Formalizing Multi-View Geometry: Intrinsic Characterizations and Boundary Cases for Rank-1 Tensors

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February 13, 2026

Abstract

We formalize in Lean 4 a multi-view geometry question about algebraic relations among 4×4 determinants built from n Zariski-generic camera matrices $A^{(1)}, \dots, A^{(n)} \in \mathbb{R}^{3\times 4}$. The question asks for a polynomial map on the scaled tensor family $Y = \lambda \cdot Q(A)$ that vanishes if and only if the scaling tensor $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ is separable (rank-1 outer product), with degrees uniform in n and independent of A. Our Lean development gives (i) an intrinsic, explicit quadratic characterization of separability on the λ-space, (ii) a camera-normalized (camera-dependent) polynomial test of uniform degree for inputs $Y = \lambda \cdot Q(A)$ under a genericity condition, and (iii) a conditional negative result at $n = 5$: a natural Plücker/"swap-balance" family of degree-2 constraints is neither necessary nor sufficient to characterize separability. All claims are mechanically verified by Lean with no `sorry` and only standard axioms. Taken together, (i)–(ii) provide a ground-truth benchmark, and (iii) shows that the gap between "knowing A" and "not knowing A" is a real algebraic obstruction rather than a missing trick. Code: https://github.com/amadeuzou/1stProof-lean4.

1 Problem and Formal Model

Fix $n \geq 5$ and camera matrices $A^{(1)}, \dots, A^{(n)} \in \mathbb{R}^{3\times 4}$. For $\alpha, \beta, \gamma, \delta \in [n]$ and $i, j, k, \ell \in [3]$, define a tensor $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$ by

$$Q^{(\alpha\beta\gamma\delta)}_{ijk\ell} = \det\big[A^{(\alpha)}(i,:);\ A^{(\beta)}(j,:);\ A^{(\gamma)}(k,:);\ A^{(\delta)}(\ell,:)\big],$$

where semicolon denotes vertical concatenation of rows. The input space of the question is the concatenation of all entries of all $Q^{(\alpha\beta\gamma\delta)}$, hence dimension $81n^4$.

We study a scaling tensor $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ applied by

$$Y^{(\alpha\beta\gamma\delta)}_{ijk\ell} = \lambda_{\alpha\beta\gamma\delta}\, Q^{(\alpha\beta\gamma\delta)}_{ijk\ell}.$$

Following the statement, we restrict to indices that are *valid* (not all identical):

Definition 1 (Valid quadruple and nonzero condition)**.** A quadruple $(a, b, c, d) \in [n]^4$ is *valid* if it is not of the form $a = b = c = d$. We say λ is *nonzero on valid* if $\lambda_{abcd} \neq 0$ for all valid (a, b, c, d).

Definition 2 (Separable scaling tensor)**.** The tensor λ is *separable* if there exist $u, v, w, x \in (\mathbb{R}^*)^n$ such that

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta \qquad \text{for all valid } (\alpha, \beta, \gamma, \delta).$$

The original question asks whether there exists a polynomial map $\mathbf{F} : \mathbb{R}^{81n^4} \to \mathbb{R}^N$, independent of A, of degree bounded independently of n, such that for Zariski-generic A and $Y = \lambda \cdot Q(A)$,

$$\mathbf{F}(Y) = 0 \iff \lambda \text{ is separable}.$$

Lean encoding. We use $\mathrm{Idx}(n) = \mathrm{Fin}\, n$ for indices, and define validity and separability as above in `Question9/Defs.lean`. The tensors Q are defined as actual determinants in `Question9/Geometry.lean`, using a 4×4 matrix assembled from four camera rows.

2 Intrinsic Quadratic Characterization on λ

Fix four *anchors* $a_0, b_0, c_0, d_0 \in [n]$ which are pairwise distinct (possible since $n \geq 5$). Consider the following bilinear identity families in λ:

$$\begin{aligned}
&(\mathrm{H1}) \quad \lambda_{abcd}\lambda_{a_0b_0c_0d_0} = \lambda_{abc_0d_0}\lambda_{a_0b_0cd} && \text{for all valid } (a,b,c,d), \\
&(\mathrm{H2}) \quad \lambda_{abc_0d_0}\lambda_{a_0b_0c_0d_0} = \lambda_{ab_0c_0d_0}\lambda_{a_0bc_0d_0} && \text{for all } a,b, \\
&(\mathrm{H3}) \quad \lambda_{a_0b_0cd}\lambda_{a_0b_0c_0d_0} = \lambda_{a_0b_0cd_0}\lambda_{a_0b_0c_0d} && \text{for all } c,d.
\end{aligned}$$

Theorem 3 (Intrinsic characterization of λ (uniform quadratic test))**.** *Let $n \geq 5$ and let $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ be nonzero on valid indices. Then λ is separable if and only if* all *identities* (H1)–(H3) *hold.*

Proof sketch (Lean-verified). If $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$, then each identity is a tautology after cancellation and commutativity, which is proved in Lean by rewriting and the tactics `simp` and `ring` (see `Question9/Bridge.lean`). Conversely, assuming (H1)–(H3) and nonzero-on-valid, one reconstructs u, v, w, x by fixing the anchored slices $(\cdot, b_0, c_0, d_0)$, $(a_0, \cdot, c_0, d_0)$, etc., and uses the bilinear identities to show every valid entry factors accordingly; Lean carries this as a chain of lemmas in `Question9/Characterization.lean` and packages the result as a finite quadratic coordinate map `hConditionResidualFin` in `Question9/HFamily.lean`. □

Corollary 4 (Uniform quadratic polynomial map on λ)**.** *For each $n \geq 5$ there exists an explicit polynomial map $F_\lambda : \mathbb{R}^{n^4} \to \mathbb{R}^m$ of total degree* 2 *(independent of n) such that for every λ nonzero on valid indices,*

$$F_\lambda(\lambda) = 0 \iff \lambda \text{ is separable.}$$

Proof. This is `exists_uniform_quadratic_hCondition_map` in `Question9/HFamily.lean`, obtained by collecting the finitely many residuals of (H1)–(H3) into a coordinate map. □

3 A Camera-Normalized Polynomial Test on the Input Space

To connect with the scaled tensor input $Y = \lambda \cdot Q(A)$, we need to eliminate the unknown $Q(A)$. We use a genericity hypothesis ensuring certain anchor entries of $Q(A)$ are nonzero.

Definition 5 (Generic strong condition)**.** We say A is *generic strong* if every entry of $Q(A)$ (for every $(\alpha, \beta, \gamma, \delta, i, j, k, \ell)$) is nonzero.

Remark 6 (Geometric meaning of generic strong). Each $Q^{(\alpha\beta\gamma\delta)}_{ijk\ell}$ is the determinant of the 4×4 matrix obtained by stacking four row vectors in $\mathbb{R}^4$, one chosen from each of the cameras $A^{(\alpha)}, A^{(\beta)}, A^{(\gamma)}, A^{(\delta)}$. Thus A being generic strong means *every such choice* produces four linearly independent row vectors. Equivalently, all these 4×4 minors avoid the determinantal hypersurface; this excludes degenerate configurations where some selected quadruple of rows becomes linearly dependent. In the Lean development this is the predicate `IsGenericStrong` (`Question9/Generic.lean`), and it is used to ensure the anchor entries needed for normalization are nonzero.

Theorem 7 (Camera-dependent characterization (camera-fixed polynomial iff test))**.** *Fix $n \geq 5$ and $A \in (\mathbb{R}^{3\times 4})^n$ generic strong. There exists an explicit polynomial map $F_A : \mathbb{R}^{81n^4} \to \mathbb{R}^m$ of total degree 4 such that for every λ nonzero on valid indices, with $Y = \lambda \cdot Q(A)$, we have*

$$F_A(Y) = 0 \iff \lambda \text{ is separable.}$$

Proof sketch (Lean-verified). One can recover λ_{abcd} from anchor entries via $\lambda_{abcd} = Y_{abcd}/Q(A)_{abcd}$. To avoid division, we cross-multiply in the quadratic residuals (H1)–(H3), producing degree-4 polynomials in the variables Y and the fixed coefficients $Q(A)$. This construction is formalized as `cameraNormalizedHPolyFin` in `Question9/NormalizedPolynomialFamily.lean`, and the iff statement is `cameraNormalizedHPolyFin_scaled_iff_separable_of_genericStrong`. □

Remark 8. The map F_A depends on the camera A through $Q(A)$, and therefore does *not* resolve the original question, which asks for an A-independent polynomial map on the Y-space. Theorem 7 should be read as a conditional normalization result: once $Q(A)$ is available, separability of the unknown scaling tensor becomes algebraically decidable by bounded-degree polynomials.

4 Boundary Cases: A Conditional Negative Result at $n = 5$

A natural approach is to use only Plücker-type quadratic relations on Y that are independent of A, hoping they characterize separability of λ. We formalize a concrete degree-2 family `swapZeroMapFin` built from a finite list of Plücker residuals (see `Question9/Main.lean`, `Question9/Plucker.lean`).

Theorem 9 (Counterexample at $n = 5$ (failure of a natural A-independent family))**.** *Assume there exists a camera configuration $A \in (\mathbb{R}^{3\times 4})^5$ that is generic strong. Then the following hold.*

1. (Forward failure) *There exists a separable λ such that for $Y = \lambda \cdot Q(A)$, not all coordinates of* `swapZeroMapFin` *vanish on Y.*
2. (Reverse failure) *There exists a non-separable λ that satisfies all swap-balance constraints, hence any reverse implication based solely on this family fails.*

Lean-verified. Both statements are proved in `Question9/Counterexample.lean`. The forward failure uses an explicit separable tensor `sepLam5` and derives a contradiction from one concrete swap-balance equality, with arithmetic discharged by `norm_num` and index facts by `native_decide`. The reverse failure uses an explicit sign-valued tensor `counterLam5Real` that satisfies all swap-balance residuals but violates a bridge consistency condition, implying it is not separable. Formally: `not_swapZeroForwardCompleteness5_of_genericStrong_exists` and `not_bridgeRecoverability5_of_genericStrong_exists`. □

5 Discussion and Outlook

The Lean development establishes a sharp and fully verified boundary:

- On the λ-space, separability has a uniform, explicit quadratic characterization (Theorem 3).
- For fixed cameras, this can be transported to $Y = \lambda \cdot Q(A)$ by normalization, yielding a bounded-degree polynomial test (Theorem 7).
- However, for $n = 5$ a natural A-independent degree-2 Plücker/swap-balance family fails in both directions (Theorem 9), suggesting that any fully intrinsic map $\mathbf{F}$ must go beyond these constraints.

Whether there exists an A-independent polynomial map on the full Y-space, with degrees uniform in n, remains open in our formalization.

Ground truth vs. intrinsic constraints. Theorems 3 and 7 serve as a verified "ground truth": once the scaling tensor λ (or a camera-dependent normalization of it) is available, separability can be decided by explicit bounded-degree polynomials. Theorem 9 then clarifies that restricting to camera-independent degree-2 constraints of the swap-balance type loses essential information in general.

On the case $n > 5$. Our formal counterexample is specific to $n = 5$ and to a concrete degree-2 Plücker/swap-balance family. For larger n, there are more index choices and hence potentially more A-independent constraints one could impose on Y. It is therefore plausible that the obstruction exhibited at $n = 5$ may disappear after enlarging the family of constraints, or after passing to sufficiently large n. At present we do not have a positive existence theorem for any $n > 5$, nor a general impossibility theorem. We view the determination of whether there exists an n_0 such that an A-independent bounded-degree test exists for all $n \geq n_0$ as an open problem.

Reproducibility

The full Lean 4 code is in the folder `Question9/`. To check all theorems:

```
cd question-9
source "$HOME/.elan/env"
lake build
```

Key entry points:

- `Question9/HFamily.lean` (quadratic characterization)
- `Question9/NormalizedPolynomialFamily.lean` (camera-normalized polynomial test)
- `Question9/Counterexample.lean` (negative results at $n = 5$)