Detailed Balance from Exchange Relations: Formal Verification of the Stationary Measure Construction for Multi-Species ASEP

Amadeu Zou
amadeuzou@gmail.com

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Abstract

We present a Lean 4 formalization of a stationary-measure construction for a ring multi-species ASEP model on restricted sectors. The core theorem proves a strict local-to-global pipeline: exchange relations of Macdonald type imply local detailed balance, which implies global balance and stationarity. We then complete the probability layer by proving nonnegativity, normalization, and nontriviality after uniformization to a discrete-time kernel. A central design principle is *interface decoupling*: special-function identities are treated as explicit assumptions, while Markov-chain conclusions are derived with no hidden analytic steps. This yields a reusable theorem schema suitable for literature transfer: any polynomial family satisfying the interface assumptions induces a verified stationary distribution of the form F^*/P^*. Code: https://github.com/amadeuzou/1stProof-lean4.

1 Introduction

The multi-species ASEP (m-ASEP) on finite rings is a standard model in integrable probability and interacting particle systems. In many constructions, explicit stationary weights are provided through exchange identities connected to Macdonald-type structures. However, the full probability-theoretic chain is often spread across separate arguments: local identities, positivity side conditions, normalization, and Markov-kernel validity.

This paper provides a machine-checked consolidation of that chain. Our formal development is not aimed at replacing the classical formulas. Instead, it provides a precise proof interface that isolates where special-function input is used and certifies the resulting stochastic conclusions.

Contributions.

1. A formal local-to-global theorem: exchange relations imply detailed balance and stationarity.
2. A formal probability completion: nonnegative rates, normalized stationary law, and nontrivial transitions.
3. A literature-facing interface theorem: under explicit assumptions on F^* and P^*, we obtain a certified discrete Markov chain with stationary law F^*/P^*.
4. A complete Lean 4 artifact with successful project build and no unfinished placeholders.

2 Model on Restricted Sectors

2.1 State space

Fix $m, n \geq 2$, and let λ be a bounded word of length n over species set $\{0, \ldots, m-1\}$, satisfying the restricted-sector conditions used in the formal model. The state space is

$$S_n(\lambda) := \{\sigma\lambda : \sigma \in \mathfrak{S}_n\},$$

implemented as a finite subtype (`SnLambdaState`).

2.2 Local transitions

For an adjacent index $a = (i, i+1)$, define

$$\mathsf{rate}_a^+ = tx_i - x_{i+1}, \qquad \mathsf{rate}_a^- = x_i - tx_{i+1}, \qquad \mathsf{ratio}_a = \frac{\mathsf{rate}_a^+}{\mathsf{rate}_a^-}.$$

The local edge kernel (`edgeKernel`) contributes only when w' is obtained from w by the adjacent swap at a. The full continuous-time rate (`transitionRate`) is the finite sum over all adjacent a.

3 Exchange Relations and Balance Equations

3.1 Exchange assumption

Definition 3.1 (Exchange relation)**.** For a weight function π, the exchange relation at adjacent inversion a is

$$\pi(s_a w) = \mathsf{ratio}_a \, \pi(w).$$

3.2 Local detailed balance

Proposition 3.2 (Local detailed balance)**.** *Under denominator nonvanishing and the exchange relation,*

$$\pi(w)\, q(w, s_a w) = \pi(s_a w)\, q(s_a w, w)$$

holds for each adjacent swap edge.

Proof sketch. The Lean theorem `Question3.local_detailed` proves a direct algebraic identity by rewriting both sides into the same rational expression in (x, t), using the explicit edge-rate formulas and the exchange ratio. $\square$

3.3 Global balance

Theorem 3.3 (Global balance from local balance)**.** *If Proposition 3.2 holds for all adjacent edges, then*

$$\sum_{u \in S_n(\lambda)} \pi(u)\, q(u, v) = \pi(v) \sum_{z \in S_n(\lambda)} q(v, z), \qquad \forall v \in S_n(\lambda).$$

Proof sketch. Summing the local edge equalities over adjacent indices and states gives the global identity. This is formalized in `Question3.stationary_global_balance_explicit`. $\square$

4 Probability Completion

4.1 Rate nonnegativity interface

Definition 4.1 (Rate-pair nonnegativity)**.** `RatePairNonneg` requires $\mathsf{rate}_a^+ \geq 0$ and $\mathsf{rate}_a^- \geq 0$ for all adjacent a.

Proposition 4.2. *Under* `RatePairNonneg`*, all edge kernels and all total transition rates are nonnegative.*

Formal anchors. See the nonnegativity lemmas in `Question3` (listed explicitly in Section 6). □

4.2 Uniformization and discrete kernel

Given a finite continuous-time rate q, define a bound $B > 0$ and

$$P(u, v) = \frac{q(u, v)}{B} + \mathbf{1}_{u=v}\left(1 - \frac{\sum_z q(u, z)}{B}\right).$$

The formal development proves:

1. $P(u, v) \geq 0$,
2. $\sum_v P(u, v) = 1$,
3. stationarity transfer from q to P,
4. existence of a strictly positive off-diagonal transition.

These are implemented by four kernel lemmas in `Macdonald.Bridge.FinalTheorem` (listed explicitly in Section 6).

5 Literature Interface and Main Theorems

5.1 Interface theorem

The structure `FstarCandidateOnRestricted` packages: restricted-sector data, exchange-side hypotheses, a candidate π, and nontrivial local dynamics. The literature-facing closure theorem is:

Theorem 5.1 (Conditional closure theorem)**.** *The literature-closure theorem states that: if a candidate satisfies exchange assumptions, rate-pair nonnegativity, and normalized nonnegative π, then there exists a discrete-time Markov chain on $S_n(\lambda)$ whose stationary law is exactly F^*/P^*, with nontrivial off-diagonal dynamics.*

5.2 Repository-final bundled statements

For the canonical restricted $q = 1$ model, the project provides explicit final theorems:

1. `question3_complete_restricted_qOne` (continuous time),
2. `question3_complete_restricted_qOne_discrete` (discrete time).

Each theorem packages: nonnegative kernel, stationary distribution, explicit F^*/P^* formula, and nontrivial transitions.

6 Lean Theorem Map and Verification Status

Lean theorem	Mathematical role
`Question3.local_detailed`	Local detailed balance identity for adjacent swaps.
`Question3.stationary_global_balance_explicit`	Global balance equation from local identities.
`Question3.transitionRate_nonneg_of_ratePairNonneg`	Kernel nonnegativity from rate-pair assumptions.
`Macdonald.Bridge.paper_main_restricted_qOne_discrete_of_literature_assumptions`	Conditional literature-interface closure theorem.
`Macdonald.Bridge.question3_complete_restricted_qOne_discrete`	Final discrete stationary-chain theorem.
`Macdonald.Bridge.question3_complete_restricted_qOne`	Final continuous-time packaged theorem.

Repository verification checklist:

1. `lake build` succeeds on the project.
2. no `sorry`, `admit`, or custom `axiom` placeholders appear in project Lean files.

7 Discussion

The principal methodological contribution is a clear separation between:

1. algebraic identities from special-function theory,
2. stochastic consequences for Markov-chain stationarity.

This separation makes the proof transportable: new polynomial models can be plugged in by proving the interface assumptions, after which the probability theorem follows automatically.

A natural next step is full in-system formalization of interpolation Macdonald polynomial constructions, so that exchange and positivity assumptions become internal derivations rather than external interfaces.

References

[1] T. M. Liggett, *Interacting Particle Systems*, Springer, 1985.

[2] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Oxford University Press, 1995.

[3] The mathlib Community, *The Lean Mathematical Library (mathlib4)*, https://github.com/leanprover-community/mathlib4.