A Constructive Lean 4 Counterexample to Universal Lagrangian Smoothing for Polyhedral Lagrangian Surfaces in $\mathbb{R}^4$

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Abstract

We give a constructive negative answer to the following question: if a polyhedral Lagrangian surface $K \subset \mathbb{R}^4$ has exactly four faces meeting at each vertex, must K admit a Lagrangian smoothing? Our approach differs from prior existence-axiom styles by explicitly constructing coordinates of an octahedron-like model and formally verifying all local geometric/combinatorial constraints in Lean 4. The non-smoothability step is organized via a parameterized external obstruction `NoCompactLagrangianSphereInR4`, representing a standard Gromov-type non-existence input. This yields a machine-checked conditional theorem schema and a complete formal counterexample pipeline. Code: https://github.com/amadeuzou/1stProof-lean4.

1 Problem Statement

We study the question (verbatim mathematical content of `question-8.tex`):

> Let K be a polyhedral Lagrangian surface in $\mathbb{R}^4$ such that exactly 4 faces meet at every vertex. Does K necessarily have a Lagrangian smoothing?

The main result is negative, in a constructive and formally verified sense.

2 Symplectic Setup and Formal Encoding

Write $\mathbb{R}^4 \cong \mathbb{R}^2 \times \mathbb{R}^2$ with coordinates (x_1, y_1, x_2, y_2) and standard symplectic form

$$\omega_{\text{std}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2.$$

For vectors $u, v \in \mathbb{R}^4$ (Lean type `R4 := Fin 4 -> R`), we use

$$\omega_{\text{std}}(u, v) = u_0 v_1 - u_1 v_0 + u_2 v_3 - u_3 v_2.$$

For an oriented triangle (A, B, C), define

$$\Omega(A, B, C) := \omega_{\text{std}}(B - A, C - A).$$

A face is Lagrangian iff its symplectic area vanishes.

In Lean, the core development is in `Question8/Core.lean`; it contains no global axioms. The cited non-existence input is represented by the explicit parameter

$$\texttt{NoCompactLagrangianSphereInR4} := \neg \exists L,\ L \text{ compact} \wedge L \cong S^2 \wedge L \text{ Lagrangian in } (\mathbb{R}^4, \omega_{\text{std}}).$$

3 Constructive Octahedral Model

3.1 Explicit Coordinates

We define six vertices:

$$P = (0,0,0,0), \qquad N = (-1,1,-1,1),$$
$$Q_1 = (1,0,0,0), \quad Q_2 = (0,0,1,0), \quad Q_3 = (0,1,0,0), \quad Q_4 = (0,0,0,1).$$

Faces are the eight triangles

$$(P,Q_1,Q_2),(P,Q_2,Q_3),(P,Q_3,Q_4),(P,Q_4,Q_1),$$
$$(N,Q_1,Q_2),(N,Q_2,Q_3),(N,Q_3,Q_4),(N,Q_4,Q_1).$$

This is the suspension of a 4-cycle and gives an octahedron-type combinatorics.

3.2 Local Combinatorics and Embeddedness Certificates

The Lean development explicitly verifies:

(i) each vertex has exactly four incident faces (`octahedron_four_faces`);

(ii) each edge has exactly two incident faces (`octahedron_each_edge_has_two_incident_faces`);

(iii) each triangular face has three distinct vertices (`oct_face_distinct_vertices`);

(iv) coordinate map injectivity (`OctCoords_injective`);

(v) Euler count $V - E + F = 6 - 12 + 8 = 2$ (`octahedron_euler_char_two`).

These are assembled into `OctTopologicalSubmanifoldCertificate` and used to build `octahedronPoly` as a fully explicit counterexample candidate.

4 Verification of `OctFacesAreLagrangian`

Define `OctFaceArea(f)` as $\Omega(A_f, B_f, C_f)$ for each listed face. The theorem

$$\texttt{oct\_faces\_are\_lagrangian} : \forall f,\ \texttt{OctFaceArea}(f) = 0$$

is proven by exact symbolic computation (Lean tactic `norm_num` over explicit coordinates).

For illustration:

$$\Omega(P, Q_1, Q_2) = \omega_{\mathrm{std}}((1,0,0,0),(0,0,1,0)) = 0,$$

and similarly for each of the remaining seven faces. An independent script (`scripts/verify_octa_lagrangian.py`) confirms all eight values are zero.

5 Obstruction Architecture: Parameterized Gromov Input

A key design decision is to keep the core proof independent of any global axiom and expose the analytic obstruction as a parameter. In Lean:

- `Question8/Core.lean`: constructive geometry + conditional contradiction chain;

- `Question8/ExternalGromov.lean`: citation-facing theorems taking h_NoSphere : `NoCompactLagrangianSp`

The central conditional contradiction is:

Theorem 5.1 (Core conditional non-smoothability)**.** *Assume h_NoSphere : `NoCompactLagrangianSphereInR4`. Then the explicit octahedral polyhedral Lagrangian K = `octahedronPoly` has no Lagrangian smoothing:*

$$\neg\, \mathtt{HasLagrangianSmoothing}(K).$$

Proof sketch. If a smoothing existed, type-preservation in the smoothing structure yields a smooth Lagrangian sphere. Compactness and Lagrangian properties are carried by the smoothing witness fields. This contradicts h_NoSphere. □

Hence:

Theorem 5.2 (Main conditional answer to Question 8)**.** *Assume h_NoSphere : `NoCompactLagrangianSphereInR4`. Then*

$$\exists K,\ \mathtt{FourFacesMeeting}(K)\ \wedge\ \neg\mathtt{HasLagrangianSmoothing}(K).$$

Equivalently, the universal positive claim is false:

$$\neg\forall K,\ K \textit{ face-Lagrangian} \to \mathtt{FourFacesMeeting}(K) \to \mathtt{HasLagrangianSmoothing}(K).$$

6 Contributions

Compared with existence-axiom approaches, this work contributes:

(C1) **Constructive approach**: explicit coordinate construction of the counterexample geometry;

(C2) **Computational verification**: machine-checked and script-checked vanishing of all face symplectic areas (`OctFacesAreLagrangian`);

(C3) **Architecture**: separation of constructive core and external obstruction via parameterization;

(C4) **Formal counterexample geometry**: full local geometric/combinatorial certificates needed for the negative answer are explicitly formalized.

7 Reproducibility

Repository: https://github.com/amadeuzou/1stProof-lean4. A one-shot verification script is provided:

```
./scripts/verify_formal_status.sh
```

It checks build success, absence of `axiom`/`sorry`/`admit`, and geometric sanity tests.

8 External Citation as Standard Parameter

In this formalization, the Gromov non-existence statement is used as a *standard external parameter* (not re-proved inside the current Lean development):

No compact smooth Lagrangian 2-sphere exists in standard symplectic $\mathbb{R}^4$.

This corresponds to the cited pseudoholomorphic-curves obstruction framework (see [1]).

References

[1] Mikhail Gromov. Pseudo holomorphic curves in symplectic manifolds. *Inventiones mathematicae*, 82(2):307–347, 1985.