Single Shell Free Water Elimination Model Implementation

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1 Introduction

This document details out an implementation of a Free Water Elimination model for Diffusion Tensor MRI. We are mainly following [Pasternak2009] with a few simplifications suggested in [Pasternak2014].

2 Bi-tensor Model

In this section we summarize the **Theory** section in [Pasternak2009].

For each voxel we have several readings. $\mathbf{S}_0$ is the signal acquired for the zero diffusion weighting from the **b0** image. $\mathbf{S}_k$ is the signal from the diffusion weighted image (DWI) when the gradient orientation $\mathbf{q}_k$ is applied. We calculate the attenuation $[\hat{\mathbf{A}}]_k = \mathbf{S}_k/\mathbf{S}_0$. The attenuation values are between 0 and 1.

For the **single compartment model** we assume that the attenuation $[\hat{\mathbf{A}}]_k$ all comes from tissue $[\mathbf{A}_{\text{tissue}}]_k$. For a Diffusion Tensor $\mathbf{D}$ this gives us

$$[\mathbf{A}_{\text{tissue}}(\mathbf{D})]_k = \exp(-b\mathbf{q}_k^T\mathbf{D}\mathbf{q}_k).$$

The b value in the equation is above is the b-value of our single shell. The gradient $\mathbf{q}_k$ is a vector of length 3 and $\mathbf{D}$ is a 3x3 symmetric matrix at each voxel. Let's denote by n the number of applied gradients that are non-colinear. The minimum n required is 6 as $\mathbf{D}$ has 6 parameters that need to be estimated at each voxel. Usually we have a large number (somewhere greater than 30 is customary).

If we had water instead of tissue at the voxels then the DTI model (above) simplifies and we get the same result in all directions. The matrix $\mathbf{D}$ becomes a scalar d and we get the same value of attenuation at every voxel.

$$[\mathbf{A}_{\text{water}}]_k = \exp(-bd),$$

where $d = 3 \cdot 10^{-3}$ mm^2/s for water at 37°C.

For the **bi-tensor model** we assume that each voxel has a two compartments. One compartment contains tissue and one contains free water. Let f be the fraction of the volume that is free water. Then we have

$$[\mathbf{A}_{\text{bi-tensor}}(\mathbf{D}, f)]_k = (f)[\mathbf{A}_{\text{tissue}}(\mathbf{D})]_k + (1-f)[\mathbf{A}_{\text{water}}]_k$$

f is the tissue fraction and a scalar that needs to be estimated for each voxel. $(0 \leq f < 1)$

Note: This notaion interchanges the values of 1 and $1-f$ in the multi-shell free water elimination implementation in DIPY. Here the free water maps are given by the values of $1-f$.

To fit the bi-tensor model we would like to estimate the best values of $\mathbf{D}$ and f that fit

$$[\hat{\mathbf{A}}]_k = [\mathbf{A}_{\text{bi-tensor}}(\mathbf{D}, f)]_k$$

We need to estimate a scalar f and a real symmetric matrix $\mathbf{D}$ at each voxel. This is solvable if $f = 0$ everywhere and we are looking at a single compartment model. However, if f is a parameter to be estimated at each voxel then there are infinitely many viable solutions. Choosing amongst them requires additional constraints.

3 Initialization

We first pick scalar values S_{tissue} and S_{water} which are typical values in the $b0$ image that we see for deep brain tissue and water respectively. Then we initialize f using

$$f = 1 - \frac{\log(S_0/S_{\text{tissue}})}{\log(S_{\text{water}}/S_{\text{tissue}})} = \frac{\log(S_0) - \log(S_{\text{tissue}})}{\log(S_{\text{water}}) - \log(S_{\text{tissue}})}$$

If $S_0 = S_{\text{water}}$ then we have $f = 0$ which means that the free water component is 1. If $S_0 = S_{\text{tissue}}$ then we have $f = 1$ and the free water component is 0.

We can also put additional constraints on f because we know that $\hat{A}_{tissue}$ is likely to have additional constraints. Lets say

$$A_{\text{min}} \leq \hat{A}_{\text{tissue}} \leq A_{\text{max}}.$$

Then we get

$$A_{\text{min}} \leq \frac{\hat{A} - \hat{A}_{\text{water}}}{f} + \hat{A}_{\text{water}} \leq A_{\text{max}}$$

This gives us

$$\frac{\min\{\hat{A}\} - \hat{A}_{\text{water}}}{A_{\text{max}} - \hat{A}_{\text{water}}} \leq f \leq \frac{\max\{\hat{A}\} - \hat{A}_{\text{water}}}{A_{\text{min}} - \hat{A}_{\text{water}}}$$

Note: The equation above in [Pasternak2009] is slightly different.

4 Variational Framework

$$L(\mathbf{D}, f) = \int_\Omega \left(\sum_{k=1}^{n} \left\| [\mathbf{A}_{\text{bi-tensor}}(\mathbf{D}, f)]_k - [\hat{\mathbf{A}}]_k \right\| + \alpha \sqrt{|\gamma(\mathbf{D})|} \right) d\Omega.$$

Here α is a parameter to control the regularization. We can split this up into two parts

$$L(\mathbf{D}, f) = \int_\Omega L_{\text{fid}}(\mathbf{D}, f) + \alpha L_{\text{reg}}(\mathbf{D}, f) d\Omega$$

5 Induced Metric

Instead of using the natural Riemannian metric on the spatial-feature space as given in [Pasternak2009], we use the Euclidean metric given in [Pasternak2014]

$$h = \begin{pmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & \beta & & & & & \\ & & & & \beta & & & & \\ & & & & & \beta & & & \\ & & & & & & \beta & & \\ & & & & & & & \beta & \\ & & & & & & & & \beta \end{pmatrix}$$

for some $\beta > 0$. Also the embedding is given by

$$\mathbf{x} = [x, y, z, D^{11}, D^{22}, D^{33}, \sqrt{2}D^{12}, \sqrt{2}D^{23}, \sqrt{2}D^{13}].$$

We can pull this metric back to get the induced metric. The components of the induced metric are given in Einstein's summation notation by

$$\gamma_{\mu\nu} = \partial_\mu \mathbf{x}^i \partial_\nu \mathbf{x}^j h_{ij}(\mathbf{x}).$$

The indices μ and ν take values 1, 2 or 3 which correspond to the x, y, and z directions. The co-ordinates of the spatial-feature space are denoted by $\mathbf{x}^i$ and the indices i and j take values 1 to 9. Since h is diagonal h_{ij} is only non-zero when $i = j$. So the double sum over i and j become a single sum over either one of the indices. Expanding this out we get

$$\gamma_{\mu\nu} = \sum_{i=1}^{3} \partial_\mu \mathbf{x}^i \partial_\nu \mathbf{x}^i + \beta \sum_{i=4}^{9} \partial_\mu \mathbf{x}^i \partial_\nu \mathbf{x}^i$$

The γ matrix is symmetric so we only need to write out 6 equations.

$$
\begin{aligned}
\gamma_{11} &= 1+\beta\sum_{i=4}^{9}\left(\mathbf{x}_x^i\right)^2 \\
\gamma_{22} &= 1+\beta\sum_{i=4}^{9}\left(\mathbf{x}_y^i\right)^2 \\
\gamma_{33} &= 1+\beta\sum_{i=4}^{9}\left(\mathbf{x}_z^i\right)^2 \\
\gamma_{12} &= \beta\sum_{i=4}^{9}\mathbf{x}_x^i\mathbf{x}_y^i \\
\gamma_{23} &= \beta\sum_{i=4}^{9}\mathbf{x}_y^i\mathbf{x}_z^i \\
\gamma_{13} &= \beta\sum_{i=4}^{9}\mathbf{x}_x^i\mathbf{x}_z^i
\end{aligned}
$$

The subscripts x, y, z denote partial derivatives w.r.t those co-ordinates. We denote the determinant of the matrix $\gamma_{\mu\nu}$ by $|\gamma|$. We also denote the inverse of the $\gamma_{\mu\nu}$ matrix by $\gamma^{\mu\nu}$. Since the inverse of an invertible matrix is the transpose of the co-factor matrix divided by the determinant, these two quantities are related by

$$\gamma^{\mu\nu} = \frac{C_{\nu\mu}}{|\gamma|}$$

The cofactor $C_{\nu\mu}$ is computed by $C_{\nu\mu} = (-1)^{\nu+\mu} M_{\nu\mu}$ where $M_{\nu\mu}$, a minor, is the determinant of the sub-matrix of γ with the ν-th row and μ-th column removed. Since γ is symmetric, its inverse matrix is also symmetric. This means that we do not need to take the transpose of the co-factor matrix and can also use the identity

$$\gamma^{\mu\nu} = \frac{C_{\mu\nu}}{|\gamma|}.$$

$$
\begin{aligned}
C_{11} &= (\gamma_{22}\gamma_{33} - \gamma_{23}^2) \\
C_{22} &= (\gamma_{11}\gamma_{33} - \gamma_{13}^2) \\
C_{33} &= (\gamma_{11}\gamma_{22} - \gamma_{12}^2) \\
C_{12} &= (-\gamma_{12}\gamma_{33} + \gamma_{13}\gamma_{23}) \\
C_{23} &= (-\gamma_{11}\gamma_{23} + \gamma_{13}\gamma_{12}) \\
C_{13} &= (\gamma_{12}\gamma_{23} - \gamma_{13}\gamma_{22})
\end{aligned}
$$

We also calculate the determinant as the expansion of the co-factors along the first row.

$$|\gamma| = \gamma_{11}C_{11} + \gamma_{12}C_{12} + \gamma_{13}C_{13}$$

6 Gradient Descent

6.1 Derivatives

We calculate the derivatives of L_{fid} for a fixed voxel

$$
\begin{aligned}
\frac{\partial}{\partial \mathbf{x}^j} L_{\text{fid}}(\mathbf{D}, f) &= \frac{\partial}{\partial \mathbf{x}^j}\left(\sum_{k=1}^{n} \left\| [\mathbf{A}_{\text{bi}}]_k - [\hat{\mathbf{A}}]_k \right\| \right) \\
&= \sum_{k=1}^{n} \frac{\partial}{\partial \mathbf{x}^j} \left\| [\mathbf{A}_{\text{bi}}]_k - [\hat{\mathbf{A}}]_k \right\| \\
&= 2\sum_{k=1}^{n} \left([\mathbf{A}_{\text{bi}}]_k - [\hat{\mathbf{A}}]_k \right) \frac{\partial}{\partial \mathbf{x}^j} [\mathbf{A}_{\text{bi}}]_k \\
&= 2\sum_{k=1}^{n} \left([\mathbf{A}_{\text{bi}}]_k - [\hat{\mathbf{A}}]_k \right) \frac{\partial}{\partial \mathbf{x}^j} \big(f[\mathbf{A}_{\text{tissue}}]_k + (1-f)\mathbf{A}_{\text{water}} \big) \\
&= 2f\sum_{k=1}^{n} \left([\mathbf{A}_{\text{bi}}]_k - [\hat{\mathbf{A}}]_k \right) \frac{\partial}{\partial \mathbf{x}^j} \exp(-b\mathbf{q}_k^T \mathbf{D} \mathbf{q}_k) \\
&= 2f\sum_{k=1}^{n} \left([\mathbf{A}_{\text{bi}}]_k - [\hat{\mathbf{A}}]_k \right) \exp(-b\mathbf{q}_k^T \mathbf{D} \mathbf{q}_k) \frac{\partial}{\partial \mathbf{x}^j} (-b\mathbf{q}_k^T \mathbf{D} \mathbf{q}_k) \\
&= -2fb\sum_{k=1}^{n} \left([\mathbf{A}_{\text{bi}}]_k - [\hat{\mathbf{A}}]_k \right) [\mathbf{A}_{\text{tissue}}]_k \left(\mathbf{q}_k \frac{\partial \mathbf{D}}{\partial \mathbf{x}^j} \mathbf{q}_k \right)
\end{aligned}
$$

Also we need to differentiate w.r.t f.

$$
\begin{aligned}
\frac{\partial}{\partial f} L_{\text{fid}}(\mathbf{D}, f) &= \frac{\partial}{\partial f}\left(\sum_{k=1}^{n} \left\| [\mathbf{A}_{\text{bi}}]_k - [\hat{\mathbf{A}}]_k \right\| \right) \\
&= \sum_{k=1}^{n} \frac{\partial}{\partial f} \left\| [\mathbf{A}_{\text{bi}}]_k - [\hat{\mathbf{A}}]_k \right\| \\
&= 2\sum_{k=1}^{n} \big([\mathbf{A}_{\text{bi}}]_k - [\hat{\mathbf{A}}]_k \big) \frac{\partial}{\partial f} [\mathbf{A}_{\text{bi}}]_k \\
&= 2\sum_{k=1}^{n} \big([\mathbf{A}_{\text{bi}}]_k - [\hat{\mathbf{A}}]_k \big) \frac{\partial}{\partial f} \big(f[\mathbf{A}_{\text{tissue}}]_k + (1-f)\mathbf{A}_{\text{water}} \big) \\
&= 2\sum_{k=1}^{n} \big([\mathbf{A}_{\text{bi}}]_k - [\hat{\mathbf{A}}]_k \big) \big([\mathbf{A}_{\text{tissue}}]_k - \mathbf{A}_{\text{water}} \big)
\end{aligned}
$$

6.2 Equations of motion

The equations of motion in [Pasternak2009][A8] simplify with this new choice of metric h as the Christoffel numbers are zero. For the 6 tensor elements $j \in 4, 5, \ldots, 9$,

$$\Delta \mathbf{x}^j = b \sum_{k=1}^{n} (\hat{\mathbf{A}} - \mathbf{A}_{\text{bi-tensor}}) \mathbf{A}_{\text{tissue}} \left(\mathbf{q}_k^T \frac{\partial \mathbf{D}}{\partial \mathbf{x}^j} \mathbf{q}_k \right) + \frac{\alpha}{\sqrt{|\gamma|}} \partial_\mu (\sqrt{|\gamma|} \gamma^{\mu\nu} \partial_\nu \mathbf{x}^j)$$

where Einstein's summation notation is used in the second term on the right hand side. For the fractional volume parameter we have

$$\Delta f = -b \sum_{k=1}^{n} (\hat{\mathbf{A}} - \mathbf{A}_{\text{bi-tensor}}) (\mathbf{A}_{\text{tissue}} - \mathbf{A}_{\text{water}})$$

6.3 Beltrami Operator

Here we write out the expression for the Beltrami operator Δ_γ

$$\Delta_\gamma \mathbf{x}^j = \frac{1}{\sqrt{|\gamma|}} \partial_\mu (\sqrt{|\gamma|} \gamma^{\mu\nu} \partial_\nu \mathbf{x}^j)$$

This expression is in Einstein Summation notation and μ and ν go from 1 to 3. So we have 9 expressions that we consider 3 at a time.

Setting $\mu = 1$ we get three terms by letting ν go from 1 to 3.

$$\begin{aligned}
& \frac{1}{\sqrt{|\gamma|}} \frac{\partial}{\partial x} (\sqrt{|\gamma|} \gamma^{11} \mathbf{x}_x^j + \sqrt{|\gamma|} \gamma^{12} \mathbf{x}_y^j + \sqrt{|\gamma|} \gamma^{13} \mathbf{x}_z^j) \\
= \; & \frac{1}{\sqrt{|\gamma|}} \frac{\partial}{\partial x} \left(\frac{C_{11}}{\sqrt{|\gamma|}} \mathbf{x}_x^j + \frac{C_{12}}{\sqrt{|\gamma|}} \mathbf{x}_y^j + \frac{C_{13}}{\sqrt{|\gamma|}} \mathbf{x}_z^j \right) \\
= \; & \frac{1}{\sqrt{|\gamma|}} \frac{\partial}{\partial x} \left(\frac{p^j}{\sqrt{|\gamma|}} \right), \quad \text{Setting} \quad p^j = C_{11} \mathbf{x}_x^j + C_{12} \mathbf{x}_y^j + C_{13} \mathbf{x}_z^j \\
= \; & \frac{p_x^j}{|\gamma|} - \frac{p^j \gamma_x}{2|\gamma|^2}, \quad \text{Using the quotient rule.}
\end{aligned}$$

Setting $\mu = 2$ and $q^j = C_{12}\mathbf{x}^j_x + C_{22}\mathbf{x}^j_y + C_{23}\mathbf{x}^j_z$, we get the next three terms as

$$\frac{q^j_x}{|\gamma|} - \frac{q^j\gamma_x}{2|\gamma|^2}$$

Similarly, setting $\mu = 3$ and $r^j = C_{13}\mathbf{x}^j_x + C_{23}\mathbf{x}^j_y + C_{33}\mathbf{x}^j_z$, we get the last three terms as

$$\frac{r^j_x}{|\gamma|} - \frac{r^j\gamma_x}{2|\gamma|^2}$$

Note: The definition of p^j and q^j is a generalization of the ones given in the discretization of the Beltrami flow given in [Kimmel] (Equations 10.4 and 10.5).

Putting it all together

$$\Delta_\gamma \mathbf{x}^j = \frac{1}{|\gamma|}(p^j_x + q^j_y + r^j_z) - \frac{1}{2|\gamma|^2}(p^j\gamma_x + q^j\gamma_y + r^j\gamma_z).$$

References

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