Geometry Comps Practice

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Problem 2019-J-II-6. Let X and Y be vector fields on $\mathbb{R}^3$, defined by

$$X = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \quad \text{and} \quad Y = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \tag{1}$$

Is there a coordinate chart $\varphi = (x_1, x_2, x_3) : U \to \mathbb{R}^3$ of the origin $0 \in \mathbb{R}^3$ such that

$$X|_U = \frac{\partial}{\partial x^1} \quad \text{and} \quad Y|_U = \frac{\partial}{\partial x^2}. \tag{2}$$

No, there exists no coordinate chart containing the origin $0 \in \mathbb{R}^3$ that satisfies the above conditions. To show this, we will compute the Lie Brackets of the given vector fields; let $\widetilde{X} = \partial/\partial x^1$ and $\widetilde{Y} = \partial/\partial x^2$. First, we observe that

$$\begin{aligned}
[X, Y] &= \frac{\partial}{\partial x}(y, z, 1) + x\frac{\partial}{\partial y}(y, z, 1) + y\frac{\partial}{\partial z}(y, z, 1) - y\frac{\partial}{\partial x}(1, x, y) - z\frac{\partial}{\partial y}(1, x, y) - \frac{\partial}{\partial z}(1, x, y) \\
&= x(1, 0, 0) + y(0, 1, 0) - y(0, 1, 0) - z(0, 0, 1) \\
&= x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}.
\end{aligned} \tag{3}$$

This means that the Lie Bracket of X and Y is not identically zero on any neighborhood of the origin. On the other hand, it is straightforward to see that the Lie Bracket of $\widetilde{X}$ and $\widetilde{Y}$ is identically zero on *all* of U. This is a contradiction. Therefore, such a coordinate chart cannot exist.