Basic Lectures on Emerging Design and Informatics III
Report 5

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Steady 1-dimensional advection-diffusion equation $T(x)$:

$$u\frac{dT}{dx} = \frac{1}{Pe}\frac{d^2T}{dx^2} \tag{1}$$

Given $u = 1.0$,

$$\frac{dT}{dx} = \frac{1}{Pe}\frac{d^2T}{dx^2} \tag{2}$$

The domain of x is $[0.0, 1.0]$, and additionally these conditions are given:

$$T(0.0) = 1.0 \tag{3}$$

$$T(1.0) = 0.0 \tag{4}$$

Task 1: Analytic solution of the differential equation

Since this is a homogenous differential equation with constant coefficients, we can be certain that T will have the form of $T(x) = e^{rx}$. If we make the substitution:

$$\frac{d(e^{rx})}{dx} = re^{rx} \tag{5}$$

$$\frac{d(e^{rx})}{dx} = r^2e^{rx} \tag{6}$$

$$re^{rx} = \frac{1}{Pe}r^2e^{rx} \tag{7}$$

$$re^{rx} - \frac{1}{Pe}r^2e^{rx} = 0 \tag{8}$$

$$e^{rx}r(1 - \frac{1}{Pe}r) = 0 \tag{9}$$

Since an exponential function never has roots,

$$r(1 - \frac{1}{Pe}r) = 0 \tag{10}$$

Thus, there are two solutions for r:

$$r_1 = 0 \tag{11}$$

$$1 - \frac{1}{Pe} r_2 = 0 \tag{12}$$

$$r_2 = Pe \tag{13}$$

and consequently we can see two possible solutions of the differential equation:

$$T_1(x) = e^{r_1 x} = 1 \tag{14}$$

$$T_2(x) = e^{r_2 x} = e^{Pe\ x} \tag{15}$$

Since all solutions of a second-order homogeneous differential equation can be expressed as a linear combination of two particular, linearly independent solutions, and T_1 and T_2 are linearly independent, we can get a general solution in the following way:

$$T(x) = aT_1(x) + bT_2(x) \tag{16}$$

$$= a + be^{Pe\ x} \tag{17}$$

Using the conditions from (3) and (4), we get the following equation system:

$$1.0 = a + be^{0.0Pe} \tag{18}$$

$$0.0 = a + be^{1.0Pe} \tag{19}$$

i.e.

$$1 = a + b \tag{20}$$

$$0 = a + be^{Pe} \tag{21}$$

$$1 = b - be^{Pe} \tag{22}$$

$$= b(1 - e^{Pe}) \tag{23}$$

$$b = \frac{1}{1 - e^{Pe}} \tag{24}$$

$$a = 1 - b \tag{25}$$

$$= 1 - \frac{1}{1 - e^{Pe}} \tag{26}$$

Thus,

$$T(x) = 1 - \frac{1}{1 - e^{Pe}} + \frac{1}{1 - e^{Pe}} e^{Pe\ x} \tag{27}$$

$$= \frac{1}{1 - e^{Pe}} \left((1 - e^{Pe}) - 1 + e^{Pe\ x}\right) \tag{28}$$

and we finally obtain

$$T(x) = \frac{e^{Pe\ x} - e^{Pe}}{1 - e^{Pe}} \tag{29}$$

Task 2-1: Discretisation of the differential equation using the central difference method

$$\frac{dT}{dx} = \frac{1}{Pe}\frac{d^2T}{dx^2} \tag{30}$$
$$\frac{T_{i+1} - T_{i-1}}{2\Delta x} = \frac{1}{Pe}\frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2} \tag{31}$$

We can see that the next value T_{i+1} depends on the previous two values (T_i, T_{i-1}), yet only T_0 and T_N are given. Thus, we will convert the equation into a form we can use in Gauss-Seidel algorithm, using the central difference method:

$$\frac{dT}{dx} = \frac{1}{Pe}\frac{d^2T}{dx^2} \tag{32}$$
$$\frac{T_{i+1} - T_{i-1}}{2\Delta x} = \frac{1}{Pe}\frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2} \tag{33}$$

$$T_{i-1}(\frac{1}{Pe}\frac{1}{(\Delta x)^2} + \frac{1}{2\Delta x}) - T_i\frac{1}{Pe}\frac{2}{(\Delta x)^2} + T_{i+1}(\frac{1}{Pe}\frac{1}{(\Delta x)^2} - \frac{1}{2\Delta x}) = 0 \tag{34}$$

Task 3-1: Discretisation of the differential equation using the forward difference method for the first-order differential, and the central difference method for the second-order differential

$$\frac{T_{i+1} - T_i}{\Delta x} = \frac{1}{Pe}\frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta x)^2} \tag{35}$$

$$T_{i-1}(\frac{1}{Pe}\frac{1}{(\Delta x)^2} + \frac{1}{2\Delta x}) - T_i\frac{1}{Pe}\frac{2}{(\Delta x)^2} + T_{i+1}(\frac{1}{Pe}\frac{1}{(\Delta x)^2} - \frac{1}{2\Delta x}) = 0 \tag{36}$$