\|\hat{i} + \hat{j} + \hat{k}\| \cdot \|2\hat{i} - \hat{j} - 3\hat{k}\|} \\ &= \frac{1(2) + (1)(-1) + (1)(-3)}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{2^2 + (-1)^2 + (-3)^2}} \\ &= \frac{-2}{\sqrt{3}\sqrt{14}} = \frac{-2}{\sqrt{42}}. \end{aligned}$$

Therefore, $\theta = \arccos \frac{-2}{\sqrt{42}}$ rad.

- Start by finding the value of the cosine of the angle between the vectors:

$$\begin{aligned} \cos \theta &= \frac{\langle 2, 5, 6 \rangle \cdot \langle -2, -4, 4 \rangle}{\|\langle 2, 5, 6 \rangle\| \cdot \|\langle -2, -4, 4 \rangle\|} \\ &= \frac{2(-2) + (5)(-4) + (6)(4)}{\sqrt{2^2 + 5^2 + 6^2} \sqrt{(-2)^2 + (-4)^2 + 4^2}} \\ &= \frac{0}{\sqrt{65}\sqrt{36}} = 0. \end{aligned}$$

Now, $\cos \theta = 0$ and $0 \leq \theta \leq \pi$, so $\theta = \pi/2$.

Exercise 11.3.3

Find the measure of the angle, in radians, formed by vectors $\vec{a} = \langle 1, 2, 0 \rangle$ and $\vec{b} = \langle 2, 4, 1 \rangle$. Round to the nearest hundredth.

The angle between two vectors can be acute ($0 < \cos \theta < 1$), obtuse ($-1 < \cos \theta < 0$), or straight ($\cos \theta = -1$). If $\cos \theta = 1$, then both vectors have the same direction. If $\cos \theta = 0$, then the vectors, when placed in standard position, form a right angle (Figure 11.3.3). We can formalize this result into a theorem regarding orthogonal (perpendicular) vectors.

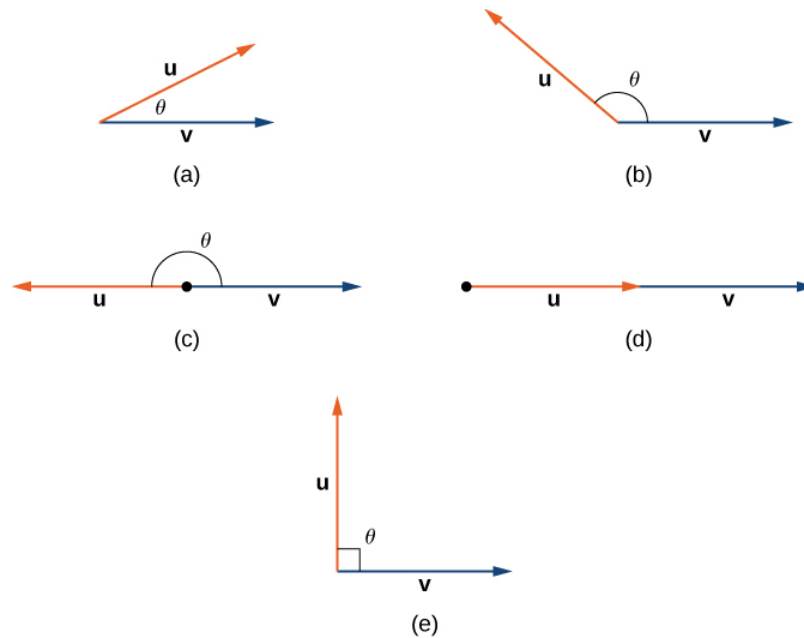


Figure 11.3.3: (a) An acute angle has $0 < \cos \theta < 1$. (b) An obtuse angle has $-1 < \cos \theta < 0$. (c) A straight line has $\cos \theta = -1$. (d) If the vectors have the same direction, $\cos \theta = 1$. (e) If the vectors are orthogonal (perpendicular), $\cos \theta = 0$.

Orthogonal Vectors

The nonzero vectors \vec{u} and \vec{v} are **orthogonal vectors** if and only if $\vec{u} \cdot \vec{v} = 0$.

Proof

Let \vec{u} and \vec{v} be nonzero vectors, and let θ denote the angle between them. First, assume $\vec{u} \cdot \vec{v} = 0$. Then

$$\|\vec{u}\| \|\vec{v}\| \cos \theta = 0. \quad (11.3.10)$$

However, $\|\vec{u}\| \neq 0$ and $\|\vec{v}\| \neq 0$, so we must have $\cos \theta = 0$. Hence, $\theta = 90^\circ$, and the vectors are orthogonal.

Now assume \vec{u} and \vec{v} are orthogonal. Then $\theta = 90^\circ$ and we have

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \|\vec{u}\| \|\vec{v}\| \cos \theta \\ &= \|\vec{u}\| \|\vec{v}\| \cos 90^\circ \\ &= \|\vec{u}\| \|\vec{v}\| (0) \\ &= 0. \end{aligned}$$

□

The terms **orthogonal**, **perpendicular**, and **normal** each indicate that mathematical objects are intersecting at right angles. The use of each term is determined mainly by its context. We say that vectors are orthogonal and lines are perpendicular. The term *normal* is used most often when measuring the angle made with a plane or other surface.

Example 11.3.4: Identifying Orthogonal Vectors

Determine whether $\vec{p} = \langle 1, 0, 5 \rangle$ and $\vec{q} = \langle 10, 3, -2 \rangle$ are orthogonal vectors.

Solution

Using the definition, we need only check the dot product of the vectors:

$$\vec{p} \cdot \vec{q} = 1(10) + (0)(3) + (5)(-2) = 10 + 0 - 10 = 0.$$

Because $\vec{p} \cdot \vec{q} = 0$, the vectors are orthogonal (Figure 11.3.4).

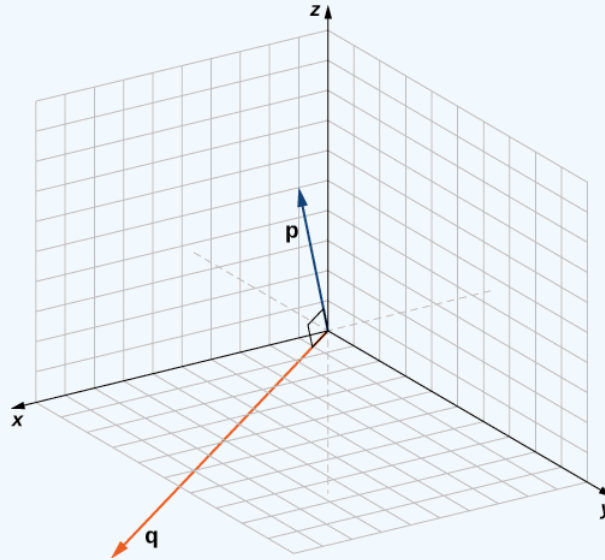


Figure 11.3.4: Vectors \vec{p} and \vec{q} form a right angle when their initial points are aligned.

Exercise 11.3.4

For which value of x is $\vec{p} = \langle 2, 8, -1 \rangle$ orthogonal to $\vec{q} = \langle x, -1, 2 \rangle$?

Example 11.3.5: Measuring the Angle Formed by Two Vectors

Let $\vec{v} = \langle 2, 3, 3 \rangle$. Find the measures of the angles formed by the following vectors.

- \vec{v} and \hat{i}
- \vec{v} and \hat{j}
- \vec{v} and \hat{k}

Solution

a. Let α be the angle formed by \vec{v} and \hat{i} :

$$\cos \alpha = \frac{\vec{v} \cdot \hat{i}}{\|\vec{v}\| \cdot \|\hat{i}\|} = \frac{\langle 2, 3, 3 \rangle \cdot \langle 1, 0, 0 \rangle}{\sqrt{2^2 + 3^2 + 3^2} \sqrt{1}} = \frac{2}{\sqrt{22}}$$

$$\alpha = \arccos \frac{2}{\sqrt{22}} \approx 1.130 \text{ rad.}$$

b. Let β represent the angle formed by \vec{v} and \hat{j} :

$$\cos \beta = \frac{\vec{v} \cdot \hat{j}}{\|\vec{v}\| \cdot \|\hat{j}\|} = \frac{\langle 2, 3, 3 \rangle \cdot \langle 0, 1, 0 \rangle}{\sqrt{2^2 + 3^2 + 3^2} \sqrt{1}} = \frac{3}{\sqrt{22}}$$

$$\beta = \arccos \frac{3}{\sqrt{22}} \approx 0.877 \text{ rad.}$$

c. Let γ represent the angle formed by \vec{v} and \hat{k} :

$$\cos \gamma = \frac{\vec{v} \cdot \hat{k}}{\|\vec{v}\| \cdot \|\hat{k}\|} = \frac{\langle 2, 3, 3 \rangle \cdot \langle 0, 0, 1 \rangle}{\sqrt{2^2 + 3^2 + 3^2} \sqrt{1}} = \frac{3}{\sqrt{22}}$$

$$\gamma = \arccos \frac{3}{\sqrt{22}} \approx 0.877 \text{ rad.}$$

Exercise 11.3.5

Let $\vec{v} = \langle 3, -5, 1 \rangle$. Find the measure of the angles formed by each pair of vectors.

- \vec{v} and \hat{i}
- \vec{v} and \hat{j}
- \vec{v} and \hat{k}

The angle a vector makes with each of the coordinate axes, called a direction angle, is very important in practical computations, especially in a field such as engineering. For example, in astronautical engineering, the angle at which a rocket is launched must be determined very precisely. A very small error in the angle can lead to the rocket going hundreds of miles off course. Direction angles are often calculated by using the dot product and the cosines of the angles, called the direction cosines. Therefore, we define both these angles and their cosines.

Definition: direction angles

The angles formed by a nonzero vector and the coordinate axes are called the **direction angles** for the vector (Figure 11.3.5). The cosines for these angles are called the **direction cosines**.

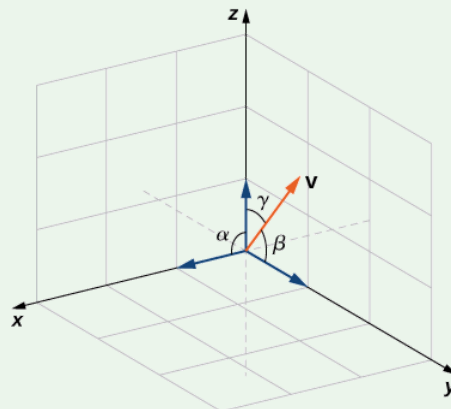


Figure 11.3.5: Angle α is formed by vector \vec{v} and unit vector \hat{i} . Angle β is formed by vector \vec{v} and unit vector \hat{j} . Angle γ is formed by vector \vec{v} and unit vector \hat{k} .

In Example, the direction cosines of $\vec{v} = \langle 2, 3, 3 \rangle$ are $\cos \alpha = \frac{2}{\sqrt{22}}$, $\cos \beta = \frac{3}{\sqrt{22}}$, and $\cos \gamma = \frac{3}{\sqrt{22}}$. The direction angles of \vec{v} are $\alpha = 1.130$ rad, $\beta = 0.877$ rad, and $\gamma = 0.877$ rad.

So far, we have focused mainly on vectors related to force, movement, and position in three-dimensional physical space. However, vectors are often used in more abstract ways. For example, suppose a fruit vendor sells apples, bananas, and oranges. On a given day, he sells 30 apples, 12 bananas, and 18 oranges. He might use a quantity vector, $\vec{q} = \langle 30, 12, 18 \rangle$, to represent the quantity of fruit he sold that day. Similarly, he might want to use a price vector, $\vec{p} = \langle 0.50, 0.25, 1 \rangle$, to indicate that he sells his apples for 50¢ each, bananas for 25¢ each, and oranges for \$1 apiece. In this example, although we could still graph these vectors, we do not interpret them as literal representations of position in the physical world. We are simply using vectors to keep track of particular pieces of information about apples, bananas, and oranges.

This idea might seem a little strange, but if we simply regard vectors as a way to order and store data, we find they can be quite a powerful tool. Going back to the fruit vendor, let's think about the dot product, $\vec{q} \cdot \vec{p}$. We compute it by multiplying the number of apples sold (30) by the price per apple (50¢), the number of bananas sold by the price per banana, and the number of oranges sold by the price per orange. We then add all these values together. So, in this example, the dot product tells us how much money the fruit vendor had in sales on that particular day.

When we use vectors in this more general way, there is no reason to limit the number of components to three. What if the fruit vendor decides to start selling grapefruit? In that case, he would want to use four-dimensional quantity and price vectors to represent the number of apples, bananas, oranges, and grapefruit sold, and their unit prices. As you might expect, to calculate

the dot product of four-dimensional vectors, we simply add the products of the components as before, but the sum has four terms instead of three.

Example 11.3.6: Using Vectors in an Economic Context

AAA Party Supply Store sells invitations, party favors, decorations, and food service items such as paper plates and napkins. When AAA buys its inventory, it pays 25¢ per package for invitations and party favors. Decorations cost AAA 50¢ each, and food service items cost 20¢ per package. AAA sells invitations for \$2.50 per package and party favors for \$1.50 per package. Decorations sell for \$4.50 each and food service items for \$1.25 per package.

During the month of May, AAA Party Supply Store sells 1258 invitations, 342 party favors, 2426 decorations, and 1354 food service items. Use vectors and dot products to calculate how much money AAA made in sales during the month of May. How much did the store make in profit?

Solution

The cost, price, and quantity vectors are

$$\vec{c} = \langle 0.25, 0.25, 0.50, 0.20 \rangle$$

$$\vec{p} = \langle 2.50, 1.50, 4.50, 1.25 \rangle$$

$$\vec{q} = \langle 1258, 342, 2426, 1354 \rangle.$$

AAA sales for the month of May can be calculated using the dot product $\vec{p} \cdot \vec{q}$. We have

$$\begin{aligned}\vec{p} \cdot \vec{q} &= \langle 2.50, 1.50, 4.50, 1.25 \rangle \cdot \langle 1258, 342, 2426, 1354 \rangle \\ &= 3145 + 513 + 10917 + 1692.5 \\ &= 16267.5.\end{aligned}$$

So, AAA took in \$16,267.50 during the month of May. To calculate the profit, we must first calculate how much AAA paid for the items sold. We use the dot product $\vec{c} \cdot \vec{q}$ to get

$$\begin{aligned}\vec{c} \cdot \vec{q} &= \langle 0.25, 0.25, 0.50, 0.20 \rangle \cdot \langle 1258, 342, 2426, 1354 \rangle \\ &= 314.5 + 85.5 + 1213 + 270.8 \\ &= 1883.8.\end{aligned}$$

So, AAA paid \$1,883.30 for the items they sold. Their profit, then, is given by

$$\vec{p} \cdot \vec{q} - \vec{c} \cdot \vec{q} = 16267.5 - 1883.8 = 14383.7.$$

Therefore, AAA Party Supply Store made \$14,383.70 in May.

Exercise 11.3.6

On June 1, AAA Party Supply Store decided to increase the price they charge for party favors to \$2 per package. They also changed suppliers for their invitations, and are now able to purchase invitations for only 10¢ per package. All their other costs and prices remain the same. If AAA sells 1408 invitations, 147 party favors, 2112 decorations, and 1894 food service items in the month of June, use vectors and dot products to calculate their total sales and profit for June.

Projections

As we have seen, addition combines two vectors to create a resultant vector. But what if we are given a vector and we need to find its component parts? We use vector projections to perform the opposite process; they can break down a vector into its components. The magnitude of a vector projection is a scalar projection. For example, if a child is pulling the handle of a wagon at a 55° angle, we can use projections to determine how much of the force on the handle is actually moving the wagon forward (11.3.6). We return to this example and learn how to solve it after we see how to calculate projections.

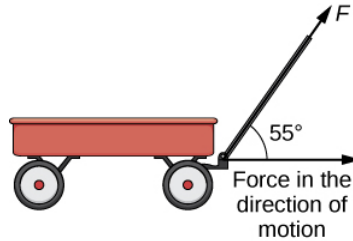


Figure 11.3.6: When a child pulls a wagon, only the horizontal component of the force propels the wagon forward.

Definition: Vector and Projection

The **vector projection** of \vec{v} onto \vec{u} is the vector labeled $\text{proj}_{\vec{u}} \vec{v}$ in Figure 11.3.7. It has the same initial point as \vec{u} and \vec{v} and the same direction as \vec{u} , and represents the component of \vec{v} that acts in the direction of \vec{u} . If θ represents the angle between \vec{u} and \vec{v} , then, by properties of triangles, we know the length of $\text{proj}_{\vec{u}} \vec{v}$ is $\|\text{proj}_{\vec{u}} \vec{v}\| = \|\vec{v}\| \cos \theta$. When expressing $\cos \theta$ in terms of the dot product, this becomes

$$\|\text{proj}_{\vec{u}} \vec{v}\| = \|\vec{v}\| \cos \theta = \|\vec{v}\| \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \right) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}. \quad (11.3.11)$$

We now multiply by a unit vector in the direction of \vec{u} to get $\text{proj}_{\vec{u}} \vec{v}$:

$$\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|} \left(\frac{1}{\|\vec{u}\|} \vec{u} \right) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}. \quad (11.3.12)$$

The length of this vector is also known as the **scalar projection** of \vec{v} onto \vec{u} and is denoted by

$$\|\text{proj}_{\vec{u}} \vec{v}\| = \text{comp}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}. \quad (11.3.13)$$

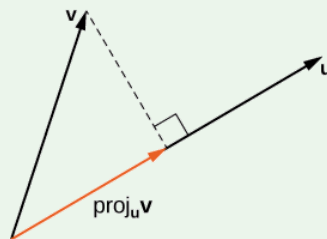


Figure 11.3.7: The projection of \vec{v} onto \vec{u} shows the component of vector \vec{v} in the direction of \vec{u} .

Example 11.3.7: Finding Projections

Find the projection of \vec{v} onto \vec{u} .

- $\vec{v} = \langle 3, 5, 1 \rangle$ and $\vec{u} = \langle -1, 4, 3 \rangle$
- $\vec{v} = 3\hat{i} - 2\hat{j}$ and $\vec{u} = \hat{i} + 6\hat{j}$

Solution

- Substitute the components of \vec{v} and \vec{u} into the formula for the projection:

$$\begin{aligned}
 \text{proj}_{\vec{u}} \vec{v} &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \\
 &= \frac{\langle -1, 4, 3 \rangle \cdot \langle 3, 5, 1 \rangle}{\|\langle -1, 4, 3 \rangle\|^2} \langle -1, 4, 3 \rangle \\
 &= \frac{-3 + 20 + 3}{(-1)^2 + 4^2 + 3^2} \langle -1, 4, 3 \rangle \\
 &= \frac{20}{26} \langle -1, 4, 3 \rangle \\
 &= \left\langle -\frac{10}{13}, \frac{40}{13}, \frac{30}{13} \right\rangle.
 \end{aligned}$$

b. To find the two-dimensional projection, simply adapt the formula to the two-dimensional case:

$$\begin{aligned}
 \text{proj}_{\vec{u}} \vec{v} &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \\
 &= \frac{(\hat{i} + 6\hat{j}) \cdot (3\hat{i} - 2\hat{j})}{\|\hat{i} + 6\hat{j}\|^2} (\hat{i} + 6\hat{j}) \\
 &= \frac{1(3) + 6(-2)}{1^2 + 6^2} (\hat{i} + 6\hat{j}) \\
 &= -\frac{9}{37} (\hat{i} + 6\hat{j}) \\
 &= -\frac{9}{37} \hat{i} - \frac{54}{37} \hat{j}.
 \end{aligned}$$

Sometimes it is useful to decompose vectors—that is, to break a vector apart into a sum. This process is called the **resolution of a vector into components**. Projections allow us to identify two orthogonal vectors having a desired sum. For example, let $\vec{v} = \langle 6, -4 \rangle$ and let $\vec{u} = \langle 3, 1 \rangle$. We want to decompose the vector \vec{v} into orthogonal components such that one of the component vectors has the same direction as \vec{u} .

We first find the component that has the same direction as \vec{u} by projecting \vec{v} onto \vec{u} . Let $\vec{p} = \text{proj}_{\vec{u}} \vec{v}$. Then, we have

$$\begin{aligned}
 \vec{p} &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \\
 &= \frac{18 - 4}{9 + 1} \vec{u} \\
 &= \frac{7}{5} \vec{u} = \frac{7}{5} \langle 3, 1 \rangle = \left\langle \frac{21}{5}, \frac{7}{5} \right\rangle.
 \end{aligned}$$

Now consider the vector $\vec{q} = \vec{v} - \vec{p}$. We have

$$\begin{aligned}
 \vec{q} &= \vec{v} - \vec{p} \\
 &= \langle 6, -4 \rangle - \left\langle \frac{21}{5}, \frac{7}{5} \right\rangle \\
 &= \left\langle \frac{9}{5}, -\frac{27}{5} \right\rangle.
 \end{aligned}$$

Clearly, by the way we defined \vec{q} , we have $\vec{v} = \vec{q} + \vec{p}$, and

$$\begin{aligned}\vec{q} \cdot \vec{p} &= \left\langle \frac{9}{5}, -\frac{27}{5} \right\rangle \cdot \left\langle \frac{21}{5}, \frac{7}{5} \right\rangle \\ &= \frac{9(21)}{25} + -\frac{27(7)}{25} \\ &= \frac{189}{25} - \frac{189}{25} = 0.\end{aligned}$$

Therefore, \vec{q} and \vec{p} are orthogonal.

Example 11.3.8: Resolving Vectors into Components

Express $\vec{v} = \langle 8, -3, -3 \rangle$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{u} = \langle 2, 3, 2 \rangle$.

Solution

Let \vec{p} represent the projection of \vec{v} onto \vec{u} :

$$\begin{aligned}\vec{p} &= \text{proj}_{\vec{u}} \vec{v} \\ &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} \\ &= \frac{\langle 2, 3, 2 \rangle \cdot \langle 8, -3, -3 \rangle}{\|\langle 2, 3, 2 \rangle\|^2} \langle 2, 3, 2 \rangle \\ &= \frac{16 - 9 - 6}{2^2 + 3^2 + 2^2} \langle 2, 3, 2 \rangle \\ &= \frac{1}{17} \langle 2, 3, 2 \rangle \\ &= \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle.\end{aligned}$$

Then,

$$\begin{aligned}\vec{q} &= \vec{v} - \vec{p} = \langle 8, -3, -3 \rangle - \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle \\ &= \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle.\end{aligned}$$

To check our work, we can use the dot product to verify that \vec{p} and \vec{q} are orthogonal vectors:

$$\begin{aligned}\vec{p} \cdot \vec{q} &= \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle \cdot \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle \\ &= \frac{268}{17} - \frac{162}{17} - \frac{106}{17} = 0.\end{aligned}$$

Then,

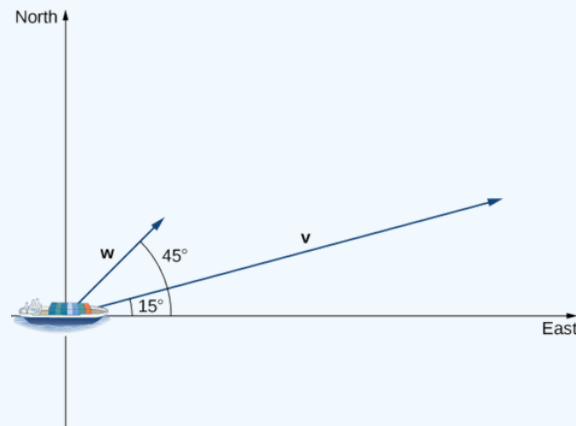
$$\vec{v} = \vec{p} + \vec{q} = \left\langle \frac{2}{17}, \frac{3}{17}, \frac{2}{17} \right\rangle + \left\langle \frac{134}{17}, -\frac{54}{17}, -\frac{53}{17} \right\rangle.$$

Exercise 11.3.7

Express $\vec{v} = 5\hat{i} - \hat{j}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\vec{u} = 4\hat{i} + 2\hat{j}$.

Example 11.3.9: Scalar Projection of Velocity

A container ship leaves port traveling 15° north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction 15° north of east? Round the answer to two decimal places.



Solution

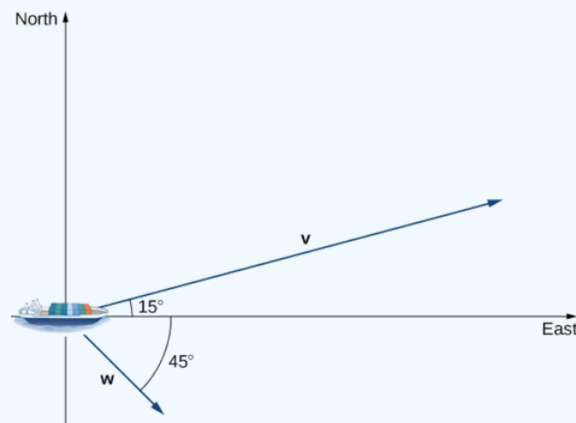
Let \vec{v} be the velocity vector generated by the engine, and let \vec{w} be the velocity vector of the current. We already know $\|\vec{v}\| = 20$ along the desired route. We just need to add in the scalar projection of \vec{w} onto \vec{v} . We get

$$\begin{aligned} \text{comp}_{\vec{v}} \vec{w} &= \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|} \\ &= \frac{\|\vec{v}\| \|\vec{w}\| \cos(30^\circ)}{\|\vec{v}\|} = \|\vec{w}\| \cos(30^\circ) = 2 \frac{\sqrt{3}}{2} = \sqrt{3} \approx 1.73 \text{ knots.} \end{aligned}$$

The ship is moving at 21.73 knots in the direction 15° north of east.

Exercise 11.3.8

Repeat the previous example, but assume the ocean current is moving southeast instead of northeast, as shown in the following figure.



Work

Now that we understand dot products, we can see how to apply them to real-life situations. The most common application of the dot product of two vectors is in the calculation of work.

From physics, we know that work is done when an object is moved by a force. When the force is constant and applied in the same direction the object moves, then we define the work done as the product of the force and the distance the object travels: $W = Fd$. We saw several examples of this type in earlier chapters. Now imagine the direction of the force is different from the direction of motion, as with the example of a child pulling a wagon. To find the work done, we need to multiply the component of the force that acts in the direction of the motion by the magnitude of the displacement. The dot product allows us to do just that. If we represent an applied force by a vector \vec{F} and the displacement of an object by a vector \vec{s} , then the **work done by the force** is the dot product of \vec{F} and \vec{s} .

Definition: Constant Force

When a constant force is applied to an object so the object moves in a straight line from point P to point Q , the work W done by the force \vec{F} , acting at an angle θ from the line of motion, is given by

$$W = \vec{F} \cdot \vec{PQ} = \|\vec{F}\| \|\vec{PQ}\| \cos \theta. \quad (11.3.14)$$

Let's revisit the problem of the child's wagon introduced earlier. Suppose a child is pulling a wagon with a force having a magnitude of 8 lb on the handle at an angle of 55° . If the child pulls the wagon 50 ft, find the work done by the force (Figure 11.3.8).

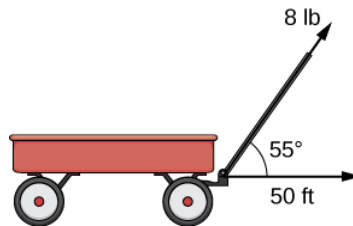


Figure 11.3.8: The horizontal component of the force is the projection of \vec{F} onto the positive x -axis.

We have

$$W = \|\vec{F}\| \|\vec{PQ}\| \cos \theta = 8(50)(\cos(55^\circ)) \approx 229 \text{ ft}\cdot\text{lb}.$$

In U.S. standard units, we measure the magnitude of force $\|\vec{F}\|$ in pounds. The magnitude of the displacement vector $\|\vec{PQ}\|$ tells us how far the object moved, and it is measured in feet. The customary unit of measure for work, then, is the foot-pound. One foot-pound is the amount of work required to move an object weighing 1 lb a distance of 1 ft straight up. In the metric system, the unit of measure for force is the newton (N), and the unit of measure of magnitude for work is a newton-meter (N·m), or a joule (J).

Example 11.3.10 : Calculating Work

A conveyor belt generates a force $\vec{F} = 5\hat{i} - 3\hat{j} + \hat{k}$ that moves a suitcase from point $(1, 1, 1)$ to point $(9, 4, 7)$ along a straight line. Find the work done by the conveyor belt. The distance is measured in meters and the force is measured in newtons.

Solution

The displacement vector \vec{PQ} has initial point $(1, 1, 1)$ and terminal point $(9, 4, 7)$:

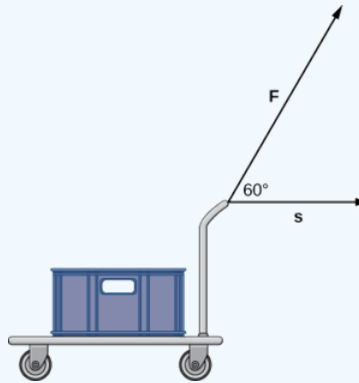
$$\vec{PQ} = \langle 9 - 1, 4 - 1, 7 - 1 \rangle = \langle 8, 3, 6 \rangle = 8\hat{i} + 3\hat{j} + 6\hat{k}.$$

Work is the dot product of force and displacement:

$$\begin{aligned}
 W &= \vec{F} \cdot \vec{PQ} \\
 &= (5\hat{i} - 3\hat{j} + \hat{k}) \cdot (8\hat{i} + 3\hat{j} + 6\hat{k}) \\
 &= 5(8) + (-3)(3) + 1(6) \\
 &= 37 \text{ N}\cdot\text{m} \\
 &= 37 \text{ J}
 \end{aligned}$$

Exercise 11.3.9

A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground. What is the work done by this force?



Key Concepts

- The dot product, or scalar product, of two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is $\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3$.
- The dot product satisfies the following properties:
 - $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
 - $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
 - $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
 - $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
- The dot product of two vectors can be expressed, alternatively, as $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$. This form of the dot product is useful for finding the measure of the angle formed by two vectors.
- Vectors \vec{u} and \vec{v} are orthogonal if $\vec{u} \cdot \vec{v} = 0$.
- The angles formed by a nonzero vector and the coordinate axes are called the *direction angles* for the vector. The cosines of these angles are known as the *direction cosines*.
- The vector projection of \vec{v} onto \vec{u} is the vector $\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}$. The magnitude of this vector is known as the *scalar projection* of \vec{v} onto \vec{u} , given by $\text{comp}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}$.
- Work is done when a force is applied to an object, causing displacement. When the force is represented by the vector \vec{F} and the displacement is represented by the vector \vec{s} , then the work done W is given by the formula $W = \vec{F} \cdot \vec{s} = \|\vec{F}\| \|\vec{s}\| \cos \theta$.

Key Equations

- Dot product of \vec{u} and \vec{v}**

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 = \|\vec{u}\| \|\vec{v}\| \cos \theta$$
- Cosine of the angle formed by \vec{u} and \vec{v}**

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

- **Vector projection of \vec{v} onto \vec{u}**

$$\text{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}$$

- **Scalar projection of \vec{v} onto \vec{u}**

$$\text{comp}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}$$

- **Work done by a force \vec{F} to move an object through displacement vector \vec{PQ}**

$$W = \vec{F} \cdot \vec{PQ} = \|\vec{F}\| \|\vec{PQ}\| \cos \theta$$

Glossary

Contributors and Attributions

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- edited for vector notation by Paul Seeburger