(1). (20 points) Please choose all correct statements(s) for each of the following questions. You will receive the full credit if and only if you selected all true statements and You will get ZERO otherwise. Each part is equally weighted.

(1):Which of the following statements are true?

(A):Every nonzero vector in R^n has nonzero magnitude.

(B):There are exactly two unit vectors perpendicular to any given nonzero vector in R^n.

(C):The angle between two nonzero vectors in R^n is less than $\pi/2$ if and only if the dot product of the vectors is positive.

(D)If u and v are vectors in R^n of the same magnitude, then the magnitude of $u - v$ is 0

Ans: A,C

(2) Which of the following statements are true?

(A)Every linear system with the same number of equations as unknowns has a unique solution.

(B):Every linear system with the sare number of equations as unknowns has at least one solution.

(C):A linear system with more equations than unknowns may have an infinite number of solutions.

(D):A linear system with fewer equations than unknowns may have no solution.

Ans: C,D

(3):Which of the following statements are true?

(A) Every matrix is row equivalent to a unique matrix in row-echelon form.

(B):Every matrix is row equivalent to a unique matrix in reduced row-echelon form.

(C): If [A|b] and [B|c] are row-equivalent partitioned matrices, the linear systems Ax = b and Bx = c have the same solution set.

(D):A linear system with square coefficient matrix A has a unique solution if and only if A is row equivalent to the identity matrix.

Ans: B,C,D

(4):Which of the following statements are true? (The statements involve matrices A,B and C, which are assumed to be of appropriate size.)

(A):If AC = BC and C is invertible, then A = B.

(B):If AB = O and B is invertible, then A = O.

(C):If AB = C and two of the matrices are invertible, then so is the third.

(D):If AB = C and two of the matrices are singular(ie :not invertible), then so is the third.

Ans: A,B,C

(5):Which of the following statements are true? (The statements involve matrices A, B and C, which are assume to be of appropriate size)

(A):If A^2 is invertible, then A^3 is invertible

(B):If A^3 is invertible, then A^2 is invertible

(C):Every elementary matrix is invertible

(D):Every invertible matrix is an elementary matrix

Ans:A,B,C

(2) (10 points)

(a) (2 points)Let r(t) pass point (1, 0, 1) and parallel to vector b = (1, 2, 3). Please write down a equation of line r(t)

$r(t) = (1, 0, 1) + t(1, 2, 3)$

(b) (3 points) Find an equation of the plane P through the points P = (1, -1, 0), Q = (2, 1, 0) and R = (-2, 2, 2). The equation should be in form of ax + by + cz = d

$4x - 2y + 9z = 6$

(c) (5 points) Find the equation of place that contains r(t) and its perpendicular to plane P (P defined in part (b)). Express your equation in form of ax + by + cz = d

$n = [4, -2, 9],\ b = [1, 2, 3]$

$v = [4, -2, 9] \times [1, 2, 3] = [24, 3, -10]$

Since r(t) is on the plane, we know the tip of $r(0) = (1, 0, 1)$ will be on the plane

$(24, 3, -10) \cdot ((x, y, z) - r(0)) = 0 \Leftrightarrow 24x + 3y - 10z = (24, 3, -10) \cdot (1, 0, 1)$

$\Leftrightarrow 24x + 3y - 10z = 14$

(3) (10 points)

(a) (4 points) State the triangle inequality and Cauchy Schwarz inequality

See your notes.

(b) (6 points) If $x^2 + y^2 + z^2 = 1$, then what's the maximum value of x + 2y + 3z (Hint: use Cauchy Schwarz inequality)

$Let\ u = [x, y, z],\ v = [1, 2, 3]$

$|u \cdot v| \le ||u|| \cdot ||v||$

$x + 2y + 3z \le \sqrt{x^2 + y^2 + z^2} \cdot \sqrt{14}$

$x + 2y + 3z \le \sqrt{1} \cdot \sqrt{14},\ since\ x^2 + y^2 + z^2 = 1$

$x + 2y + 3z \le \sqrt{14}$

(4) (10 points)

(a) (5 points) Compute the angle between w = [-1, 1] and v = [1, 1]

$use\ ||v||||w||cos\theta = v \cdot w$

$\theta = \pi/2$

(b) Write down the Rotation matrix R such that Rv = w

$\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ⇔ $cos\theta - sin\theta = -1,\ sin\theta + cos\theta = 1$

$(1 - sin\theta) - sin\theta = -1,\ plugging\ eq(2)\ into\ eq(1)$
⇔ $sin\theta = 1$
⇔ $\theta = \pi/2$

Therefore, $R_\theta = \begin{bmatrix} cos(\pi/2) & -sin(\pi/2) \\ sin(\pi/2) & cos(\pi/2) \end{bmatrix}$

(5) (20 points)

(a) (3 points) Write down the definition of basis of subspace of R^n
See ur notes

(b) (4 points) Let $v_1 = [2, -1, -3]$ $v_2 = [1, 2, 2]$ $v_3 = [3, 1, 1]$. Check if they are basis of R^3
***** = Can't post the soln to this because its on an assignment that's still due lmao**
Basic idea: Place vectors as columns in a matrix, then row reduce, if the reduced matrix is row eqv to the identity, then the vectors are linearly independent and because there are three vectors they form a basis of R^3

(c) (6 points) Let A =

$-- v_1 --$
$-- v_2 --$
$-- v_3 --$

Where v_1, v_2, v_3 are defined in part (b). Is A invertible? Explain your reasons. If so, find A^{-1} and write it in terms of product of elementary matrix
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Basic idea: The matrix A defined in this question is the transpose of the matrix in the previous question. Is the previous matrix invertible? Is there any relationship between the transpose of a matrix and its inverse?
To write A^{-1} *in terms of a product of elementary matrices, remember that all elementary row operations have a corresponding elementary matrix and applying the row op is equivalent to multiplying by the elementary matrix.*

(d) Suppose A =

$-- v_1 --$
$-- v_2 --$
3 1 a

$x = [x_1,\ x_3, x_3]$ and $c = [4, 6, b]$ Find conditions on a and b such that Ax = c has (i) no solutions, (ii) a unique solution and (iii) infinitely many solutions

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(6) (10 points)

(a) (2 points) Define vector subspace of R^n

See your notes

(b) (4 points) Prove or disprove. If U and V are subspaces of R^n, then $U \cap V$ is a subspace of R^n

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(c) (4 points) Prove or disprove. If U and V are subspaces of R^n, then $U \cup V$ is a subspace of R^n

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(7) (12 points) Let C be a square Matrix

(a) (3 points) Check if $S = C + C^T$ is symmetric

$(C + C^T)^T = C^T + (C^T)^T = C^T + C = C + C^T$

Therefore S is symmetric

(b) (3 points) Check if $N = C - C^T$ is skew symmetric. Is trace of the skew symmetric matrix always zero? Justify your answer

$(C - C^T)^T = C^T - (C^T)^T = C^T - C = -(C - C^T)$

Therefore S is skew symmetric

We know, $tr(S) = tr(S^T)$ *, but since S is skew symmetric* $S^T = -S$ *, so* $tr(S) = tr(-S)$,

$tr(S) = -tr(S)$ *, by properties of trace, therefore* $tr(S) = 0$

(c) (6 points) Prove that every square matrix can be written down as sum of skew symmetric matrix and symmetric matrix

Let A be an arbitrary square matrix.

We know from (a) that $A + A^T$ *is symmetric and from (b) that* $A - A^T$ *is skew symmetric.*

$\frac{1}{2}(A + A^T)$ *is still symmetric and* $\frac{1}{2}(A - A^T)$ *is still skew symmetric*

Notice $\frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = A$

Therefore, every matrix can be written down as sum of skew symmetric matrix and symmetric matrix

(8) (8 points)

(a) (3 points) Define the trace of a matrix A

See ur notes

(b) (5 points) Let $A \varepsilon M_{n\times n}(R)$ such that $A^3 = I$. Let B be a matrix with $AB = -BA$. Show $tr(B) = 0$

$AB = -BA \Leftrightarrow ABAA = -BAAA \Leftrightarrow ABA^2 = -B$ *, sinze* $A^3 = I$

$\Leftrightarrow tr(ABA^2) = tr(-B)$

$\Leftrightarrow tr((AB)(AA)) = -tr(B)$ *, by associativity of matrix mult*

$\Leftrightarrow$ $tr((AA)(AB)) = -tr(B)$*, since* $tr(AB) = tr(BA)$

$\Leftrightarrow$ $tr(A^3 B) = -tr(B)$*, by associativity of matrix mult*

$\Leftrightarrow$ $tr(B) = -tr(B)$

$\Leftrightarrow$ $tr(B) = 0$*, as no real number is equal to its negative aside from 0*