True/False from my quizzes

1. Consider the R \ {0} of non-zero real number with the usual division. Every element of R \ {0} is invertible

   False, for every element of a set to be invertible, it must have an identity but R \ {0} doesn't have an identity.

2. Every vector space with a non-zero vector has at least two distinct subspaces

   True, recall every vector space has the zero vector and the set with just the zero vector is a subspace. And a vector space V is a subspace of itself.

3. Let V be a vector space. Consider the set X = {v1,v2,...vn}. Suppose that every vector in V is unique linear combination of vectors in X. Then is a basis for V.

   True, this is another definition of basis for V

4. Let V and W be vector spaces. For every vector c in W, the function $T_c : V \to W$ defined by $T_c(v) = c$ for all v in V is a linear transformation.

   False, doesn't hold if c is a non-zero vector

5. Let B, B', B'' be ordered bases for the vector space V then $C_{B\to B''} = C_{B'\to B''} C_{B\to B'}$

   True

6. Suppose A and C are n by n matrices. Suppose further that C is invertible. Let v be an eigenvector of A. Then $C^{-1}v$ and Cv are eigenvectors of $C^{-1}AC$ and CAC^{-1} respectively

   True, recall the definition of eigenvector $Av = \lambda v$, *so show*
   $C^{-1}AC(C^{-1}v) = \lambda(C^{-1}v)$

7. Let V be a vector space. Let <,> be an inner product in V. For any k in R, define <,>' by $<u, v>' = k<u, v>$ Then <,>' is also an inner product

   False, let k=0, then the inner product violates property (3), positive definite

8. Every non-trivial subspace of R^n has an orthonormal basis.

   True, every subspace has a basis B, we can apply gram-smith on this to get an orthonormal basis.

Past Graded Homework Questions

12. A is 7×7, $A + I$ has nullity 3, $(A + I)^k$ has nullity 5 for $k \geq 2$; $A + iI$ has nullity 1, $(A + iI)^j$ has nullity 2 for $j \geq 2$.

. Since $A + I$ has nullity 3, it annihilates three Jordan basis vectors, say $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_4$:
$\mathbf{b}_1 \longrightarrow 0$, $\mathbf{b}_2 \longrightarrow 0$, $\mathbf{b}_4 \longrightarrow 0$.

Since $(A - 3I)^2$ has nullity 5, it annihilates two more Jordan basis vector, starting strings of length 2, say
$\mathbf{b}_3 \longrightarrow \mathbf{b}_2 \longrightarrow 0$, $\mathbf{b}_5 \longrightarrow \mathbf{b}_4 \longrightarrow 0$, $\mathbf{b}_1 \longrightarrow 0$.

Since $A + iI$ has nullity 1, it annihilates one Jordan basis vector, say $\mathbf{b}_6$: $\mathbf{b}_6 \longrightarrow 0$.

Since $(A + iI)^j$ has nullity 2 for $j \geq 2$, these powers of the matrix annihilate one more Jordan basis vector, which starts a string of length 2, say $\mathbf{b}_7 \longrightarrow \mathbf{b}_6 \longrightarrow 0$. Since $n = 7$, and we have found the string structure for 7 vectors in a Jordan basis, we find that a Jordan canonocal form for A is

$$J = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -i \end{bmatrix}.$$

Problem 3. *Let $A \in M_9(\mathbb{C})$, with eigenvalues λ, α and γ with algebraic multiplicity 4, 2 and 3 and geometric multiplicities 2, 1 and 1 respectively.*

(1) *We say two Jordan forms are equivalent if they only differ by the order of the Jordan blocks. Give all possible nonequivalent Jordan canonical forms for A with the given information.*
(2) *Moreover suppose we know $\operatorname{rank}(A - \lambda I)^3 = \operatorname{rank}(A - \lambda I)^4$ and $\operatorname{rank}(A - \lambda I)^2 \neq \operatorname{rank}(A - \lambda I)^3$. Give all nonequivalent possible Jordan canonical forms for A with the given information.*
(3) *Suppose a Jordan canonical form of a linear transformation T, written in the Jordan basis $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_5\}$ is*

$$J = \begin{bmatrix} \gamma & 1 & & & \\ & \gamma & & & \\ & & \alpha & 1 & \\ & & & \alpha & \\ & & & & \gamma \end{bmatrix},$$

where γ and α are distinct scalars. Which of the sets below is a linearly independent subset of $\mathbb{C}^5$? Justify.

(a) $\{T(\vec{b}_4), T(\vec{b}_3), \vec{b}_2\}$
(b) $\{\vec{b}_4, T(\vec{b}_4), \vec{b}_5\}$
(c) $\{\vec{b}_4, T(\vec{b}_4), \vec{b}_3\}$

Solution.

(1) *Since the geometric multiplicity of* γ *is 1, therefore it has 1 Jordan block which is* 3×3, *3 being the algebraic multiplicity of* γ. *Likewise,* α *has 1 Jordan block of size* 2×2. λ *has 2 Jordan blocks, and the sizes must add to 4, so they are either* 3×3 *and* 1×1, *or both are* 2×2. *This creates two possibilities for the Jordan form, ignoring order, which are: (1) The matrix with block diagonals* $\begin{bmatrix} \gamma & 1 & 0 \\ 0 & \gamma & 1 \\ 0 & 0 & \gamma \end{bmatrix}$, $\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$, $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, $[\lambda]$, *or (2)* $\begin{bmatrix} \gamma & 1 & 0 \\ 0 & \gamma & 1 \\ 0 & 0 & \gamma \end{bmatrix}$, $\begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$, $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

(2) *Let* $\rho_i = \operatorname{rank}(A - \lambda I)^{i-1} - \operatorname{rank}(A - \lambda I)^i$, *which is the number of linearly independent generalized eigenvectors associated to* λ *of rank* i. *Then,* $\rho_3 = \operatorname{rank}(A - \lambda I)^2 - \operatorname{rank}(A - \lambda I)^3 \neq 0$, *so there exists a generalized eigenvector of rank 3, i.e. there exists a Jordan chain of size 3, i.e. there exists a Jordan block of size 3. Note that we are using that* $\rho_4 = 0$ *to conclude this. Since there is a* 3×3 *Jordan block associated to* λ, *as noted earlier, the second and final Jordan block associated to* λ *must be* $[\lambda]$. *This means, with this information, there is precisely one Jordan form, which is (1) in the part above.*

(3) *First, we identify the Jordan chains. There are 3 Jordan blocks. The first is* 2×2 *and hence corresponds to a Jordan chain (written in reverse order) of the form* $((T - \gamma id)\vec{v}, \vec{v})$. *Here,* $\vec{v} = \vec{b}_2$, *so it is actually of the form* $((T - \gamma id)\vec{b}_2, \vec{b}_2)$. *Moreover,* $(T - \gamma id)^2 \vec{b}_2 = \vec{0}$ *by property of Jordan chains. Likewise, the second Jordan block has chain, in reverse order,* $((T - \alpha id)\vec{b}_4, \vec{b}_4)$ *and* $(T - \alpha id)^2 \vec{b}_4 = \vec{0}$, *and the third Jordan block has chain* $(\vec{b}_5)$. *Hence,* $(\vec{b}_1, \ldots, \vec{b}_5) = ((T - \gamma id)\vec{b}_2, \vec{b}_2, (T - \alpha id)\vec{b}_4, \vec{b}_4, \vec{b}_5)$.

(a) *Earlier work shows that* $\vec{b}_3 = (T - \alpha id)\vec{b}_4$, *so that* $T(\vec{b}_4) = \vec{b}_3 + \alpha \vec{b}_4$. *Also,* $(T - \alpha id)^2 \vec{b}_4 = \vec{0}$ *can be rewritten as* $(T - \alpha id)\vec{b}_3 = \vec{0}$, *so that* $T(\vec{b}_3) = \alpha \vec{b}_3$. *Hence, the set in question equals* $\{\vec{b}_3 + \alpha \vec{b}_4, \alpha \vec{b}_3, \vec{b}_2\}$. *So, if* $\alpha = 0$, *this set is dependent, and otherwise the independence of* $\{\vec{b}_1, \ldots, \vec{b}_5\}$ *implies the independence of this set.*

(b) *The work above shows that this set equals* $\{\vec{b}_4, \vec{b}_3 + \alpha \vec{b}_4, \vec{b}_5\}$, *which is again linearly independent by the same argument.*

(c) *Likewise, the set equals* $\{\vec{b}_4, \vec{b}_3 + \alpha \vec{b}_4, \vec{b}_3\}$. *There is the dependence relation* $-\alpha \vec{b}_4 + (\vec{b}_3 + \alpha \vec{b}_4) - \vec{b}_3 = \vec{0}$, *so this set is dependent.*

Problem 3. *Prove that for any* $m \times n$ *matrix* A *with real entries, there is an orthonormal basis* $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ *of* $\mathbb{R}^n$ *such that the vectors* $A\vec{v}_1, \ldots, A\vec{v}_n$ *are orthogonal. Note that some of the vectors* $A\vec{v}_i$ *may be* $\vec{0}$.[1]

Solution. *Let* A *be an* $m \times n$ *matrix. Then* $A^T A$ *is symmetric, and therefore is orthogonally diagonalizable by the spectral theorem. Let* $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ *be an orthonormal basis of* $\mathbb{R}^n$ *consisting of eigenvectors of* $A^T A$, *say* $A^T A \vec{v}_i = \lambda_i \vec{v}_i$ *for each* $1 \leq i \leq n$. *Then for all* $1 \leq i \neq j \leq n$,

$$A\vec{v}_i \cdot A\vec{v}_j \;=\; \vec{v}_i^T A^T A \vec{v}_j \;=\; \vec{v}_i^T \lambda_j \vec{v}_j \;=\; \lambda_j(\vec{v}_i \cdot \vec{v}_j) \;=\; 0.$$

This shows that $\{A\vec{v}_1, \ldots, A\vec{v}_n\}$ *is an orthogonal set, as desired.*

Problem 2. *An $n \times n$ matrix A is* orthogonal *if $A^TA = I_n$,* idempotent *if $A^2 = A$,* nilpotent *if $A^k = 0$ for some positive integer k, and* skew-symmetric *if $A^T = -A$. Let A be an $n \times n$ matrix. Making sure to justify your claims, find all possible values of $\det A$ if A is:*

(1) *orthogonal;*
(2) *idempotent;*
(3) *nilpotent;*
(4) *skew-symmetric, assuming n is odd;*
(5) *skew-symmetric, assuming $n = 2$.*

Solution.

(1) *Suppose A is an $n \times n$ orthogonal matrix, so $A^TA = I_n$. Then*
$$\det(A^TA) \;=\; \det(A)\det(A^T) \;=\; \det(A)^2 \;=\; \det(I_n) \;=\; 1,$$
which shows that $\det(A) = \pm 1$.

(2) *Suppose A is an $n \times n$ idempotent matrix, so $A^2 = A$. Then*
$$\det(A) \;=\; \det(A^2) \;=\; \det(A)^2,$$
which shows that $\det(A) = 0$ or $\det(A) = 1$.

(3) *Suppose A is an $n \times n$ nilpotent matrix, and let k be a positive integer such that $A^k = 0$. Then*
$$0 \;=\; \det(0) \;=\; \det(A^k) \;=\; (\det A)^k,$$
which shows that $\det(A) = 0$.

(4) *Suppose A is an $n \times n$ skew-symmetric matrix where n is odd, so $A^T = -A$. Then*
$$\det(A) \;=\; \det(A^T) \;=\; \det(-A) \;=\; (-1)^n\det(A) \;=\; -\det(A),$$
which shows that $\det(A) = 0$.

(5) *If A is a 2×2 skew-symmetric matrix, then by the definition of skew-symmetric we see that $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$ for some $b \in \mathbb{R}$, which means $\det(A) = 0 - b(-b) = b^2$. Thus given $d \in \mathbb{R}$, we have that d is the determinant of some 2×2 skew-symmetric matrix if and only if $d \geq 0$.*

Problem 3. *Let $n \geq 2$, and let A be the $n \times n$ matrix whose jth column is $\vec{e}_{j+1}$ for $j < n$ and whose last column has entries $-c_0, \ldots, -c_{n-1}$, as pictured below:*
$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{bmatrix}.$$

Viewing t as a variable, show that $\det(A - tI_n) = (-1)^n(c_0 + c_1t + \cdots + c_{n-1}t^{n-1} + t^n)$.

Remark: *The determinant of the matrix $A - tI_n$ is the* characteristic polynomial *of A, and its roots are the eigenvalues of A. So this problem shows how we can find, for any polynomial p with leading coefficient 1, a matrix whose characteristic polynomial is p or $-p$.*

Solution. *We prove the claim for all matrices of the given form by induction on the size of the matrix. For the induction base $n = 2$, we have*

$$\det \begin{bmatrix} -t & -c_0 \\ 1 & -c_1 - t \end{bmatrix} = (-t)(-c_1 - t) - 1(-c_0) = (-1)^2(c_0 + c_1 t + t^2).$$

Now let $n > 2$ be arbitrary, suppose the claim has been proven for all matrices of the given form of size $n-1$, and let A be as given. Then using a Laplace expansion along the first row, together with the induction hypothesis, we have

$$\det A = -t(-1)^{n-1}\big(c_1 + c_2 t + \cdots + c_{n-1}t^{n-2} + t^{n-1}\big) + (-1)^{n-1}(-c_0)\det B,$$

where B is the $(n-1) \times (n-1)$ matrix obtained from A by deleting the first row and last column of A. Since B is an upper triangular matrix with 1s along the main diagonal, we have $\det B = 1$ and thus

$$\begin{aligned} \det A &= (-1)^{n-1}\big(-c_0 - c_1 t - c_2 t^2 - \cdots - c_{n-1}t^{n-1} - t^n\big) \\ &= (-1)^n(c_0 + c_1 t + \cdots + c_{n-1}t^{n-1} + t^n) \end{aligned}$$

as desired, completing the induction.

Problem 4. *Let $(\vec{v}, \vec{w})$ be a linearly independent pair of vectors in $\mathbb{R}^3$. Define the map $T : \mathbb{R}^3 \to \mathbb{R}^2$ by*

$$T(\vec{x}) = \begin{bmatrix} \det\big(\begin{bmatrix} \vec{v} & \vec{w} & \vec{x} \end{bmatrix}\big) \\ \det\big(\begin{bmatrix} \vec{w} & \vec{v} & \vec{x} \end{bmatrix}\big) \end{bmatrix} \in \mathbb{R}^2.$$

1. *Show that T is a linear transformation.*
2. *Find a basis of $\ker(T)$. Briefly justify your answer.*
3. *Find a basis of $\operatorname{im}(T)$. Briefly justify your answer.*
4. *Show that if $\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{x} = \vec{w} \cdot \vec{x} = 0$, then $\|T(\vec{x})\| = \sqrt{2}\|\vec{v}\|\|\vec{w}\|\|\vec{x}\|$.*

Solution.

(1) *Using the multilinearity property of determinants, for all $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $c \in \mathbb{R}$ we have*

$$\begin{aligned} T(\vec{x} + \vec{y}) &= \begin{bmatrix} \det\big(\begin{bmatrix} \vec{v} & \vec{w} & \vec{x}+\vec{y} \end{bmatrix}\big) \\ \det\big(\begin{bmatrix} \vec{w} & \vec{v} & \vec{x}+\vec{y} \end{bmatrix}\big) \end{bmatrix} = \begin{bmatrix} \det\big(\begin{bmatrix} \vec{v} & \vec{w} & \vec{x} \end{bmatrix}\big) + \det\big(\begin{bmatrix} \vec{v} & \vec{w} & \vec{y} \end{bmatrix}\big) \\ \det\big(\begin{bmatrix} \vec{w} & \vec{v} & \vec{x} \end{bmatrix}\big) + \det\big(\begin{bmatrix} \vec{w} & \vec{v} & \vec{y} \end{bmatrix}\big) \end{bmatrix} \\ &= \begin{bmatrix} \det\big(\begin{bmatrix} \vec{v} & \vec{w} & \vec{x} \end{bmatrix}\big) \\ \det\big(\begin{bmatrix} \vec{w} & \vec{v} & \vec{x} \end{bmatrix}\big) \end{bmatrix} + \begin{bmatrix} \det\big(\begin{bmatrix} \vec{v} & \vec{w} & \vec{y} \end{bmatrix}\big) \\ \det\big(\begin{bmatrix} \vec{w} & \vec{v} & \vec{y} \end{bmatrix}\big) \end{bmatrix} = T(\vec{x}) + T(\vec{y}) \end{aligned}$$

and

$$\begin{aligned} T(c\vec{x}) &= \begin{bmatrix} \det\big(\begin{bmatrix} \vec{v} & \vec{w} & c\vec{x} \end{bmatrix}\big) \\ \det\big(\begin{bmatrix} \vec{w} & \vec{v} & c\vec{x} \end{bmatrix}\big) \end{bmatrix} = \begin{bmatrix} c\det\big(\begin{bmatrix} \vec{v} & \vec{w} & \vec{x} \end{bmatrix}\big) \\ c\det\big(\begin{bmatrix} \vec{w} & \vec{v} & \vec{x} \end{bmatrix}\big) \end{bmatrix} \\ &= c\begin{bmatrix} \det\big(\begin{bmatrix} \vec{v} & \vec{w} & \vec{x} \end{bmatrix}\big) \\ \det\big(\begin{bmatrix} \vec{w} & \vec{v} & \vec{x} \end{bmatrix}\big) \end{bmatrix} = cT(\vec{x}). \end{aligned}$$

This shows that T is a linear transformation.

(2) $\{\vec{v},\vec{w}\}$ *is a basis of* $\ker(T)$. *To see this, first note that* $T(\vec{v}) = T(\vec{w}) = \vec{0}$, *since the determinant of a matrix with repeated columns is zero. On the other hand, if* $T(\vec{x}) = 0$ *then* $\det\begin{bmatrix}\vec{v} & \vec{w} & \vec{x}\end{bmatrix} = 0$, *which implies that* $\{\vec{v},\vec{w},\vec{x}\}$ *is linearly dependent and thus* $\vec{x} \in Span(\vec{v},\vec{w})$, *since* $\{\vec{v},\vec{w}\}$ *is linearly independent. It follows that* $\{\vec{v},\vec{w}\}$ *is a basis of* $\ker(T)$.

(3) *By the rank-nullity theorem and part (b), we know that* $\dim \operatorname{im}(T) = 1$, *so it will suffice to find a single non-zero vector in* $\operatorname{im}(T)$. *Since* $\det\begin{bmatrix}\vec{v} & \vec{w} & \vec{x}\end{bmatrix} = -\det\begin{bmatrix}\vec{w} & \vec{v} & \vec{x}\end{bmatrix}$ *for all* $\vec{x}$, *every vector in* $\operatorname{im}(T)$ *will have the form* $\begin{bmatrix} a \\ -a\end{bmatrix}$, *where* $a \in \mathbb{R}$. *Thus* $\left\{\begin{bmatrix} 1 \\ -1\end{bmatrix}\right\}$ *is a basis of* $\operatorname{im}(T)$.

(4) *First note that if* $\vec{x} = \vec{0}$, *then the claim is trivially true, so assume* $\vec{x} \neq \vec{0}$. *Write* $A = \begin{bmatrix}\vec{v} & \vec{w} & \vec{x}\end{bmatrix}$ *and* $Q = \begin{bmatrix}\frac{\vec{v}}{\|\vec{v}\|} & \frac{\vec{w}}{\|\vec{w}\|} & \frac{\vec{x}}{\|\vec{x}\|}\end{bmatrix}$, *so* $T(\vec{x}) = \begin{bmatrix}\det A \\ -\det A\end{bmatrix}$ *and* $\det A = \|\vec{v}\|\|\vec{w}\|\|\vec{x}\| \det Q$. *Since* $\vec{v}$, $\vec{w}$, *and* $\vec{x}$ *are mutually orthogonal,* Q *has orthonormal columns and hence is an orthogonal matrix, so* $\det Q = \pm 1$. *Thus*

$$\|T(\vec{x})\|^2 = (\det A)^2 + (-\det A)^2 = 2(\det A)^2 = 2\big(\|\vec{v}\|\|\vec{w}\|\|\vec{x}\| \det Q\big)^2,$$

so $\|T(\vec{x})\| = \sqrt{2}\|\vec{v}\|\|\vec{w}\|\|\vec{x}\||\det Q| = \sqrt{2}\|\vec{v}\|\|\vec{w}\|\|\vec{x}\|$.

or

We know that $\left|\det\begin{bmatrix}\vec{v} & \vec{w} & \vec{x}\end{bmatrix}\right|$ *is the volume of the parallelepiped determined by* $\vec{v},\vec{w},\vec{x}$. *Since these vectors are mutually orthogonal, this parallelepiped is just a box, with volume*

$$\textit{length} \times \textit{width} \times \textit{height} = \|\vec{v}\|\|\vec{w}\|\|\vec{x}\|.$$

Therefore,

$$\begin{aligned}\|T(\vec{x})\|^2 &= \big(\det\begin{bmatrix}\vec{v} & \vec{w} & \vec{x}\end{bmatrix}\big)^2 + \big(\det\begin{bmatrix}\vec{w} & \vec{v} & \vec{x}\end{bmatrix}\big)^2 = \big(\det\begin{bmatrix}\vec{v} & \vec{w} & \vec{x}\end{bmatrix}\big)^2 + \big(-\det\begin{bmatrix}\vec{v} & \vec{w} & \vec{x}\end{bmatrix}\big)^2 \\ &= 2\big(\det\begin{bmatrix}\vec{v} & \vec{w} & \vec{x}\end{bmatrix}\big)^2 = 2\big|\det\begin{bmatrix}\vec{v} & \vec{w} & \vec{x}\end{bmatrix}\big|^2 = 2\big(\|\vec{v}\|\|\vec{w}\|\|\vec{x}\|\big)^2.\end{aligned}$$

Taking square roots completes the argument.

Other possible solutions include using QR-factorization.

Problem 3. *In Homework 7 question 4 you proved that by choosing a basis* $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$, *we can associate a symmetric matrix* B *to a finite dimensional inner product space* V *such that for all* $\vec{v},\vec{w} \in V$

$$\langle \vec{v},\vec{w}\rangle = [\vec{v}]_{\mathcal{B}}^T B [\vec{w}]_{\mathcal{B}}$$

Further more you showed that the ij-th entry of B *is* $\langle \vec{b}_i, \vec{b}_j\rangle$.

Suppose that $\mathcal{A} = (\vec{a}_1, \ldots, \vec{a}_n)$ *is another basis of* V, *and let* A *be the corresponding matrix whose ij-entry is* $\langle \vec{a}_i, \vec{a}_j\rangle$. *Show that* $B = C^T A C$, *where* $C = C_{\mathcal{B}\to\mathcal{A}}$ *is the change of coordinates matrix from* $\mathcal{B}$ *to* $\mathcal{A}$.

Solution. *Let* C *be the change of basis matrix* $C_{\mathcal{B}\to\mathcal{A}}$. *Then* $\langle \vec{x},\vec{y}\rangle = [\vec{x}]_{\mathcal{A}}^T A [\vec{y}]_{\mathcal{A}}$. *Substitute* $C[\vec{x}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{A}}$ *and* $C[\vec{y}]_{\mathcal{B}} = [\vec{y}]_{\mathcal{A}}$ *to get*

$$\langle \vec{x},\vec{y}\rangle = [C[\vec{x}]_{\mathcal{B}}]^T A [C[\vec{y}]_{\mathcal{B}}] = [\vec{x}]_{\mathcal{B}}^T C^T A C [\vec{y}]_{\mathcal{B}}.$$

As noted in the problem statement (i.e. GH7 #4), it must then be that the ij-th entry of $C^T A C$ *is equal to* $\langle \vec{b}_i, \vec{b}_j\rangle$, *which is the ij-th entry of* B. *That is,* $C^T A C = B$.

Problem 4. *Let* A *be an* $n \times d$ *matrix.*

(1) *Use the formula* $(\operatorname{im} A)^{\perp} = \ker A^T$ *to prove that* $\operatorname{rank}(A) = \operatorname{rank}(A^T)$.

(2) *Prove that* $\operatorname{rank}(A) = \operatorname{rank}(A^T A)$.

(3) *Prove or disprove:* $\operatorname{rank}(A^T A) = \operatorname{rank}(A A^T)$. [1]

Solution.

(1) *We want to show that* $\operatorname{im} A$ *and* $\operatorname{im} A^T$ *have the same dimension. Let's use Rank-Nullity on the matrix* A^T *to compute its rank. The matrix* A *is* $n \times d$ *so its source has dimension* d*. This means* $\operatorname{rank} A + \dim \ker A = d$ *and* $\operatorname{rank} A^T + \dim \ker A^T = n$ *. We know* $\operatorname{im} A^{\perp} = \ker A^T$*, so this formula becomes* $\operatorname{rank} A^T + \dim(\operatorname{im} A^{\perp}) = n$*. We know that any subspace and its orthogonal complement have dimensions summing to the dimension of the ambient space. Applying this to* $\operatorname{im} A$*, which is a subspace of* $\mathbb{R}^n$*, we see that* $\dim \operatorname{im} A + \dim(\operatorname{im} A)^{\perp} = n$*. So both* $\operatorname{rank} A$ *and* $\operatorname{rank} A^T$ *equal* $n - \dim(\operatorname{im} A)^{\perp}$*. So* $\operatorname{rank} A = \operatorname{rank} A^T$*.*

(2) *Use rank nullity. Both* A *and* $A^T A$ *have source of dimension* d*. Recall that* $(\operatorname{im} A)^{\perp} = \ker A^T$*. Let* $\vec{x} \in \ker(A^T A)$*, so that* $\vec{x} \in \mathbb{R}^d$ *and* $A^T A \vec{x} = \vec{0}$*. This is equivalent to saying that* $A\vec{x} \in \ker A^T$*, which as noted is equivalent to* $A\vec{x} \in (\operatorname{im} A)^{\perp}$*. But,* $\operatorname{im} A \cap (\operatorname{im} A)^{\perp} = \{\vec{0}\}$*, so the last statement is equivalent to* $A\vec{x} = \vec{0}$*, i.e.* $\vec{x} \in \ker A$*. This shows that* $\ker(A^T A) = \ker A$*, and hence* A *and* $A^T A$ *have the same nullity. By rank nullity, both matrices have the same rank.*

(3) *From (2) and (1), we know* $\operatorname{rank}(A^T A) = \operatorname{rank} A = \operatorname{rank} A^T = \operatorname{rank}(A^T)^T A^T = \operatorname{rank} A A^T$*.*

Problem 5 (Bonus 2 points). *Characterize all the orthogonal linear transformation from* $\mathbb{R}^2$ *to* $\mathbb{R}^2$ *(with respect to the dot product).*

Solution. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be an orthogonal linear transformation (with respect to the dot product). Recall (e.g. TUT8) that, by being orthogonal,* T *maps the standard basis* $\mathcal{E} = (\vec{e}_1, \vec{e}_2)$ *of* $\mathbb{R}^2$ *to an orthonormal basis* $\mathcal{B} = (T(\vec{e}_1), T(\vec{e}_2))$ *of* $\mathbb{R}^2$*. Recall that, if we let* $T(\vec{e}_1) = [a, b]$ *and* $T(\vec{e}_2) = [c, d]$*, then* $[T]_{\mathcal{E}} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$*. Since* $\mathcal{B}$ *is orthonormal, therefore in particular* $\|[a, b]\| = 1$*, so* $a^2 + b^2 = 1$*. This means that* (a, b) *lies on the circle of radius 1 centred at* $(0, 0)$*. Consequently, there exists an angle* θ *such that* $a = \cos\theta$ *and* $b = \sin\theta$*. Next, again since* $\mathcal{B}$ *is orthonormal, therefore* (c, d) *lies on the same circle and moreover* $[a, b]$ *and* $[c, d]$ *are perpendicular. This means that* (c, d) *is a rotation of* $\pi/2$ *either counter-clockwise or clockwise from* (a, b) *on the circle. That is, either* $[c, d] = [\cos(\theta + \pi/2), \sin(\theta + \pi/2)] = [-\sin\theta, \cos\theta]$*, or* $[c, d] = [\cos(\theta - \pi/2), \sin(\theta - \pi/2)] = [\sin\theta, -\cos\theta]$*. Consequently, all such orthogonal linear transformations* T *satisfy either* $[T]_{\mathcal{E}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *or* $[T]_{\mathcal{E}} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$.

Problem 3. *Recall from Section 3.1 exercise 15 that the set of complex numbers together with the standard complex number addition and real scalar multiplication is a real vector space. Consider the map that associates to each pair* (z, w) *of complex numbers the real number*

$$\langle z, w \rangle = \frac{\overline{z} w + z \overline{w}}{2}, \tag{3.1}$$

where for any complex number $z = x + iy$*, the notation* $\overline{z}$ *denotes the complex conjugate* $\overline{z} = x - iy$*.*

(1) *What is the dimension of the real vector space* $\mathbb{C}$*? Justify your answer.*
(2) *Show that* (3.1) *defines an inner product on* $\mathbb{C}$*.*
(3) *Find a basis of* $\mathbb{C}$ *that is orthonormal with respect to this inner product.*
(4) *Let* $\langle -, - \rangle$ *be the inner product on* $\mathbb{C}$ *defined above, and let* $\cdot$ *be the usual dot product on* $\mathbb{R}^2$*. Prove that* $(\mathbb{C}, \langle -, - \rangle)$ *and* $(\mathbb{R}^2, \cdot)$ *are* isomorphic as inner product spaces*. This means that there is an isomorphism* $T : \mathbb{R}^2 \to \mathbb{C}$ *such that for all* $\vec{x}, \vec{y} \in \mathbb{R}^2$*,*

$$\vec{x} \cdot \vec{y} = \langle T(\vec{x}), T(\vec{y}) \rangle.$$ [1]

Solution.

(1) *As a real vector space, note that* $\mathbb{C} = \mathrm{sp}\{1, i\}$ *and that this set is linearly independent, so the dimension is 2. To prove linear independence, assume that* $r + si = 0$. *Then,* $-r = si$, *so squaring yields* $r^2 = -s^2$, *implying* $r^2 = s^2 = 0$ *since* r, s *are real, and hence* $r = s = 0$.

(2) *We verify linearity:* $\langle z + z', w\rangle = \frac{\overline{(z+z')}w+(z+z')\overline{w}}{2} = \frac{\overline{z}w+\overline{z'}w+z\overline{w}+z'\overline{w}}{2} = \frac{\overline{z}w+z\overline{w}}{2} + \frac{\overline{z'}w+z'\overline{w}}{2} = \langle z, w\rangle + \langle z', w\rangle$. *Also* $\langle kz, w\rangle = \frac{\overline{kz}w+kz\overline{w}}{2} = \frac{k(\overline{z}w+z\overline{w})}{2} = k\langle z, w\rangle$. *We verify symmetry:* $\langle z, w\rangle = \frac{\overline{z}w+z\overline{w}}{2} = \frac{\overline{w}z+w\overline{z}}{2} = \langle w, z\rangle$. *And positive definiteness:* $\langle z, z\rangle = \frac{\overline{z}z+z\overline{z}}{2} = z\overline{z} = x^2 + y^2 \geq 0$, *and equals 0 iff* $x^2 = y^2 = 0$, *i.e.* $x = y = 0$, *i.e.* $z = 0$.

(3) *The vector space is two dimensional. It is easy to check that* $\{1, i\}$ *is an orthonormal basis.*

(4) *Define* $T : \mathbb{R}^2 \to \mathbb{C}$ *by* $T\big([a\ b]^T\big) = a + bi$. *Then* T *is a isomorphism, and for all* $\vec{x} = [x_1\ x_2]^T$, $\vec{y} = [y_1\ y_2]^T \in \mathbb{R}^2$ *we have*

$$\begin{aligned}\langle T(\vec{x}), T(\vec{y})\rangle &= \langle x_1 + x_2 i, y_1 + y_2 i\rangle = \frac{(x_1 - x_2 i)(y_1 + y_2 i) + (x_1 + x_2 i))(y_1 - y_2 i)}{2} \\ &= x_1 y_1 + x_2 y_2 = \vec{x} \cdot \vec{y}.\end{aligned}$$

Problem 4. *In this problem you will prove the following theorem:*

Theorem: *Let* V *be a finite dimensional inner product space with basis* $\mathcal{B} = \{\vec{b}_1, \cdots \vec{b}_n\}$ *and an inner product denoted by* $\langle\ ,\ \rangle_V$. *Then there exists a symmetric matrix* B *such that for all vectors* $\vec{v}, \vec{w} \in V$,

$$\langle \vec{v}, \vec{w}\rangle_V = [\vec{v}]_{\mathcal{B}}^T\, B\, [\vec{w}]_{\mathcal{B}}.$$

The matrix $B = I_n$ *if and only if* $\mathcal{B}$ *is orthonormal.*

(1) *Let* $T_{\mathcal{B}} : V \to \mathbb{R}^n$ *be the coordinate isomorphism with respect to the basis* $\mathcal{B}$. *Define the map* $\langle\ ,\ \rangle_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *by* $\langle \vec{x}, \vec{y}\rangle_{\mathbb{R}^n} := \langle T_{\mathcal{B}}^{-1}(\vec{x}), T_{\mathcal{B}}^{-1}(\vec{y})\rangle_V$. *Prove that* $\langle\ ,\ \rangle_{\mathbb{R}^n}$ *is an inner product on* $\mathbb{R}^n$.

(2) *Show that there exists a symmetric matrix* B *such that for all vectors* $\vec{v}, \vec{w} \in V$,

$$\langle \vec{v}, \vec{w}\rangle_V = [\vec{v}]_{\mathcal{B}}^T\, B\, [\vec{w}]_{\mathcal{B}}.^2$$

(3) *What is the ij-th entry of* B *in terms of the inner product of the vectors in* $\mathcal{B}$.

(4) *Prove that* $\langle \vec{v}, \vec{w}\rangle_V = [\vec{v}]_{\mathcal{B}} \cdot [\vec{w}]_{\mathcal{B}}$ *if and only if* $\mathcal{B}$ *is an orthonormal basis.*

Solution.

(1) *Using linearity of* $T_{\mathcal{B}}^{-1}$ *and that* $\langle\cdot,\cdot\rangle_V$ *is an inner product,* $\langle\vec{x}+r\vec{y},\vec{z}\rangle_{\mathbb{R}^n} = \langle T_{\mathcal{B}}^{-1}(\vec{x}+r\vec{y}), T_{\mathcal{B}}^{-1}(\vec{z})\rangle_V = \langle T_{\mathcal{B}}^{-1}(\vec{x})+rT_{\mathcal{B}}^{-1}(\vec{y}), T_{\mathcal{B}}^{-1}(\vec{z})\rangle_V = \langle T_{\mathcal{B}}^{-1}(\vec{x}), T_{\mathcal{B}}^{-1}(\vec{z})\rangle_V + r\langle T_{\mathcal{B}}^{-1}(\vec{y}), T_{\mathcal{B}}^{-1}(\vec{z})\rangle_V = \langle\vec{x},\vec{z}\rangle_{\mathbb{R}^n} + r\langle\vec{y},\vec{z}\rangle_{\mathbb{R}^n}$. *For symmetry,* $\langle\vec{x},\vec{y}\rangle_{\mathbb{R}^n} = \langle T_{\mathcal{B}}^{-1}(\vec{x}), T_{\mathcal{B}}^{-1}(\vec{y})\rangle_V = \langle T_{\mathcal{B}}^{-1}(\vec{y}), T_{\mathcal{B}}^{-1}(\vec{x})\rangle_V = \langle\vec{y},\vec{x}\rangle_{\mathbb{R}^n}$. *For positive-definiteness,* $\langle\vec{x},\vec{x}\rangle_{\mathbb{R}^n} = \langle T_{\mathcal{B}}^{-1}(\vec{x}), T_{\mathcal{B}}^{-1}(\vec{x})\rangle_V \geq 0$, *and equals 0 iff* $T_{\mathcal{B}}^{-1}(\vec{x}) = \vec{0}$, *i.e.* $\vec{x} = \vec{0}$.

(2) *From GH6 we know that there exists a symmetric matrix* B *such that for all* $\vec{x},\vec{y}\in\mathbb{R}^n$, $\langle\vec{x},\vec{y}\rangle_{\mathbb{R}^n} = \vec{x}^TB\vec{y}$. *Hence,* $\langle T_{\mathcal{B}}^{-1}(\vec{x}), T_{\mathcal{B}}^{-1}(\vec{y})\rangle_V = \vec{x}^TB\vec{y}$. *Since this formula holds for all* $\vec{x},\vec{y}\in\mathbb{R}^n$, *replace* $\vec{x}$ *with* $T_{\mathcal{B}}(\vec{v}) = [\vec{v}]_{\mathcal{B}}$ *and* $\vec{y}$ *with* $T_{\mathcal{B}}(\vec{w}) = [\vec{w}]_{\mathcal{B}}$. *Then, the last equation becomes* $\langle\vec{v},\vec{w}\rangle_V = \langle T_{\mathcal{B}}^{-1}(T(\vec{v})), T_{\mathcal{B}}^{-1}(T(\vec{w}))\rangle_V = T(\vec{v})^TBT(\vec{w}) = [\vec{v}]_{\mathcal{B}}^TB[\vec{w}]_{\mathcal{B}}$.

(3) *From GH6 we know that* $B_{ij} = \langle\vec{e}_i,\vec{e}_j\rangle_{\mathbb{R}^n} = \langle T_{\mathcal{B}}^{-1}(\vec{e}_i), T_{\mathcal{B}}^{-1}(\vec{e}_j)\rangle_V = \langle\vec{b}_i,\vec{b}_j\rangle_V$, *since* $T_{\mathcal{B}}(\vec{b}_i) = [\vec{b}_i]_{\mathcal{B}} = \vec{e}_i$.

(4) *First suppose that* $\langle\vec{v},\vec{w}\rangle_V = [\vec{v}]_{\mathcal{B}}\cdot[\vec{w}]_{\mathcal{B}}$ *for all* $\vec{v},\vec{w}\in V$. *Then, in particular,* $\langle\vec{b}_i,\vec{b}_j\rangle_V = [\vec{b}_i]_{\mathcal{B}}\cdot[\vec{b}_j]_{\mathcal{B}} = \vec{e}_i\cdot\vec{e}_j$ *which is 0 if* $i\neq j$ *and is 1 if* $i=j$, *thereby showing that* $\mathcal{B}$ *is orthonormal. Coversely, suppose that* $\mathcal{B}$ *is orthonormal. Let* $\vec{v} = r_1\vec{b}_1+\cdots+r_n\vec{b}_n$ *and* $\vec{w} = t_1\vec{b}_1+\cdots+t_n\vec{b}_n$. *Then, using linearity in both arguments of the inner product,* $\langle\vec{v},\vec{w}\rangle_V = \sum_{i=1}^n\sum_{j=1}^n r_it_j\langle\vec{b}_i,\vec{b}_j\rangle_V$. *Using that* $\mathcal{B}$ *is orthonormal shows that for each* $1\leq i\leq n$, *the only nonzero term* $r_it_j\langle\vec{b}_i,\vec{b}_j\rangle_V$ *is when* $j=i$. *Hence, the sum equals* $\sum_{i=1}^n r_it_i\langle\vec{b}_i,\vec{b}_i\rangle_V = \sum_{i=1}^n r_it_i\cdot 1 = [r_1,\ldots,r_n]\cdot[t_1,\ldots,t_n] = [\vec{v}]_{\mathcal{B}}\cdot[\vec{w}]_{\mathcal{B}}$.

Problem 5 (Bonus 1 points). *Show that the distance between two orthonornal vectors is* $\sqrt{2}$ *in any inner product space.*

Solution. Let $\vec{v},\vec{w}$ be two orthonormal vectors, so that $\langle\vec{v},\vec{w}\rangle = 0$ and $||\vec{v}|| = 1 = ||\vec{w}||$. Then, $||\vec{v}-\vec{w}||^2 = \langle\vec{v}-\vec{w},\vec{v}-\vec{w}\rangle$. Using linearity in both arguments, this equals $\langle\vec{v},\vec{v}\rangle-\langle\vec{v},\vec{w}\rangle-\langle\vec{w},\vec{v}\rangle+\langle\vec{w},\vec{w}\rangle$. Using symmetry and $\vec{v},\vec{w}$ being orthonormal, this sum equals $||\vec{v}||^2-0-0+||\vec{w}||^2 = 1+1 = 2$. Hence, $||\vec{v}-\vec{w}||^2 = 2$, and taking square roots yields $||\vec{v}-\vec{w}|| = \sqrt{2}$.

Problem 3. *Let* $\langle\cdot,\cdot\rangle$ *be an inner product on* $\mathbb{R}^n$. *In this question you will show that there exists a* $n\times n$ *matrix* A *such that*

$$\langle\vec{x},\vec{y}\rangle = \vec{x}^TA\vec{y} \tag{3.1}$$

for all $\vec{x},\vec{y}\in\mathbb{R}^n$. *Further you will show that* A *must be symmetric.*

(1) *Suppose that such a matrix exists. That is suppose that there exists a* $n\times n$ *matrix* $A = (a_{ij})$ *for which (3.1) holds. Calculate* $\langle\vec{e}_i,\vec{e}_j\rangle$ *in terms of entries of* A.
(2) *Describe* A *in terms of* $\langle\vec{e}_i,\vec{e}_j\rangle$ $1\leq i,j\leq n$.
(3) *Prove that there exists an* $n\times n$ *matrix* A *for which (3.1) holds.*[1]
(4) *Show that* A *in (3.1) is symmetric, i.e.,* $A^T = A$.
(5) *Classify all the inner products in* $\mathbb{R}^2$.[2]

Solution.

(1) $\langle \vec{e}_i, \vec{e}_j \rangle = \vec{e}_i^T A \vec{e}_j = \vec{e}_i^T \begin{bmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{bmatrix} = a_{ij}$.

(2) *From (1),* $A = (a_{ij}) = (\langle \vec{e}_i, \vec{e}_j \rangle)$.

(3) *From (1), (2) we see that any such matrix A must equal* $(\langle \vec{e}_i, \vec{e}_j \rangle)$. *Hence, it remains to show that this choice of matrix satisfies (3.1). Recall that inner products satisfy* $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ *and* $\langle r\vec{v}, \vec{w} \rangle = r\langle \vec{v}, \vec{w} \rangle$, *and likewise* $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$ *and* $\langle \vec{v}, r\vec{w} \rangle = r\langle \vec{v}, \vec{w} \rangle$ *(e.g. we use symmetry of (real) inner products on linearity to get the last two). First, note that* $\langle \vec{x}, \vec{y} \rangle = \langle \sum_{i=1}^n x_i \vec{e}_i, \sum_{j=1}^n y_j \vec{e}_j \rangle$. *Linearity in the first argument shows the latter equals* $\sum_{i=1}^n x_i \langle \vec{e}_i, \sum_{j=1}^n y_j \vec{e}_j \rangle$, *and linearity in the second argument shows this equals* $\sum_{i=1}^n x_i \sum_{j=1}^n y_j \langle e_i, e_j \rangle$. *It remains to show that* $\vec{x}^T A \vec{y}$ *equals this. Note that* $\vec{x}^T A \vec{y} = \vec{x}^T \begin{bmatrix} \sum_{j=1}^n y_j \langle \vec{e}_1, \vec{e}_j \rangle \\ \vdots \\ \sum_{j=1}^n y_j \langle \vec{e}_n, \vec{e}_j \rangle \end{bmatrix} = \sum_{i=1}^n x_i \sum_{j=1}^n y_j \langle \vec{e}_i, \vec{e}_j \rangle$, *as desired.*

(4) *Since the inner product is symmetric, therefore, if we let* $B = A^T = (b_{ij})$, $b_{ij} = a_{ji} = \langle \vec{e}_j, \vec{e}_i \rangle = \langle \vec{e}_i, \vec{e}_j \rangle = a_{ij}$. *Since* $b_{ij} = a_{ij}$ *for all* i, j, *therefore* $A^T = B = A$.

(5) *From (1) through (4), we know that an inner product on* $\mathbb{R}^2$ *is of the form (3.1) for a symmetric matrix A. If we show that given a symmetric matrix A, with the additional property that A is positive-definite, i.e. that* $\vec{x}^T A \vec{x} > 0$ *whenever* $\vec{x} \neq \vec{0}$, *then we have shown that the classification is that any inner product* $\langle , \rangle$ *on* $\mathbb{R}^2$ *is of the form* $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y}$ *for some symmetric positive-definite matrix A. Let's show this. Note that* $\langle \vec{x} + \vec{y}, \vec{z} \rangle = (\vec{x} + \vec{y})^T A \vec{z} = (\vec{x}^T + \vec{y}^T) A \vec{z} = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$, *and that* $\langle r\vec{x}, \vec{y} \rangle = (r\vec{x})^T A \vec{y} = r(\vec{x}^T A \vec{y}) = r\langle \vec{x}, \vec{y} \rangle$. *The remaining work will be for general n (i.e. instead of just* $n = 2$, *i.e.* $\mathbb{R}^2$*) because the argument is the same.The work in (3) almost immediately shows that* $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^n x_i \sum_{j=1}^n x_j a_{ij}$. *Using the fact that we can change the order of summation for any finite sum, and using that A is symmetric, we have the latter equaling* $\sum_{i=1}^n \sum_{j=1}^n x_i y_j a_{ij} = \sum_{j=1}^n \sum_{i=1}^n x_i y_j a_{ij} = \sum_{j=1}^n y_j \sum_{i=1}^n x_i a_{ji} = \langle \vec{y}, \vec{x} \rangle$, *thereby showing symmetry of the proposed inner product. The last property, positive-definiteness, holds due to positive-definiteness of A:* $\langle \vec{x}, \vec{x} \rangle = \vec{x}^T A \vec{x} = 0$ *iff* $\vec{x} = \vec{0}$.

Problem 4. *Let V be an inner product space of dimension n, and let U and W be two m-dimensional subspaces of V. Assume that* $\vec{u} \perp W$ *for some* $\vec{u} \in U$, *where* $\vec{u} \neq \vec{0}$ *(that is* $\langle \vec{u}, \vec{w} \rangle = 0$ *for all* $\vec{w} \in W$*). Prove that* $\vec{w} \perp U$ *for some* $\vec{0} \neq \vec{w} \in W$.[3]

Solution. *Let* $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ *and* $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ *be bases for* U *and* W*, respectively. By hypothesis, there exists a vector* $\boldsymbol{u} \in U$ *that is orthogonal to every element in the basis for* W*. Suppose that* $\boldsymbol{u} = c_1\boldsymbol{v}_1 + \cdots + c_m\boldsymbol{v}_m$*, so that for every* $1 \leq i \leq m$*, we have*

$$\langle (c_1\boldsymbol{v}_1 + \cdots + c_m\boldsymbol{v}_m), \boldsymbol{w}_i\rangle = \langle c_1(\boldsymbol{v}_1, \boldsymbol{w}_i)\rangle + \cdots + \langle c_m(\boldsymbol{v}_m, \boldsymbol{w}_i)\rangle = 0.$$

Consider the following $m \times m$ *matrix*

$$A = \begin{bmatrix} \langle \boldsymbol{v}_1, \boldsymbol{w}_1\rangle & \langle \boldsymbol{v}_2, \boldsymbol{w}_1\rangle & \cdots & \langle \boldsymbol{v}_m, \boldsymbol{w}_1\rangle \\ \langle \boldsymbol{v}_1, \boldsymbol{w}_2\rangle & \langle \boldsymbol{v}_2, \boldsymbol{w}_2\rangle & \cdots & \langle \boldsymbol{v}_m, \boldsymbol{w}_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \boldsymbol{v}_1, \boldsymbol{w}_m\rangle & \langle \boldsymbol{v}_2, \boldsymbol{w}_m\rangle & \cdots & \langle \boldsymbol{v}_m, \boldsymbol{w}_m\rangle \end{bmatrix},$$

and observe that $[c_1, c_2, \ldots, c_m]^T \in NulA$ *by the observations above. In particular, the Invertible Matrix Theorem (*$A_{n\times n}$ *in invertible if and only if it has a trivial null space) applies to say that* A *is not invertible as its kernel is not trivial, and so neither is* A^T*. Taking transposes yields*

$$A^T = \begin{bmatrix} \langle \boldsymbol{v}_1, \boldsymbol{w}_1\rangle & \langle \boldsymbol{v}_1, \boldsymbol{w}_2\rangle & \cdots & \langle \boldsymbol{v}_1, \boldsymbol{w}_m\rangle \\ \langle \boldsymbol{v}_2, \boldsymbol{w}_1\rangle & \rangle \boldsymbol{v}_2, \boldsymbol{w}_2\rangle & \cdots & \langle \boldsymbol{v}_2, \boldsymbol{w}_m\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \boldsymbol{v}_m, \boldsymbol{w}_1\rangle & \langle \boldsymbol{v}_m, \boldsymbol{w}_2\rangle & \cdots & \langle \boldsymbol{v}_m, \boldsymbol{w}_m\rangle \end{bmatrix} = \begin{bmatrix} \langle \boldsymbol{w}_1, \boldsymbol{v}_1\rangle & \langle \boldsymbol{w}_2, \boldsymbol{v}_1\rangle & \cdots & \langle \boldsymbol{w}_m, \boldsymbol{v}_1\rangle \\ \langle \boldsymbol{w}_1, \boldsymbol{v}_2\rangle & \langle \boldsymbol{w}_2, \boldsymbol{v}_2\rangle & \cdots & \langle \boldsymbol{w}_m, \boldsymbol{v}_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \boldsymbol{w}_1, \boldsymbol{v}_m\rangle & \langle \boldsymbol{w}_2, \boldsymbol{v}_m\rangle & \cdots & \langle \boldsymbol{w}_m, \boldsymbol{v}_m\rangle \end{bmatrix}.$$

By the Invertible Matrix Theorem, there is a non-zero vector in $NulA^T$*, say* $[d_1, d_2, \ldots, d_m]^T$*. Thus, for all* $1 \leq i \leq m$*, we have*

$$d_1\langle \boldsymbol{w}_1, \boldsymbol{v}_i\rangle + \cdots + d_m\langle \boldsymbol{w}_m, \boldsymbol{v}_i\rangle = \langle (d_1\boldsymbol{w}_1 + \cdots + d_m\boldsymbol{w}_m), \boldsymbol{v}_i\rangle = 0.$$

Therefore, the vector $\boldsymbol{w} = d_1\boldsymbol{w}_1 + \cdots + d_m\boldsymbol{w}_m$ *is orthogonal to every element in the basis for* U*.*