Geometry Crash Course

Aniruddha Madhava

December 21, 2025

Contents

1 Smooth Manifolds

1.1 Topological Manifolds

- **Def. (Topological Manifold)** A topological space M with the following properties:

  1. M is Hausdorff;
  2. M is second countable (i.e., has a countable basis for its topology);
  3. M is locally Euclidean of dimension n (i.e., for each $p \in M$, there exists a neighborhood $U \subset M$, an open set $\tilde{U} \subset \mathbb{R}^n$, and a homeomorphism $\varphi : U \to \tilde{U}$).

Exercise 1.1. Show that equivalent definitions of locally Euclidean spaces are obtained if instead of requiring U to be homeomorphic to an open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Let M be a topological space that satisfies conditions (1) and (2). ($\Leftarrow$) Suppose that for each $p \in M$, there exists a neighborhood U of p that is homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself. Since each of these are open subsets of $\mathbb{R}^n$, it follows that M satisfies condition (3). ($\Rightarrow$) Suppose that M satisfies conditions (1) - (3). Suppose that for some $p \in U \subset M$, $U \cong_\varphi \widetilde{U} \subseteq \mathbb{R}^n$. Since every open subset of $\mathbb{R}^n$ is the countable union of open balls in $\mathbb{R}^n$, suppose that $\widetilde{U} = \bigcup_1^\infty B_j$. Pick some ball B_{j_0} containing $\varphi(p)$. Then $V = \varphi^{-1}(B_{j_0})$ is an open neighborhood of p in M that is homeomorphic, under the map $\varphi|V : V \to B_{j_0}$, to the open ball B_{j_0}.

- **Def. (Coordinate Chart)** A pair (U, φ), where U is an open subset of M and $\varphi : U \to \widetilde{U}$ is a homeomorphism from U to an open subset $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^n$.

- **Def. (Precompact Subset)** Let X be a topological space. A subset $K \subset X$ is said to be *precompact* (or *relatively compact*) in X if its closure in X is compact. E.g., the subsets $(-1,1), (2,3], (4,5) \cup \{6\}$ are all precompact in $\mathbb{R}$, but the subset $(-1, \infty)$ is not.

- **Lem 1.6. (Topological Manifolds have Precompact Basis)** Every topological manifold has a countable basis of precompact coordinate balls.

  Let M be a topological n-manifold. Suppose $\varphi : M \to \widetilde{U} \subset \mathbb{R}^n$ is a global coordinate map. Let $\mathcal{B}$ be the collection of all open balls $B_r(x) \subset \mathbb{R}^n$ such that (1) r is rational, (2) x has rational coordinates, and (3) $\overline{B}_r(x) \subset \widetilde{U}$. By definition, each such ball is precompact in $\widetilde{U}$ and $\mathcal{B}$ is a countable basis for the topology of $\widetilde{U}$. Since φ is a homeomorphism, the collection $\mathcal{B}^{-1} = \{\varphi^{-1}(B) : B \in \mathcal{B}\}$ is a countable basis for the topology of M. Moreover, each of the sets in this collection is precompact in M: for each $B \in \mathcal{B}$, $\overline{\varphi^{-1}(B)} = \varphi^{-1}(\overline{B}) \subset M$; since φ^{-1} is continuous, $\varphi^{-1}(\overline{B})$ is compact in M. The restrictions of φ are the coordinate maps. In this case, we assumed that M had a global coordinate map, which might not necessarily be true in general.

  So, let M be an arbitrary topological n-manifold. By definition, every point of M is contained in the domain of a chart. Since every open cover of a second countable space has a countable subcover, M is covered by countably many charts $\{(U_i, \varphi_i)\}$. By the preceding argument, for each i, U_i has a countable basis of precompact coordinate balls, and the union of all these balls is a countable basis for the topology on M. Suppose $V \subset U_i$ is one of these precompact balls. Since the closure of V in U_i is compact, the closure must be closed in M. Hence, the closure of V in M is the same as the closure of V in U_i, so that V is precompact in M.

1.2 Smooth Manifolds

- **Def. (Transition Map between Charts)** Let M be a topological n-manifold. Let $(U, \varphi), (V, \varphi)$ be two charts such that $U \cap V \neq \varnothing$. The composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is called the *transition map* from φ to ψ.

- **Def. (Smoothly Compatible Charts)** Two charts (U, φ) and (V, ψ) are said to be *smoothly compatible* if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.

- **Def. ((Smooth) Atlases)** Let M be a topological n-manifold. (1) An *atlas* $\mathcal{A}$ for M is a collection of charts whose domains cover M; (2) $\mathcal{A}$ is said to be a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible.

- **Def. (Maximal Atlas):** A smooth atlas $\mathcal{A}$ on M is said to be *maximal* iff it is not contained in any strictly larger smooth atlas. I.e., any chart that is smoothly compatible with every chart in $\mathcal{A}$ is already contained in $\mathcal{A}$. A *smooth structure* on M is a maximal atlas.

- **Lem 1.10. (Smooth Atlases)** Let M be a topological manifold.
  (a) Every smooth atlas for M is contained in a unique maximal smooth atlas.
  (b) Two smooth atlases for M determine the same maximal smooth atlas if and only if their union is a smooth atlas.

  The proof of (b) was left as an exercise (see below). The proof of (a) is given. Let $\mathcal{A}$ be a smooth atlas for M, and let $\overline{\mathcal{A}}$ denote the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$. To show that $\overline{\mathcal{A}}$ is a smooth atlas, we need to show that any two charts of $\overline{\mathcal{A}}$ are smoothly compatible with each other, which to say that for any $(U, \varphi), (V, \psi) \in \overline{\mathcal{A}}$, $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is smooth.

  Let $x = \varphi(p) = \varphi(U \cap V)$ be arbitrary. Because the domains of the charts in $\mathcal{A}$ cover M, there is some chart (W, θ) in $\mathcal{A}$ such that $p \in W$. Since every chart in $\overline{\mathcal{A}}$ is smoothly compatible with (W, θ), both of the maps $\theta \circ \varphi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth where they are defined. Since $p \in U \cap V \cap W$, it follows that
  $$\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \phi^{-1}) \tag{1}$$
  is smooth on a neighborhood of x. Hence, $\psi \circ \varphi^{-1}$ is smooth in a neighborhood of each point in $\varphi(U \cap V)$. This concludes that $\overline{\mathcal{A}}$ is a smooth atlas. Now, we need to show that $\overline{\mathcal{A}}$ is maximal. But this is straightforward to see: any chart that is smoothly compatible with every chart contained in $\overline{\mathcal{A}}$ must be smoothly compatible with every chart contained in $\mathcal{A}$, and hence, must be contained in $\overline{\mathcal{A}}$. Therefore, $\overline{\mathcal{A}}$ is maximal. Uniqueness also follows in a straightforward way: suppose $\mathcal{B}$ is another maximal atlas containing $\mathcal{A}$. Then since every chart in $\mathcal{B}$ is smoothly compatible with every chart in $\mathcal{A}$, it follows that $\mathcal{B} \subset \overline{\mathcal{A}}$. Hence by maximality of $\mathcal{B}$, $\mathcal{B} = \overline{\mathcal{A}}$.

Exercise 1.4. Prove Lemma 1.10(b).

Let M be a topological n-manifold, $\mathcal{A}_1, \mathcal{A}_2$ be two smooth atlas on M, and $\overline{\mathcal{A}}_1, \overline{\mathcal{A}}_2$ the maximal smooth atlases determined by the two smooth atlases, respectively. This means that among all the smooth atlases that contain $\mathcal{A}_{1,2}$, $\overline{\mathcal{A}}_{1,2}$ are maximal, respectively. ($\Rightarrow$) Suppose that $\overline{\mathcal{A}}_1 = \overline{\mathcal{A}}_2$. This means that every chart contained in $\overline{\mathcal{A}}_2$ is smoothly compatible with every chart in $\mathcal{A}_1$; since $\mathcal{A}_2 \subset \overline{\mathcal{A}}_2$, this implies that every chart of $\mathcal{A}_2$ is smoothly compatible with every chart of $\mathcal{A}_1$. Likewise, since every chart in $\overline{\mathcal{A}}_1$ is smoothy compatible with every chart in $\mathcal{A}_2$, and $\mathcal{A}_1 \subset \overline{\mathcal{A}}_1$, it follows that every chart in $\mathcal{A}_1$ is smoothly compatible with every chart in $\mathcal{A}_2$. Hence, it follows that every pair of charts in $\mathcal{A}_1 \cup \mathcal{A}_2$ is smoothly compatible, showing that the union is a smooth atlas. ($\Leftarrow$) Suppose that $\mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas. This implies that every chart in $\mathcal{A}_2$ is smoothly compatible with every chart in $\mathcal{A}_1$, and thus, $\mathcal{A}_2 \subset \overline{\mathcal{A}}_1$; by maximality of $\overline{\mathcal{A}}_2$, $\overline{\mathcal{A}}_1 \subseteq \overline{\mathcal{A}}_2$. Likewise, we can show that $\mathcal{A}_1 \subset \overline{\mathcal{A}}_2$; by maximality of $\overline{\mathcal{A}}_2$, $\overline{\mathcal{A}}_2 \subseteq \overline{\mathcal{A}}_1$. Therefore, $\overline{\mathcal{A}}_1 = \overline{\mathcal{A}}_2$.

2 Smooth Maps

2.1 Smooth Functions and Smooth Maps

- **Def. (Smooth Function)** Let M be a smooth n-manifold. A function $f : M \to \mathbb{R}^k$ is *smooth* if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contains p and such that the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\widetilde{U} = \varphi(U) \subset \mathbb{R}^n$.

Exercise 2.3. Suppose M is a smooth manifold and $f : M \to \mathbb{R}^k$ is a smooth function. Show that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth for *every* smooth chart (U, φ) for M.

Suppose M is a smooth manifold and $f : M \to \mathbb{R}^k$ is a smooth function. Let (U, φ) be a smooth chart for M. By definition of a smooth function, for every $p \in U$, there exists a smooth chart (V_p, ψ_p) for M containing p in its domain such that $f \circ \psi_p^{-1} : \psi_p(V_p) \to \mathbb{R}^k$ is smooth. Since

$$U = \bigcup_{p \in U} (U \cap V_p) \implies \varphi(U) = \varphi\left(\bigcup_{p \in U} (U \cap V_p)\right) = \bigcup_{p \in U} \varphi(U \cap V_p), \tag{2}$$

it suffices to show that $f \circ \varphi^{-1}$ is smooth on $\varphi(U \cap V_p)$ for each p. Indeed, since (V_p, ψ_p) and (U, φ) are smoothly compatible for all p, $\psi_p \circ \varphi^{-1} : \varphi(U \cap V_p) \to \psi_p(U \cap V_p)$ is smooth. Since $f \circ \psi_p^{-1}$ is smooth on $\psi_p(V_p)$, it must be smooth on the subset $\psi_p(U \cap V_p)$. Therefore,

$$f \circ \varphi^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1}) : \varphi(U \cap V_p) \to \mathbb{R}^k \tag{3}$$

is smooth for all p. Thus, we conclude that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth.

- **Def. (Coordinate Representation)** Given a function $f : M \to \mathbb{R}^k$ and a chart (U, φ) for M, the function $\hat{f} : \varphi(U) \to \mathbb{R}^k$ defined by $\hat{f}(x) = f \circ \varphi^{-1}(x)$ is called the *coordinate representation* of f. By definition, f is smooth iff its coordinate representation is *smooth* in some smooth chart of M; but by the preceding exercise, the coordinate representation of f is smooth in *every* smooth chart of M.

- **Def. (Smooth Map between Manifolds)** Let M, N be smooth manifolds, and let $F : M \to N$ be any map. F is a *smooth map* if for every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.

Exercise 2.4. Let M and N be smooth manifolds, and let $F : M \to N$ be a map. If every point $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, show that F is smooth. Conversely, if F is smooth, show that its restriction to any open subset is smooth.

Let M and N be smooth manifolds, and let $F : M \to N$ be a map.

(i) Let $p \in M$, and let W be a neighborhood of p such that $F|_W$ is smooth. This means that there exist smooth charts (U, φ), where $p \in U \subset W$, and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and the composite function $\psi \circ (F|_W) \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth. Since $U \subset W$, it follows that $(F|_W)|_U = F|_U$. This means that $\psi \circ F|_U \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth. Hence, since p was arbitrary, we conclude that F is smooth.

(ii) Now assume that F is smooth, and let W be an arbitrary open subset of M. By definition of smoothness, for each $p \in W$, there exist smooth charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$ such that $F(U) \subset V$ and the composite function $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth. Since $\varphi(U \cap W)$ is an open subset of $\varphi(U)$, it follows that $\psi \circ F \circ \varphi^{-1}$ is smooth on $\varphi(U \cap W)$; that is, $F|_{(U \cap W)}$ is smooth. Hence, we have shown that for every $p \in W$, there exists a neighborhood of p such that the restriction of F to this neighborhood is smooth. Therefore, we conclude that $F|_W$ is smooth.

- **Lem 2.1. (Constructing Smooth Maps)** Let M and N be smooth manifolds, and let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Suppose that for each $a \in A$, we are given a smooth map $F_\alpha : U_\alpha \to N$ such that the maps agree on overlaps $F_\alpha|_{U_\alpha\cap U_\beta} = F_\beta|_{U_\alpha\cap U_\beta}$ for all α and β. Then there exists a unique smooth map $F : M \to N$ such that $F|_{U_\alpha} = F_\alpha$ for each $\alpha \in A$.

- **Lem. 2.2 (Smoothness Implies Continuity)** Every smooth map between smooth manifolds is continuous.

  Suppose $F : M \to N$ is smooth. By definition of smoothness, for each $p \in M$, we can choose smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \varphi(V)$ is a smooth map, and hence continuous. Since $\varphi : U \to \varphi(U)$ and $\psi : V \to \psi(V)$ are homeomorphisms, this implies in turn that

  $$F|_U = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi : U \to V, \tag{4}$$

  which is a composition of continuous maps, is continuous. Hence, since F is continuous in a neighborhood of each point, it is continuous on M.

- **Def. (Coordinate Representation)** Let $F : M \to N$ be a smooth map, and (U, φ), (V, ψ) be any smooth charts for M and N, respectively. Then we call $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ the coordinate representation of F with respect to the given coordinates.

Exercise 2.6. Suppose $F : M \to N$ is a smooth map between smooth manifolds. Show that the coordinate representation of F with respect to any pair of smooth charts for M and N is smooth.

Let $F : M \to N$ be a smooth map between smooth manifolds, and let (U, φ), (V, ψ) be any pair of smooth charts for M and N. Without loss of generality, assume that $F(U) \subset V$. Our task is to show that $\psi\circ F\circ\varphi^{-1}$ is smooth. Let $p \in U$. Since F is smooth, there exist smooth charts (W, θ) and (R, ϑ) containing p and $F(p)$, respectively, such that $F(W) \subset V \cap R$ and the composite function $\vartheta \circ F \circ \theta^{-1} : \theta(W) \to \vartheta(R)$ is smooth. Since $U \cap W$ is nonempty and the corresponding charts are smoothly compatible, the transition map $\theta \circ \varphi^{-1} : \varphi(U \cap W) \to \theta(U \cap W)$ is smooth. Likewise, the transition map $\psi \circ \vartheta^{-1}$ is smooth. Hence, the composite function:

$$\psi \circ F \circ \varphi^{-1} = (\psi \circ \vartheta^{-1}) \circ (\vartheta \circ F \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1}) \tag{5}$$

is smooth on $\varphi(U \cap W)$. By locality of smoothness, since for each $p \in U$, there exists a neighborhood on which $\psi \circ F \circ \varphi^{-1}$ is smooth, we conclude that the coordinate representation of F with respect to the given coordinates is smooth.

2.2 Smooth Covering Maps

- **Def. (Covering Map)** A surjective continuous map $\pi : \widetilde{M} \to M$ between connected, locally path connected spaces with the property that for every $p \in M$, there exists a neighborhood U that is *evenly covered* (i.e., U is connected, and each component of $\pi^{-1}(U)$ is mapped hormeomorphically onto U by π).

- **Def. (Smooth Covering Map)** Let $\widetilde{M}$ and M be connected smooth manifolds. A smooth covering map $\pi : \widetilde{M} \to M$ is a smooth surjective map with the property that every $p \in M$ has a connected neighborhood U such that each component of $\pi^{-1}(U)$ is mapped *diffeomorphically* onto U by π. In this instance also, we say that U is evenly covered.

- **Prop 2.9. (Properties of Smooth Coverings)**
  (a) Any smooth covering map is a local diffeomorphism and an open map.
  (b) An injective smooth covering map is a diffeomorphism.
  (c) A topological covering map is a smooth covering map if and only if it is a local diffeomorphism.

Exercise 2.12. If $\pi_1 : \widetilde{M}_1 \to M_1$ and $\pi_2 : \widetilde{M}_2 \to M_2$ are smooth covering maps, show that $\pi_1 \times \pi_2 : \widetilde{M}_1 \times \widetilde{M}_2 \to M_1 \times M_2$ is a smooth covering map.

Since $\widetilde{M}_{1,2}$ and $M_{1,2}$ are all connected smooth manifolds, $\widetilde{M}_1 \times \widetilde{M}_2$ and $M_1 \times M_2$ are all connected smooth manifolds. Now let $(p, q) \in M_1 \times M_2$. Since π_1 is surjective, there exists $\tilde{p} \in \widetilde{M}_1$ such that $\pi_1(\tilde{p}) = p$; likewise, there exists $\tilde{q} \in \widetilde{M}_2$ such that $\pi_2(\tilde{q}) = q$. Hence, $\pi_1 \times \pi_2 : (\tilde{p}, \tilde{q}) \mapsto (p, q)$, which shows that $\pi_1 \times \pi_2$ is surjective. Likewise, since π_1, π_2 are smooth, $\pi_1 \times \pi_2$ is smooth. Now we need to verify the evenly covered property for $\pi_1 \times \pi_2$.
Let $(p, q) \in M_1 \times M_2$. By the definition of smooth covering maps, there exist connected neighborhoods $p \in U \subset M_1$ and $q \in V \subset M_2$ such that each component of $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$ is mapped diffeomorphically onto U and V by π_1 and π_2, respectively. Since the product of connected open sets is connected, $U \times V$ is a connected neighborhood of (p, q). Then since $(\pi_1 \times \pi_2)^{-1}(U \times V) = \pi_1^{-1}(U) \times \pi_2^{-1}(V)$, the components of $(\pi_1 \times \pi_2)^{-1}(U \times V)$ are just the products of the components of $\pi_1^{-1}(U)$ with the components of $\pi_2^{-1}(V)$. Hence, since π_1 (π_2) maps each component of $\pi_1^{-1}(U)$ ($\pi_2^{-1}(V)$) diffeomorphically onto U (V), it follows that $\pi_1 \times \pi_2$ maps each component of $\pi_1^{-1}(U) \times \pi_2^{-1}(V)$ diffeomorphically onto $U \times V$. Therefore, $\pi_1 \times \pi_2$ is a smooth covering map.

- **Def. (Section of a Continuous Map)** If $\pi : \widetilde{M} \to M$ is any continuous map, a *section* of π is a continuous map $\sigma : M \to \widetilde{M}$ such that $\pi \circ \sigma = \mathrm{Id}_M$:

$$\begin{tikzcd} \widetilde{M} \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

Figure 1: Section of π.

- **Def. (Local Section of a Continuous Map)** A continuous map $\sigma : U \subset M \to \widetilde{M}$ such that $\pi \circ \sigma = \mathrm{Id}_U$.

2.3 Proper Maps

- **Def. (Proper Maps)** Let M, N be topological spaces. $F : M \to N$ is *proper* if for every compact set $K \subset N$, $F^{-1}(K)$ is compact.
- **Lem. 2.14 (Sufficient Condition for Proper Map I)** Suppose M is a compact space and N is Hausdorff space. Then every continuous map $F : M \to N$ is proper.

  Let $K \subset N$ be compact; since N is Hausdorff, K is closed. Then by continuity of F, $F^{-1}(K)$ is closed in M. Since M is compact, $F^{-1}(K)$ must be compact in K.
- **Def. (Saturated Subset)** A subset $A \subset M$ is said to be saturated with respect to a map $F : M \to N$ if $A = F^{-1}(F(A))$.
- **Lem. 2.15. (Sufficient Condition for Proper Map II)** Suppose $F : M \to N$ is a proper map between topological spaces, and $A \subset M$ is any subset that is saturated with respect to F. hen $F|_A : A \to F(A)$ is proper.

  Let $K \subset F(A)$ be compact. Since A is saturated, $(F|_A)^{-1}(K) = F^{-1}(K)$, which is compact since F is proper.

3 Tangent Vectors

- **Def. (Derivation at a Point)** Let $a \in \mathbb{R}^n$. A linear map $X : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a *derivation at a* iff it satisfies the following product rule:

$$X(fg) = f(a)Xg + g(a)Xf. \tag{6}$$

- **Lem 3.1. (Properties of Derivations)** Suppose $a \in \mathbb{R}^n$ and $X \in T_a(\mathbb{R}^n)$.
  (a) If f is a constant function, then $Xf = 0$.
  (b) If $f(a) = g(a) = 0$, then $X(fg) = 0$.

  (a) It suffices to show that if $f \equiv 1$, then $Xf = 0$. Indeed,

$$Xf = X(ff) = f(a)Xf + f(a)Xf = 2f(a)Xf = 2Xf, \tag{7}$$

  whence $Xf = 0$.
  (b) From the product rule, $X(fg) = f(a)Xg + g(a)Xf = 0 + 0 = 0$.

3.1 Tangent Vectors on a Manifold

- **Def. (Derivations on Manifolds)** Let M be a smooth manifold and $p \in M$. A linear map $X : C^\infty(M) \to \mathbb{R}$ is called a *derivation at p* if it satisfies

$$X(fg) = f(p)Xg + g(p)Xf \tag{8}$$

  for all $f, g \in C^\infty(M)$. The set of all derivations at p is called the *tangent space* to M at p, and is denoted by T_pM.

- **Lem 3.4. (Properties of Tangent Vectors on Manifolds)** Let M be a smooth manifold, and suppose $p \in M$ and $X \in T_pM$.
  (a) If f is a constant function, then $Xf = 0$.
  (b) If $f(p) = g(p) = 0$, then $X(fg) = 0$.

3.2 Pushforwards

- **Def. (Pushforward associated with a Map)** Let M, N be smooth manifolds and $F : M \to N$ a smooth map. For each $p \in M$, we define a map $F_* : T_pM \to T_{F(p)}(N)$, called the pushforward associated with F as follows:

$$(F_*X)(f) = X(f \circ F). \tag{9}$$

  It is straightforward to see that the pushforward is linear. It is also a derivation at p:

$$\begin{aligned}(F_*X)(fg) &= X(fg \circ F) = X((f \circ F)(g \circ F)) \\ &= (f \circ F)(p)X(g \circ F) + (g \circ f)(p)X(f \circ F) \\ &= (f \circ F)(p)(F_*X)(g) + (g \circ F)(p)(F_*X)(f).\end{aligned} \tag{10}$$

- **Lem 3.5. (Properties of Pushforwards)** Let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.
  (a) $F_* : T_pM \to T_{F(p)}N$ is linear.
  (b) $(G \circ F)_* = G_* \circ F_* : T_pM \to T_{(G \circ F)(p)}P$.
  (c) $(\mathrm{Id}_M)_* = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.
  (d) If F is a diffeomorphism, then $F_* : T_pM \to T_{F(p)}N$ is an isomorphism.

Exercise 3.2. Prove Lemma 3.5.

(a) Let $f \in C^\infty(N)$, $X, Y \in T_p(M)$, $c_{1,2} \in \mathbb{R}^n$. Then

$$\begin{aligned}(F_*(c_1X + c_2Y))(f) &= (c_1X + c_2Y)(f \circ F) \\ &= c_1X(f \circ F) + c_2Y(f \circ F) = c_1F_*(X)(f) + c_2F_*(Y)(f).\end{aligned} \tag{11}$$

(b) Let $f \in C^\infty(N)$, and $X \in T_p(M)$. Then

$$\begin{aligned}((G \circ F)_*X)(f) = X(f \circ (G \circ F)) &= X((f \circ G) \circ F) \\ &= (F_*X)(f \circ G) \\ &= (G_*(F_*X))(f) = ((G_* \circ F_*)X)(f).\end{aligned} \tag{12}$$

(c) Let $f \in C^\infty(N)$, and $X \in T_pM$. Then

$$(\mathrm{Id}_{M*}X)(f) = X(f \circ \mathrm{Id}_M) = X(f). \tag{13}$$

- **Prop. 3.6. (Tangent Space is Local)** Suppose M is a smooth manifold, $p \in M$, and $X \in T_pM$. If f and g are smooth functions in M that agree on some neighborhood of p, then $Xf = Xg$.

  Let $h = f - g$. It suffices to show that $Xh = 0$ by linearity of X whenever H vanishes is a neighborhood of p. Let $\psi \in C^\infty(M)$ be a smooth function that is identically 1 on the support of h and supported in $M \setminus \{p\}$. Because $\psi \equiv 1$ where h is nonzero, the product ψh is identically equal to h. Since $h(p) = \psi(p) = 0$, Lemma 3.4(b) implies that $Xh = X(\psi h) = 0$.

3.3 Computation in Coordinates

- **Def. (Basis for T_pM in Coordinates)** Let (U, φ) be a smooth coordinate chart on M; in particular, $\varphi : U \to \widetilde{U} \subset \mathbb{R}^n$ is a diffeomorphism. This implies that $\varphi_* : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism. We've seen that $T_{\varphi(p)}\mathbb{R}^n$ has as a basis consisting of all the derivations $\partial_{x^i}|_{\varphi(p)}$, $i = 1, \dots, n$. Therefore, the pushforward of these vectors under $(\varphi^{-1})_*$ form a basis for T_pM. We use the following notation:

$$\left.\frac{\partial}{\partial x^i}\right|_p = (\varphi^{-1})_* \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}. \tag{14}$$

  Indeed, if $f : U \to \mathbb{R}$ is smooth, then

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} (f \circ \varphi^{-1}) = \frac{\partial \hat{f}}{\partial x^i}(\hat{p}), \tag{15}$$

  where $\hat{f}$ is the coordinate representation of f, and $\hat{p} = (p^1, \dots, p^n) = \varphi(p)$ is the coordinate representation of p.

- **Def. (Pushforward in Coordinates I)** Consider a smooth map $F : U \to V$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open subsets of Euclidean spaces. Let $p \in U$. We will use $(x^1, \dots, x^n)$ to denote the coordinates in the domain and $(y^1, \dots, y^m)$ to denote the coordinates in the range. Then using the chain rule,

$$\begin{aligned}\left(F_* \left.\frac{\partial}{\partial x^i}\right|_p\right) f &= \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) \\ &= \frac{\partial f}{\partial y^j}(F(p)) \frac{\partial F^j}{\partial x^i}(p) \\ &= \left(\frac{\partial F^j}{\partial x^i} \left.\frac{\partial}{\partial y^j}\right|_{F(p)}\right) f.\end{aligned} \tag{16}$$

Since f was arbitrary, we conclude that

$$F_* \frac{\partial}{\partial x^i}\bigg|_p = \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\bigg|_{F(p)}. \tag{17}$$

In other words, the matrix of F_* in terms of the standard coordinate basis is given by

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}. \tag{18}$$

This is precisely the Jacobian matrix of F.

- **Def. (Pushforward in Coordinates II)** Let $F : M \to N$ be an arbitrary smooth map. Choosing smooth coordinate charts (U, φ) for M near p and (V, ψ) for N near $F(p)$, we obtain the coordinate representation $\widehat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$. Now we apply the chain rule:

$$\begin{aligned} F_* \frac{\partial}{\partial x^i}\bigg|_p &= F_*\left((\varphi^{-1})_* \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right) = (F \circ \varphi^{-1})_*\left(\frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right) \\ &= (\psi^{-1})_*\left(\widehat{F}_* \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right) = (\psi^{-1})_*\left(\frac{\partial \widehat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j}\bigg|_{\widehat{F}(\varphi(p))}\right) = \frac{\partial \widehat{F}^j}{\partial x^i}(\widehat{p}) \frac{\partial}{\partial y^j}\bigg|_{F(p)}. \end{aligned} \tag{19}$$

In other words, the pushforward of F is precisely the Jacobian matrix of its coordinate representation.

- **Obs. (Transformation of Vectors)** Suppose (U, φ) and (V, ψ) are two smooth charts on M, and let $p \in U \cap V$. We have two bases for the tangent space at p, namely $\{\partial/\partial x^i|_p\}$, where the coordinate functions of φ are (x^i), and $\{\partial/\partial \tilde{x}^i|_p\}$, where the coordinate functions of ψ are $(\tilde{x}^i)$. By **Def. (Pushforward in Coordinates I)**, we have

$$(\psi \circ \varphi^{-1})_* \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} = \frac{\partial \tilde{x}^j(x)}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j}\bigg|_{\psi(p)}. \tag{20}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial x^i}\bigg|_p &= (\varphi^{-1})_* \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} = (\psi^{-1} \circ \psi \circ \varphi^{-1})_* \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)} \\ &= (\psi^{-1})_*\left((\psi \circ \varphi^{-1})_* \frac{\partial}{\partial x^i}\bigg|_{\varphi(p)}\right) \\ &= (\psi^{-1})_*\left(\frac{\partial \widetilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \widetilde{x}^j}\bigg|_{\psi(p)}\right) = \frac{\partial \widetilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j}\bigg|_p. \end{aligned} \tag{21}$$

In particular, for any $X \in T_pM$, if

$$X = X^i \frac{\partial}{\partial x^i}\bigg|_p = \widetilde{X}^i \frac{\partial}{\partial \widetilde{x}^i}\bigg|_p, \tag{22}$$

then by the above result,

$$\widetilde{X}^i = X^i \frac{\partial \widetilde{x}^j}{\partial x^i}(\varphi(p)) = X^i \frac{\partial \widetilde{x}^j}{\partial x^i}(\widehat{p}), \tag{23}$$

where $\widehat{p} = \varphi(p)$ is the representation of p in x^i-coordinates.

4 Vector Fields

- **Def. (Tangent Bundle)** Let M be a smooth manifold. Then the *tangent bundle* of M is the disjoint union of the tangent spaces at all points of M:

$$TM := \coprod_{p\in M} T_pM. \tag{24}$$

A typical element of the tangent bundle is of the form (p, X), where $p \in M$ and $X \in T_pM$.

- **Lem. 4.1: (Tangent Bundle is a Manifold)** For any smooth n-manifold M, the tangent bundle TM has a natural topology and smooth structure that make it into a $2n$-dimensional smooth manifold. With this structure, the canonical projection map $\pi : TM \to M$, defined as the map $\pi : (p, X) \mapsto p$, is a smooth map.

We start by defining the smooth charts that will give TM its smooth structure. For some given smooth chart (U, φ) for M, let $(x^1, \ldots, x^n)$ denote the coordinate functions of φ, and define the map $\widetilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\widetilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n\right). \tag{25}$$

More precisely, the image set of $\widetilde{\varphi}$ is the set $\varphi(U)\times\mathbb{R}^n$, which is an open subset of $\mathbb{R}^{2n}$. [!! Complete Later !!]

Exercise 4.2. Suppose $F : M \to N$ is a smooth map. By examining the local expression (3.6) for F_* in coordinates, show that $F_* : TM \to TN$ is a smooth map.

Let $F : M \to N$ be a smooth map, and consider its pushforward $F_* : TM \to TN$. Our goal is to show that F_* is a smooth map. Let $p \in M$; by smoothness there exist smooth charts (U, φ) containing p in its domain, and (V, ψ) containing $F(p)$ in its domain such that $F(U) \subset V$ and the composite function $\psi \circ F \circ \varphi^{-1}$ is smooth. Let (x^i) denote the coordinate functions of φ and (y^j) denote the coordinate functions of ψ. Let $(\pi^{-1}(U), \widetilde{\varphi})$ and $(\pi^{-1}(V), \widetilde{\psi})$ be the corresponding smooth charts for TM and TN, respectively, where π is the canonical projection map from the tangent bundle of a manifold onto the manifold. These charts are equipped with the standard coordinates (x^i, v^i) and (y^j, w^j), respectively. Then in coordinates, the local expression for F_* is given by,

$$F_* \frac{\partial}{\partial x^i}\bigg|_p = \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\bigg|_{F(p)}. \tag{26}$$

This implies that

$$F_* : (p, v) \longmapsto \left(F^j(p), v^i \frac{\partial F^j}{\partial x^i}(p)\right). \tag{27}$$

Since F is smooth, each coordinate function F^j must be smooth. Likewise, each $\partial F^j/\partial x^i(p)$. Since the map $(x^i, v^i) \mapsto v^i \frac{\partial F^j}{\partial x^i}(x)$ is the finite sum of smooth functions, it must be smooth as well. Hence, we conclude that F_* is smooth.

- **Def. (Vector Field)** A section of the map $\pi : TM \to M$; i.e., a vector field is a continuous map $Y : M \to TM$, usually written $p \mapsto Y_p$, with the property that

$$\pi \circ Y = \mathrm{Id}_M. \tag{28}$$

- **Def. (Smooth Vector Field)** A smooth vector field.

- **Lem. 4.2 (Smoothness Criterion for Vector Fields)** Let M be a smooth manifold, and let $Y : M \to TM$ be a rough vector field. If $(U, (x^i))$ is *any* smooth coordinate chart on M, then Y is smooth on U if and only if its component functions with respect to this chart are smooth.

  Let (x^i, v^i) be the standard coordinates on $\pi^{-1}(U) \subset TM$ associated with the chart $(U, (x^i))$. By definition of the standard coordinate representation of Y,

$$\widehat{Y}(x) = \left(x^1, \dots, x^n, Y^1(x), \dots, Y^n(x)\right), \tag{29}$$

  where Y^i is the ith component function of Y in x^i-coordinates. Hence, smoothness of Y is equivalent to smoothness of the component functions.

- **Lem 4.5. (Extending a Tangent Vector)** Let M be a smooth manifold. If $p \in M$ and $X \in T_pM$, there is a smooth vector field $\widetilde{X}$ on M such that $\widetilde{X}_p = X$.

  Let (x^i) be smooth coordinates on a neighborhood U of p, and let $X^i \partial/\partial x^i|_p$ be the coordinate expression for X. Let ψ be a smooth bump function supported in U and with $\psi(p) = 1$. Then the vector field $\widetilde{X}$ defined by

$$\widetilde{X}_q = \begin{cases} \psi(q) X^i \frac{\partial}{\partial x^i}\Big|_q, & q \in U, \\ 0, & q \notin \operatorname{supp} \psi \end{cases} \tag{30}$$

  is a smooth vector field whose value at p is equal to X.

- **Def. (Set of all Smooth Vector Fields)** Let $\mathfrak{T}(M)$ denote the set of all smooth vector fields on M; $\mathfrak{T}(M)$ is a vector space under pointwise addition and scalar multiplication:

$$(aY + bZ)_p = aY_p + bZ_p. \tag{31}$$

  If $f \in C^\infty(M)$ and $Y \in \mathfrak{T}(M)$, we define $fY : M \to TM$ by

$$(fY)_p = f(p)Y_p. \tag{32}$$

Exercise 4.3. If Y and Z are smooth vector fields on M and $f, g \in C^\infty(M)$, show that $fY + gZ$ is a smooth vector field.

Let $(U, (x^i))$ be a smooth coordinate chart on M. Then in these coordinates,

$$Y = Y^i \frac{\partial}{\partial x^i}, \qquad Z = Z^i \frac{\partial}{\partial x^i}. \tag{33}$$

Then

$$fY + gZ = (fY^i + gZ^i) \frac{\partial}{\partial x^i}. \tag{34}$$

Since f, g, Y^i, Z^i are all smooth, and the product/sum of smooth functions is smooth, $fY^i + gZ^i$ is smooth for all i. Hence, since the component functions of $fY + gZ$ are smooth on any smooth coordinate chart on M, it follows that $fY + gZ$ is a smooth vector field on M.

- **Def. (Action of Vector Field on Functions)** If $Y \in \mathfrak{T}(M)$ and f is a smooth real-valued function defined on an open set $U \subset M$, we obtain a new function $Yf : U \to \mathbb{R}$ defined by

$$Yf(p) = Y_p f. \tag{35}$$

5 Cotangent Bundle

5.1 Covectors

- **Def. (Covector)** Let V be a finite-dimensional vector space. A *covector* on V is a real-valued linear functional on V; i.e., a linear map $\omega : V \to \mathbb{R}$. The vector space of all covectors on V is denoted by V^* and called the *dual* space to V.

- **Prop. 6.1. (Dual Basis)** Let V be a finite-dimensional vector space. If $(E_1, \dots, E_n)$ is any basis for V, then the covectors $(\varepsilon^1, \dots, \varepsilon^n)$, defined by

$$\varepsilon^i(E_j) = \delta^i_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{36}$$

form a basis for V^*, called the dual basis to (E_i). Therefore, $\dim V^* = \dim V$.

Exercise 6.1. Prove Proposition 6.1.

Assume the given hypotheses of Proposition 6.1. We must show that (ε^i) is a linearly independent collection of covectors that spans V^*; we start by showing linear independence. Suppose $a_1, \dots, a_n \in \mathbb{R}$ are scalars such that

$$a_1\varepsilon^1 + \cdots + a_n\varepsilon^n = 0. \tag{37}$$

Then allowing the left side to act on the V-basis vector E_j, for some $j \in \{1, \dots, n\}$,

$$0 = (a_1\varepsilon^1 + \cdots + a_n\varepsilon^n)(E_j) = \sum_1^n a_i\varepsilon^i(E_j) = a_j. \tag{38}$$

Since this is true for all $j \in \{1, \dots, n\}$, we conclude that each $a_j = 0$. Therefore, (ε^i) is linearly independent. Now let $\omega \in V^*$. For each $i = 1, \dots, n$, let $\omega(E_i) = a_i \in \mathbb{R}$. Then we claim that $\omega = a_i\varepsilon^i$ (where, we follow Einstein Summation Convention as usual). Indeed,

$$\begin{aligned} \omega(v) &= \omega(v^iE_i) = a_iv^i. \\ a_i\varepsilon^i(v) &= a_i\varepsilon^i(v^jE_j) = a_iv^j\varepsilon^i(E_j) = a_iv^j\delta^i_j = a_iv^i. \end{aligned} \tag{39}$$

Hence, it follows that (ε^i) spans V^*. Altogether, we have shown that this collection forms a basis for the dual space.

- **Def. (Dual Map)** Suppose V and W are vector spaces, and $A : V \to W$ is a linear map. Define a linear map $A^* : W^* \to V^*$, called the *dual map* of A by,

$$(A^*\omega)(X) = \omega(AX) \quad \text{for } \omega \in W^*, X \in V. \tag{40}$$

Exercise 6.2. Show that $A^*\omega$ is actually a linear functional on V, and that A^* is a linear map.

Let V, W be vector spaces, $A : V \to W$ a linear map, and $A^* : W^* \to V^*$ the dual map of A.

(i) Let $\omega \in W^*$ be a fixed covector, and let $X, Y \in V$, $a_1, a_2 \in \mathbb{R}$. Then

$$\begin{aligned} (A^*\omega)(a_1X + a_2Y) &= \omega(A(a_1X + a_2Y)) \\ &= \omega(a_1AX + a_2AY) \\ &= \omega(a_1AX) + \omega(a_2AY) \\ &= a_1\omega(AX) + a_2\omega(AY) \\ &= a_1(A^*\omega)(X) + a_2(A^*\omega)(Y), \end{aligned} \tag{41}$$

where the second inequality follows from linearity of A, and the third and fourth inequalities follow from linearity of ω. Hence, $A^*\omega$ is a linear functional for each $\omega \in W^*$.

(ii) Now let $X \in V$ be fixed, and let $\omega_1, \omega_2 \in V^*$, $a_1, a_2 \in \mathbb{R}$. Then

$$\begin{aligned}(A^*(a_1\omega_1 + a_2\omega_2))(X) &= (a_1\omega_1 + a_2\omega_2)(AX) \\ &= a_1\omega_1(AX) + a_2\omega_2(AX) \\ &= (a_1(A^*\omega_1) + a_2(A^*\omega_2))(X).\end{aligned} \tag{42}$$

Since X was arbitrary, it follows that A^* is a linear map.

- **Prop. 6.2. (Properties of Dual Maps)** The dual map satisfies the following properties:
  (a) $(A \circ B)^* = B^* \circ A^*$.
  (b) $(\mathrm{Id}_V)^* : V^* \to V^*$ is the identity map of V^*.

Exercise 6.3. Prove the preceding proposition.

(a) Let $B : V \to W$ and $A : W \to Y$ be linear maps, and A^*, B^* their corresponding dual maps. Let $\omega \in Y^*$ and $X \in V$. Then

$$\begin{aligned}((B^* \circ A^*)\omega)\,(X) &= B^*(A^*\omega)X) \\ &= A^*\omega(BX) = \omega(ABX) = \omega((A \circ B)X) \\ &= ((A \circ B)^*\omega)(X).\end{aligned} \tag{43}$$

Since X, ω were arbitrary, $(A \circ B)^* = B^* \circ A^*$.

(b) Let $\omega \in V^*$, and $X \in V$. Then

$$((\mathrm{Id}_V)^*\,\omega)(X) = \omega\,(\mathrm{Id}_V\,X) = \omega(X). \tag{44}$$

Since X was arbitrary, we conclude that $(\mathrm{Id}_V)^*\,\omega = \omega$ for all $\omega \in V^*$.

- **Def. (Natural Basis-Independent Map)** For each vector space V, there is a natural, basis-independent map $\xi : V \to V^{**}$, defined as follows: for each vector $X \in V$, define a linear functional $\xi(X) : V^* \to \mathbb{R}$ by

$$\xi(X)(\omega) = \omega(X), \qquad \text{for } \omega \in V^*. \tag{45}$$

Exercise 6.4. Let V be a vector space.
(a) For any $X \in V$, show that $\xi(X)(\omega)$ depends linearly on ω, so that $\xi(X) \in V^{**}$.
(b) Show that the map $\xi : V \to V^{**}$ is linear.

Let V be a vector space.

(a) Fix $X \in V$, and let $\omega_1, \omega_2 \in V^*$, $a_1, a_2 \in \mathbb{R}$. Then

$$\begin{aligned}\xi(X)(a_1\omega_1 + a_2\omega_2) &= (a_1\omega_1 + a_2\omega_2)(X) = a_1\omega_1(X) + a_2\omega_2(X) \\ &= a_1\xi(X)(\omega_1) + a_2\xi(X)(\omega_2).\end{aligned} \tag{46}$$

Hence, $\xi(X) \in V^{**}$.

(b) Fix $\omega \in V^*$, and let $X_1, X_2 \in V$, $a_1, a_2 \in \mathbb{R}$. Then

$$\begin{aligned}\xi(a_1X_1 + a_2X_2)(\omega) &= \omega(a_1x_1 + a_2x_2) = a_1\omega(X_1) + a_2\omega(X_2) \\ &= a_1\xi(X_1)(\omega) + a_2\xi(X_2)(\omega).\end{aligned} \tag{47}$$

Hence, since $\omega \in V^*$ was arbitrary, we conclude that $\xi : V \to V^{**}$ is linear.

- **Prop. 6.4 (Dual Dual Space is Isomorphic)** Let V be a finite-dimensional vector space. The map $\xi : V \to V^{**}$ is an isomorphism.

Since V and V^{**} have the same dimension, it suffices to check that ξ is injective. Suppose $X \in V \setminus \{0\}$. Extend X to a basis $(X = E_1, \ldots, E_n)$, and let $(\varepsilon^1, \ldots, \varepsilon^n)$ be the corresponding dual basis. Then

$$\xi(X)(\varepsilon^1) = \varepsilon^1(X) = \varepsilon^1(E_1) = 1 \neq 0, \tag{48}$$

so that $\xi(X) \neq 0$. Hence, the kernel is trivial, which proves injectivity.

5.2 Tangent Covectors on Manifolds

- **Def. (Cotangent Space)** Let M be a smooth manifold. For each $p \in M$, define the *cotangent space* at p, denoted by T_p^*M, to be the dual space to T_pM: $T_p^*M = (T_pM)^*$.

- **Obs. (Transformation Law for Covectors)** Suppose (U, φ) and (V, ψ) are two smooth charts on M, and let $p \in U \cap V$. As we saw before, we have two bases for the tangent space at p, namely $\{\partial/\partial x^i|_p\}$, where the coordinate functions of φ are (x^i), and $\{\partial/\partial \widetilde{x}^i|_p\}$, whre the coordinate functions of ψ are $(\widetilde{x}^i)$. Let (dx^i) and $(d\widetilde{x}^i)$ be the corresponding dual bases for the cotangent space at p. In particular, we have

$$\omega = \omega_i dx^i = \widetilde{\omega}_j d\widetilde{x}^j \iff \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) dx^i = \widetilde{\omega}_j d\widetilde{x}^j. \tag{49}$$

Using the transformation law for vectors,

$$\omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) dx^i = \omega\left(\frac{\partial \widetilde{x}^j}{\partial x^i}(p)\left.\frac{\partial}{\partial \widetilde{x}^j}\right|_p\right) dx^i = \widetilde{\omega}_j \frac{\partial \widetilde{x}^j}{\partial x^i}(p) dx^i = \widetilde{\omega}_j d\widetilde{x}^j. \tag{50}$$

Therefore, we conclude that

$$d\widetilde{x}^j = \frac{\partial \widetilde{x}^j}{\partial x^i} dx^i. \tag{51}$$

Contrast this with the transformation law for vectors.

5.3 The Cotangent Bundle

- **Def. (Cotangent Bundle)** The disjoint union

$$T^*M = \coprod_{p \in M} T_p^*M. \tag{52}$$

- **Lem. 6.6. (Smoothness Criteria for Covector Fields)** Let M be a smooth manifold, and let $\omega : M \to T^*M$ be a rough section.
  (a) If $\omega = \omega_i \lambda^i$ is the coordinate representation for ω in any smooth chart $(U, (x^i))$ for M, then ω is smooth on U if and only if its component functions ω_i are smooth
  (b) ω is smooth if and only if for every smooth vector field X on an open subset $U \subset M$, the function $\langle \omega, X \rangle : U \to \mathbb{R}$ defined by

$$\langle \omega, X \rangle (p) = \langle \omega_p, X_p \rangle = \omega_p(X_p) \tag{53}$$

  is smooth.

Exercise 6.5. Prove Lemma 6.6.

5.4 The Differential of a Function

Exercise 6.6. Let $f(x, y) = x^2$ on $\mathbb{R}^2$, and let X be the vector field

$$X = \text{grad}(f) = 2x\frac{\partial}{\partial x}. \tag{54}$$

Compute the coordinate expression of X in polar coordinates (on some open set on which they are defined) using (6.4) and show that it is *not* equal to

$$\frac{\partial f}{\partial r}\frac{\partial}{\partial r}+\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}. \tag{55}$$

Recall that (6.4) stated the following:

$$\left.\frac{\partial}{\partial x^i}\right|_p=\frac{\partial \widetilde{x}^j}{\partial x^i}(p)\left.\frac{\partial}{\partial \widetilde{x}^j}\right|_p. \tag{56}$$

Remember that polar coordinates are given by $(x,y)=(r\cos(\theta),r\sin(\theta))$. In particular, by (6.4),

$$\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}. \tag{57}$$

Using the polar coordinates,

$$\begin{aligned}\frac{\partial r}{\partial x}&=\frac{\partial}{\partial x}(\sqrt{x^2+y^2})\\&=\frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x=\frac{x}{r}=\cos(\theta).\\\frac{\partial \theta}{\partial x}&=\frac{\partial}{\partial x}(\arctan(x^{-1}y))\\&=-\frac{y}{x^2+y^2}=-\frac{r\sin(\theta)}{r^2}=-\frac{\sin(\theta)}{r}.\end{aligned} \tag{58}$$

Hence, this implies that

$$X=2r\cos^2(\theta)\frac{\partial}{\partial r}-2\sin(\theta)\cos(\theta)\frac{\partial}{\partial \theta}. \tag{59}$$

On the other hand,

$$\frac{\partial f}{\partial r}\frac{\partial}{\partial r}+\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}=2r\cos^2(\theta)\frac{\partial}{\partial r}-2r^2\sin(\theta)\cos(\theta)\frac{\partial}{\partial \theta}\neq X. \tag{60}$$

- **Def. (Differential of a Function)** Let f be a smooth real-valued function on a smooth manifold M. Define the covector field df, called the *differential* of f, by

$$df_p(X_p)=X_pf \quad \text{for } X_p\in T_pM. \tag{61}$$

- **Lem. 6.7. (Differential is Smooth Covector Field)** The differential of a smooth function is a smooth covector field.

- **Obs. (Differential in Coordinates)** Let (x^i) be smooth coordinates on an open subset $U\subset M$, and let (λ_i) be the corresponding coframe on U. Suppose that in coordinates, $df_p=A_i(p)\lambda^i|_p$ for some functions $A_i:U\to\mathbb{R}$. This implies the following:

$$\begin{aligned}A_i(p)=A_i(p)\lambda^i|_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)&=df_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)\\&=\frac{\partial f}{\partial x^i}(p).\end{aligned} \tag{62}$$

  This implies that

$$df_p=\frac{\partial f}{\partial x^i}(p)\lambda^i|_p. \tag{63}$$

  Taking f to be $x^j:U\to\mathbb{R}$, we obtain

$$dx^j|_p=\frac{\partial x^j}{\partial x^i}(p)\lambda^i|_p=\lambda^j|_p. \tag{64}$$

Therefore, this proves that

$$df|_p = \frac{\partial f}{\partial x^i}(p)dx^i|_p \iff df = \frac{\partial f}{\partial x^i}dx^i. \tag{65}$$

- **Prop. 6.9 (Properties of Differentials)** Let M be a smooth manifold, and let $f, g \in C^\infty(M)$.
  (a) For any constants a, b, $d(af + bg) = a\,df + b\,dg$.
  (b) $d(fg) = f\,dg + g\,df$.
  (c) $d(f/g) = (g\,df - f\,dg)/g^2$ on the set where $g \neq 0$.
  (d) If $J \subset \mathbb{R}$ is an interval containing the image of f, and $h : J \to \mathbb{R}$ is a smooth function, then $d(h \circ f) = (h' \circ f)\,df$.
  (e) If f is constant, then $df = 0$.

Exercise 6.7. Prove Proposition 6.9.

Let $a, b \in \mathbb{R}$, and $f, g \in C^\infty(M)$.

(a) Let $U \subseteq M$ be open, (x^i) smooth coordinates on U, and (dx^i) the corresponding coordinate coframe. Then

$$\begin{aligned} d(af + bg) &= \frac{\partial(af + bg)}{\partial x^i}dx^i = \left[\frac{\partial(af)}{\partial x^i} + \frac{\partial(bg)}{\partial x^i}\right]dx^i \\ &= \left[a\frac{\partial f}{\partial x^i} + b\frac{\partial g}{\partial x^i}\right]dx^i = a\,df + b\,dg. \end{aligned} \tag{66}$$

(b) We will work in coordinates as in (a). Then we observe that

$$\begin{aligned} d(fg) &= \frac{\partial(fg)}{\partial x^i}dx^i = \left[g\frac{\partial f}{\partial x^i} + f\frac{\partial g}{\partial x^i}\right]dx^i \\ &= g\,df + f\,dg. \end{aligned} \tag{67}$$

(c) Let $E = \{x \in M : g(x) \neq 0\}$. Then let $U \subseteq E$ be an open subset, (x^i) be smooth coordiates, and (dx^i) the corresponding coframe. Then

$$\begin{aligned} d(f/g) &= \frac{\partial(f/g)}{\partial x^i}dx^i = \frac{g\frac{\partial f}{\partial x^i} - f\frac{\partial g}{\partial x^i}}{g^2}dx^i \\ &= \frac{g\,df - f\,dg}{g^2}. \end{aligned} \tag{68}$$

(d) Let $J \subset \mathbb{R}$ be an interval containing the image of f, and $h : J \to \mathbb{R}$ be smooth. Let $U \subseteq M$ be open, (x^i) smooth coordinates on U, and (dx^i) the corresponding coordinate coframe. Then

$$d(h \circ f) = \frac{\partial(h \circ f)}{\partial x^i}dx^i = (h' \circ f) \cdot \frac{\partial f}{\partial x^i}dx^i = (h' \circ f)\,df, \tag{69}$$

where the second equality follows from the chain rule.

(e) It suffices to show that if $f = 1$, then $df = 0$. Indeed,

$$df = d(ff) = f\,df + f\,df = 2f\,df = 2\,df. \tag{70}$$

Hence, this proves that $df = 0$.

- **Prop. 6.10. (Functions with Vanishing Differentials)** If f is a smooth real-valued function on a smooth manifold M, then $df = 0$ if and only if f is constant on each component of M.

  It suffices to assume that M is connected and to show that $df = 0$ if and only if f is constant. Indeed, assume f is constant. Then by Prop. 6.9(e), $df = 0$. Now suppose $df = 0$, $p \in M$, and let $\mathcal{C} = \{q \in M : f(p) = f(q)\}$. If q is any point in $\mathcal{C}$, then let U be a smooth coordinate ball centered at q. By virtue of the differential being zero, we must have $\partial f/\partial x^i = 0$ in U for each i. This implies that f is constant on U. Hence, $\mathcal{C}$ is open. On the other hand, by continuity of f, $\mathcal{C}$ is closed. Since the only open and closed sets in a connected set are the empty set and M, it follows that $\mathcal{C} = M$; i.e., f is constant on M.

5.5 Pullbacks

- **Def. (Pullback of a Smooth Map)** Let $F : M \to N$ be a smooth map, and $F_* : T_pM \to T_{F(p)}N$ its pushforward. Then the pushforward induces a dual linear map $F_* : T^*_{F(p)}N \to T^*_pM$ defined by

$$(F^*\omega)(X_p) = \omega(F_*X), \quad \text{for } \omega \in T^*_{F(p)}N, X \in T_pM. \tag{71}$$

- **Obs. (Pullback in Coordinates)** Let $p \in M$ be arbitrary, and choose smooth coordinates (x^i) for M near p and (y^j) for N near $G(p)$. Then

$$G^*\omega = G^*(\omega_j dy^j) = (\omega_j \circ G)dG^j. \tag{72}$$

- **Ex. 6.14. (Example of Pullback)** Let $G : \mathbb{R}^3 \to \mathbb{R}^2$ be the map defined by

$$(u, v) = G(x, y, z) = (x^2y, y\sin(z)), \tag{73}$$

and let $\omega \in \mathfrak{T}^*(\mathbb{R}^2)$ be the covector field

$$\omega = u\,dv + v\,du. \tag{74}$$

First, we shall compute the differentials.

$$\begin{aligned} du &= \frac{\partial u}{\partial x^i}dx^i = 2xy\,dx + x^2\,dy. \\ dv &= \frac{\partial v}{\partial x^i}dx^i = \sin(z)\,dy + y\cos(z)\,dz. \end{aligned} \tag{75}$$

Therefore,

$$\begin{aligned} G^*\omega &= x^2y\,[\sin(z)\,dy + y\cos(z)\,dz] + y\sin(z)\left[2xy\,dx + x^2\,dy\right] \\ &= 2xy^2\sin(z)\,dx + 2xy^2\sin(z)\,dy + x^2y^2\cos(z)\,dz. \end{aligned} \tag{76}$$

5.6 Line Integrals

- **Prop. 6.16 (Diffeomorphism Invariance of the Integral)** Let ω be a smooth covector field on the compact interval $[a, b] \subset \mathbb{R}$. If $\varphi : [c, d] \to [a, b]$ is an increasing diffeomorphism (meaning that $t_1 < t_2$ implies $\varphi(t_1) < \varphi(t_2)$), then

$$\int_{[c,d]} \varphi^*\omega = \int_{[a,b]} \omega. \tag{77}$$

Let s be the standard coordinates on $[c, d]$ and t be the standard coordinates on $[a, b]$. We may write $\omega_t = f(t)\,dt$ for some smooth function $f : [a, b] \to \mathbb{R}$. Then using the pullback expression in local coordinates,

$$(\varphi^*\omega)_s = f(\varphi(s))\varphi'(s)\,ds. \tag{78}$$

Therefore,

$$\int_{[c,d]} \varphi^*\omega = \int_c^d f(\varphi(s))\varphi'(s)\,ds \underbrace{=}_{t:=\varphi(s)} \int_a^b f(t)\,dt = \int_{[a,b]} \omega. \tag{79}$$

Exercise 6.8. If $\varphi : [c, d] \to [a, b]$ is a decreasing diffeomorphism, show that $\int_{[c,d]} \varphi^*\omega = -\int_{[a,b]} \omega$.

The proof follows almost nearly identically to the proof from above. Suppose that s is the standard coordinate on $[c,d]$, and let t be the standard coordinate on $[a,b]$. We may assume that $\omega_t = f(t)\,dt$ for some smooth function $f : [a,b] \to \mathbb{R}$. Note that because of the decreasing property, $\varphi(c) = b$ and $\varphi(d) = a$. Hence,

$$\int_{[c,d]} \varphi^*\omega = \int_c^d f(\varphi(s))\varphi'(s)\,ds = \int_b^a f(t)\,dt = -\int_{[a,b]} f(t)\,dt = -\int_{[a,b]} \omega. \tag{80}$$

- **Def. (Curve Segment)** Let M be a smooth manifold. A *curve segment* is a continuous curve $\gamma : [a,b] \to M$ whose domain is a compact interval. It is a *smooth curve segment* if it has a smooth extension to an open interval containing $[a,b]$. A *piecewise smooth curve segment* is a piecewise smooth curve segment.
- **Def. (Line Integral)** Let $\gamma : [a,b] \to M$ be a smooth curve segment and ω a smooth covector field on M. The *line integral* of ω over γ is defined to be the real number
$$\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega. \tag{81}$$
If γ is *piecewise smooth*, then
$$\int_\gamma \omega = \sum_{i=1}^n \int_{[a_{i-1},a_i]} \gamma^*\omega, \tag{82}$$
where $\{a_i\}_0^n$ is a partition of $[a,b]$.
- **Prop. 6.18. (Properties of Line Integrals)** Let M be a smooth manifold. Suppose $\gamma : [a,b] \to M$ is a piecewise smooth curve segment and $\omega, \omega_1, \omega_2 \in \mathfrak{T}^*(M)$.
  (a) For any $c_1, c_2 \in \mathbb{R}$,
  $$\int_\gamma (c_1\omega_1 + c_2\omega_2) = c_1\int_\gamma \omega_1 + c_2\int_\gamma \omega_2. \tag{83}$$
  (b) If γ is a constant map, then $\int_\gamma \omega = 0$.
  (c) If $a < c < b$, then
  $$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega, \tag{84}$$
  where $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma_{[c,b]}$.

Exercise 6.9. Prove Proposition 6.18

Assume all of the hypotheses given in the statement of the proposition. Let $a = a_0 < a_1 < \cdots < a_n = b$ be a partition of $[a,b]$ such that γ is smooth on each subinterval.
(a) By linearity of pullbacks,

$$\begin{aligned}\int_\gamma (c_1\omega_1 + c_2\omega_2) &= \sum_{i=1}^n \int_{[a_{i-1},a_i]} \gamma^*(c_1\omega_1 + c_2\omega_2) = \sum_{i=1}^n \int_{[a,b]} [\gamma^*(c_1\omega_1) + \gamma^*(c_2\omega_2)] \\ &= \sum_{i=1}^n \left[\int_{[a_{i-1},a_i]} \gamma^*(c_1\omega_1) + \int_{[a_{i-1},a_i]} \gamma^*(c_2\omega_2)\right] \\ &= c_1\int_\gamma \omega_1 + c_2\int_\gamma \omega_2.\end{aligned} \tag{85}$$

(c) Let $a < c < b$; let $a = a_0' < a_1' < \cdots < a_n' = c$ and $c = a_{n+1}' < \cdots < a_m' = b$ be partitions of $[a,c]$ and $[c,b]$, respectively. Clearly $\{a_i'\}_{i=0}^m$ is also a partition of $[a,b]$. Then

$$\int_\gamma \omega = \sum_{i=1}^m \int_{[a_{i-1}',a_i']} \gamma^*\omega = \sum_{i=1}^n \int_{[a_{i-1}',a_i']} \gamma^*\omega + \sum_{i=n+1}^m \int_{[a_{i-1}',a_i']} \gamma^*\omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega. \tag{86}$$

(b) It suffices to assume that γ is a smooth curve segment. Let s be the standard coordinates on $[a,b]$. Then in local coordinates, $\gamma^*\omega = \omega(\gamma(s))\gamma'(s)\ ds = 0$ since $\gamma'(s) = 0$ for all s. Hence,

$$\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega = \int_{[a,b]} 0\ ds = 0. \tag{87}$$

5.7 Conservative Vector Fields

- **Def. (Exact Smooth Covector Field)** Let ω be a smooth covector field on a smooth manifold M. ω is *exact* if it is the differential of some $f \in C^\infty(M)$. The function f is called a *potential* for ω.
- **Def. (Conservative Covector Field)** A smooth covector field ω is *conservative* if the line integral of ω over *any* closed piecewise smooth curve segment is zero.
- **Lem. 6.23. (Conservative Covector Field Criterion I)** A smooth covector field ω is conservative if and only if the line integral of ω depends only on the endpoints of the curve, i.e., $\int_\gamma \omega = \int_{\widetilde{\gamma}} \omega$ whenever γ and $\widetilde{\gamma}$ are piecewise smooth curves are piecewise smooth curve segments with the same starting and ending points.

Exercise 6.10. Prove Lemma 6.23. [Observe that this would be much harder to prove if we defined conservative fields in terms of smooth curves instead of piecewise smooth ones.]

6 Submersions, Immersions, and Embeddings

6.1 Maps of Constant Rank

- **Def. (Rank of a Smooth Map)** Let M and N be smooth manifolds, and $F : M \to N$ a smooth map. The *rank* of F at $p \in M$ is the rank of the linear map $F_* : T_pM \to T_{F(p)}N$; this is equivalent to the rank of the matrix of partial derivatives of F in any smooth chart, or to the dimension of $\operatorname{Im} F_* \subset T_{F(p)}N$. I.e., the rank is equivalent to the maximum number of linearly independent rows/columns of the corresponding matrix.

- **Def. (Submersion)** A smooth map $F : M \to N$ such that F_* is surjective at each point, which is to say that $\operatorname{rank} F = \dim N$.

- **Def. (Immersion)** A smooth map $F : M \to N$ such that F_* is injective at each point; equivalently $\operatorname{rank} F = \dim M$.

- **Def. (Smooth Embedding)** An immersion $F : M \to N$ such that $F : M \to F(M) \subset N$ is a homeomorphism.

Exercise 7.2. Show that a composition of submersions is a submersion, a composition of immersions is an immersion, and a composition of smooth embeddings is a smooth embedding.

(i) Let $F : M \to N$ and $G : N \to P$ be submersions, where M is a smooth m-manifold, N is a smooth n-manifold, and P is a smooth p-manifold. Let U, V, T be smooth coordinate charts for M, N, and P, respectively, such that (wlog) $F(U) \subset V$ and $G(V) \subset T$. Then since $(G \circ F)_* = G_* \circ F_*$, in local coordinates, $(G \circ F)_*$ corresponds to the matrix product of an $p \times n$ matrix with an $n \times m$ matrix (the $p \times n$ matrix representing G_*, and the $n \times m$ matrix representing F_*); the rank of the $p \times n$ matrix is p, while the rank of the $n \times m$ matrix is n. Hence, by the properties of the rank of a matrix product, the matrix representation of $G_* \circ F_*$ has rank p, which proves that $(G \circ F)$ is a submersion.

(ii) Let $F : M \to N$ and $G : N \to P$ be immersions. Since F_* is injective and G_* is injective, and the composition of injective functions is injective,

$$(G \circ F)_* = G_* \circ F_* \tag{88}$$

is injective. Hence, $G \circ F$ is an immersion.

(iii) Let $F : M \to N$ and $G : N \to P$ be smooth embeddings. From (ii), $G \circ F$ is an immersion. Finally, since the composition of homeomorphisms is always another homeomorphism, we conclude that $G \circ F$ is a smooth embedding.

- **Prop. 7.4. (Smooth Embedding Criteria)** Suppose $F : M \to N$ is an injective immersion. If either of the following condition holds, then F is a smooth embedding with closed image:
  (a) M is compact.
  (b) F is a proper map.

7 Tensors

7.1 The Algebra of Tensors

- **Def. (Multilinear Function)** Suppose $V_1, \ldots, V_n$ and W are vector spaces. A map $F : V_1 \times \cdots \times V_k \to W$ is said to be *multlinear* if it is linear as a function of each variable separately:

$$F(v_1, \ldots, av_i + a'v_i', \ldots, v_k) = aF(v_1, \ldots, v_i, \ldots, v_k) + a'F(v_1, \ldots, v_i', \ldots, v_k). \tag{89}$$

- **Def. (Covariant k-Tensor)** Let V be a finite-dimensional real vector space, and let k be a natural number. A *covariant k-tensor* on V is a real-valued multilinear function of k elements of V:

$$T : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R}. \tag{90}$$

  The number k is called the *rank* of T.

- **Def. (Tensor Product)** We can build up covariant tensors of larger ranks as follows: let V be a finite-dimensional real vector space and let $S \in T^k(V)$, $T \in T^l(V)$. Define a map $S \otimes T : \underbrace{V \times \cdots \times V}_{k+l \text{ copies}} \to \mathbb{R}$ by

$$S \otimes T(X_1, \ldots, X_{k+l}) = S(X_1, \ldots, X_k)T(X_{k+1}, \ldots, X_{k+l}). \tag{91}$$

  We will check multilinearity. WLOG, assume $i \le k$. Then

$$\begin{aligned} S \otimes T(X_1, \ldots, aX_i + a'X_i', \ldots, X_k, \cdots, X_{k+1}) &= S(X_1, \ldots, aX_i + a'X_i', \ldots, X_k)T(X_{k+1}, \ldots, X_{k+l}) \\ &= \left[aS(X_1, \ldots, X_i, \ldots, X_k) + a'S(X_1, \ldots, X_i', \ldots, X_k)\right] T(X_{k+1}, \ldots, X_{k+l}) \\ &= a(S \otimes T)(X_1, \ldots, X_i, \ldots, X_{k+1}) + a'(S \otimes T)(X_1, \ldots, X_i', \ldots, X_{k+l}). \end{aligned} \tag{92}$$

  Hence, $S \otimes T$ is a covariant $(k + l)$-tensor.

Exercise 11.1. Show that the tensor product operation is bilinear and associative. More precisely, show that $S \otimes T$ depends linearly on each of the tensors S and T, and that $(R \otimes S) \otimes T = R \otimes (S \otimes T)$.

Let P, R, S, T be k, k, l, l-tensors, respectively. Then

$$\begin{aligned} (a_1P + a_2R) \otimes (a_3S + a_4T)(X_1, \ldots, X_{k+l}) &= (a_1P + a_2R)(X_1, \ldots, X_k) \cdot (a_3S + a_4T)(X_{k+1}, \ldots, X_{k+l}) \\ &= a_1P(X_1, \ldots, X_k) \cdot (a_3S + a_4T)(X_{k+1}, \ldots, X_{k+l}) \\ &\quad + a_2R(X_1, \ldots, X_k)(a_3S + a_4T)(X_{k+1}, \ldots, X_{k+l}) \\ &= a_1a_3P(X_1, \ldots, X_k)S(X_{k+1}, \ldots, X_{k+l}) \\ &\quad + a_1a_4P(X_1, \ldots, X_k)T(X_{k+1}, \ldots, X_{k+l}) \\ &\quad + a_2a_3P(X_1, \ldots, X_k)S(X_{k+1}, \ldots, X_{k+l}) \\ &\quad + a_2a_4P(X_1, \ldots, X_k)T(X_{k+1}, \ldots, X_{k+l}). \end{aligned} \tag{93}$$

Using the definition of the tensor products, we can simply the final expressions to see that the tensor product is, indeed, linear in each of the tensor terms.

- **Prop. 11.2. (Basis for T^kV)** Let V be a real vector space of dimension n, let (E_i) be any basis for V, and let (ε^i) be the dual basis. The set of all k-tensors of the form $\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}$ for $1 \le i_1 \le \cdots \le i_k \le n$ is a basis for T^kV, which therefore has dimension n^k.

  Let $\mathcal{B}$ denote the set $\{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} : 1 \le i_1 \le \cdots \le i_k \le n\}$. It suffices to show that $\mathcal{B}$ is linearly independent and spans T^kV. Let $T \in T^k(V)$. For any k-tuple of integers $(i_1, \ldots, i_k)$, where $1 \le i_j \le n$ for all $j = 1, \ldots, k$, define the number $T_{i_1 \cdots i_k}$ as follows:

$$T_{i_1 \cdots i_k} = T(E_{i_1}, \ldots, E_{i_k}). \tag{94}$$

We will show that $T = T_{i_1 \cdots i_k} \varepsilon^{i_1} \otimes \varepsilon^{i_k}$. Indeed,

$$\begin{aligned} T_{i_1 \cdots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}(E_{j_1}, \ldots, E_{j_k}) &= T_{i_1 \cdots i_k} \varepsilon^{i_1}(E_{j_1}) \cdots \varepsilon^{i_k}(E_{j_k}) \\ &= T_{j_1 \cdots j_k} \\ &= T(E_{j_1}, \ldots, E_{j_k}). \end{aligned} \tag{95}$$

By multilinearity, since a tensor is completely determined by its action on sequences of basis vectors, this proves the claim that $\mathcal{B}$ spans $T^k V$. Now, we must prove linear independence. But this is straightforward to show by letting $T_{i_1 \cdots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} = 0$ act on a sequence of basis vectors.

- **Def. (Free Vector Space)** Let S be a set. The *free vector space on* S, denoted by $\mathbb{R}\langle S \rangle$, is the set of all finite formal linear combinations of S with real coefficients. More precisely, a finite formal linear combination is a function $\mathcal{F} : S \to \mathbb{R}$ such that $\mathcal{F}(s) = 0$ for all but finitely many $s \in S$.

Exercise 11.2 (Characteristic Property of Free Vector Spaces). Let S be a set and W a vector space. Show that any map $F : S \to W$ has a unique extension to a linear map $\overline{F} : \mathbb{R}\langle S \rangle \to W$.

Let S be a set, W a vector space, and $F : S \to W$ an arbitrary map. Define the map $\overline{F} : \mathbb{R}\langle S \rangle \to W$ as follows: given a formal sum $\sum_{s \in S} \alpha_s s$, where $\alpha_s = 0$ for all but finitely many elements $s \in S$, let

$$\overline{F}\left(\sum_{s \in S} \alpha_s s\right) = \sum_{s \in S} \alpha_s F(s). \tag{96}$$

Since each $\alpha_s \in \mathbb{R}$ and $F(s) \in W$, it follows that $\sum_{s \in S} \alpha_s F(s) \in W$. First, we must show that $\overline{F}$ is a linear map. Let $\sum_{s \in S} \alpha_s s, \sum_{s \in S} \beta_s s \in \mathbb{R}\langle S \rangle$, where α_s, β_s are zero for all but finitely elements (not necessarily the same) of S. Then

$$\begin{aligned} \overline{F}\left(\sum_{s \in S} \alpha_s s + \sum_{s \in S} \beta_s s\right) &= \overline{F}\left(\sum_{s \in S} (\alpha_s + \beta_s) s\right) \\ &= \sum_{s \in S} (\alpha_s + \beta_s) F(s) \\ &= \sum_{s \in S} \alpha_s F(s) + \sum_{s \in S} \eta_s F(s) \\ &= \overline{F}\left(\sum_{s \in S} \alpha_s s\right) + \overline{F}\left(\sum_{s \in S} \beta_s s\right). \end{aligned} \tag{97}$$

Hence, $\overline{F}$ is linear. The proof of uniqueness follows as proceeds: if F extends to two linear maps $\overline{F}_1$ and $\overline{F}_2$, let these linear maps act on each element of S. By construction of these maps, it follows that $\overline{F}_1(s) = \overline{F}_2(s)$ for all $s \in S$. Since S is a basis for $\mathbb{R}\langle S \rangle$ and $\overline{F}_{1,2}$ are completely determined by their actions on the basis elements, we conclude that $\overline{F}_1 = \overline{F}_2$, and so this extension is unique.

- **Def. (Tensor Product of Vector Spaces)** Let V and W be finite-dimensional real vector spaces, and let $\mathcal{R}$ be the subspace of the free vector space $\mathbb{R}\langle V \times W \rangle$ spanned by all elements of the following forms:

$$\begin{aligned} \alpha(v, w) - (\alpha v, w), \\ \alpha(v, w) - (v, \alpha w), \\ (v, w) + (v', w) - (v + v', w) \\ (v, w) + (v, w') - (v, w + w;), \end{aligned} \tag{98}$$

for $\alpha \in \mathbb{R}$, $v, v' \in V$, and $w, w' \in W$. Define the *tensor product of* V and W, denoted $V \otimes W$ to be the quotient space $\mathbb{R}\langle V \times W \rangle / \mathcal{R}$. The equivalence class of an element $(v, w) \in V \otimes W$ is denoted by $v \otimes w$, and is called the *tensor product* of v and w.

- **Prop. 11.3. (Characteristic Property of Tensor Products)** Let V and W be finite dimensional real vector spaces. IF $A : V \times W \to X$ is a bilinear map into any vector space X, there is a unique linear map $\widetilde{A} : V \otimes W \to X$ such that the following diagram commutes:

$$\begin{tikzcd} V \times W \arrow[r, "A"] \arrow[d, "\pi"] & X \\ V \otimes W, \arrow[ur, "\widetilde{A}"] & \end{tikzcd}$$

  where $\pi(v, w) = v \otimes w$.

- **Prop. 11.4. (Other Properties of Tensor Products)** Let V, W, and X be finite-dimensional real vector spaces.
  (a) The tensor product $V^* \otimes W^*$ is canonically isomorphic to the space $B(V, W)$ of bilinear maps from $V \times W$ into $\mathbb{R}$.
  (b) If (E_i) is a basis for V and (F_j) is a basis for W, then the set of all elements of the form $E_i \otimes F_j$ is a basis for $V \otimes W$, which therefore has dimension $(\dim V)(\dim W)$.
  (c) There is a unique isomorphism $V \otimes (W \otimes X) \to (V \otimes W) \otimes X$ sending $v \otimes (w \otimes x)$ to $(v \otimes w) \otimes x$.

- **Cor. 11.5. (Space of Covariant k-Tensors and Tensor Products)** If V is a finite-dimensional real vector space, the space $T^k(V)$ of covariant k-tensors on V is canonically isomorphic to the k-fold tensor product $V^* \otimes \cdots \otimes V^*$.

- **Def. (Space of Contravariant k-Tensors)** Let V be a finite-dimensional real vector space, and define the space of all *contravariant* k-tensors on V to be the space

$$T_k(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}}. \tag{99}$$

  Because of the canonical identification $V = V^{**}$, one may think of an element of $T_k V$ as a multilinear function from $V^* \times \cdots V^*$ into $\mathbb{R}$.

7.2 Tensors and Tensor Fields on Manifolds

- **Def. (Various Tensor Bundles)** Let M be a smooth manifold. Define the following:
  (a) *Bundle of covariant k-tensors on M:*

$$T^k M = \coprod_{p \in M} T^k(T_p M). \tag{100}$$

  (b) *Bundle of contravariant l-tensors on M:*

$$T_l M = \coprod_{p \in M} T_l(T_p M). \tag{101}$$

  (c) *Bundle of mixed tensors of type $\binom{k}{l}$ on M:*

$$T^k_l M = \coprod_{p \in M} T^k_l(T_p M). \tag{102}$$

- **Def. (Smooth Tensor Fields)** A *smooth tensor field* is a smooth section of the above tensor bundles.

- **Obs. (Smooth Tensor Fields in Coordinates)** Given any smooth local coordinates (x^i) on M, sections of the above bundles can be written as:

$$\sigma = \begin{cases} \sigma_{i_1\cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}, & \sigma \in \mathcal{T}^k(M); \\ \sigma^{j_1\cdots j_l} \dfrac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \dfrac{\partial}{\partial x^{j_l}}, & \sigma \in \mathcal{T}_l(M). \\ \sigma^{j_1\cdots j_l}_{i_1\cdots i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k} \otimes \dfrac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \dfrac{\partial}{\partial x^{j_l}}, & \sigma \in \mathcal{T}^k_l(M). \end{cases} \tag{103}$$

- **Lem. 11.6. (Equivalent Conditions for Smooth Tensor Fields)** Let M be a smooth manifold, and let $\sigma : M \to T^k M$ be a rough section. The following are equivalent:
  (a) σ is smooth.
  (b) In any smooth coordinate chart, the composition functions of σ are smooth.
  (c) If $X_1, \ldots, X_k$ are smooth vector fields defined on an open subset $U \subset M$, then the function $\sigma(X_1, \ldots, X_k) : U \to \mathbb{R}$, defined by

$$\sigma(X_1, \ldots, X_k)(p) = \sigma_p(X_1|_p, \ldots, X_k|_p) \tag{104}$$

  is smooth.

7.3 Pullbacks of Smooth Tensor Fields

- **Def. (Pullback of a Smooth Map in Relation to Tensor Fields)** If $F : M \to N$ is a smooth map, for each integer $k \geq 0$ and each $p \in M$, we obtain a map $F_* : T^k(T_{F(p)}N) \to T^k(T_p M)$ called the pullback by

$$(F^* S)(X_1, \ldots, X_k) = S(F_* X_1, \ldots, F_* X_k). \tag{105}$$

- **Prop. 11.8. (Properties of Tensor Pullbacks)** Suppose $F : M \to N$ and $G : N \to P$ are smooth maps, $p \in M$, $S \in T^k(T_{F(p)}N)$, and $T \in T^l(T_{F(p)}N)$.
  (a)

8 Homotopy and the Fundamental Group

8.1 Homotopy

- **Def. (Homotopy of Maps)** Let X and Y be *topological* spaces, and $f, g \in C(X, Y)$. Then a *homotopy* from f to g is a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

- **Def. (Homotopy Relative to a Subset)** Let X and Y be topological spaces, and $A \subset X$ an arbitrary subspace. A homotopy H between maps $f, g : X \to Y$ is called a *homotopy relative to* A if

$$H(x, t) = f(x), \qquad \text{for all } x \in A,\, t \in I. \tag{106}$$

- **Def. (Path Homotopy)** Given two paths f, g on X, a *path homotopy* from f to g is a homotopy between the paths relative to the subset $\{0, 1\} \subset I$.

- **Def. (Fundamental Group)** The *fundamental group of* X based at $q \in X$, denoted by $\pi_1(X, q)$ is the set of all path classes of loops based at q, with operation defined by concatenation.

- **Def. (Simply Connected Topological Space)** Let X be a topological space. If X is path connected and $\pi_1(X)$ is trivial, then X is said to be *simply connected.*

Exercise 7.2. Let X be a topological space.
(a) Let $f, g : I \to X$ be two paths from p to q. Show that $f \sim g$ if and only if $f \cdot g^{-1} \sim c_p$.
(b) Show that X is simply connected if and only if any two paths in X with the same initial and terminal points are path homotopic.

Let X be a topological space.
(a) Let $f, g : I \to X$ be two paths from p to q. Suppose $f \sim g$. Then since $[f] = [g]$,

$$[g] \cdot [g^{-1}] = [c_p] \implies [f] \cdot [g^{-1}] = [c_p] \implies f \cdot g^{-1} \sim c_p. \tag{107}$$

On the other hand, if $f \cdot g^{-1} \sim c_q$, then

$$[f] \cdot [g^{-1}] = [c_p] = [g] \cdot [g^{-1}] \implies [f] = [g] \implies f \sim g. \tag{108}$$

(b) Suppose that X is a simply connected space, and let $f, g : I \to X$ be two paths from p to q. Then the product $f \cdot g^{-1}$ is well-defined and is a loop based at p. By simple connectedness, $f \cdot g^{-1} \sim c_p$. Hence, by part (a), we conclude that $f \sim g$. Now suppose that X is path connected and that any two paths in X that have the same initial and terminal points are path homotopic. Let γ be an arbitrary loop based at $p \in X$. Then by hypothesis, $\gamma \sim c_p$. Hence, $\pi_1(X, p)$ is trivial. By path connectedness, $\pi_1(X)$ is trivial, and so X is simply connected.

8.2 Homomorphisms Induced by Continuous Maps

- **Lem. 7.14. (Path Homotopy is Preserved by Composition with Continuous Maps)** The path homotopy relation is preserved by composition with continuous maps. That is, if $f_0, f_1 : I \to X$ are path homotopic and $\varphi : X \to Y$ is continuous, then $\varphi \circ f_0$ and $\varphi \circ f_1$ are path homotopic.

Exercise 7.5. Prove Lemma 7.14.

Suppose that $f_0, f_1 : I \to X$ are path homotopic, and that $\varphi : X \to Y$ is continuous. Let $H : I \times I \to X$ be the path homotopy from f_0 to f_1, and consider the map $\varphi \circ H : I \times I \to Y$. Since H and φ are continuous on their respective domains, it follows that $\varphi \circ H$ is continuous. Moreover, for any $s \in I$,

$$(\varphi \circ H)(s, 0) = (\varphi \circ f_0)(s), \quad \text{and} \quad (\varphi \circ H)(s, 1) = (\varphi \circ f_1)(s). \tag{109}$$

Hence, $\varphi \circ H$ is a path homotopy from $\varphi \circ f_0$ to $\varphi \circ f_1$.

- **Def. (Homomorphism Induced by a Continuous Map)** Let X and Y be topological spaces, and $\varphi : X \to Y$ a continuous map. The map $\varphi_* : \pi_1(X, q) \to \pi_1(X, \varphi(q))$ defined by $\varphi_*([f]) = [\varphi \circ f]$ is a group homomorphism, and is called the *homomorphism induced by* φ.

- **Prop. 7.16 (Properties of the Induced Homomorphsm)**
  (a) Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be continuous maps. Then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.
  (b) If $\mathrm{Id}_X : X \to X$ denotes the identity map of X, then for any $q \in X$, $(\mathrm{Id}_X)_*$ is the identity map of $\pi_1(X, q)$.

  (a) Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be continuous maps, $p \in X$, and $[f] \in \pi_1(X, p)$. Then

  $$(\psi_* \circ \varphi_*)([f]) = \psi_*([\varphi \circ f]) = [(\psi \circ \varphi) \circ f] = (\psi \circ \varphi)_*([f]). \tag{110}$$

  Since this is true for all $[f] \in \pi_1(X, p)$, $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.
  (b) Let $[f] \in \pi_1(X, q)$. Then

  $$(\mathrm{Id}_X)_* ([f]) = [\mathrm{Id}_X \circ f] = [f]. \tag{111}$$

  Since this is true for all $[f] \in \pi_1(X, q)$, we conclude that $(\mathrm{Id}_X)_*$ is the identity map of $\pi_1(X, q)$.

- **Cor. 7.17 (Induced Isomorphism)** Homeomorphic spaces have isomorphic fundamental groups; namely, if $\varphi : X \to Y$ is a homeomorphism, then $\varphi_* : \pi_1(X, q) \to \pi_1(X, q)$ is an isomorphism.

- **Def. (Retraction of a Space)** Let X be a topological space, and A a subspace of X. A continuous map $r : X \to A$ is called a *retraction* if $r|_A = \mathrm{Id}_A$. Equivalently, r is a retraction if $r \circ \iota_A = \mathrm{Id}_A$, where $\iota_A : A \hookrightarrow X$ is the inclusion map. If there exists a retraction from X to A, then we say that A is a *retract* of X.

- **Prop. 7.18. (Injective Induced Homomorphism)** Suppose A is a retract of X. If $r : X \to A$ is any retraction, then for any $q \in A$, $(\iota_A)_* : \pi_1(A, q) \to \pi_1(X, q)$ is injective and $r_* : \pi_1(X, q) \to \pi_1(A, q)$ is surjective.

  Since $r \circ \iota_A = \mathrm{Id}_A$, $r_* \circ (\iota_A)_*$ is the identity on $\pi_1(A, q)$), from which it follows that $(\iota_A)_*$ is injective and r_* is surjective.

8.3 Homotopy Equivalence

- **Def. (Homotopy Equivalences)** Let $\varphi : X \to Y$ and $\psi : Y \to X$.
  (a) ψ is a *homotopy inverse* for φ if $\psi \circ \varphi \simeq \mathrm{Id}_X$ and $\varphi \circ \psi \simeq \mathrm{Id}_Y$.
  (b) If φ has a homotopy inverse ψ, then φ is called a *homotopy equivalence*, and we say that X is *homotopically equivalent to* Y, or that X has the same *homotopy type* as Y. We denote $X \simeq Y$.

- **Def. (Deformation Retract)** A subspace $A \subset X$ is said to be a *deformation retract* if there exists a retraction $r : X \to A$ such that the identity of X is homotopic to $\iota_A \circ r$; the homotopy $H : \mathrm{Id}_X \simeq \iota_A \circ r$ is called a *deformation retraction*. Intuitively, this means that points in A *end up* at the same position they started at. A deformation retraction is *strong* iff $\mathrm{Id}_X \simeq_A (r \circ \iota_A)$, which is to say that the points of A remain *fixed* throughout the retraction.[1]

- **Def. (Contractible Space)** Let X be any topological space. X is said to be *contractible* iff the identity map of X is homotopic to a constant map (i.e., if Id_X is nullhomotopic).

[1] See https://encycla.com/Deformation_retraction for a `gif` of a (strong) deformation retraction.

9 Differential Forms

9.1 The Geometry of Volume Measurement

- **Lem. 12.1. (Intuition Behind Using Alternating Tensors for Integration)** Suppose Ω is a k-tensor on a vector space with the property that $\Omega(X_1, \dots, X_k) = 0$ whenever $X_1, \dots, X_k$ are linearly dependent. Then Ω is alternating.

  Let Ω be a k-tensor on a vector space with the above property. Remember that an alternating k-tensor is a multilinear function $\Omega : V \times \cdots \times V \to \mathbb{R}$ such that

$$\Omega(X_1, \dots, X_i, \dots, X_j, \dots, X_k) + \Omega(X_1, \dots, X_j, \dots, X_i, \dots, X_k) = 0. \tag{112}$$

  By our hypothesis, whenever two arguments of Ω are the same, we ought to get zero. Therefore,

$$\begin{aligned} 0 &= \Omega(X_1, \dots, X_i + X_j, \dots, X_i + X_j, \dots, X_n) \\ &= \Omega(X_1, \dots, X_i, \dots, X_i, \dots, X_n) + \Omega(X_1, \dots, X_i, \dots, X_j, \dots, X_n) \\ &\quad + \Omega(X_1, \dots, X_j, \dots, X_i, \dots, X_n) + \Omega(X_1, \dots, X_j, \dots, X_j, \dots, X_n) \\ &= \Omega(X_1, \dots, X_i, \dots, X_j, \dots, X_n) + \Omega(X_1, \dots, X_j, \dots, X_i, \dots, X_n). \end{aligned} \tag{113}$$

  Hence, Ω is alternating.

9.2 The Algebra of Alternating Tensors

- **Obs. (Alternating 2-Tensors)** Any 2-tensor T can be expressed as the sum of an alternating tensor and a symmetric one.

  Let T be an alternating tensor. Then we observe that

$$\begin{aligned} T(X,Y) &= \frac{1}{2}\left(T(X,Y) - T(Y,X)\right) + \frac{1}{2}\left(T(X,Y) + T(Y,X)\right) \\ &= A(X,Y) + S(X,Y). \end{aligned} \tag{114}$$

  We claim that A is an alternating tensor, and S is a symmetric tensor. To see this, note that

$$\begin{aligned} A(X,Y) + A(Y,X) &= \frac{1}{2}\left(T(X,Y) - T(Y,X)\right) + \frac{1}{2}\left(T(Y,X) - T(X,Y)\right) = 0. \\ S(X,Y) - S(Y,X) &= \frac{1}{2}(T(X,Y) + T(Y,X)) - \frac{1}{2}\left(T(X,Y) + T(Y,X)\right) = 0. \end{aligned} \tag{115}$$

- **Def. (Alternating Projection)** Define the *alternating projection*, $\mathrm{Alt} : T^k(V) \to \Lambda^k(V)$ as follows:

$$\mathrm{Alt}(T)(X_1, \dots, X_k) = \frac{1}{k!}\sum_{\sigma \in S_k} (\operatorname{sgn}\sigma) T(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \iff \mathrm{Alt}\, T = \frac{1}{k!}\sum_{\sigma \in S_k} (\operatorname{sgn}\sigma)({}^{\sigma}T) \tag{116}$$

- **Ex. (Examples of Alternating Projections)** Let T be any 1-tensor. Then $\mathrm{Alt}\, T = T$. If T is a 2-tensor, then $\mathrm{Alt}\, T(X,Y) = (1/2)(T(X,Y) - T(Y,X))$. If T is a 3-tensor, then

$$\begin{aligned} \mathrm{Alt}\, T(X,Y,Z) = \frac{1}{6}&\left(T(X,Y,Z) - T(X,Z,Y) - T(Z,Y,X)\right. \\ &\left. T(Z,X,Y) + T(Y,Z,X) - T(Y,X,Z)\right). \end{aligned} \tag{117}$$

- **Lem. 12.3 (Properties of the Alternating Projection)**
  (a) For any tensor T, $\mathrm{Alt}\, T$ is alternating.
  (b) T is alternating if and only if $\mathrm{Alt}\, T = T$.

Exercise 12.2. Prove Lemma 12.3.

(a) Let T be an arbitrary k-tensor, and $\operatorname{Alt} T$ its alternating projection. Let τ be the transposition $(i\ j)$. Then

$$\begin{aligned}\operatorname{Alt} T(X_1,\dots,X_j,\dots,X_i,\dots,X_k) &= \operatorname{Alt} T(X_{\tau(1)},\dots,X_{\tau(i)},\dots,X_{\tau(j)},\dots,X_k)\\ &= \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma\tau)T(X_{\sigma\tau(1)},\dots,X_{\sigma\tau(k)})\\ &= -\frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)T(X_{\sigma\tau(1)},\dots,X_{\sigma\tau(k)})\\ &= -\frac{1}{k!}\sum_{\sigma'\in S_k}(\operatorname{sgn}\sigma')T(X_{\sigma'(1)},\dots,X_{\sigma'(k)})\\ &= -\operatorname{Alt} T(X_1,\dots,X_i,\dots,X_j,\dots,X_k).\end{aligned} \tag{118}$$

Hence, the alternating projection is indeed alternating.

(b) If $\operatorname{Alt} T = T$, then by (a), T is alternating.

- **Def. (Multi-Index)** Let k be a positive integer. An ordered k-tuple $I = (i_1,\dots,i_k)$ of positive integers is called an *multi-index* of length k. If $\sigma\in S_k$ is a permutation, then we write

$$I_\sigma = (i_{\sigma(1)},\dots,i_{\sigma(k)}). \tag{119}$$

- **Def. (Generalized Kronecker delta)** Let I and J be multi-indices of length k. Then we may define

$$\delta_I^J = \begin{cases}\operatorname{sgn}\sigma, & \text{if neither } I \text{ nor } J \text{ has a repeated index}\\ & \quad\text{and } J = I_\sigma \text{ for some } \sigma\in S_k\\ 0 & \text{if } I \text{ or } J \text{ has a repeated index}\\ & \quad\text{or } J \text{ is not a permutation of } I.\end{cases} \tag{120}$$

- **Def. (Elementary Alternating Tensor/k-Covector)** Let V be an n-dimensional vector space, and suppose $(\varepsilon^1,\dots,\varepsilon^n)$ be a basis for V^*. We will define a collection of alternating tensors on V that generalize the determinant function on $\mathbb{R}^n$. For each multi-index $I = (i_1,\dots,i_k)$ of length k such that $1\le i_1,\dots,i_k\le n$, define a covariant k-tensor ε^I by

$$\begin{aligned}\varepsilon^I(X_1,\dots,X_k) &= \det\begin{pmatrix}\varepsilon^{i_1}(X_1) & \cdots & \varepsilon^{i_1}(X_k)\\ \vdots & & \vdots\\ \varepsilon^{i_k}(X_1) & \cdots & \varepsilon^{i_k}(X_k)\end{pmatrix}\\ &= \det\begin{pmatrix}X_1^{i_1} & \cdots & X_k^{i_1}\\ \vdots & & \vdots\\ X_1^{i_k} & \cdots & X_k^{i_k}\end{pmatrix}.\end{aligned} \tag{121}$$

I.e., if $\mathbb{X}$ denotes the matrix whose columns are the components of the vectors $X_1,\dots,X_k$ with respect to the basis (E_i) dual to the basis (ε^i), then $\varepsilon^I(X_1,\dots,X_k)$ is the determinant of the $k\times k$ minor consisting of rows $i_1,\dots,i_k$ of $\mathbb{X}$. Since the determinant is an alternating tensor, ε^I must also be an alternating tensor. We call ε^I an *elementary alternating tensor* or *elementary k-covector*.

- **Def. (Example of an Elementary k-Covector)** Let (e^1,e^2,e^3) be the standard dual basis for $(\mathbb{R}^3)^*$. Then

$$\mathbb{X} = \begin{pmatrix}X^1 & Y^1\\ X^2 & Y^2\\ X^3 & Y^3\end{pmatrix} \implies \varepsilon^{13}(X,Y) = \det\begin{pmatrix}X^1 & Y^1\\ X^3 & Y^3\end{pmatrix} = X^1Y^3 - Y^1X^3. \tag{122}$$

- **Lem. 12.4. (Properties of Elementary Alternating Tensor)** Let (E_i) be a basis for V, let (ε^i) be the dual basis for V^*, and let ε^I be as defined above.
  (a) If I has a repeated index, then $\varepsilon^I = 0$.
  (b) If $J = I_\sigma$ for some $\sigma \in S_k$, then $\varepsilon^I = (\operatorname{sgn}\sigma)\varepsilon^J$.
  (c) $\varepsilon^I(E_{j_1}, \ldots, E_{j_k}) = \delta^I_J$.

- **Prop. 12.5. (Basis for $\Lambda^k V$)** Let V be an n-dimensional vector space. If (ε^i) is any basis for V^*, then for each positive integer $k \le n$, the collection of k-covectors

$$\mathcal{E} = \left\{\varepsilon^I : I \text{ is an increasing multi-index of length } k\right\} \tag{123}$$

is a basis for $\Lambda^k V$. Therefore,

$$\dim \Lambda^k V = \begin{cases} \dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}, & k \le n \\ 0, & n < k. \end{cases} \tag{124}$$

- **Lem. 12. 6. (The Space $\Lambda^n(V)$)** Suppose V is an n-dimensional vector space and $\omega \in \Lambda^n(V)$. If $T : V \to V$ is any linear map and $X_1, \ldots, X_n$ are arbitrary vectors in V, then

$$\omega(TX_1, \ldots, TX_n) = (\det T)\omega(X_1, \ldots, X_n). \tag{125}$$

Let (E_i) be any basis for V, and let (ε^i) be the corresponding dual basis for V^*. Let (T_i^j) denote the matrix of T with respect to this basis, and let $T_i = TE_i = T_i^j E_j$. It suffices to prove this relationship holds when $X_i = E_i$ for each i. By Proposition 12.5, $\dim \Lambda^n V = 1$. This implies that $\omega = c\varepsilon^{1\cdots n}$ for some real number n. Then we obeerve that

$$\begin{aligned} (\det T)c\varepsilon^{1\cdots n}(E_1, \ldots, E_n) &= c\det T. \\ c\varepsilon^{1\cdots n}(TE_1, \ldots, TE_n) &= c\varepsilon^{1\cdots n}(T_1, \ldots, T_n) = c\det(\varepsilon^j(T_i)) = c\det T_i^j. \end{aligned} \tag{126}$$

Hence, this concludes the proof.

9.3 The Wedge Product

- **Def. (Wedge/Exterior Product)** Let $\omega \in \Lambda^k V$ and $\eta \in \Lambda^l V$. Define the *wedge product*, or *exterior product*, of ω and η to be the alternating $(k+l)$-tensor:

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!}\operatorname{Alt}(\omega \otimes \eta). \tag{127}$$

- **Lem. 12.7. (Wedge Product of Multi-Indices)** Let $(\varepsilon^1, \ldots, \varepsilon^n)$ be a basis for V^*. For any multi-indices $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$,

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}, \tag{128}$$

where $IJ = (i_1, \ldots, i_k, j_1, \ldots, j_l)$ is the concatenated multi-index.

- **Prop. 12.8. (Properties of the Wedge Product)**
  (a) BILINEARITY:

$$\begin{aligned} (a\omega' + a'\omega') \wedge \eta &= a(\omega \wedge \eta) + a'(\omega' \wedge \eta). \\ \eta \wedge (a\omega + a'\omega') &= a(\eta \wedge \omega) + a'(\eta \wedge \omega'). \end{aligned} \tag{129}$$

  (b) ASSOCIATIVITY:

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi. \tag{130}$$

(c) ANTICOMMUTATIVITY: For any $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$,

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega. \tag{131}$$

(d)

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} = \varepsilon^I. \tag{132}$$

(e) For any covectors $\omega^1, \ldots, \omega^k$ and vectors $X_1, \ldots, X_k$,

$$\omega^1 \wedge \cdots \wedge \omega^k (X_1, \ldots, X_k) = \det(\omega^j(X_i)). \tag{133}$$

9.4 Differential Forms on Manifolds

- **Def. (Space of all Alternating k-Tensors)** Let M be an n-dimensional smooth manifold. The subset of $T^k M$ consisting of alternating tensors is denoted by $\Lambda^k M$:

$$\Lambda^k M = \coprod_{p \in M} \Lambda^k(T_p M). \tag{134}$$

- **Note:** Look over Cartan's Lemma (Exercise 12-17 on pg. 323).

10 Orientations

10.1 Orientations of Vector Spaces

- **Def. (Consistently Oriented Ordered Bases)** Let V be a vector space of dimension $n \geq 1$. We say that two ordered bases $(V_1, \ldots, V_n)$ and $(\widetilde{E}_1, \ldots, \widetilde{E}_n)$ are *consistently oriented* if the transition matrix (B_i^j) defined by,
$$E_i = B_i^j \widetilde{E}_j \tag{135}$$
has positive determinant.

Exercise 13.1. Show that being consistently ordered is an equivalence relation on the set of all ordered bases for V, and show that there are exactly two equivalence classes.

Let V be a vector space of dimension $n \geq 1$, and let $\mathcal{V}$ denote the set of all ordered bases for V. Define the binary relation $\sim$ on V as follows: $(E_i) \sim (\widetilde{E}_i)$ if and only if (E_i) and $(\widetilde{E}_i)$ are consistently oriented. Then, we observe the following:

(i) For any $(E_i) \in \mathcal{V}$, clearly $(E_i) \sim (E_i)$ since the transition matrix is the identity matrix, which has determinant $1 > 0$.

(ii) Suppose $(E_i) \sim (F_i)$, where the transition matrix (B_i^j) defined by $E_i = B_i^j F_j$ has positive determinant. Multiplying both sides by the inverse of the transition matrix,
$$F_j = D_j^i E_i, \tag{136}$$
where D_j^i is the inverse of B_i^j. Since $1 = \det I = \det B_i^j D_j^i = \det B_i^j \det D_j^i$, and $\det B_i^j > 0$, it follows that $\det D_j^i > 0$. Hence, $(F_i) \sim (E_i)$.

(iii) Suppose $(E_i) \sim (F_i)$ and $(F_i) \sim (G_i)$; let $F_i = B_i^j E_j$ and $G_k = C_k^l F_l$. Then
$$G_k = C_k^l F_l = C_k^l B_l^j E_j. \tag{137}$$
Sinc $\det B_l^j, \det C_k^l > 0$, it follows that $\det C_k^l B_l^j > 0$. Hence, $(E_i) \sim (G_k)$.

The above observations prove that $\sim$ is an equivalence relation on $\mathcal{V}$. Now pick two bases $(E_i), (F_i) \in \mathcal{V}$ such that they are *not* consistently oriented. That is, if B_i^j is the corresponding transition matrix such that $F_i = B_i^j E_j$, then B_i^j has *negative* determinant. Now let (G_i) be an arbitrary ordered basis for V. If (G_i) and (E_i) are consistently oriented, then it follows trivially that $(G_i) \in [(E_i)]$. On the other hand, if (G_i) is not consistently oriented with (E_i) such that the transition matrix defined by $E_k = C_k^l G_l$ has negative determinant, then $(G_i) \sim (F_i)$ since
$$F_i = B_i^j E_j = B_i^j C_j^l G_l, \tag{138}$$
and $\det B_i^j C_j^l = \det B_i^j \det C_j^l > 0$. Therefore, there are exactly two equivalence classes.

- **Def. (Orientation for a Vector Space)** Let V be a vector space of dimension $n \geq 1$. We define an *orientation* for V as an equivalence class of ordered bases.

- **Lem. 13.2. (Orientations and Alternating Tensors)** Let V be a vector space of dimension $n \geq 1$ and suppose Ω is a nonzero element of $\Lambda^n(V)$. The set of ordered bases $(E_1, \ldots, E_n)$ such that $\Omega(E_1, \ldots, E_n) > 0$ is an orientation for V.

10.2 Orientations of Manifolds

- **Def. (Pointwise Orientation)** Let M be a smooth manifld. We define a *pointwise orientation* on M to be a choice of orientation of each tangent space.
- **Def. (Oriented Local Frames)** Suppose that M is a smooth n-manifold with a given pointwise orientation. A local frame (E_i) for M is (*positively*) *oriented* if $(E_1|_p, \ldots, E_n|_p)$ is a positively oriented basis for T_pM at each point $p \in U$. A *negatively oriented* frame is defined analogously.
- **Def. (Continuous Pointwise Orientation)** A pointwise orientation for M is said to be *continuous* if every point of M is contained in the domain of an oriented local frame. Such a pointwise orientation is called an *orientation* of M.

Exercise 13.2. If M is an oriented manifold of dimension $n \geq 1$, show that every local frame with connected domain is either positively oriented or negatively oriented. Show that the connectedness assumption is necessary.

- **Def. (Smooth Oriented Coordinate Chart)** A smooth oriented coordinate chart on an oriented manifold is said to be (*positively*) *oriented* if the coordinate frame $(\partial/\partial x^i)$ is positively oriented. A collection of smooth charts $\{(U_\alpha, \varphi_\alpha)\}$ is said to be *consistently oriented* if for each α, β, the transition map $\varphi_\alpha \circ \varphi_\beta^{-1}$ has positive Jacobian determinant everywhere on $\varphi_\alpha(U_\alpha \cap U_\beta)$.

11 de Rham Theory

11.1 de Rham Cohomology

- **Def. (Closed and Exact Differential Forms)** Let M be a smooth manifold, and ω a differential form on M. ω is *closed* if $d\omega = 0$ and *exact* if $\omega = d\tau$ for some form τ of degree $\deg\omega - 1$.

- **Def. (de Rham Cohomology Group in degree k)** Let $Z^k(M)$ be the vector space of all closed k-forms, and $B^k(M)$ the space of all exact k-forms on a smooth manifold M; since $d^2 = 0$, $B^k(M)$ is a subspace of $Z^k(M)$. Therefore, the quotient space $H^k_{dR}(M) := Z^k(M)/B^k(M)$ is called the *de Rham cohomology group of degree* k, and measures the extent to which closed forms "fail" to be exact forms.

- **Prop. 24.1. (de Rham Cohomology Group in degree 0)** Let M be a smooth manifold, with r connected components. Then its de Rham cohomology group in degree 0 is $H^0_{\text{dr}}(M) = \mathbb{R}^r$. An element of $H^0(M)$ is specified by an ordered r-tuple of real numbers, each real number representing a constant function on a connected component of M.

  Since there are no nonzero exact 0-forms, it follows that $H^0_{\text{dr}}(M) = Z^0(M) = \{\text{closed 0-forms}\}$. Suppose f is a closed 0-form on M; i.e., f is a C^∞ function on M such that $df = 0$. On a smooth chart $(U, (x^i))$, the differential of f was found to be

$$df = \frac{\partial f}{\partial x^i} dx^i. \tag{139}$$

  Hence, $df = 0$ on U iff $\partial_{x^i} f = 0$ on U, which is equivalent to requiring that f be locally constant on U. Therefore, the closed 0-forms on M are exactly the locally constant functions on M. Such a function must then be constant one each connected component of M. If M has r connected components, then a locally constant function on M can be specified by an ordered set of r real numbers. Therefore, $Z^0(M) = \mathbb{R}^r \implies H^0_{\text{dr}}(M) = \mathbb{R}^r$.

- **Prop. 24.2. (de Rham Cohomology Group in degree $k > n$)** On a manifold of dimension n, $H^k_{\text{dr}}(M)$ vanishes for $k > n$.

  At any point $p \in M$, T_pM is a vector space of dimension n. If ω is a k-form on M, then $\omega_p \in \Lambda_k(T_pM)$, the space of alternating k-linear functions on T_pM. But if $k > n$, then any collection of k vectors is linearly dependent, and hence $\Lambda_k(T_pM) = 0$. Therefore, for $k > n$, the only k-form on M is the 0-form.

- **Ex. 24.3. ($H^0_{\text{dr}}(\mathbb{R})$)** Consider the de Rham cohomology group in degree 0 of $\mathbb{R}$. By Proposition 24.1, since $\mathbb{R}$ is connected (i.e., has one connected component), $H^0_{\text{dr}}(\mathbb{R}) = \mathbb{R}$.

- **Ex. 24.3b ($H^k_{\text{dr}}(\mathbb{R})$)** For dimensional reasons, there are no nonzero 2-forms on $\mathbb{R}$. Hence, every 1-form on $\mathbb{R}$ is closed. A 1-form $f(x)dx$ on $\mathbb{R}$ is exact if and only if there exists a C^∞ function on $\mathbb{R}$ such that $f(x)dx = dg = g'(x)dx$. But such a function always exists:

$$g(x) = \int_0^x f(t)\, dt. \tag{140}$$

  Hence, every 1-form on $\mathbb{R}$ is closed, which proves that $H^1(\mathbb{R}) = 0$. Also using Proposition 24.2, we conclude that

$$H^k_{\text{dr}}(\mathbb{R}) = \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k \geq 1. \end{cases} \tag{141}$$

- **Ex. 24.4. (de Rham Cohomology of a Circle)** Let S^1 be the unit circle in the plane. Since S^1 is connected, by Proposition 24.1, $H^0_{\text{dr}}(S^1) = \mathbb{R}$. On the other hand, since S^1 is one-dimensional, $H^k(S^1) = 0$ for all $k > 1$. It remains for us to determine $H^1(S^1)$. Consider the map $h : \mathbb{R} \to S^1$

defined by $h(t) = (\cos(t), \sin(t))$. Let $i : [0, 2\pi] \to \mathbb{R}$ be the inclusion map. Restricting the domain of h to $[0, 2\pi]$ gives us a parametrization $F : h \circ i : [0, 2\pi] \to S^1$ of the circle. Consider the 1-form $\omega = -ydx + xdy$ on S^1. Then it is straightforward to see that $F^*\omega = i^*h^*\omega = i^*dt = dt$. Then

$$\int_{S^1} \omega = \int_{F([0,2\pi])} \omega = \int_{[0,2\pi]} F^*\omega = \int_0^{2\pi} dt = 2\pi. \tag{142}$$

Since S^1 is 1-dimensional, all 1-forms on S^1 are closed, so that $\Omega^1(S^1) = Z^1(S^1)$. The integration of 1-forms on S^1 defines a linear map

$$\varphi : Z^1(S^1) = \Omega^1(S^1) \to \mathbb{R}, \qquad \varphi(\alpha) = \int_{S^1} \alpha. \tag{143}$$

Since $\varphi(\omega) = 2\pi \neq 0$, the linear map $\varphi : \Omega^1(S^1) \to \mathbb{R}$ is onto. Now suppose α is an exact form: $\alpha = df$, for some $f \in C^\infty(S^1)$. Then by Stokes's Theorem,

$$\int_{S^1} \alpha = \int_{S^1} df = \int_{\partial S^1} f = 0, \tag{144}$$

since S^1 is a smooth manifold without boundary. Therefore, every exact 1-form is contained in $\ker \varphi$. On the other hand, suppose α is a 1-form in $\ker \varphi$. In particular, suppose $\alpha = f\omega$ is a smooth 1-form on S^1 such that $\varphi(\alpha) = 0$. Let $\overline{f} = h^*f = f \circ h \in \Omega^0(\mathbb{R})$. The $\overline{f}$ is 2π-periodic and

$$0 = \int_{S^1} \alpha = \int_{F([0,2\pi])} \alpha = \int_{[0,2\pi]} F^*\alpha = \int_{[0,2\pi]} (i^*h^*f)(t) \cdot F^*\omega = \int_0^{2\pi} \overline{f}(t)dt. \tag{145}$$

Therefore, we see that

$$H^1_{\mathrm{dr}}(S^1) = Z^1(S^1)/\ker \varphi \cong \operatorname{img} \varphi = \mathbb{R}. \tag{146}$$

This concludes the example.

11.2 Diffeomorphism Invariance

- **Lem. 24.6. (Pullback of Differential Forms)** Let N and M be smooth manifolds, and $F : M \to N$ a smooth map. Then the pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ sends closed forms to closed forms, and exact forms to exact forms.

  Suppose ω is closed. By commutativity of F^* with d,

  $$dF^*\omega = F^*d\omega = 0. \tag{147}$$

  Hence, $F^*\omega$ is closed. Now, suppose $\omega = d\tau$ is exact. Then

  $$F^*\omega = F^*d\tau = dF^*\tau, \tag{148}$$

  so that $F^*\omega$ is exact.

- **Def. (Pullback in Cohomology)** The pullback map F^* of differential forms induces a linear map of quotient spaces, denoted by $F^\#$:

  $$F^\# : \frac{Z^k(M)}{B^k(M)} \to \frac{Z^k(N)}{B^k(N)}, \quad F^\#([\omega]) = [F^*(\omega)], \tag{149}$$

  called the *pullback map in cohomology*. In particular, $F^\#$ defines a contravariant functor. Moreover, *if $F : M \to N$ is a diffeomorphism, then $F^\# : H^k(M) \to H^k(N)$ is an isomorphism of vector spaces.*

11.3 The Ring Structure on de Rham Cohomology

- **Lem. (Product Structure in Cohomology)** The wedge product of differential forms on a manifold M gives the vector space $\Omega^*(M)$ of differential forms a product structure. This product structure induces a product structure in cohomology: if $[\omega] \in H^k(M)$ and $[\tau] \in H^\ell(M)$, define

$$[\omega] \wedge [\tau] = [\omega \wedge \tau] \in H_{\text{dr}}^{k+\ell}(M). \tag{150}$$

Exercise 24.1. Prove that a nowhere vanishing 1-form on a compact manifold cannot be exact.

Let M be a smooth compact manifold, and ω a nowhere vanishing 1-form on M. Assume to the contrary that ω is exact; i.e., let f be a 0-form (i.e., a smooth map $f : M \to \mathbb{R}$) such that $\omega = df$. Since M is compact, f must attain a maximum value at some $p \in M$. But then if $(U, (x^i))$ is a smooth coordinate chart containing p,

$$\omega = df = \frac{\partial f}{\partial x^i} dx^i. \tag{151}$$

Since f is smooth, all of the partial derivatives $\partial_{x^i} f$ vanish at the maximum, which means ω must vanish at p; this is a contradiction. Hence, any nowhere vanishing 1-form on a compact manifold cannot be exact.

Exercise 24.2. Suppose a manifold M has infinitely many connected components. Compute its de Rham cohomology vector space $H^0_{\text{dr}}(M)$ in degree 0. (*Hint*: By second countability, the number of connected components of a manifold is countable.)

Let M be a smooth manifold with infinitely many connected components; by the second countability axiom for manifolds, M has countably many connected components. Since M has no nonzero exact 0-forms, it follows that $H^0_{\text{dr}}(M) \cong Z^0(M) = \{\text{closed 0-forms on } M\}$. Let ω be a closed 0-form on M. Then in any smooth coordinate chart $(U, (x^i))$,

$$0 = df = \frac{\partial f}{\partial x^i} dx^i, \tag{152}$$

so that f is locally constant on U. This means that the closed 0-forms on M are exactly the locally constant functions on M. Such a function must then be constant on each connected component of M. If M has countable infinitely many connected components, then a locally constant function on M can be specified by an ordered set of countable infinitely many numbers. Therefore, $Z^0(M) = \mathbb{R}^\omega$, so that $H^0_{\text{dr}}(M) \cong \mathbb{R}^\omega$.

12 The Long Exact Sequence in Cohomology

12.1 Exact Sequences

- **Def. (Cochain Complex)** A collection of vector space $\{C^k\}_{k\in\mathbb{Z}}$ together with a sequence of linear maps $d_k : C^k \to C^{k+1}$,
$$\cdots \to C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots, \tag{153}$$
such that $d_k \circ d_{k-1} = 0$ for all k. The collection of linear maps $\{d_k\}$ is called the *differential* of the cochain complex $\mathcal{C}$.

- **Def. (de Rham Complex)** Let M be a smooth manifold. The vector space $\Omega^*(M)$ of differential forms on M together with the exterior derivative d is a cochain complex, denoted the *de Rham complex of M*:
$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots. \tag{154}$$

- **Def. (Exact Sequences)** A sequence of homomorphisms of vector spaces $A \underset{f}{\overset{B}{\to}} \underset{g}{\overset{C}{\to}}$ is said to be *exact at B* if $\operatorname{im} f = \ker g$. A sequence of homomorphisms
$$A^0 \xrightarrow{f_0} A^1 \xrightarrow{f_1} A^2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \tag{155}$$
that is exact at every term except the first and the last is an *exact sequence*.

- **Def. (Short Exact Sequence)** A five-term exact sequence of the form $0 \to A \to B \to 0$.

12.2 Cohomology of Cochain Complexes

13 Problems

13.1 Smooth Maps

Problem 2-5. Let M be a nonempty smooth manifold of dimension $n \geq 1$. Show that $C^\infty(M)$ is infinite dimensional.

Let M be a nonempty smooth manifold of dimension $n \geq 1$. Let (U, φ) be a smooth chart for M, and let $x_1, \ldots, x_k$ be k distinct points contained in U. For each $j = 1, \ldots, k$, define the smooth function real-valued function f_j with compact support inside $\varphi(U)$ as follows: $f_j(x_m) = \delta_{mj}$. Then for each j, define the function $g_j : M \to \mathbb{R}$ as follows:

$$g_j(x) = \begin{cases} f_j(\varphi(x)), & x \in U. \\ 0, & x \in M \setminus U. \end{cases} \tag{156}$$

Then since U is open, it follows that g_j is smooth for each j. Hence, we have obtained a linearly independent subset of $C^\infty(M)$ consisting of k vectors. Since k was arbitrary, we conclude that $C^\infty(M)$ is infinite dimensional.

Problem 2-6. For any topological space M, let $C(M)$ denote the algebra of continuous functions $f : M \to \mathbb{R}$. If $F : M \to N$ is a continuous map, define $F^* : C(N) \to C(M)$ by $F^* = f \circ F$,

(a) Show that F^* is a linear map.

(b) If M and N are smooth manifolds, show that F is smooth if and only if $F^*(C^\infty(N)) \subset C^\infty(M)$.

(c) If $F : M \to N$ is a homeomorphism between smooth manifolds, show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

(a) Let $a, b \in \mathbb{R}$ and $f, g \in C(N)$. Then

$$F^*(af + bg) = (af + bg) \circ F = a(f \circ F) + b(g \circ F) = aF^*(f) + bF^*(g). \tag{157}$$

(b) Let M and N be smooth manifolds. Assume that F is smooth, and let $f \in C^\infty(N)$. Then

$$F^*(f) = (f \circ F) : M \to \mathbb{R} \tag{158}$$

is smooth since it is the composition of smooth functions. Hence, $F^*(C^\infty(N)) \subset C^\infty(M)$. Now we need to show the converse. Suppose $F^*(C^\infty(N)) \subset C^\infty(M)$. Let (U, φ) and (V, ψ) be smooth charts for M and N, respectively such that $F(U) \subset V$. Let $\psi = (\psi^i)$, where each coordinate function $\psi^i : V \to \mathbb{R}$ is smooth. Note we can extend ψ^i to a smooth function on N by means of a smooth bump function. By our hypothesis, $F^*(\psi^i) = \psi^i \circ F$ is smooth. Then since $\varphi^{-1} : \varphi(U) \to M$ is smooth,

$$\psi^i \circ F \circ \varphi^{-1} \tag{159}$$

is smooth for each i, which means that $\psi \circ F \circ \varphi^{-1}$ is smooth. Hence, we conclude that F is smooth on M.

(c) Let $F : M \to N$ be a homeomorphism between smooth manifolds. Suppose F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$. Then since $F^*(C^\infty(N)) \subset C^\infty(M)$, F is a smooth map. Let G be the inverse function of F. Then G^* is the inverse function of F^*, and so G^* restricts to an isomorphism from $C^\infty(M)$ to $C^\infty(N)$. In particular, this implies that G is a smooth map. Hence, F is a diffeomorphism. Now assume that F is a diffeomorphism. Let G be its inverse map. Since $G : N \to M$ is smooth, $G^*(C^\infty(M)) \subset C^\infty(N)$. Let $g \in C^\infty(M)$ and let $C^\infty(N) \ni f = G^*(g)$. Then

$$F^*(f) = f \circ F = g \circ G \circ F = g, \tag{160}$$

so that F^* is surjective. Now we show injectivity of F^*. Suppose $F^*(f) = F^*(g) \iff f \circ F = g \circ F$. Then $(f \circ F) \circ G = (g \circ F) \circ G \iff f = g$. Hence, F^* is injective. Using (a), we conclude that F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

13.2 Tangent Vectors

Problem 3-1. Suppose M and N are smooth manifolds with M connected, and $F : M \to N$ is a smooth map such that $F_* : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$. Show that F is a constant map.

Let M and N be smooth manifolds with M connected, and $F : M \to N$ be a smooth map such that F_* is the zero map for each $p \in M$. Let $p \in M$, and define the subset

$$\mathcal{C} = \{q \in M : F(q) = F(p)\}. \tag{161}$$

Clearly this subset is nonempty since it at least contains $p \in M$. If $q \in \mathcal{M}$, let U be a smooth coordinate chart containing q. By hypothesis, for all $r \in U$, F_* is the zero map; in local coordinates, this is possible iff all of the partial derivatives of the coordinate representation of F is zero at each $r \in U$. But this means that F is constant on U. Hence, $U \subset \mathcal{C}$, which means that $\mathcal{C}$ is an open subset of M. By continuity of F, $\mathcal{C}$ is also a closed subset of M. Since M is connected and $\mathcal{C}$ is nonempty, it then follows that $\mathcal{C} = M$. Hence, F is a constant map.

Problem 3-3. If a nonempty smooth n-manifold is diffeomorphic to an m-manifold, show that $n = m$.

Let M be a nonempty m-manifold and N a nonempty n-manifold; let $F : M \to N$ be a diffeomorphism. Then since F is a local diffeomorphism, for each $p \in M$, $F_* : T_pM \to T_{F(p)}N$ is an isomorphism. Since $\dim T_pM = m$ and $\dim T_{F(p)}N = n$ for every $p \in M$, it then follows that $m = n$.

Problem 3-4. Let $C \subset \mathbb{R}^2$ be the unit circle, and let $S \subset \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin:

$$S = \{(x, y) : \max(|x|, |y|) = 1\}.$$

Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(C) = S$, but there is no *diffeomorphism* with the same property. [Hint: Consider what F does to the tangent vector to a suitable curve in C.]

Let $C \subset \mathbb{R}^2$ be the unit circle and $S \subset \mathbb{R}^2$ the boundary of the square of side 2 centered at the origin. Consider the map $G : S \to C$ defined as follows

$$G(x, y) = \frac{(x, y)}{\sqrt{x^2 + y^2}} \in S. \tag{162}$$

First, we show that G is injective. Suppose $G(x_1, y_1) = G(x_2, y_2)$. Since $\sqrt{x_1^2 + y_1^2}, \sqrt{x_2^2 + y_2^2}$ are nonzero, multiplying both sides by $\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$, we get $(x_1, y_1) = (x_2, y_2)$. Hence, G is injective. Now we show that G is surjective. Let $(\tilde{x}, \tilde{y}) \in C$. We give a rough sketch for surjectivity, but the idea is clear. Consider the ray connecting the origin $(0, 0)$ to the point $(\tilde{x}, \tilde{y})$. Extend this ray indefinitely. Then this ray must intersect S at some point (x_0, y_0). Since G radially projects all of the points in S inwards onto C, it follows that $G(x_0, y_0) = (\tilde{x}, \tilde{y})$. Hence, G is bijective. By calculus, G is continuous. Since continuous bijections from compact spaces onto Hausdorff spaces is a homeomorphism, G is a homeomorphism. Notably, its inverse F must also be a homeomorphism, proving the claim. However, there can be no diffeomorphism between C and S. Suppose F was such a diffeomorphism, and let a be one of the corners of the square, and $p = F^{-1}(a)$. Since F is a diffeomorphism, $T_pC \cong T_aS$ under the isomorphism F_*. As we showed before T_pC is 1-dimensional. On the other hand, T_cS is not well-defined. But this is a contradiction. Therefore, C and S are not diffeomorphic.

13.3 The Cotangent Bundle

Problem 6-1.

(a) If V and W are finite-dimensional vector spaces and $A : V \to W$ is any linear map, show that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "A"] \arrow[d, "\xi_V"] & W \arrow[d, "\xi_W"] \\ V^{**} \arrow[r, "(A^*)^*"] & W^{**}, \end{tikzcd}$$

where ξ_V and ξ_W denote the isomorphisms defined by (6.3) for V and W, respectively.

(a) Assume all of the given hypotheses. Let $X \in V$ and let $\omega \in W^*$. Then

$$\xi_W(AX)(\omega) = \omega(AX). \tag{163}$$

On the other hand, since $A^*\omega \in V^*$, $\xi_V(X)(A^*\omega) = A^*\omega(X)$.

Problem 6-2.

(a) If $F : M \to N$ is a smooth map, show that $F^* : T^*N \to T^*M$ is a smooth bundle map.

(b) Show that the assignment $M \mapsto T^*M$, $F \mapsto F^*$ defines a contravariant functor from the category of smooth manifolds to the category of smooth vector bundles.

13.4 Comps

Problem 2017-J-II-1. Let $f : M \to \mathbb{R}$ be a smooth function on a smooth manifold M. In an arbitrary smooth local coordinate chart $x : U \to \mathbb{R}^n$ of M, define

$$\mathcal{D}f := \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}. \tag{164}$$

Does $\mathcal{D}f$ give a well-defined vector field on M?

Let $f : M \to \mathbb{R}$ be a smooth function on a smooth manifold M, and define $\mathcal{D}f$ as prescribed above. For any given smooth local coordinate chart $(U, (x^i))$, by smoothness of f, all of the partial derivatives $\partial f / \partial x^i$ are smooth so that the component functions of $\mathcal{D}f$ are all smooth; hence $\mathcal{D}f$ is smooth in each smooth local coordinate chart. However, we need to check if $\mathcal{D}f$ transforms like a vector field. Suppose $p \in (U, (x^i)) \cap (V, (\widetilde{x}^i))$. Then

$$\begin{aligned} \mathcal{D}f &= \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}\bigg|_p \\ &= \frac{\partial f}{\partial \widetilde{x}^j}(p) \frac{\partial \widetilde{x}^j}{\partial x^i}(\hat{p}) \cdot \frac{\partial \widetilde{x}^k}{\partial x^i}(\hat{p}) \frac{\partial}{\partial \widetilde{x}^k}\bigg|_p \\ &\neq \frac{\partial f}{\partial \widetilde{x}^k}(p) \frac{\partial}{\partial \widetilde{x}^k}\bigg|_p, \end{aligned} \tag{165}$$

where $\hat{p}$ is the coordinate representation of p in the (x^i) coordinates, and we used the contravariant vector transformation law in the second line. Therefore, since $\mathcal{D}f$ does not transform as a vector field on M, it cannot be a well-defined vector field on M.

Problem 2023-J-II-4. Prove that $S^2 \times S^2$ is not diffeomorphic to $M_1 \times M_2 \times M_3$, where M_1, M_2, M_3 are smooth manifolds of nonzero dimension.

Assume to the contrary that $S^2 \times S^2$ is diffeomorphic to $M_1 \times M_2 \times M_3$; since diffeomorphisms preserve dimensions, $\dim(S^2 \times S^2) = \dim S^2 + \dim S^2 = 4$, and $\dim M_{1,2,3} \neq 0$, without loss of generality, we must have $\dim M_1 = \dim M_2 = 1$ and $\dim M_3 = 2$. Additionally, since S^2 is compact and connected, $\bigtimes_{j=1}^{3} M_j$ must be compact and connected, which then implies that each M_j must be compact and connected. Moreover, since diffeomorphisms induce isomorphisms between fundamental groups, we must have

$$\pi_1(S^2 \times S^2) \cong \pi_1\left(\bigtimes_{j=1}^{3} M_j\right). \tag{166}$$

On the left side, since S^2 is simply connected, $\pi_1(S^2 \times S^2)$ is trivial. On the right side, since the only compact, connected, smooth 1-manifold, up to diffeomorphism, is S^1, and $\pi_1(S^1) \cong \mathbb{Z}$,

$$\pi_1\left(\bigcup_{j=1}^{3} M_j\right) = \mathbb{Z} \times \mathbb{Z} \times \pi_1(M_3), \tag{167}$$

which is clearly not isomorphic to the trivial group. But this is a contradiction. Hence, by contradiction, $S^2 \times S^2$ *cannot* be diffeomorphic to $M_1 \times M_2 \times M_3$.

Problem 2024-J-I-5. Let α be a closed 1-form on $\mathbb{R}\mathrm{P}^n$, $n > 1$. Show that if $f : [0,1] \to \mathbb{R}\mathrm{P}^n$ is a smooth function such that $f(0) = f(1)$, then

$$\int_{[0,1]} f^*\alpha = 0.$$

Include all calculations that are relevant to your solution.

Let α be a closed 1-form on $\mathbb{R}\mathrm{P}^n$, $n > 1$, and $f : [0,1] \to \mathbb{R}\mathrm{P}^n$ a smooth function such that $f(0) = f(1)$. We should show this computation in light of the above problem, but we know that the k^{th} de Rham Cohomology group of $\mathbb{R}\mathrm{P}^n$ vanishes for all $0 < k < n$. Since

$$H^1(\mathbb{R}\mathrm{P}^n) = \frac{\{\text{closed 1-forms on } \mathbb{R}\mathrm{P}^n\}}{\{\text{exact 1-forms on } \mathbb{R}\mathrm{P}^n\}} = 0, \tag{168}$$

it follows that a 1-form on $\mathbb{R}\mathrm{P}^n$ is closed iff it is exact. So, since α is a closed 1-form, there exists a smooth function g on $\mathbb{R}\mathrm{P}^n$ such that $\alpha = dg$. Then $f^*\alpha = f^*dg = d(g \circ f)$. Therefore,

$$(*) \coloneqq \int_{[0,1]} f^*\alpha = \int_0^1 d(g \circ f) = g(f(1)) - g(f(0)). \tag{169}$$

By hypothesis, since $f(1) = f(0)$, $g(f(1)) = g(f(0))$. Therefore, $(*) = 0$, which concludes the proof.