MATA37
Seminar
Midterm
Review.

"The first principle is that you must not fool yourself - and you are the easiest person to fool." - Richard Feyman.

A2 (14) Suppose • f is continuous (i)
• $\exists m, M \in \mathbb{R}^+ \ni: \quad m \leq f(x) \leq M \quad \forall x \in [a,b]$. (ii)

w.t.s. $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$. Note $x_k^* \in [x_{k-1}, x_k]$.

By supposition (i) we can use Riemann def^n.
Let $P = \{t_i\}_{i=0}^{n}$ be a Riemann Partition.
By supposition (ii),

$$m \leq f(x) \leq M$$

$$m(b-a) \leq f(x)(b-a) \leq M(b-a) \quad , \text{ ~~b-a~~ by def}^n \text{ of } \leq$$

$$\frac{m(b-a)}{n} \leq \frac{f(x)(b-a)}{n} \leq \frac{M(b-a)}{n} \quad , \text{ by def}^n \text{ of } \leq$$

$$\frac{m(b-a)}{n} \leq f(x_k^*)\Delta x \leq \frac{M(b-a)}{n} \quad , \text{ by supposition (ii)}$$

$$\sum_{k=1}^{n} \frac{m(b-a)}{n} \leq \sum_{k=1}^{n} f(x_k^*)\Delta x \leq \sum_{k=1}^{n} \frac{M(b-a)}{n} \quad , \text{ by def}^n \text{ of } \Sigma \text{ and } \leq$$

$$\frac{m(b-a)}{n} \cdot \sum_{k=1}^{n} 1 \leq \sum_{k=1}^{n} f(x_k^*)\Delta x \leq \frac{M(b-a)}{n} \sum_{k=1}^{n} 1$$

$$\frac{m(b-a)}{n} \cdot n \leq \sum_{k=1}^{n} f(x_k^*)\Delta x \leq \frac{M(b-a)}{n} \cdot n$$

$$m(b-a) \leq \sum_{k=1}^{n} f(x_k^*)\Delta x \leq M(b-a)$$

$$\lim_{n\to\infty} m(b-a) \leq \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \leq \lim_{n\to\infty} M(b-a)$$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \quad \text{by Riemann def}^n.$$

$\blacksquare$

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Darboux: Use integrability Reformulation to prove not integrability of f;

$$f(x) = \begin{cases} \alpha & , x \in \mathbb{Q} \\ \beta & , x \notin \mathbb{Q} \end{cases} \quad \text{where } \alpha < \beta \text{ and } \alpha, \beta \in \mathbb{R}^+.$$

w.t.s. $\neg(\forall \varepsilon > 0\ \exists$ a partition P of $[a,b]$ $\ni: \mathcal{U}(f,P) - \mathcal{L}(f,P) < \varepsilon)$

$\exists \varepsilon > 0\ \forall$ partition P of $[a,b]$ $\ni: \mathcal{U}(f,P) - \mathcal{L}(f,P) \geq \varepsilon$.

Let P be any partition of $[a,b]$.

Choose $\varepsilon = (\beta - \alpha)/2$

Recall: • $m_i = \inf \{ f(x_i) \mid x_i \in [x_{i-1}, x_i] \}$

$= \inf \{ \alpha, \beta \} = \alpha$ ~~$\forall i \in \{1, \dots, n\}$~~

• $M_i = \sup \{ f(x_i) \mid x_i \in [x_{i-1}, x_i] \}$

$= \sup \{ \alpha, \beta \} = \beta$

• $\mathcal{L}(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = \alpha(1-0)$

$= \alpha$

• $\mathcal{U}(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = \beta(1-0)$

$= \beta$.

Sum telescoping and both $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ dense in $\mathbb{R}$

So,

$\mathcal{U}(f,P) - \mathcal{L}(f,P) = \beta - \alpha \geq \varepsilon$, as wanted.

$\blacksquare$

FTOC : Evaluate $\left(\dfrac{\int_{\lambda}^{\tan(x)} \frac{1}{\tan^2(t)+\ln(t)}\, dt}{\int_{\mu} \frac{1}{1+\sin^2(t)}\, dt} \right)' \quad -(\alpha)$

(Let all hypothesis hold). Just the procedure to evaluate.

Let $f(t) = \dfrac{1}{1+\sin^2(t)}$ and $g(t) = \dfrac{1}{\tan^2(t)+\ln t)}$ and

$\tau(x) = \tan(x)$.

So, $\displaystyle\int_{\mu}^{\int_{\lambda}^{\tan(x)} \frac{1}{\tan^2(t)+\ln(t)} dt} \frac{1}{1+\sin^2(t)}\, dt = \int_{\mu}^{\int_{\lambda}^{\tau(x)} g(t)\, dt} f(t)\, dt$

Let, $\mathcal{G}(x) = \displaystyle\int_{\lambda}^{x} g(t)\, dt$ and $\mathcal{F}(x) = \displaystyle\int_{\mu}^{x} f(t)\, dt$

Observe, $(\alpha) = \mathcal{F}(\mathcal{G}(\tau(x)))$.

Now, $\alpha' = \left(\mathcal{F}(\mathcal{G}(\tau(x)))\right)'$

$= \mathcal{F}'(\mathcal{G}(\tau(x))) \cdot \mathcal{G}'(\tau(x)) \cdot \tau'(x)$ by chain

$= f(\mathcal{G}(\tau(x))) \cdot g(\tau(x)) \cdot \sec^2(x)$ by FTOC(I).

$= \dfrac{1}{1+\sin^2\left(\int_{\lambda}^{\tan(x)} g(t)\, dt\right)} \cdot \dfrac{\sec^2(x)}{\tan^2(\tan(x))+\ln(\tan(x))}$.

$$\int \frac{1}{(w-1)^2}\,dw = \int (w-1)^{-2}\,dw$$
$$= -\frac{1}{w-1} + C$$

(A6) 4(d) $\displaystyle\int_1^\infty \frac{1}{(w-1)^2}\,dw.$

Let $f(w) = \frac{1}{(w-1)^2}$.

$f(w)$ is continuous at $w=0$. So,

$$\int_1^\infty f(w)\,dw = \int_0^\infty f(w)\,dw + \int_1^0 f(w)\,dw \text{; union integral property}$$

$$= \lim_{\alpha\to\infty}\int_0^{\alpha} f(w)\,dw + \lim_{\beta\to 1^-}\int_\beta^0 f(w)\,dw$$

$$= \lim_{\alpha\to\infty}\left[-\frac{1}{w-1}\right]_0^{\alpha} + \lim_{\beta\to 1^-}\left[-\frac{1}{w-1}\right]_\beta^0$$

$$= \lim_{\alpha\to\infty}\left[\frac{1}{\alpha-1} - 1\right] + \lim_{\beta\to 1^-}\left[1 - \frac{1}{\beta-1}\right]$$

$$= (0-1) + (1-\infty)$$

$$= -1 - \infty$$

Thus, diverges...