Question 1.

Circle the statements below that are equivalent to $c \to (a \to b)$.

1. $(a \vee \neg b) \to \neg c$
2. $\neg c \to \neg(a \to b)$
3. c is necessary for $\neg a$ or b
4. c is sufficient for $\neg a$ or b
5. a and c are sufficient for b.
6. a is necessary for b both of which are necessary for c.
7. $(c \wedge a) \to b$
8. $(a \to b) \to \neg c$
9. All of the above.
10. None of the above.

Soln:
4, 5, 7

Question 2.

Let $P(x)$ be the predicate x *is prime.*
Let $Q(x, y)$ be the predicate for x *divides* y.

Consider the statement:

For every x that is not prime, there is some prime y that divides it.

a) Write the statement in predicate logic

Solution:

$$\forall x, \neg P(x) \to \exists y, (P(y) \wedge Q(y, x))$$

b) Negate the statement

Solution:

$$\exists x, \neg P(x) \wedge \forall y, \neg P(y) \vee \neg Q(y, x)$$

c) Write the english translation of statement from b)

Solution:

There is a number that is not prime and has no prime divisors.

Question 3.

Fill in the blanks with either $\exists x$ or $\forall x$. Marks will be given for showing your work.

$$\exists x \in X, (a(x) \to b(x)) \iff (\_\_\_\_\_ \in X, a(x)) \to (\_\_\_\_\_ \in X, b(x)).$$

Sample Soln.

$$\exists x \in X, (a(x) \to b(x)) \iff (\_\_\_\_\_ \in X, a(x)) \to (\_\_\_\_\_ \in X, b(x)).$$

$$\begin{aligned} \exists x \in X \neg a(x) \vee b(x) &\Leftrightarrow \exists x \in X \neg a(x) \vee \exists x \in X, b(x) \\ &\Leftrightarrow \neg \forall x \in X a(x) \vee \exists x \in X, b(x) \\ &\Leftrightarrow (\forall x \in X a(x)) \to (\exists x \in X, b(x)) \end{aligned}$$

Question 4.

Answer the following questions to construct a direct proof that $\forall n \in \mathbb{N}, n > 1$,

$$\forall a \in \mathbb{N}, \forall b \in \mathbb{N},\ ab \bmod n = (a \bmod n)(b \bmod n) \bmod n$$

Solutions:

$a = nq_1 + r_1$ and $b = nq_2 + r_2$.

Then $ab = (nq_1 + r_1)(nq_1 + r_1) = (n^2q_1^2 + 2nq_1) + r_1r_2$

so $ab \bmod n = ((n^2q_1^2 + 2nq_1) + r_1r_2) \bmod n = r_1r_2 \bmod n$
and $(a \bmod n)(b \bmod n) \bmod n = r_1r_2 \bmod n$.

Question 5.

Prove $\sqrt[3]{2}$ is irrational.

Solutions:

Proof by contradiction. Suppose $\sqrt[3]{2}$ is rational then exists integers p and $q \neq 0$ such that $\sqrt[3]{2} = \frac{p}{q}$. Then $2q^3 = p^3$. (**)

Therefore for each prime factor x_i of p, there are a multiple of 3 x_i on the righthand side. Notice that if 2 is a prime factor of q then there are the lefthand side has $3k+1$ prime factors equal to 2 for $k \in \mathbb{N}$. $\star$

Therefore we have a contradiction by the Fun. Thm. of Arithmetic.

Question 6.

We have seen the *Pigeon Hole Principle* in class. We will now prove the *average version of the Pigeon Hold Principle*:

> If z is the average of the collection of numbers
>
> $$x_1, x_2, \ldots, x_n$$
>
> then at least one number x_i in the list is greater than or equal to z.

Recall that the average is $\frac{1}{n}\sum_{i=1}^{n} x_i$.

Sample Solution. Proof by contrapositive.

If $x_i < z$ for all x_i then $\frac{1}{n}\sum_{i=1}^{n} x_i < \frac{1}{n}\sum_{i=1}^{n} z = z$. Therefore z is greater than the average completing the proof by contrapositive.

Question 7.

Prove the following statement $S(n)$ for all $n \in \mathbb{N}, n \geq 0$ using simple induction:

$$S(n) : 3^{2n} - 1 \text{ is divisible by } 8$$

Sample Solutions.

Base CAse: n $= 0$. $3^0 - 1 = 0$ which is divisible by 8.

I.H. Assume that $S(k)$ holds for $0 \leq k$.

I.S. Prove $S(k+1)$.

$3^{2k+2} - 1 = 3^2 \cdot 3^{2k} - 1 = 9 \cdot 3^{2k} - 9 + 8 = 9(3^{2k} - 1) + 8$ by I.H. $3^{2k} - 1$ is divisible by 8 therefore $9(3^{2k} - 1) + 8$ is divisible by 8.

Question 8.

Suppose that we have access to an unlimited number of 5 and 11 cent stamps. Prove, using simple induction, that we can use these stamps to make any amount of postage that is at least 40 cents.

P(n): $\exists\ x, y \in \mathbb{N},\ 5x + 11y = n$

WTP: $\forall\ n \in \mathbb{N},\ n \geq 40,\ P(n)$

Base Case:

let n = 40

let x = 8, let y = 0

⇒ 5x + 11y

= 5(8) + 11(0)

= 40

= n

⇒ P(n) is true for n=40, so the base case holds.

Inductive Step:

Let n ≥ 40 be given where n ∈ $\mathbb{N}$, and assume P(n) holds.

So $\exists\ x', y' \in \mathbb{N},\ 5x' + 11y' = n$ **[IH]**

WTP $\exists\ x, y \in \mathbb{N},\ 5x + 11y = n + 1$

By IH, we have x', y' such that 5x' + 11y' = n

We have two cases, when $0 \leq x < 2$ and x > 2

Case 1: Suppose $0 \leq x < 2$

Case 1.1: x=0

⇒ 11y' = n (n ∈ $\mathbb{N}$)

⇒ n ≥ 44

⇒ y' ≥ 4

Case 1.2: x=1

⇒ 11y'+5 = n (n ∈ $\mathbb{N}$)

⇒ 11y' +5 ≥ 40

⇒ 11y' ≥ 35 [11(3) = 33 < 35]

⇒ y' ≥ 4

In both cases, we have y' ≥ 4.

Let x = x' + 9, y = y' - 4

⇒ 5x + 11y

= 5(x' + 9) + 11(y' - 4)

= 5x' + 45 + 11y' - 44

= n + 1 , as wanted

So

Case 2: Suppose x > 2

Let x = x' - 2, y = y' + 1

⇒ 5x + 11y

$= 5(x' - 2) + 11(y' + 1)$
$= 5x' - 10 + 11y' + 11$
$= n + 1$, as wanted

So P(n) is true in both case 1 and case 2.
So, by the PSI, since Base Case and Inductive Step hold, $\forall\ n \in \mathbb{N}, n \geq 40, P(n)$

Question 9.

$$\text{Suppose that } f(n) = \begin{cases} 0, & n = 0, \\ 1, & n = 1, \\ 2, & n = 2, \\ 3f(n-1) + f(n-2) - 3f(n-3), & n > 2. \end{cases}$$

Use complete induction to show that, for all $n \in \mathbb{N}$, $f(n) = \dfrac{3^n - (-1)^n}{4}$.

Solution:

P(n): $f(n) = \dfrac{3^n - (-1)^n}{4}$

WTP: $\forall\ n \in \mathbb{N},\ P(n)$

Let b=0 and k=3

Base Case:

n = 0 LHS: f(0) = 0 RHS: $f(0) = \frac{3^n - (-1)^n}{4} = \frac{3^0 - (-1)^0}{4} = \frac{1-1}{4} = \frac{0}{4} = 0 \Rightarrow LHS = RHS$

n = 1 LHS: f(1) = 1 RHS: $f(1) = \frac{3^n - (-1)^n}{4} = \frac{3^1 - (-1)^1}{4} = \frac{3+1}{4} = \frac{4}{4} = 1 \Rightarrow LHS = RHS$

n = 2 LHS: f(2) = 2 RHS: $f(2) = \frac{3^n - (-1)^n}{4} = \frac{3^2 - (-1)^2}{4} = \frac{9-1}{4} = \frac{8}{4} = 2 \Rightarrow LHS = RHS$

$\Rightarrow$ P(n) holds for $b \leq n \leq b + k - 1$, so the base cases hold

Inductive Step:

Let n ≥ 3 be given where $n \in \mathbb{N}$, and assume:

$\forall\ j,\ 0 \leq j < n,\ f(j) = \dfrac{3^j - (-1)^j}{4}$ **[IH]**

WTP: $f(n) = \dfrac{3^n - (-1)^n}{4}$

$$f(n) = 3f(n-1) + f(n-2) - 3f(n-3)$$

$$\textbf{[IH]} = 3\left(\frac{3^{n-1} - (-1)^{n-1}}{4}\right) + \left(\frac{3^{n-2} - (-1)^{n-2}}{4}\right) - 3\left(\frac{3^{n-3} - (-1)^{n-3}}{4}\right)$$

$$= 3\left(\frac{3^n 3^{-1} - (-1)^n(-1)^{-1}}{4}\right) + \left(\frac{3^n 3^{-2} - (-1)^n(-1)^{-2}}{4}\right) - 3\left(\frac{3^n 3^{-3} - (-1)^n(-1)^{-3}}{4}\right)$$

$$= \left(\frac{33^n 3^{-1} - 3(-1)^n(-1)^{-1}}{4}\right) + \left(\frac{3^n 3^{-2} - (-1)^n(-1)^{-2}}{4}\right) - \left(\frac{33^n 3^{-3} - 3(-1)^n(-1)^{-3}}{4}\right)$$

$$= \frac{33^n 3^{-1} - 3(-1)^n(-1)^{-1} + 3^n 3^{-2} - (-1)^n(-1)^{-2} - 33^n 3^{-3} + 3(-1)^n(-1)^{-3}}{4}$$

$$= \frac{3^n - 3(-1)^n(-1) + 3^n 3^{-2} - (-1)^n(1) - 33^n 3^{-3} + 3(-1)^n(-1)}{4}$$

$$= \frac{3^n + 3(-1)^n + 3^n 3^{-2} - (-1)^n - 33^n 3^{-3} - 3(-1)^n}{4}$$
$$= \frac{3^n(1 + 3^{-2} - 3^{-3}3) - (-1)^n}{4}$$
$$= \frac{3^n(1 + 3^{-2} - 3^{-2}) - (-1)^n}{4}$$
$$= \frac{3^n - (-1)^n}{4}$$, as wanted

So P(n) holds $\forall\ n \in \mathbb{N},\ n \geq 3$

So since base case and induction step hold, $\forall\ n \in \mathbb{N},\ f(n) = \frac{3^n - (-1)^n}{4}$ by complete induction.