# Comps Practice

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## Contents

## Comps Lemma

**Problem Comps Lemma.** Let $M, N$ be smooth, connected, $n$-manifolds, and $f : M \to N$ a (smooth) immersion. If $M$ is compact and nonempty, then $N$ is compact and $f$ is a (smooth) covering map.

Let $M, N$ be smooth, connected $n$-manifolds and $f : M \to N$ an immersion. Assume that $M$ is compact and nonempty. Since $\dim N = n$ and $f$ is an immersion, $\operatorname{rank} df_p = n$ at every $p \in M$. Hence, by the Inverse Function Theorem, $f$ is a local diffeomorphism. Since local diffeomorphisms are open maps, $f(M)$ is open in $N$. On the other hand, since the continuous image of compact sets is compact, $f(M)$ is compact in $N$. Since $N$ is Hausdorff, $f(M)$ is closed in $N$. Since $N$ is connected, $f(M) = N$. Therefore, $N$ is compact.
Now, let $q \in N$, and consider $f^{-1}(q) \subset M$. For each $x \in f^{-1}(q)$, let $U_x$ be an open neighborhood of $M$ containing $x$. Since $M$ is Hausdorff, we can shrink each $U_x$ so that these neighborhoods are pairwise disjoint. This means that each $x \in f^{-1}(q)$ is isolated, and hence $f^{-1}(q)$ is discrete. Since $M$ is compact, we conclude that $f^{-1}(q)$ must be finite; let $f^{-1}(q) = \{x_1, \dots, x_s\}$. As noted above, for each $j = 1, \dots, s$, let $U_j$ be a neighborhood of $x_j$ such that $f|_{U_j} : U_j \to V_j \subset N$ is a diffeomorphism. Then by the Hausdorff condition on $M$, shrink each $U_j$ so that $U_i \cap U_j = \varnothing$ for all $i \neq j$; $f$ remains a diffeomorphism on these shrunken neighborhoods. Setting $V = \bigcap_1^s f(U_j)$ and taking $\widetilde{U}_j = f^{-1}(V) \cap U_j$ gives us an evenly covered neighborhood of $q$ in $N$.

**Problem (Comps Lemma - Local Homeomorphisms).** Let $M, N$ be smooth, connected $n$-manifolds and $f : M \to N$ a local homeomorphism. If $M$ is compact and nonempty, then $N$ is compact and $f$ is a covering map.

**Problem (Comps Lemma - Submersions).** Let $M, N$ be smooth, connected $n$-manifolds and $F : M \to N$ a submersion. If $M$ is compact and nonempty, then $N$ is compact and $F$ is a covering map.

Let $M, N$ be smooth, connected $n$-manifolds and $F : M \to N$ a submersion. Also assume $M$ is compact and nonempty. Since submersions are open maps, $f(M)$ is open in $N$. On the other hand, since $F$ is continuous, continuous images of compacts sets are compact, and compact subsets of Hausdorff spaces are closed, $F(M)$ is closed in $N$. Hence, since $N$ is connected and $F(M)$ is nonempty, $F(M) = N$. This proves that $N$ is compact. We also claim that $F$ is a local diffeomorphism. Since $F$ is a submersion, at every $p \in M$, $dF_p : T_pM \to T_{f(p)}N$ is surjective. Since $\dim M = \dim N = n$, it follows that $dF_p$ is bijective. Hence, by the Inverse Function Theorem, $F$ is a local diffeomorphism.
All that remains to be seen is that $F$ is a covering map. Let $q \in N$ and consider the closed subset $F^{-1}(q) \subset M$. Since $F$ is a local diffeomorphism, for each $x \in F^{-1}(q)$, there exists a neighborhood $U_x$ such that $F|_{U_x}$ is a local diffeomorphism. Since $M$ is Hausdorff, we may shrink these neighborhoods so that they are pairwise disjoint. This means that each $x \in F^{-1}(q)$ is isolated, and hence, $f^{-1}(q)$ is discrete. Since $M$ is compact, $f^{-1}(q)$ is finite; let $f^{-1}(q) = \{x_1, \dots, x_s\}$. For each $j = 1, \dots, s$, let $U_j$ be a neighborhood of $x_j$ such that $F|_{U_j}$ is a diffeomorphism. Since $M$ is Hausdorff, we shrink these neighborhoods such that they are pairwise disjoint; $F$ remains a diffeomorphism on each shrunken $U_j$. Set $V = \bigcap_1^s f(U_j)$, and let $\widetilde{U}_j = f^{-1}(V) \cap U_j$. Hence, $V$ is an evenly covered neighborhood of $q \in N$, which concludes the proof that $F$ is a covering map.

## Steinhaus Theorem

**Problem (Steinhaus Theorem).** Let $E$ be a Lebesgue measurable subset of $\mathbb{R}^n$ such that $m^n(E) > 0$, and let $v_1, \dots, v_N$ be a finite collection of vectors in $\mathbb{R}^n$. Then there exists $R > 0$, depending on $E$, and $M = \max\{|v_1|, \dots, |v_N|\}$ such that for all $0 < r < R$, there exists $p \in S$ so that the $(N+1)$-points, $p, p + rv_1, \dots, p + rv_1 + \cdots + rv_n \in S$.

Let $E$ be a measurable subset of $\mathbb{R}^n$ with positive Lebesgue measure. We recall that the Lebesgue measure is *regular* (which means it is both *inner* and *outer* regular). By inner regularity, there exists a compact set $K_1 \subset E$ such that $m^n(K_1) > 0$. Let $\beta < (2^N - 1)^{-1}$; by outer regularity, there exists an open set $U$ containing $K_1$ such that

$$m^n(U) \leq (1+\beta)m^n(K_1). \tag{1}$$

Since $K_1$ is compact, $d_1 = d(K_1, U^c) > 0$. Let $R = d_1/M$, and choose an arbitrary $r$ such that $0 < r < R$. First, observe that the set $K_1 + rv_1$ is contained in $U$, since otherwise,

$$d(K_1, U^c) \leq |rv_1| \leq rM < d_1. \tag{2}$$

Therefore, $K_1 \cup (K_1 + rv_1) \subset U$, and so

$$m^n(U) \geq m^n(K_1 \cup (K_1 + rv_1)) = m^n(K_1) + m^n(K_1 + rv_1) - m^n(K_1 \cap (K_1 + rv_1)). \tag{3}$$

Since the Lebesgue measure is translation invariant,

$$m^n(K_1 \cap (K_1 + rv_1)) \geq 2m^n(K_1) - m^n(U) \geq 2m^n(K_1) - m^n(K_1) - \beta m^n(K_1) = (1-\beta)m^n(K_1). \tag{4}$$

Since $\beta < 1$, it follows that $m^n(K_1 \cap (K_1 + rv_1)) > 0$, and so $K_1 \cap (K_1 + rv_1) \neq \varnothing$. Now we proceed by induction. For each $i = 1, \dots, N$, let $K_{i+1} = K_i \cap (K_i + rv_i)$. Each $K_i + rv_i$ must be contained in $U$ (by a generalization of the argument made above) and each $K_{i+1} \subset K_i \subset U$. We claim that for each $i$, $m^n(K_{i+1}) \geq (1 - (2^i - 1)\beta)m^n(K_1)$. We have already proven the base case $i = 1$. So assume the result holds for some $1 \leq m < N$. Then

$$m^n(U) \geq m^n(K_i \cup (K_i + rv_i)) = m^n(K_i) + m^n(K_i + rv_i) - m^n(K_i \cap (K_i + rv_i)). \tag{5}$$

By translation invariance of the Lebesgue measure,

$$\begin{aligned} m^n(K_{i+1}) = m^n(K_i + rv_i) \geq 2m^n(K_i) - m^n(U) &\geq 2(1-(2^i-1)\beta)m^n(K_1) - (1+\beta)m^n(K_1) \\ &= m^n(K_1) - 2^{i+1}\beta m^n(K_1) + 2\beta m^n(K_1) - \beta m^n(K_1) \\ &= (1-(2^{i+1}-1)\beta)m^n(K_1). \end{aligned} \tag{6}$$

Hence, since $\beta < (2^N - 1)^{-1}$, we obtain a nested sequence of compact subsets $\varnothing \neq K_{N+1} \subset K_N \subset \cdots \subset K_1 \subset U$. Let $q \in K_{N+1}$ be arbitrary. Since $K_{N+1} = K_N \cap (K_N + rv_N)$, the point $q - rv_N$ is contained in $K_N$. Then since $K_N = K_{N-1} \cap (K_{N-1} \cap rv_{N-1})$, $q - rv_N - rv_{N-1} \in K_{N-1}$. Proceeding inductively, we obtain the sequence $\{q, q - rv_N, q - rv_N - rv_{N-1}, \dots, q - rv_N - \cdots - rv_1\} \subset K_1 \subset E$. Hence, the proof concludes.

## January 2025

**Problem 2025-J-I-1 (Algebra).** Let $R$ be a UFD (unique factorization domain). Let $F$ be its quotient field. Let $p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in F[x]$ be a monic polynomial with coefficients in $R$ admitting a root $a \in F$. Prove that $a \in R$.

Let $R$ be a UFD, and $F$ its quotient field. Let $p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in F[x]$ be a monic polynomial with coefficients in $R$ admitting a root $a \in F$. Let $a = c/d$, where $c, d \in R \setminus \{0\}$ so that $\gcd(c, d) = 1$. By definition of a root, we must have

$$0 = p(a) = \left(\frac{c}{d}\right)^n + b_{n-1}\left(\frac{c}{d}\right)^{n-1} + \cdots + b_0. \tag{7}$$

Multiplying both sides by $d^n$,

$$c^n + d(b_{n-1}c^{n-1} + b_{n-2}c^{n-2}d \cdots + b_0 d^{n-1}) = 0 \implies c^n = -d(b_{n-1}c^{n-1} + \cdots + b_0 d^{n-1}). \tag{8}$$

From this, we observe that $d \mid c^n$. If $d$ is not a unit in $R$, then every nonidentity irreducible divisor of $d$ is an irreducible divisor of $c^n$, and hence an irreducible divisor of $c$. But this contradicts our hypothesis that $\gcd(c, d) = 1$. Hence, $d$ has to be a unit of $R$. If $v \in R \setminus \{0\}$ such that $dv = vd = 1$, then

$$a = \frac{c}{d} = \frac{c}{d} \cdot \frac{v}{v} = cv \in R. \tag{9}$$

Hence, this concludes the proof.

**Problem 2025-J-I-2 (Real Analysis).** Let $\{f_n\}_{n\geq 1}$ be a sequence of Lebesgue-measurable functions on $[0,1]$. Suppose that

$$\int_0^1 f_n^2 dm \leq \frac{1}{n^2} \quad \text{for all } n \geq 1.$$

Prove that $f_n$ converges to 0 a.e. on $[0,1]$.

Let $\{f_n\}_{n\geq 1}$ be a sequence of Lebesgue-measurable functions on $[0,1]$ so that

$$\int_0^1 f_n^2 dm \leq \frac{1}{n^2} \quad \text{for all } n \geq 1. \tag{10}$$

Consider the sequence $\{\sum_1^m f_n^2\}$, which is increasing and converges a.e. to $\sum_1^\infty f_n^2$. Hence, by the Monotone Convergence Theorem,

$$\sum_1^\infty \int_0^1 f_n^2 = \lim_{m\to\infty} \sum_1^m \int_0^1 f_n^2 = \lim_{m\to\infty} \int_0^1 \sum_1^m f_n^2 = \int_0^1 \sum_1^\infty f_n^2 \leq \int_0^1 \sum_1^\infty \frac{1}{n^2} < \infty. \tag{11}$$

Therefore, $\sum_1^\infty f_n^2 \in L^1(\mathbb{R})$, which means that $\sum_1^\infty f_n^2 < \infty$ a.e. on $[0,1]$. Hence, $\sum_{n=1}^\infty f_n^2$ converges a.e. on $[0,1]$. This implies that $f_n^2 \to 0$ a.e. on $[0,1]$, and hence $f_n \to 0$ a.e. on $[0,1]$.

**Problem 2025-J-I-3 (Geometry/Topology).** Let $M$ be an orientable, connected, and compact smooth $n$-manifold with boundary. Show that there is no (smooth) retraction to the boundary, that is, there does not exist a smooth map $f : M \to \partial M$ such that $f(x) = x$ when $x \in \partial M$.

Let $M$ be an orientable, connected, and compact smooth $n$-manifold with boundary. Assume to the contrary that there exists a smooth map $f : M \to \partial M$ such that $f(x) = x$ when $x \in \partial M$. Let $\omega \in \Omega^{n-1}(\partial M)$ be a volume form for the boundary of $M$. Since volume forms are closed (hence, $\omega$ is closed), we have by Stokes's theorem

$$0 = \int_M f^* d\omega = \int_M d(f^*\omega) = \int_{\partial M} f^*\omega = \int_{\partial M} \omega > 0, \tag{12}$$

which is a contradiction. Hence, by contradiction, there cannot exist a smooth retraction to the boundary.

**Problem 2025-J-II-3 (Algebra).** Let $V$ be a vector space of dimension $n$ over $\mathbb{Q}$. Let $T : V \to V$ be a linear transformation with minimal polynomial $x^4 - x^2 - 2$ over $\mathbb{Q}$. Show that $n$ must be even.

Consider $V$ as a module over the ring $\mathbb{Q}[x]$ by letting a polynomial $f(x) \in \mathbb{Q}[x]$ act as the linear operator $f(T)$. Since $\dim V = n$, this module is finitely generated. By the structure theorem for finitely generated modules over principal ideal domains, $V$ is isomorphic to a direct sum of modules of the form $\mathbb{Q}[x]/(p(x))^e$, where $p(x) \in \mathbb{Q}[x]$ is irreducible. Moreover, each $p(x)$ must divide the minimal polynomial of $T$. We note that over $\mathbb{Q}$,

$$x^4 - x^2 - 2 = (x^2 - 2)(x^2 + 1), \tag{13}$$

where both factors are irreducible over $\mathbb{Q}$. Therefore, the only choices for $p(x)$ are $x^2 - 2$ and $x^2 + 1$. Therefore, $\mathbb{Q}[x]/(p(x))^e$ has dimension $\deg p \cdot e = 2e$ for each choice of $p$. Since 2 divides these dimensions, we conclude that 2 must divide $n$. Hence, $n$ is even.

**Problem 2025-J-II-4 (Topology).** Let $\Sigma_2$ be a compact oriented surface of genus 2. Is there a submersion $f : \Sigma_2 \to S^1 \times S^1$, where $S^1$ denotes the unit circle?

Assume to the contrary that there exists a submersion $f : \Sigma_2 \to S^1 \times S^1$, where $S^1$ denotes the unit circle. Since $\dim \Sigma_2 = \dim S^1 \times S^1 = 2$, $df_p$ must have constant rank 2 at every $p \in \Sigma_2$. Hence, $f$ is a local diffeomorphism. Since $f$ is a local diffeomorphism, $f(\Sigma_2)$ is compact in $S^1 \times S^1$; since $S^1 \times S^1$ is Hausdorff, $f(\Sigma_2)$ must be closed in $S^1 \times S^1$. On the other hand, since local diffeomorphisms are open maps, $f(\Sigma_2)$ is open in $S^1 \times S^1$. Therefore, since $S^1 \times S^1$ is connected, $f(\Sigma_2) = S^1 \times S^1$; i.e., $f$ is surjective. Therefore, $f$ is a covering map. This means that the induced homomorphism, $f_* : \pi_1(\Sigma_2) \to \pi_1(S^1 \times S^1)$ is injective, and so $f_*(\pi_1(\Sigma_2)) \cong \operatorname{img} f_* \leq \pi_1(S^1 \times S^1)$. However, $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$ is an abelian group and cannot have any nonabelian subgroups, whereas $\pi_1(\Sigma_2)$ is nonabelian. Hence, by contradiction, $f$ cannot be a submersion.

**Problem 2025-J-II-5 (Analysis).** Let $V$ be a topological vector space whose topology is Hausdorff. Let $X_1$ and $X_2$ be two Banach spaces, and assume there exist continuous linear bijections $F_1 : X_1 \to V$ and $F_2 : X_2 \to V$. Show that there is a continuous linear bijection $G : X_1 \to X_2$.

Assume the given hypotheses. Let $G = F_2^{-1} \circ F_2$. Since $F_1, F_2$ are bijections, we conclude that $G$ is a bijection. Likewise, since $F_1, F_2$ are linear, $G$ must also be linear. It suffices to prove that $G$ is continuous. By the Closed Graph Theorem, continuity of $G$ is equivalent to the graph of $G$ being a closed subspace of $X_1 \times X_2$. Let $\{x_n\} \subset X_1$ be a sequence in $X_1$ such that $x_n \to x$ and $y_n = Gx_n \to y$. We need to show that $y = Gx$. By continuity of $F_1$, $F_1 x_n \to Fx$. By continuity of $F_2$,

$$F_2 y = \lim F_2 y_n = \lim F_2 G x_n = \lim F_1 x_n = F_1 x. \tag{14}$$

Since $F_2$ is bijective, $y = F_2^{-1} F_1 x = Gx$. Hence, the graph of $G$ is closed, which implies that $G$ is continuous.

## August 2025

**Problem 2025-A-I-1 (Geometry/Topology).** Let $S$ be a closed orientable surface of genus 4 and $C$ be an embedded circle that partitions $S$ into two subsurfaces of genus 2. Does $S$ retract to $C$?

We claim that the answer is no; assume to the contrary that there exists a retraction $r : S \to C$. Let $i : C \hookrightarrow S$ be the inclusion map so that $r \circ i = \mathrm{id}_C$. Now since $C$ is an embedded circle, $H_1(C)$ (i.e., the first homology) is isomorphic to $H_1(S^1) = \mathbb{Z}$. On the other hand, since $C$ is separating in $S$, its homology class in $H_1(S)$ is the zero element. Hence, the induced map $i_* : H_1(C) \to H_1(S)$ is the zero map. But this is impossible since if $i_*$ is the zero map,

$$0 = r_* \circ i_* = (r \circ i)_* = \mathrm{id}_{H_1}(C), \tag{15}$$

which is a contradiction. Hence, no such retraction can exist.

**Problem 2025-A-I-6 (Algebra).** Let $f(x)$ be an irreducible polynomial of degree $n$ over a field $F$, and let $g(x)$ be any polynomial in $F[x]$. Prove that every irreducible factor of the composition $f(g(x))$ has degree divisible by $n$.

Let $h(x)$ be an irreducible factor of $f(g(x))$ in $F[x]$ and let $\alpha$ be the root of $h(x)$ in some algebraic closure of $F$. Since $h$ is irreducible and $\alpha$ is a root, the minimum polynomial of $\alpha$ over $F$ is $h$. Therefore,

$$\deg h = [F(\alpha) : F]. \tag{16}$$

Now since $\alpha$ is a root of $h(x) = f(g(x))$, $f(g(\alpha)) = 0$. In particular, $g(\alpha)$ is a root of $f$. Since $f$ is irreducible of degree $n$ over $F$, the minimal polynomial of $g(\alpha)$ over $\alpha$ is $f$. Hence,

$$[F(g(\alpha)) : F] = n. \tag{17}$$

Since $F \subset F(g(\alpha)) \subset F(\alpha)$, by the Tower Law,

$$\deg h = [F : (\alpha) : F] = [F(\alpha) : F(g(\alpha))] \cdot [F(g(\alpha)) : F] = n[F(\alpha) : F(g(\alpha))], \tag{18}$$

so that $n \mid \deg h$. Hence, this concludes the proof.

**Problem 2025-A-II-2 (Geometry/Topology).** Consider the plane distribution in $\mathbb{R}^3$ spanned by two vector fields

$$V = \partial_x + 2xy\partial_z, \qquad W = x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z. \tag{19}$$

(i) Show that this distribution is integrable.

(ii) Does the pair of vector fields $V$ and $W$ generate a coordinate system on integral surfaces? If not, find a pair that can play this role for the local integral surfaces passing through points $(0, 0, z_0)$.

(i) Let $D$ be the plane distribution in $\mathbb{R}^3$ spanned by the two vector fields $V$ and $W$ given above. Then by the Frobenius Theorem, $D$ is integrable if and only if $D$ is involutive, which is true if and only if the Lie Bracket of $V$ and $W$ is a smooth section of $D$ at each $p \in \mathbb{R}^3$. We observe that:

$$\begin{aligned} V(W) &= (\partial_x + 2xy\partial_z)\,(x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z) \\ &= \partial_x + (4xy + 2x)\partial_z. \\ W(V) &= \left(x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z\right)(\partial_x + 2xy\partial_z) \\ &= 2xy\partial_z + 2x\partial_z. \end{aligned} \tag{20}$$

Therefore, for any $p \in \mathbb{R}^3$,

$$[V, W] = V(W) - W(V) = \partial_x + 2xy\partial_z = V. \tag{21}$$

Since $V$ is a smooth section of $D$, we conclude that $D$ is involutive, and hence integrable.

(ii) Let $\mathcal{S}$ be an integral surface, and assume there are coordinates $(u, v)$ on $\mathcal{S}$ such that $V|_{\mathcal{S}} = \partial_u$ and $W|_{\mathcal{S}} = \partial_v$. Then we observe that $[V|_{\mathcal{S}}, W|_{\mathcal{S}}] = \partial_u(\partial_v) - \partial_v(\partial_u) = 0$. On the other hand,

$$[V|_{\mathcal{S}}, W|_{\mathcal{S}}] = ([V, W])|_{\mathcal{S}} = V|_{\mathcal{S}} \neq 0, \tag{22}$$

which is a contradiction. Therefore, $V$ and $W$ cannot generate a coordinate system on integral surfaces. However, consider the fields $\widetilde{V} = V$ and $\widetilde{W} = W - xV$ on $\mathbb{R}^3$. Then since

$$[\widetilde{V}, \widetilde{W}] = V(W - xV) - (W - xV)(V) = VW - xVV - W(V) + xVV = 0, \tag{23}$$

and so this pair generates a coordinate system on all integral surfaces.

## January 2024

**Problem 2024-J-I-1 (Algebra).** For distinct odd primes $p$ and $q$, prove that every finite group of order $2pq$ is a semidirect product of a normal subgroup of order $pq$ and a subgroup of order 2.

Let $G$ be a group of order $2pq$, where $p, q$ are distinct odd primes. Without loss of generality, assume $q > p$. By Sylow's Theorem,

$$n_q \in \{1, 2, p, 2p\} \cap \{1, q+1, \ldots\} = 1, \tag{24}$$

since $q > 2$ and $q > p$. Therefore, $G$ has a unique, normal, Sylow $q$-subgroup, which we denote as $Q$. Let $P$ be a Sylow $p$-subgroup of $G$. By the Second Isomorphism Theorem, we conclude that $N = PQ$ is a subgroup of $G$ of order $|P||Q| = pq$. Since $|G : N| = 2pq/(pq) = 2$, where 2 is the smallest prime dividing $|G|$, we conclude that $N$ is a normal subgroup of $G$. Next, by Cauchy's Theorem, $G$ contains an element of order 2. Let $M$ be the subgroup generated by this element, which also must have order 2. By Lagrange's Theorem, $N \cap M = \{e\}$. Next,

$$|NM| = \frac{|N||M|}{|N \cap M|} = |N||M| = 2pq = |G|, \tag{25}$$

so that $G = NM$. Therefore, we conclude that $G = N \rtimes M$.

**Problem 2024-J-I-2 (Geometry/Topology).** Let $p : E \to B$ be a covering space map, with $B$ and $E$ path connected. Choose a point $e_0 \in E$ and $b_0 \in B$ such that $p(e_0) = b_0$. This gives us a subgroup $H = p_*\pi_1(E, e_0)$ of the fundamental group $G = \pi_1(B, b_0)$.
Construct a bijection between the fiber $p^{-1}(b_0)$ and the set of right cosets of $H$ and prove that this is indeed a bijection. Prove that the number of sheets of $p$ equals the index $(G : H)$.

Assume all of the given hypotheses. Let $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ be the lifting correspondence induced by $p$ defined by $\phi([f]) = \tilde{f}(1)$, where $\tilde{f}$ is the lift of $f$, and let $\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$ be the map induced by $\phi$. It suffices to prove that $\Phi$ is a bijection.

(i) Since $E$ is path connected and $p : E \to B$ is a covering map, the lifting correspondence $\phi$ must be surjective. Hence, since $\Phi$ is induced by $\phi$, it follows that $\Phi$ is also surjective.

(ii) Now we will show that $\Phi$ is injective. Let $f$ and $g$ be two paths in $B$, and $\widetilde{f}, \widetilde{g}$ their liftings to paths in $E$ that begin at $e_0$. We must show that $\widetilde{f}(1) = \widetilde{g}(1)$ iff $[f] \in H * [g]$.

($\Leftarrow$) Suppose $[f] = [h * g]$, where $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ in $E$ based at $e_0$. Since $\tilde{g}$ is a path in $E$ that *begins* at $e_0$, the product $\tilde{h} * \tilde{g}$ is well-defined. Since $[f] = [h * g]$, it follows that $\tilde{f}$ and $\tilde{h} * \tilde{g}$ must end at the same point. Hence, $\tilde{f}$ and $\tilde{g}$ end at the same point. Therefore, $\phi([f]) = \phi([g])$.

($\Rightarrow$) Suppose $\phi([f]) = \phi([g])$, which means that $\tilde{f}(1) = \tilde{g}(1)$. Then the product of $\tilde{f}$ with the reverse of $\tilde{g}$ is well-defined and is a loop $\tilde{h}$ in $E$ based at $e_0$. By direct computation, $[\tilde{h} * \tilde{g}] = [\tilde{f}]$. If $\tilde{F}$ is a path homotopy between $\tilde{h} * \tilde{g}$ and $\tilde{f}$, then $p \circ \tilde{F}$ is a path homotopy between $h * g$ and $f$, which means that $[f] \in H * [g]$. Hence, this concludes the proof that $\Phi$ is injective.

Hence, $|p^{-1}(b_0)| = |G/H| = (G : H)$.

**Problem 2024-J-I-3 (Complex Analysis).** Suppose $f$ is continuous on the plane and holomorphic on $\mathbb{C} \setminus \mathbb{R}$. Prove that $f$ is holomorphic on the whole plane.

Let $f : \mathbb{C} \to \mathbb{C}$ be continuous on $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \mathbb{R}$. We show that $f$ is holomorphic on all of $\mathbb{C}$.
By Morera's Theorem, it suffices to prove that

$$\oint_\gamma f(z)\, dz = 0$$

for every closed piecewise $C^1$ curve $\gamma \subset \mathbb{C}$.
If $\gamma$ lies entirely in the upper or lower half-plane, then $f$ is holomorphic on a neighborhood of $\gamma$, and by the Cauchy–Goursat theorem,

$$\oint_\gamma f(z)\,dz = 0.$$

Now suppose that $\gamma$ intersects the real axis. For $\varepsilon > 0$, construct a closed piecewise $C^1$ curve $\gamma_\varepsilon$ by modifying $\gamma$ so that it avoids the real axis by small detours of height $\pm\varepsilon$. Then $\gamma_\varepsilon \subset \mathbb{C} \setminus \mathbb{R}$, so $f$ is holomorphic on a neighborhood of $\gamma_\varepsilon$, and hence

$$\oint_{\gamma_\varepsilon} f(z)\,dz = 0.$$

Since $f$ is continuous on $\mathbb{C}$, it is uniformly continuous on compact sets, and the total length of the detours tends to 0 as $\varepsilon \to 0$. Therefore,

$$\lim_{\varepsilon\to 0} \oint_{\gamma_\varepsilon} f(z)\,dz = \oint_\gamma f(z)\,dz.$$

Thus $\oint_\gamma f(z)\,dz = 0$.

Since this holds for every closed piecewise $C^1$ curve in $\mathbb{C}$, Morera's Theorem implies that $f$ is holomorphic on all of $\mathbb{C}$.

**Problem 2024-J-I-4 (Algebra).** For each field $K$, prove that the polynomial ring $K[x, y]$ in two variables is not a principal ideal domain.

Let $K$ be a field, and consider the polynomial ring $K[x, y]$. Let $(x, y)$ be the proper ideal of $K[x, y]$ generated by the monomials $x$ and $y$. Assume to the contrary that $(x, y) = (f(x, y))$ where $f(x, y) \in K[x, y]$ is not a unit of the polynomial ring. Since $x \in (f(x, y))$, $f(x, y) \mid x$. By our assumption that $f$ is not a unit, it follows that $f(x, y)$ is an associate of $x$. Likewise, $f(x, y)$ must be an associate of $y$. But this is impossible since $x$ and $y$ are not associates of each other. This forces $f(x, y)$ to be a unit, which means that $(f(x, y)) = K[x, y]$. But this contradicts the fact that $(x, y) = (f(x, y))$ is a proper ideal. Hence, by contradiction, $(x, y)$ is not a principal ideal, and so $K[x, y]$ is not a principal ideal domain.

**Problem 2024-J-I-5 (Geometry/Topology).** Let $\alpha$ be a closed 1-form on $\mathbb{RP}^n$, $n > 1$. Show that if $f : [0, 1] \to \mathbb{RP}^n$ is a smooth function such that $f(0) = f(1)$, then

$$\int_{[0,1]} f^*\alpha = 0.$$

Include all calculations that are relevant to your solution.

We recall that $H^k(\mathbb{RP}^n) = 0$ for all $0 < k < n$ so that $H^1(\mathbb{RP}^n) = 0$ if $n > 1$. In particular, this means that $\alpha$ is also an exact 1-form on $\mathbb{RP}^n$. Let $g$ be a smooth function on $\mathbb{RP}^n$ so that $\alpha = dg$. Then

$$\int_0^1 f^*\alpha = \int_0^1 f^*dg = \int_0^1 d(f^*g) = g(f(1)) - g(f(0)) = 0, \tag{26}$$

where the last equality follows from the fact that $f(1) = f(0)$. Hence, the proof concludes.

**Problem 2024-J-I-6 (Real Analysis).** Let $f$ and $g$ be Lebesgue-measurable functions on $\mathbb{R}$. Define the convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y)\,dy$$

for all $x$ such that the integral exists. Prove that if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ with $p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, then $f * g$ is a bounded continuous function on $\mathbb{R}$.

Assume the given hypotheses. Then by Hölder's inequality, for any $x \in \mathbb{R}$,

$$|(f * g)(x)| \le \int_{\mathbb{R}} |f(x - y)g(y)|\,dy \le \|f(x - \cdot)\|_p \|g\|_q. \tag{27}$$

Since $L^p$ norms are translation invariant, $\|f(x-\cdot)\|_p = \|f\|_p$. Hence, $|(f*g)(x)| \le \|f\|_p \|g\|_q = M < \infty$ for all $x \in \mathbb{R}$. Hence, we conclude that $f*g$ is a bounded function on $\mathbb{R}$. Next, let $\tau_z$ be the translation operator defined by $\tau_z f = f(x-z)$. Since translation operators are continuous in the $L^p$ norms, $\|\tau_z f - f\| \to 0$ as $z \to 0$, which implies that

$$\begin{aligned} \|\tau_z(f*g) - (f*g)\|_\infty &= \|(\tau_z f - f)*g\|_\infty && (28)\\ &\le \|\tau_z f - f\|_p \|g\|_q \longrightarrow 0 \text{ as } z \to 0. && (29) \end{aligned}$$

Hence, $f*g$ is uniformly continuous, and therefore continuous on $\mathbb{R}$. Note that the inequality used in the second line of the above equation comes from *Young's convolution inequality*, which states the following:

> **(Young's Convolution Inequality)** Let $f \in L^p$, $g \in L^q$, and $p^{-1}+q^{-1} = r^{-1}+1$. Then $\|f*g\|_r \le \|f\|_p \|g\|_q$.

In our case, we had $r = \infty$ so that $r^{-1} = 0$.

**Problem 2024-J-II-2.** Suppose $E \subset \mathbb{R}^2$ is a set of positive Lebesgue measure. Show that there are points $a, b, c$ in $E$ such that their connecting segments form a right angle, i.e., $a-b$ is perpendicular to $c-b$ (as vectors in $\mathbb{R}^2$).

Let $E \subset \mathbb{R}^2$ be a set of positive Lebesgue measure; let $m^2$ denote the Lebesgue measure on $\mathbb{R}^2$. Let $\{v_1, v_2, v_3\}$ be a collection of vectors in $\mathbb{R}^2$ such that $v_1 \perp v_2$, and $v_3 = -v_1$. Without loss of generality, assume that $\|v_j\| = 1$ for all $j = 1, \ldots, 3$. By inner regularity of the Lebesgue measure, there exists a compact subset $K_1 \subset E$ such that $m^2(K_1) > 0$. Taking $\beta < 1/7$, by outer regularity of the Lebesgue measure, there exists an open set $U$ containing $K_1$ such that $m^2(U) \le (1+\beta)m^2(K_1)$.
Since $K_1$ is compact, $d_1 = d(K_1, U^c) > 0$. Hence, let $R = d_1$. Fix some $r \in (0, R)$ and consider the set $K_1 + rv_1$. We claim that $K_1 + rv_1 \subset U$ since if otherwise,

$$d(K_1, U^c) \le |rv_1| = r < d_1, \text{ which is a contradiction.} \tag{30}$$

Hence, $K_1 \cup (K_1 + rv_1) \subset U$, which means that

$$m^2(U) \ge m^2(K_1 \cup (K_1+rv_1)) = m^2(K_1) + m^2(K_1+rv_1) - m^2(K_1 \cap (K_1+rv_1)). \tag{31}$$

By translation invariance of the Lebesgue measure, $m^2(K_1) + m^2(K_1+rv_1) = 2m^2(K_1)$ so that

$$m^2(K_1 \cap (K_1+rv_1)) = 2m^2(K_1) - m^2(U) \ge (1-\beta)m^2(K_1). \tag{32}$$

Since $\beta < 1$, $m^2(K_1 \cap (K_1+rv_1)) > 0$ so that the set is nonempty. For $i = 1, \ldots, 3$, define $K_{i+1} = K_i \cap (K_i + rv_i)$. Generalizing the argument from above shows that each $K_{i+1} \subset U$. We claim that $m^2(K_{i+1}) \ge (1-(2^i-1)\beta)m^2(K_1)$ for each $i$; the above work establishes the result for $i = 1$. Now assume the result holds for some $1 \le j < 3$. Then

$$m^2(U) \ge m^2(K_j \cup (K_j+rv_j)) = m^2(K_j) + m^2(K_j+rv_j) - m^2(K_j \cap (K_j+rv_j)) = 2m^2(K_j) - m^2(K_j \cap (K_j+rv_j)). \tag{33}$$

Therefore,

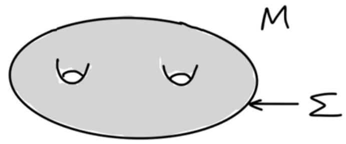
$$\begin{aligned} m^2(K_j \cap (K_j+rv_j)) &= 2m^2(K_j) - m^2(U) \\ &\ge 2m^2(K_1) - 2^{j+1}\beta m^2(K_1) + 2\beta m^2(K_1) - m^2(K_1) - \beta m^2(K_1) \\ &= (1-(2^{j+1}-1)\beta)m^2(K_1). \end{aligned} \tag{34}$$

Since $\beta < (2^3-1)^{-1} = 7^{-1}$, we conclude that each $K_i$ is nonempty. Hence, we obtain a nested sequence $\emptyset \ne K_4 \subset \cdots \subset K_1 \subset E$. Let $q \in K_4$; since $K_4 = K_3 \cap (K_3 + rv_3)$, $q - rv_3 \in K_3$. Following inductively, we obtain a sequence of points $\{p, p+rv_1, p+rv_1+rv_2, p+rv_1+rv_2+rv_3\} \subset E$, with $p \in K_1$, and $p + rv_j \in K_j$ for $j = 1, 2, 3$ (note we have renamed $q - rv_1 - \cdots - rv_3 = p$, and so on). Let $a = p$, $b = p + rv_1$, and $c = p + rv_1 + rv_2$. Then $a - b = -rv_1$ and $c - b = rv_2$. By hypothesis on $v_1$ and $v_2$, $a-b$ is orthogonal to $c-b$.

**Problem 2024-J-II-3 (Geometry/Topology).** Let $\Sigma$ be a genus 2 surface embedded in $\mathbb{R}^3$ as shown in the picture. Let $M$ be the closure of the *unbounded* component of $\mathbb{R}^3 \setminus \Sigma$; in other words, $M$ is the part of $\mathbb{R}^3$ which is *not* enclosed by $\Sigma$.

(a) Compute $\pi_1(M)$.

(b) Is $\Sigma$ a retract of $M$?




(a)

**Problem 2024-J-II-5 (Real Analysis).** Let $P$ be the vector space over $\mathbb{R}$ of (finite degree) polynomials in the variable $x \in (-\infty, \infty)$. Show that $P$ cannot be a Banach space with respect to any norm, that is, if $\|\cdot\|$ is some norm on $P$, then $P$ is not complete under this norm. Hint: You may use the Baire Category Theorem.

We recall the Baire Category Theorem:

> **(Baire Category Theorem)** Let $X$ be a complete metric space.
> (a) If $\{U_n\}_1^\infty$ is a sequence of open dense subsets of $X$, then $\bigcap_1^\infty U_n$ is dense in $X$.
> (b) $X$ is not a countable union of nowhere dense sets.

For each positive integer $n$, let $P_n$ be the vector space of all polynomials of degree $\leq n$ so that $P = \bigcup_{n\in\mathbb{N}} P_n$. Let $\|\cdot\|$ be a norm on $P$ and assume to the contrary that $P$ is complete under this norm; this means that $P$ is a complete metric space. Since $X$ cannot be the countable union of nowhere dense sets, it follows that there exists some positive integer $n_0$ so that $P_{n_0}$ is not nowhere dense; i.e., the closure of $P_{n_0}$ has nonempty interior. Since any finite dimensional vector subspace of a normed vector space is closed, it follows that $P_{n_0}$ is closed in $P$; i.e., $\overline{P}_{n_0} = P_{n_0}$. Hence, by our hypothesis, $P_{n_0}$ has nonempty interior. Let $p \in P_{n_0}$ and $B(r,p)$ a ball of radius $r > 0$ centered at $p$ that is contained *entirely within* $P_{n_0}$. Let $u \in P \setminus \{0\}$ be arbitrary, and set

$$u' = p + \frac{r \cdot u}{2\,\|u\|} \implies u' \in B(r,p) \subset P_{n_0}. \tag{35}$$

But since $P_{n_0}$ is a vector space, this implies that $u \in P_{n_0}$. Since $u$ was arbitrary in $P$, this means that $P_{n_0} = P$, which is a contradiction. Hence, every $P_n$ must have empty interior, which then contradicts the Baire Category Theorem. Hence, $P$ cannot be a Banach space with respect to any norm.

**Problem 2024-J-II-6 (Geometry/Topology).** Let $M$ be a smooth $n$-manifold, and let $\varphi$ be a differential $k$-form on $M$ which is closed, in the sense that $d\varphi = 0$. At each point $p \in M$, define

$$D_p = \left\{ v \in T_pM : v \lrcorner \varphi = 0 \right\}, \tag{36}$$

where $\lrcorner$ denotes the interior product. Assume $\ell \coloneqq \dim D_p$, so that $D \subset TM$ is a rank-$\ell$ vector subbundle of the tangent bundle of $M$. Prove that $D$ is an integrable distribution of $\ell$-planes, in the sense of the Frobenius Theorem.

By the Frobenius Theorem, it suffices to prove that $D$ is involutive, which is to say that if $X, Y$ are smooth sections of $D$, then $[X,Y]$ is also a smooth section of $D$. Indeed, let $X, Y$ be smooth sections of $D$, which means that $X \lrcorner \varphi, Y \lrcorner \varphi = 0$. Observe that,

$$[X,Y] \lrcorner \varphi = \mathcal{L}_X(Y \lrcorner \varphi) - Y \lrcorner (\mathcal{L}_X \varphi). \tag{37}$$

By hypothesis, $Y \lrcorner \varphi = 0$ so that $\mathcal{L}_X(Y \lrcorner \varphi) = 0$. On the other hand, by Cartan's Formula,

$$\mathcal{L}_X \varphi = d(X \lrcorner \varphi) + X \lrcorner d\varphi = 0, \tag{38}$$

by the hypotheses. Hence, this shows that $[X, Y] \lrcorner \varphi = 0$, and so $[X, Y]$ is a smooth section of $D$. Therefore, $D$ is involutive, which means that it is Frobenius integrable.

**Problem 2024-J-II-4 (Algebra).** Let $\alpha = \sqrt{2+\sqrt{3}} \in \mathbb{C}$. Let $K$ be the smallest Galois extension of $\mathbb{Q}$ which contains $\alpha$. Describe the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

Let $\alpha = \sqrt{2+\sqrt{3}} \in \mathbb{C}$, and $K$ the smallest Galois extension of $\mathbb{Q}$ that contains $\alpha$. We start by finding the minimal polynomial of $\alpha$. We observe that

$$\alpha^2 = 2 + \sqrt{3} \implies (\alpha^2 - 2)^2 - 3 = 0. \tag{39}$$

Simplifying,

$$\alpha^4 - 4\alpha^2 + 1 = 0. \tag{40}$$

I.e., the polynomial $x^4 - 4x^2 + 1$ is the minimal polynomial of $\alpha$. Solving this polynomial over an algebraic closure of $\mathbb{Q}$, we obtain the four roots, $\pm\sqrt{2+\sqrt{3}}, \pm\sqrt{2-\sqrt{3}}$. Hence, the elements of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ are the identity permutation, the permutation $\sigma$ that fixes $\pm\sqrt{2-\sqrt{3}}$ and permutes $\pm\sqrt{2+\sqrt{3}}$, the permutation $\tau$ that fixes $\pm\sqrt{2+\sqrt{3}}$ and permutes $\pm\sqrt{2-\sqrt{3}}$, and the permutation $\sigma\tau$. Labeling these roots as $\alpha_1, \ldots, \alpha_4$, we see that $\mathrm{Gal}(K/\mathbb{Q}) \cong \{1, (1\,2), (3\,4), (1\,2)(3\,4)\} \cong V \subset S_4$, where $V$ is the Klein-4 subgroup.

## August 2024

**Problem 2024-A-I-1 (Geometry/Topology).** Let $M$ be a smooth compact manifold without boundary, and let $\varphi$ be a smooth closed 1-form on $M$ that has the property that $\varphi \neq 0$ at every point of $M$. Prove that the first de Rham cohomology $H^1_{\mathrm{dr}}(M)$ of the given manifold is non-zero.

Let $M$ be a smooth compact manifold without boundary and let $\varphi$ be a smooth closed 1-form on $M$ that has the property that $\varphi \neq 0$ at every point of $M$. Suppose that $\varphi$ is exact; i.e., assume there exists a smooth function $f$ on $M$ such that $\varphi = df$. By the Extreme Value Theorem, since $M$ is compact, $f$ must have either a maximum or minimum value at some point $p \in M$. Since all of the first-order partial derivatives of $f$ must vanish at the point $p$ where $f$ attains its maximum/minimum value, $df|_p = 0$. This means that $\varphi$ must also vanish at $p$, which contradicts our hypothesis that $\varphi$ is nowhere vanishing. Hence, by contradiction, $\varphi$ cannot be an exact form. Since $H^1_{\mathrm{dr}}(M) := \{\text{closed 1-forms on } M\}/\{\text{exact 1-forms on } M\}$ and we have shown the existence of a closed 1-form that is *not* an exact 1-form, we conclude that $H^1_{\mathrm{dr}}(M)$ is non-zero.

**Problem 2024-A-I-2 (Geometry/Topology).** Suppose that $f : \Sigma_2 \to \Sigma_1$ is a continuous map between a genus 2 closed orientable surface $\Sigma_2$ and a torus $\Sigma_1$. Prove that $f$ is not a local homeomorphism. In other words, show that there exists a point $x \in \Sigma_2$ which does not have an open neighborhood $U \subset \Sigma_2$ on which the restriction $f|_U$ is a homeomorphism between $U$ and $f(U)$.

Before presenting our argument, we will state and prove a quick technical lemma.

> (**Modified Comps Lemma**) Let $M$ and $N$ be smooth connected manifolds, and $f : M \to N$ a local homeomorphism. If $M$ is compact and nonempty, then $N$ is compact and $f$ is a covering map.
>
> *Proof.* Let $M$ and $N$ be smooth connected manifolds, and $f : M \to N$ a local homeomorphism. Since $f$ is an open map, $f(M)$ is open in $M$. Next since the continuous image of

a compact set is compact and a compact subset of a Hausdorff space is closed, $f(M)$ is closed in $N$. Hence, since $N$ is connected, $f(M) = N$, which means $N$ is connected and $f$ is surjective.

Now let $q \in N$, and consider the closed subset $f^{-1}(q) \subset M$. For each $x \in f^{-1}(q)$, there exists a neighborhood $U_x$ such that $f|_{U_x}$ is a homeomorphism. Since $M$ is Hausdorff, we may shrink these neighborhoods so that they are pairwise disjoint. Hence, each $x \in f^{-1}(q)$ is isolated, which means $f^{-1}(q)$ is discrete. Since discrete subspaces of compact spaces is necessarily finite, $f^{-1}(q)$ is finite; let $\{x_1, \dots, x_s\} = f^{-1}(q)$. As stated above, for each $j = 1, \dots, s$, we may find a neighborhood $U_j'$ such that $f|_{U_j'}$ is a homeomorphism. Using Hausdorff-ness of $M$, we may shrink these neighborhoods to obtain the collection $\{\widetilde{U}_j\}_1^s$ of pairwise disjoint open neighborhoods. Set $V = \bigcap_1^s U_j$, which is then an evenly covered neighborhood of $q$. Therefore, $f$ is a covering map. □

Now assume to the contrary that $f : \Sigma_2 \to \Sigma_1$ is a local homeomorphism; by the modified Comps Lemma, $f$ is a covering map. Moreover, $\Sigma_2$ must be a $k$-sheeted covering space for some finite positive integer $k$, which means that $\chi(\Sigma_2) = k \cdot \chi(\Sigma_1)$. However, this is impossible since $\chi(\Sigma_1) = 0$, while $\chi(\Sigma_2) = 2 - 2(2) = 2 - 4 = -2$. Therefore, $f$ cannot be a local homeomorphism.

**Problem 2024-A-I-5 (Algebra).** Determine whether or not the complex number $i = \sqrt{-1}$ is in the field $\mathbb{Q}(\alpha)$, where $\alpha$ is any complex number subject to the relation $\alpha^3 + \alpha + 1 = 0$. Justify your answer.

The polynomial $x^3 + x + 1$ has no roots in $\mathbb{Q}$ (by the rational root test), and so is irreducible (since it is a cubic). This means that $\mathbb{Q}(\alpha)$ is an extension of degree 3 over $\mathbb{Q}$. Therefore, it cannot contain the field $\mathbb{Q}(i)$, which has degree 2 over $\mathbb{Q}$ (since the minimal polynomial of $i$ is $x^2 + 1$) since $2 \nmid 3$.

**Problem 2024-A-II-1 (Geometry/Topology).** Recall that $S^n$ denotes the unit sphere in $\mathbb{R}^{n+1}$. Also recall that a smooth map is called a smooth submersion if its differential is everywhere surjective. Prove or disprove each of the following statements:

(a) There is a smooth submersion $F : S^3 \to S^1$.

(b) There is a smooth submersion $F : S^3 \to S^2$.

(a) [!! Complete Later !!]

**Problem 2024-A-II-2 (Geometry/Topology).** On $\mathbb{R}^5$, equipped with standard coordinates $(v, w, x, y, z)$, consider the 1-form

$$\theta = dz + v\,dx + w\,dy.$$

Are there two smooth functions $f, g : \mathbb{R}^5 \to \mathbb{R}$ such that $\theta = f\,dg$? Justify your answer by means of concrete solutions.

We claim that there do *not* exist smooth functions $f, g : \mathbb{R}^5 \to \mathbb{R}$ such that $\theta = f\,dg$. Assume to the contrary. First, we observe that if $\theta = f dg$, then

$$d\theta = d(f dg) = df \wedge dg \implies \theta \wedge d\theta = f dg \wedge df \wedge dg = 0. \tag{41}$$

I.e., if $\theta = f dg$, then $\theta \wedge d\theta$ must be identically zero. However, since $\theta = dz + v dx + w dy$, we note that

$$d\theta = d^2 z + d(v dx) + d(w dy) = dv \wedge dx + dw \wedge dy \implies \theta \wedge d\theta = dz \wedge dv \wedge dx + dz \wedge dw \wedge dy + v dx \wedge dw \wedge dy + w dy \wedge dv \wedge dx, \tag{42}$$

which is nowhere vanishing on $\mathbb{R}^5$. Hence, by contradiction, there cannot exist two smooth functions $f, g : \mathbb{R}^5 \to \mathbb{R}$ such that $\theta = f dg$.

## January 2023

**Problem 2023-J-II-4 (Geometry/Topology).** Prove that $S^2 \times S^2$ is not diffeomorphic to $M_1 \times M_2 \times M_3$, where $M_1, M_2, M_3$ are smooth manifolds of nonzero dimension.

We begin with a technical lemma, that we will use to prove the desired result.

(**Comps Lemma**) Let $M, N$ be smooth, connected $n$-manifolds and $f : M \to N$ a (smooth) immersion. If $M$ is compact and nonempty, then $N$ is compact and $f$ is a (smooth) covering map.

*Proof.* Let $M, N$ be smooth connected $n$-manifolds, $f : M \to N$ an immersion, and $M$ compact and nonempty. Since $\dim N = n$ everywhere and $f$ is an immersion, $df_p : T_pM \to T_{f(p)}N$ has constant rank $n$ everywhere. Hence, by the Inverse Function Theorem, $f$ is a local diffeomorphism. Since local diffeomorphisms are open maps, $f(M)$ is open in $N$. Next since the continuous image of compact sets is compact, $f(M)$ is compact in $N$. Since $N$ is Hausdorff, $f(M)$ must be closed in $N$. Therefore, since $N$ is connected, we conclude that $f(M) = N$. This means that $N$ is compact and $f$ is surjective. All that remains is to show that $f$ is a covering map.

Let $q \in N$, and consider $f^{-1}(q)$, which is closed in $M$. For each $x \in f^{-1}(q)$, there exists a neighborhood $U_x$ of $x$ such that $f|_{U_j}$ is a diffeomorphism. Since $M$ is Hausdorff, we may shrink these neighborhoods so that they are pairwise disjoint. This means that each $x \in f^{-1}(q)$ is isolated. Hence, $f^{-1}(q)$ is discrete in $M$. Since discrete subspaces of compact spaces must be finite, it follows that $f^{-1}(q)$ is finite; let $f^{-1}(q) = \{x_1, \ldots, x_s\}$. As stated above, for each $j = 1, \ldots, s$, we can find a neighborhood $U_j$ of $x_j$ such that $f|_{U_j} : U_j \to V_j \subset N$ is a diffeomorphism. Since $M$ is Hausdorff, we may shrink these neighborhoods so that $U_i \cap U_j = \varnothing$ for all $i \neq j$; $f$ restricted to each of these new $U_j$'s remains a diffeomorphism. Set $V = \bigcap_1^s f(U_j)$, and define $\widetilde{U}_j = f^{-1}(V) \cap U_j$. For each $j$, $f : \widetilde{U}_j \to V$ is a diffeomorphism and $V = \bigsqcup_1^s f(U_j)$. Hence, $V$ is an evenly covered neighborhood of $q$, so that $f$ is a covering map. □

Now, assume to the contrary that $f : S^2 \times S^2 \to M_1 \times M_2 \times M_3$ is a diffeomorphism; since diffeomorphisms preserve dimensions and $M_1, M_2, M_3$ have nonzero dimensions, it follows, without loss of generality, that $M_1, M_2$ are 1-dimensional and $M_3$ is 2-dimensional. Since diffeomorphisms of manifolds are immersions, by the Comps Lemma, $M_1 \times M_2 \times M_3$ must be compact and connected; by projecting onto each manifold, $M_1, M_2, M_3$ must be compact and connected. Moreover, the induced group homomorphism $f_* : \pi_1(S^2 \times S^2) \to \pi_1(M_1 \times M_2 \times M_3) = \pi_1(M_1) \times \pi_1(M_2) \times \pi_1(M_3)$ must be an isomorphism. Since $S^2$ is simply connected,

$$\pi_1(S^2 \times S^2) = \pi_1(S^2) \times \pi_1(S^2) = \{0\}. \tag{43}$$

On the other hand, since the only compact connected 1-manifold, up to diffeomorphism, is the unit circle $S^1$, and $\pi_1(S^1) \cong \mathbb{Z}$ is not trivial, $\pi_1(M_1 \times M_2 \times M_3)$ is not trivial. But this contradicts our claim that $f_*$ is an isomorphism. Hence, by contradiction, $f$ cannot be a diffeomorphism.

**Problem 2023-J-II-3 (Geometry/Topology).** Consider the form $\omega = (x^2 + x + y)dy \wedge dz$ on $\mathbb{R}^3$. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, and $i : S^2 \to \mathbb{R}^3$ be the inclusion map.

(a) Calculate $\int_{S^2} i^*\omega$.

(b) Construct a closed form $\alpha$ on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$, or show that such a form $\alpha$ does not exist.

(a) (**Method 1**) Consider the form $\omega = (x^2 + x + y)dy \wedge dz$ on $\mathbb{R}^3$, and let $i : S^2 \hookrightarrow \mathbb{R}^3$ be the inclusion map. Let $D = [0, \pi] \times [0, 2\pi]$, and $F : D \to S^2$ be the coordinate map defined by

$$F(\varphi, \theta) = (\sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi)). \tag{44}$$

Taking $D_1 = [0,\pi]\times[0,\pi]$ and $D_2 = [0,\pi]\times[\pi,2\pi]$, and letting $F_1 = F|_{D_1}$ and $F_2 = F|D_2$, we observe that

$$\int_{S^2} i^*\omega = \int_{D_1} F_1^* i^*\omega + \int_{D_1} F_2^*\omega = \int_{D_1} (i\circ F_1)^*\omega + \int_{D_2} (i\circ F_2^*)\omega = \int_D F^*\omega, \tag{45}$$

where the last equality follows from the fact that $i\circ F_{1,2} = F_{1,2}$. We observe that

$$F^*dy = \cos(\varphi)\sin(\theta)d\varphi + \sin(\varphi)\cos(\theta)d\theta \qquad \text{and} \qquad F^*dz = -\sin(\varphi)d\varphi. \tag{46}$$

Therefore,

$$F^*\omega = \left[\sin^2(\varphi)\cos^2(\theta) + \sin(\varphi)\cos(\theta) + \sin(\varphi)\sin(\theta)\right]\sin^2(\varphi)\cos(\theta)d\varphi\wedge d\theta. \tag{47}$$

From this, we conclude that

$$\int_{S^2} i^*\omega = \int_0^{2\pi}\int_0^{\pi}\left[\sin^2(\varphi)\cos^2(\theta) + \sin(\varphi)\cos(\theta) + \sin(\varphi)\sin(\theta)\right]\sin^2(\varphi)\cos(\theta)d\varphi d\theta = \frac{4\pi}{3}. \tag{48}$$

(**Method 2**) Using Stokes Theorem,

$$\int_{S^2} i^*\omega = \int_{B^3} d\omega, \tag{49}$$

where $B^3$ indicates the 3-ball (recall that $S^1 = \partial B^3$). We compute, $d\omega = (2x+1)dx\wedge dy\wedge dz$ so that

$$\int_{S^2} i^*\omega = \int_{B^3} d\omega = \int_{B^3} 2xdxdydz + \int_{B^3} dxdydz = \int_{B^3} dxdydz = \frac{4\pi}{3}, \tag{50}$$

where the first integral after the second inequality is zero due to symmetry.

(b) Suppose there exists a closed form $\alpha$ on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$. Since $\alpha$ is closed, $d\alpha = 0$. Hence,

$$\int_{S^2} i^*\alpha = \int_{B^3} d(i^*\alpha) = \int_{B^3} i^*d\alpha = 0 \neq \frac{4\pi}{3} = \int_{S^2} i^*\omega, \tag{51}$$

which is a contradiction. Hence, such a closed form cannot exist.

**Problem 2023-J-I-5 (Algebra).** Consider the following irreducible polynomial over $\mathbb{Q}$: $p(x) = x^4 - 3x^2 - 1$.

(a) Describe the splitting field of $p(x)$.

(b) Consider the Galois group of $p(x)$. Compute its order and determine if it is abelian.

(a) Let $p(x) = x^4 - 3x^2 - 1$. By the rational root test, $p(x)$ has no roots over $\mathbb{Q}$. Moreover, it is straightforward to check that $p(x)$ is not the product of irreducible quadratics with rational coefficients. Hence, $p(x)$ is irreducible over $\mathbb{Q}$. We start by finding the roots of $p(x)$; let $u = x^2$. Then

$$u^2 - 3u - 1 = 0 \implies u = \frac{3\pm\sqrt{13}}{2} \implies x = \pm\sqrt{\frac{3\pm\sqrt{13}}{2}}. \tag{52}$$

Let

$$\alpha = \sqrt{\frac{3+\sqrt{13}}{2}}, \qquad \beta = \sqrt{\frac{3-\sqrt{13}}{2}}. \tag{53}$$

Observe that $\alpha^2\beta^2 = -1$ so that $\beta = \pm\frac{i}{\alpha}$. Therefore, the splitting field of $p(x)$ is

$$\mathbb{Q}(\alpha, i). \tag{54}$$

Observe that the minimal polynomial of $i$ is $x^2+1$, which is irreducible over $\mathbb{Q}(\alpha)$ so that $[\mathbb{Q}(\alpha,i):\mathbb{Q}(\alpha)] = 2$. On the other hand, the minimal polynomial of $\alpha$ is a degree 4 polynomial so that $[\mathbb{Q}(\alpha):\mathbb{Q}] = 4$. Hence, by the tower law, $[\mathbb{Q}(\alpha,i):\mathbb{Q}] = 8$.

(b) By the last work in (a), the order of the Galois group of $p(x)$ is 8. Now, we will determine the Galois group of $p(x)$. Recall that elements of $\mathrm{Gal}(\mathbb{Q}(\alpha,i)/\mathbb{Q})$ are automorphisms $\varphi$ of the field $\mathbb{Q}(\alpha,i)$ with the constraints that: (1) $\varphi$ fixes $\mathbb{Q}$, (2) $\varphi(\alpha)$ must be another root of the minimal polynomial of $\alpha$ over $\mathbb{Q}$, and (3) $\varphi(i)$ must be another root of $x^2+1$. We will explicitly work through each of the elements.

(i) $\sigma : i \mapsto -i, \alpha \mapsto \alpha$. This permutation has order 2 since $\sigma^2(\alpha) = \sigma(\alpha) = \alpha$ and $\sigma^2(i) = \sigma(-i) = i$.

(ii) $\tau : i \mapsto i, \alpha \mapsto -\alpha$. Once again, this permutation has order 2.

(iii) $\rho : i \mapsto -i, \alpha \mapsto \beta = \frac{i}{\alpha}$. To compute the order of this permutation, observe that

$$\rho^2(\alpha) = \rho(i\alpha^{-1}) = (-i)\cdot\frac{1}{i/\alpha} = -\alpha \implies \rho^4(\alpha) = \rho^2(-\alpha) = \alpha. \tag{55}$$

Likewise, $\rho^4(i) = \rho^2(i) = i$. Hence, $\rho$ has order 4.

Now, consider the three elements given above. We compute

$$\sigma\rho\sigma(i) = \sigma\rho(-i) = \sigma(i) = -i = \rho^{-1}(i). \tag{56}$$

Likewise,

$$\sigma\rho\sigma(\alpha) = \sigma\rho(\alpha) = \sigma(i)\sigma(\alpha)^{-1} = -\frac{i}{\alpha} = \rho^{-1}(\alpha). \tag{57}$$

Therefore, $\sigma\rho\sigma = \rho^{-1}$. Hence,

$$\mathrm{Gal}(\mathbb{Q}(\alpha,i)/\mathbb{Q}) = \{1,\sigma,\rho,\rho^2,\rho^3,\sigma\rho,\sigma\rho^2,\sigma\rho^3\} \cong D_8. \tag{58}$$

Since the dihedral group is not abelian, we conclude that the Galois group for $p(x)$ is non-abelian.

**Problem 2023-J-I-5 (Algebra I).** Determine the Galois group of $x^3 - x^2 - 4$.

Let $p(x) = x^3 - x^2 - 4$. We start by finding the roots of $p(x)$ over some algebraic closure of $\mathbb{Q}$. Observe that 2 is a solution. Using polnomial long division,

$$p(x) = (x-2)(x^2+x+2) \implies x = 2, \frac{-1 \pm \sqrt{-7}}{2}. \tag{59}$$

Hence, the splitting field of $p(x)$ is $\mathbb{Q}(\sqrt{7}i)$. Now since $\mathrm{Gal}(\mathbb{Q}(\sqrt{7}i)/\mathbb{Q})$ is the group of automorphisms of the splitting field $\mathbb{Q}(\sqrt{7}i)$ that preserve $\mathbb{Q}$. Since there are exactly two automorphisms (namely, the identity permutation fixing $\sqrt{7}i$ and the conjugation map $\sqrt{7}i \mapsto -\sqrt{7}i$), we conclude that $\mathrm{Gal}(\mathbb{Q}(\sqrt{7}i)/\mathbb{Q}) \cong \mathbb{Z}_2$.

**Problem 2023-J-I-5 (Algebra II).** Determine the Galois group of $x^3 - 2x + 4$.

Let $p(x) = x^3 - 2x + 4$. We start by finding the roots of $p(x)$ over some algebraic closure of $\mathbb{Q}$. Clearly $-2$ is a root of $p(x)$. Using polynomial long division,

$$p(x) = (x+2)(x^2-2x+2) \implies x = -2, 1 \pm \sqrt{-1}. \tag{60}$$

Hence, the splitting field of $p(x)$ is $\mathbb{Q}(i)$, which is a quadratic extension of $\mathbb{Q}$. Now since $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ is the group of automorphisms of the splitting field $\mathbb{Q}(i)$ that preserve $\mathbb{Q}$, and there exactly two such automorphisms (namely, the identity fixing $i$, and the conjugation map $i \mapsto -i$), we conclude that $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$.

**Problem 2023-J-I-5 (Algebra III).** Determine the Galois group of $x^3 - x + 1$.

Let $p(x) = x^3 - x + 1$. We start by finding the roots of $x$ over some algebraic closure of $\mathbb{Q}$. Since the only possible rational roots of $p$ over $\mathbb{Q}$ are $\pm 1$ by the Rational Root Test, and neither of these are actually roots of $p$, we conclude that $p$ is irreducible. Hence, a root of $f(x)$ generates an extension of degree 3 so that the degree of the splitting field of $F$ is divisible by 3. Since the Galois group is a subgroup of $S_3$, either $\mathrm{Gal}(\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)/\mathbb{Q}) \cong A_3$ or $\mathrm{Gal}(\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)/\mathbb{Q}) \cong S_3$. Since $p$ is already a depressed cubic, we calculate its discriminant to be $-4(-1)^3 - 27(1)^2 = -23$. Since the discriminant is not a perfect square in $\mathbb{Q}$, we conclude that $\mathrm{Gal}(\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)/\mathbb{Q}) \cong S_3$.

**Problem 2023-J-I-4 (Geometry/Topology).** Let $\omega$ be a smooth nowhere vanishing 1-form on a smooth 3-manifold $M^3$.

(a) Show that the distribution defined at each point $p \in M$ by

$$\ker \omega_p = \left\{ v \in T_p M^3 : \omega_p(v) = 0 \right\} \tag{61}$$

is integrable if and only if $\omega \wedge d\omega = 0$.

(b) Give an example of a codimension one distribution on $\mathbb{R}^3$ that is not integrable.

(a) We recall that a distribution $D$ is Frobenius integrable if and only if given two smooth sections $X, Y$ of $D$, the Lie Bracket $[X, Y]$ is also a smooth section of $D$. Therefore, let $X, Y$ be smooth sections of $D$, which means that $\omega(X), \omega(Y) = 0$ by definition of $D$. We recall that

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) = -\omega([X, Y]), \tag{62}$$

where the first two terms are identically zero by our hypothesis. Therefore, $D$ is integrable if and only if $[X, Y]$ is a smooth section of $D$ if and only if $\omega([X, Y]) = 0$. Now, if $D$ were integrable, then for any field $Z$ on $\mathbb{R}^3$,

$$\omega \wedge d\omega(X, Y, Z) = \omega(Z) d\omega(X, Y) = 0, \tag{63}$$

where the other terms vanish by assumption on $X$ and $Y$. Hence, since $X, Y \in \ker \omega$ were arbitrary and $Z$ was arbitrary, $\omega \wedge d\omega = 0$. On the other hand, if $\omega \wedge d\omega = 0$, let $p \in M$, $Z_p \in T_p M$ with $\omega_p(Z_p) \neq 0$ and $X_p, Y_p \in \ker \omega_p$. Then

$$0 = (\omega \wedge d\omega)_p(X_p, Y_p, Z_p) = \omega_p(Z_p) d\omega_p(X_p, Y_p). \tag{64}$$

Hece, $d\omega_p(X_p, Y_p) = 0$. This means that for smooth sections $X, Y$ of $\ker \omega$, $d\omega(X, Y) = 0$, and so $D$ is integrable.

(b) Consider the smooth nowhere vanishing 1-form $\omega = y dx + dy + dz$ on $\mathbb{R}^3$, and let $D$ be the distribution on $\mathbb{R}^3$ defined at each point $p \in M$ by $D_p = \ker \omega_p$. By the rank-nullity theorem, $\dim D = \dim T_p \mathbb{R}^3 - \operatorname{rank} \omega = 3 - 1 = 2$. Hence, $\operatorname{codim} D = 3 - 2 = 1$. Next, we observe that $d\omega = dy \wedge dx$, which is identically not zero. Then $\omega \wedge d\omega = dz \wedge dy \wedge dx$, which is also not identically zero. Hence, by the conclusion in (a), $D$ is not integrable.

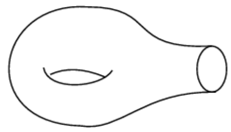
## August 2023

**Problem 2023-A-I-1 (Algebra).** Let $V$ be an $n$-dimensional vector space over a field $F$. An element $A \in \operatorname{End} V$ is called *nilpotent* if $A^k = 0$ for some $k > 1$. Prove that $A$ is nilpotent if and only if

$$\mathrm{Tr}(\Lambda^i A) = 0, \quad i = 1, \ldots, n$$

where $\Lambda^i A$ denotes the induced action of $A$ on the wedge product $\Lambda^i V$ for each $i$.

**Problem 2023-A-I-5 (Geometry/Topology).** Let $T$ be the 2-torus $S^1 \times S^1$ with an open 2-disk removed:



Show that there is no continuous retraction $r$ onto its boundary (i.e., no continuous map $r : T \to \partial T$ satisfying $r^2 = r$).

Let $T$ be the 2-torus $S^1 \times S^1$ with an open 2-disk removed, $\iota : \partial T \to T$ the inclusion map, and assume to the contrary that $r : T \to \partial T$ is a continuous retraction. Then the composition $r_* \circ \iota_* : \pi_1(\partial T) \to \pi_1(\partial T)$ must be the identity map. Since $\partial T \cong S^1$, $\pi_1(\partial T) = \mathbb{Z}$, and is generated by the element 1. By a direct computation, since $\partial_1(T) = \mathbb{Z} * \mathbb{Z}$ is the free product on two generators $a$ and $b$ $\iota_*$ maps 1 to the element $aba^{-1}b^{-1}$. But then $r_*$ maps the commutator into the abelian group $\mathbb{Z}$, where the commutator must be zero. This contradicts our claim that $r_* \circ \iota_*$ is the identity map. Hence, by contradiction, there cannot be any continuous retraction of $T$ onto its boundary.

**Problem 2023-A-I-6 (Complex Analysis).** Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk. Is there a holomorphic function $f$ with $f(\mathbb{D}) = \mathbb{D}$, $f(0) = f'(0) = 2/3$? If so, give a formula. If not, prove that it cannot exist.

The problem lends itself nicely to an application of the Schwarz-Pick Theorem:

> (**Schwarz-Pick Theorem**) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. If $|f(z)| \leq 1$ for all $z$, and $f(a) = b$ for some $a, b \in \mathbb{D}$, then
> $$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}.$$

Now assume that a holomorphic function $f$ with $f(\mathbb{D}) = \mathbb{D}$, $f(0) = f'(0) = 2/3$ exists. Then by the Schwarz-Pick Lemma,

$$\frac{2}{3} \leq \frac{1 - 4/9}{1 - 0} = \frac{5}{9} < \frac{2}{3}, \tag{65}$$

which is a contradiction. Hence, no such holomorphic function can exist.

**Problem 2023-A-I-2 (Geometry/Topology).** Let $f : T^2 \to S^2$ be a smooth map from the 2-torus to the 2-sphere. Can $f$ be an immersion? If the answer is yes, give an explicit example. If the answer is no, then give a proof.

We begin by stating and proving a technical lemma, which we will then use in our argument.

> (**Comps Lemma**) Let $M$ and $N$ be smooth connected $n$-manifolds, and $f : M \to N$ a (smooth) immersion. If $M$ is compact and nonempty, then $N$ is compact and $f$ is a (smooth) covering map.
>
> *Proof.* Let $M$ and $N$ be smooth connected $n$-manifolds, and $f : M \to N$ an immersion. Since $\dim M = \dim N = n$, and $f$ is an immersion, the map $df_p : T_pM \to T_{f(p)}N$ has constant rank $n$ at every $p \in M$. Hence, by the Inverse Function Theorem, $f$ is a local diffeomorphism. Since local diffeomorphisms are open maps, $f(M)$ is open in $N$. On the other hand, since continuous images of compact sets are compact, $f(M)$ is compact in $N$; since $N$ is Hausdorff, $f(M)$ is closed in $N$. Since $N$ is connected, it follows that $f(M) = N$. Therefore, $N$ is compact. All that remains is to show is that $f$ is a covering map.
>
> Let $q \in N$; by continuity of $f$, $f^{-1}(q)$ is a closed subset of $M$. For each $x \in f^{-1}(q)$, there exists an open neighborhood $U_x$ of $x$ such that $f|_{U_x}$ is a diffeomorphism. Since $M$ is Hausdorff,

we can shrink these neighborhoods so that they are pairwise disjoint. This means that each $x \in f^{-1}(q)$ is isolated, implying that $f^{-1}(q)$ is discrete. Since $M$ is compact, it follows that $f^{-1}(q)$ is finite; let $f^{-1}(q) = \{x_1, \dots, x_s\}$. As stated above, for each $j = 1, \dots, s$, we may find an open neighborhood $U'_j$ so that $f|U'_j$ is a diffeomorphism. Moreover, we can shrink these neighborhoods to obtain a pairwise disjoint collection $\{\widetilde{U}_j\}_1^s$ of neighborhoods. Set $V = \bigcap_1^s f(\widetilde{U}_j)$. Then taking $U_j = f^{-1}(V) \cap \widetilde{U}_j$, $V$ is an evenly covered neighborhood of $p$, so that $f$ is a covering map. □

Now assume to the contrary that there exists an immersion $f : T^2 \to S^2$. By the Comps Lemma, $f$ must be a covering map. Hence, the induced homomorphism of groups $f_* : \pi_1(T^2) \to \pi_1(S^2)$ must be injective. Since $S^2$ is simply connected, $\pi_1(S^2) \cong \{0\}$. However, $\pi_1(T^2)$ is not a trivial group (in fact, $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$). This means that $f_*$ cannot be injective. Therefore, by contradiction, $f$ cannot be an immersion. Hence, there exist no immersions from $T^2$ to $S^2$.

**Problem 2023-A-II-1 (Algebra).** A field extension $K/L$ is called algebraic, if every element in $K$ satisfies a polynomial equation with coefficients in $L$. Let $F, K, L$ be fields such that $F \supset K \supset L$, and $F/K$ and $K/L$ are algebraic extensions. Prove that $F/L$ is also an algebraic extension.

Since subfields of subfields is a subfield, $L$ is a subfield of $F$. Hence, it suffices to show that every element in $F$ satisfies a polynomial equation with coefficients in $L$. Let $a \in F$, and let

$$k(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_0 \in K[x] \tag{66}$$

such that $k(a) = 0$; this follows since $F/K$ is an algebraic extension. Each $k_j \in K$, and hence is algebraic over $L$. Therefore, $L' = L(k_0, \dots, k_n)$ is a finite extension of $L$. Since $k(a) = 0$ and $k(x)$ now has its coefficients in $L'$, it follows that $a$ is algebraic over $L'$ so that $L'(a)$ is a finite extension of $L$. Then since

$$[L(a) : L] = [L(a) : L'][L' : L], \tag{67}$$

it follows that $L(a)$ is a finite extension of $L$. Therefore, $a$ is algebraic over $L$. Since $a$ was arbitrary, $F/L$ is an algebraic extension.

**Problem 2023-A-I-2 (Geometry/Topology).** Let $f : T^2 \to S^2$ be a smooth map from the 2-torus to the 2-sphere. Can $f$ be an immersion? If the answer is yes, given an explicit example. If the answer is no, then give a proof.

There cannot be an immersion $f : T^2 \to S^2$. To prove our answer, we will state and proof a technical lemma.

(**Comps Lemma**) Let $M, N$ be smooth, connected, $n$-manifolds and $f : M \to N$ a (smooth) immersion. If $M$ is compact and nonempty, then $f$ is a (smooth) covering map.

*Proof.* Let $M, N$ be smooth connected $n$-manifolds, $M$ compact, and $f : M \to N$ an immersion. Since $\dim N = n$ everywhere and $f$ is an immersion, $df_p : T_pM \to T_{f(p)}N$ has constant rank $n$ everywhere. Hence, by the Inverse Function Theorem, $f$ is a local diffeomorphism. Let $q \in N$ so that $f^{-1}(q) \subset M$ is closed. For each $x \in f^{-1}(q)$, there exists a neighborhood $U_x$ such that $f|_{U_x} : U_x \to V_x \subset N$ is a diffeomorphism. Since $M$ is Hausdorff, we can shrink these neighborhoods so that they are pairwise disjoint. Since every $x \in f^{-1}(q)$ is now isolated, it follows that $f^{-1}(q)$ is discrete. Since $M$ is compact, we conclude that $f^{-1}(q)$ must be finite; let $f^{-1}(q) = \{x_1, \dots, x_s\}$. As stated above, for each $j = 1, \dots, s$, we can find a neighborhood $U_j$ of $x_j$ so that $f|_{U_j} : U_j \to V_j \subset N$ is a diffeomorphism. Again, since $M$ is Hausdorff, we can shrink these neigborhoods so that $U_i \cap U_j = \varnothing$ for all $i \neq j$; $f$ restricted to each of these shrunken neighborhoods remains a diffeomorphism. Now set $V = \bigcap_1^s f(U_j)$, and define $\widetilde{U}_j \subset M$ by $\widetilde{U}_j = f^{-1}(V) \cap U_j$ for each $j = 1, \dots, s$. Hence, $V$ is an evenly covered neighborhood of $q \in N$, which means $f$ is a covering map. That $f$ is surjective comes from recognizing that $f(M) = N$ due to connectedness of $N$. □

Now, assume $f : T^2 \to S^2$ is an immersion. Since $T^2, S^2$ are smooth, connected 2-manifolds, and $T^2$ is compact and nonempty, by the Comps Lemma, $f$ is a covering map. Hence, the induced homomorphism $f_* : \pi_1(T^2) \to \pi_1(S^2)$ is injective. Since $S^2$ is simply connected, $\pi_1(S^2) \cong \{0\}$. On the other hand, $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$. Since the order of $\pi_1(T^2)$ is more than one, $f_*$ cannot be injective. Hence, $f$ cannot be an immersion.

**Problem 2023-A-II-5 (Geometry/Topology).** Let $(t, x, y, z)$ be the standard coordinate system on $\mathbb{R}^4$, and let $\phi$ be the non-zero smooth 1-form on $\mathbb{R}^4$ defined by

$$\phi = dt + ydx + zdy.$$

Let $D$ be the 3-plane field on $\mathbb{R}^4$ that consists of tangent vectors $V$ such that $\phi(V) = 0$. Is $D$ Frobenius integrable? Support your answer with a proof.

Let $D$ be the 3-plane field on $\mathbb{R}^4$ defined as follows: for each $p \in \mathbb{R}^4$,

$$D_p = \left\{v \in T_p\mathbb{R}^4 : \phi(v) = 0\right\} =: \ker \phi_p. \tag{68}$$

Hence, by the Frobenius Theorem, $D$ is Frobenius integrable if and only if $\phi \wedge d\phi = 0$. We compute:

$$d\phi = d(dt + y\,dx + z\,dy) = d^2t + dy \wedge dx + dz \wedge dy = dy \wedge dx + dz \wedge dy. \tag{69}$$

Therefore,

$$\phi \wedge d\phi = dt \wedge dy \wedge dx + dt \wedge dz \wedge dy + ydx \wedge dz \wedge dy. \tag{70}$$

Since $\phi \wedge d\phi$ is nowhere vanishing on $\mathbb{R}^4$, $D$ is not Frobenius integrable.

**Problem 2023-A-I-1 (Algebra).** Let $V$ be a $n$-dimensional vector space over a field $F$. An element $A \in \operatorname{End} V$ is called *nilpotent*, if $A^k = 0$ for some $k > 1$. Prove that $A$ is nilpotent if and only if

$$\operatorname{Tr}(\Lambda^i A) = 0, \quad i = 1, \ldots, n, \tag{71}$$

where $\Lambda^i A$ denotes the induced action of $A$ on the wedge product $\Lambda^i V$ for each $i$.

Let $V$ be a $n$-dimensional vector space over a field $F$, and let $A \in \operatorname{End} V$. Recall that $\Lambda^i A$, the induced action of $A$ on the wedge product $\Lambda^i V$, is defined to be

$$(\Lambda^i A)(v_1 \wedge \cdots \wedge v_i) = Av_1 \wedge \cdots \wedge Av_i, \qquad v_j \in V \text{ for all } j = 1, \ldots, i. \tag{72}$$

Over an algebraic closure of $F$, $A$ has eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose $A$ is diagonalizable, with the set of eigenvectors given by $\{v_1, \ldots, v_n\}$. Then for each $i = 1, \ldots, n$, since the collection

$$\left\{v_{j_1} \wedge \cdots \wedge v_{j_i} : 1 \le j_1 < \cdots < j_i \le n\right\}$$

is a basis of $\Lambda^i V$, and for each $i$-tuple, $\Lambda^i A(v_{j_1} \wedge \cdots \wedge v_{j_i}) = Av_{j_1} \wedge \cdots \wedge Av_{j_i} = (\lambda_{j_1} \cdots \lambda_{j_i})(v_{j_1} \wedge \cdots \wedge v_{j_i})$, it follows that the eigenvalues of $\Lambda^i A$ are the set of all products of the form $\lambda_{j_1} \cdots \lambda_{j_i}$ for $1 \le j_1 < \cdots < j_i \le n$, counting for multiplicity. Hence,

$$\operatorname{Tr}(\Lambda^i A) = \sum_{1 \le j_1 < \cdots < j_i \le n} \lambda_{j_1} \cdots \lambda_{j_i}. \tag{73}$$

If $A$ is not diagonalizable, since the eigenvalues of $\Lambda^i A$ depend only on the eigenvalues of $A$, we may assume $A$ is in Jordan normal form. Indeed, if $A = PJP^{-1}$, then

$$\Lambda^i(A) = \Lambda^i(PJP^{-1}) = \Lambda^i(P)\Lambda^i(J)\Lambda^i(P^{-1}), \tag{74}$$

so $\Lambda^i A$ and $\Lambda^i J$ are similar and therefore have the same eigenvalues. Thus it suffices to compute the eigenvalues of $\Lambda^i J$, which are exactly the products $\lambda_{j_1} \cdots \lambda_{j_i}$ of the eigenvalues of $A$.

If $A$ is nilpotent so that $A^k = 0$ for some $k > 1$, then since $0 = A^k v = \lambda^k v$ for all eigenvectors $v$ of $A$, it follows that every eigenvalue of $A$ is zero. Therefore, the above expression implies that $\mathrm{Tr}(\Lambda^i A) = 0$ for all $i = 1, \dots, n$. On the other hand, expanding the characteristic polynomial for $A$ is given by:

$$p_A(t) = \det(tI - A) = t^n - \mathrm{Tr}(\Lambda^1 A)t^{n-1} + \cdots + (-1)^n \mathrm{Tr}(\Lambda^n A). \tag{75}$$

If $\mathrm{Tr}(\Lambda^i A) = 0$ for all $i = 1, \dots, n$, then we conclude that the characteristic polynomial of $A$ is precisely $t^n$. Therefore, $A$'s eigenvalues are all zero. Hence, the minimal polynomial of $A$ is of the form $t^k$ for some $k \le n$. This implies that $A^k = 0$, and so $A$ is nilpotent.

**Problem 2023-A-II-6 (Complex Analysis).** Find the number of solutions (counting multiplicity) to $z^8 - 5z^6 + 2z^3 - z - 1 = 0$ that lie inside the unit disk.

Recall Rouché's Formula, which states that

> For any two complex-valued functions $f$ and $g$ holomorphic inside some region $K$ with closed and simple contour $\partial K$, if $|g(z)| < |f(z)|$ on $\partial K$, then $f$ and $f+g$ have the same number of zeros inside $K$, where each zero is counted as many times as its multiplicity.

Pick $f(z) = 5z^6$ and set $h(z) = z^8 + 2z^3 - z - 1$ so that $p(z) = z^8 - 5z^6 + 2z^3 - z - 1 = h(z) - f(z)$. On the unit disk $\partial S^1$, we observe that

$$\begin{aligned} |f(z)| &= \left|5z^6\right| = 5 \\ &= 1 + 2 + 1 + 1 \\ &= |z^8| + 2|z^3| + |z| + |1| \\ &\ge |h(z)|. \end{aligned} \tag{76}$$

Hence, $p(z) = h(z) - f(z)$ has the same number of zeros, counting multiplicity, as $f(z)$. Since $f(z)$ has six zeros in the unit disk, we conclude that $p(z)$ must also have six zeros inside the unit disk.

**Problem 2023-A-II-4 (Real Analysis).** Let $\mu$ be a (positive) Borel probability measure on $[0,1]$, such that for all $t \in [0,1]$ we have $\mu(\{t\}) = 0$. Let $\mu_n$ be a (positive) Borel probability measure on $[0,1]$ for $n = 1, 2, \dots$. Suppose $\mu_n \to \mu$ in the weak* topology. Let $F(t) = \mu([0,t])$ and $F_n(t) = \mu_n([0,t])$. Prove that $F_n \to F$ uniformly.

## January 2022

**Problem 2022-J-I-3 (Algebra).** Show that a group of order 1,000,000 contains a proper normal subgroup (i.e., is not simple).

Let $G$ be a group of order $1{,}000{,}000 = 10^6 = 2^6 \cdot 5^6$. By Sylow's Theorem,

$$\begin{aligned} &n_2 \in \{1, 5, 5^2, 5^3, 5^4, 5^5, 5^6\} \cap \{2k+1 : k \in \mathbb{N}\}, \\ &n_5 \in \{1, 2, 4, 8, 16, 32, 64\} \cap \{5k+1 : k \in \mathbb{N}\} = \{1, 16\}. \end{aligned} \tag{77}$$

If $n_5 = 1$, then we are done since the unique Sylow 5-subgroup must necessarily be normal. So suppose $n_5 = 16$, and let $G$ act on $\mathrm{Syl}_5(G)$ by conjugation. This induces a homomorphism $\varphi : G \to \varphi(G) \le S_{16}$. However, $|G| = 10^6 \nmid 16! = |S_{16}|$. This means that $\varphi$ cannot be an injective homomorphism since if otherwise, $|\varphi(G)| = |G|$, but this is impossible since $|G| \nmid |S_{16}|$. Therefore, $\ker\varphi$ is a nontrivial normal subgroup of $G$. If $\ker\varphi = G$, then every Sylow 5-subgroup of $G$ is normal and is, in fact, unique, which contradicts our hypothesis that $n_5 = 16$. Hence, $\ker\varphi$ is a proper nontrivial normal subgroup of $G$, which means that $G$ cannot be simple.

## August 2022

**Problem A-II-I (Real Analysis).** Suppose $E \subset \mathbb{R}^2$ has positive Lebesgue area. Show that $E$ contains 3 points that form the vertices of an equilateral triangle.

Let $E \subset \mathbb{R}^2$ be a set of positive Lebesgue measure (we will denote by $m^2$ the Lebesgue measure on $\mathbb{R}^2$). Let $\{v_1, v_2\}$ be a collection of unit vectors in $\mathbb{R}^2$ so that the angle between $v_1$ and $v_2$ is 120°, and let $\beta < 1/3$. By inner regularity of the Lebesgue measure, there exists a compact set $K_1 \subset E$ so that $m^2(K_1) > 0$. Then by *outer* regularity of the Lebesgue measure, there exists an open set $U$ containing $K_1$ such that $m^2(U) \leq (1+\beta)m^2(K_1)$.
Since $K_1$ is compact, $d_1 = d(K_1, U^c)$ is positive; so let $R = d_1$, pick an arbitrary $r \in (0, R)$, and consider the set $K_1 + rv_1$. $K_1 + rv_1$ has to be contained within $U$ since otherwise,

$$d(K_1, U^c) < |rv_1| = r < d_1, \text{ which is a contradiction.} \tag{78}$$

Hence, $K_1 \cup (K_1 + rv_1) \subset U$, which means

$$m^2(U) \geq m^2(K_1 \cup (K_1+rv_1)) = m^2(K_1)+m^2(K_1+rv_1)-m^2(K_1\cap(K_1+rv_1)) = 2m^2(K_1)-m^2(K_1\cap(K_1+rv_1)), \tag{79}$$

where the last equality follows from translation invariance of the Lebesgue measure. Hence, $m^2(K_1 \cap (K_1 + rv_1)) = 2m^2(K_1) - m^2(U) \geq (1-\beta)m^2(K_1) > 0$. Therefore, $K_2 := K_1 \cap (K_1 + rv_1)$ is nonempty. Now define $K_3 = K_2 \cap (K_2 + rv_2)$. Using the same reasoning as above, we observe that $K_3 \neq \varnothing$ and $K_3 \subset K_2$. Hence, we obtain a nested sequence of sets $\varnothing \neq K_3 \subset K_2 \subset K_1 \subset E$. Let $M \in K_3$. Since $K_3 = K_2 \cap (K_2 + rv_1)$, $N = q - rv_2 \in K_2$. Likewise, $O = q - rv_2 - rv_1 \in K_1$. These three points form the vertices of a triangle. Then since

$$\|M - N\| = r, \qquad \|N - O\| = r, \qquad \|M - O\| = \|r(v_2 + v_1)\| = r\,\|v_2 + v_1\| = r. \tag{80}$$

**Problem 2022-A-II-4 (Algebra).** Let $G$ be a finite group in which $(ab)^p = a^p b^p$ for every $a, b \in G$, where $p$ is a prime dividing $|G|$. Prove that the Sylow $p$-subgroup of $G$ is normal in $G$ (and is in fact unique).

Let $G$ be a finite group in which $(ab)^p = a^p b^p$ for every $a, b \in G$, where $p$ is a prime dividing $|G|$. Consider the map $\varphi : G \to G$ defined by $\varphi(g) = g^p$. This map is a homomorphism since for any $g, h \in G$,

$$\varphi(gh) = (gh)^p = g^p h^p = \varphi(g)\varphi(h), \tag{81}$$

where the second equality follows from the hypothesis. Consider the map

$$\varphi^k := \underbrace{\varphi \circ \cdots \circ \varphi}_{k \text{ copies}}, \tag{82}$$

which must also be a homomorphism since the composition of homomorphisms is a homomorphism. The kernel of $\varphi^k$ consists exactly of those elements $x \in G$ whose order is a power of $p$ (i.e., $x^{p^r} = 1$ for some positive integer $r$) since

$$\varphi^k(x) = x^{p^k} = x^{p^{r+(k-r)}} = \left(x^{p^r}\right)^{p^{k-r}} = 1^{p^{k-r}} = 1. \tag{83}$$

Hence, since every element with order equal to some order of $p$ belongs in a Sylow $p$-subgroup of $G$,

$$\ker \varphi^k = \bigcup_{P \in \mathrm{Syl}_p(G)} P. \tag{84}$$

Moreover, $\ker \varphi^k$ must be a $p$-subgroup of $G$ since if not, there exists a prime $p' \neq p$ dividing $|\ker \varphi^k|$, which means by Cauchy's Theorem that $\ker \varphi^k$ contains an element of order $p'$ (which is impossible). Hence, since $\ker \varphi^k$ is a $p$-subgroup of $G$ containing a Sylow $p$-subgroup, by maximality of Sylow $p$-subgroups, $\ker \varphi^k$ must be a Sylow $p$-subgroup of $G$. Hence, $G$ has a unique Sylow $p$-subgroup. And since kernels of homomorphisms are normal subgroups, this Sylow $p$-subgroup must be normal.

## August 2021

**Problem 2021-A-I-6 (Geometry/Topology).** What connected spaces can be finitely-sheeted covering spaces of a sphere with three handles?

We claim that the finitely-sheeted covering spaces of a sphere with three handles are exactly the closed orientable connected surfaces of genus of the form $2k+1$ for some positive integer $k$. Let $M$ be a $k$-sheeted covering space of a sphere with three handles. If $M$ were nonorientable, then since covering maps are local diffeomorphisms and local diffeomorphisms preserve orientability, the sphere with three handles must also be nonorientable, which is a contradiction. Hence, $M$ has to be orientable. Next, since $M$ is a $k$-sheeted covering space of the sphere with three handles, which has Euler characteristic $2-2(3) = -4$, we must have

$$2 - 2g_M = \chi(M) = -4k \implies g_M - 1 = 2k \implies g_M = 2k + 1. \tag{85}$$

**Problem 2021-A-II-1 (Geometry/Topology).** Let $M$ be a compact manifold (without boundary) and $\pi : M \to S^1$ a submersion onto the circle. Show that the de Rham group $H^1_{\text{dr}}(M) \neq 0$.

Let $M$ be a compact manifold (without boundary) and $\pi : M \to S^1$ a submersion onto the circle. Assume to the contrary that $H^1_{\text{dr}}(M) \neq 0$ which means that every closed form on $M$ is an exact form. Since $H^1_{\text{dr}}(S^1) \cong \mathbb{R}$, let $[\omega]$ be a generator of this cohomology group, where $\omega$ is a nowhere vanishing closed 1-form on $S^1$. Since $\pi$ is a submersion, the 1-form $\pi^*\omega$ must also be a nowhere vanishing closed form on $M$. By our hypothesis on the de Rham cohomology group in degree one of $M$, $\pi^*\omega$ is exact, which means there exists a smooth function $f$ such that $\pi^*\omega = df$. Since $M$ is compact and $f$ is smooth, $f$ must attain either a maximum or minimum value at some $p_0 \in M$. This means that $df_{p_0} = 0$. But this contradicts our claim that $\pi^*\omega$ is nowhere vanishing. Hence, by contradiction, $H^1_{\text{dr}}(M) \neq 0$.

## January 2020

**Problem 2020-J-I-1 (Algebra).** Let $G$ be a finite non-abelian group, and let $Z(G)$ denote its center. Prove that $|Z(G)| \leq \frac{1}{4}|G|$, and then give an example where equality holds.

Let $G$ be a finite non-abelian group, and let $Z(G)$ denote its center. Assume to the contrary that $|Z(G)| > \frac{1}{4}|G| \implies |G|/|Z(G)| < 4$. Since $|Z(G)| \mid |G|$, $|G|/|Z(G)|$ is a positive integer. Therefore, one of the three must necessarily be true: (1) $|G|/|Z(G)| = 1$, (2) $|G|/|Z(G)| = 2$, (3) $|G|/|Z(G)| = 3$. If (1) were true, then since $|Z(G)| = |G|$, $G$ has to be abelian, which contradicts our hypothesis. If (2) were true, then $G/Z \cong \mathbb{Z}/2\mathbb{Z}$ which is cyclic. Hence, $G$ would have to be abelian, which is a contradiction. Finally, if (3) were true, then $G/Z \cong \mathbb{Z}/3\mathbb{Z}$ which is cyclic. Hence, $G$ would have to be abelian, which ia a contradiction. Hence, $|Z(G)| \not> \frac{1}{4}|G|$, which means $|Z(G)| \leq \frac{1}{4}|G|$.

**Problem 2020-J-I-4 (Geometry/Topology).** Let $\theta$ be a closed smooth 1-form on a compact $C^\infty$ manifold $M$ with empty boundary, and let $v$ be a smooth vector field on $M$. Prove that the Lie derivative $\mathcal{L}_v\theta$ vanishes at some point of $M$.

Let $\theta$ be a closed smooth 1-form on a compact $C^\infty$ manifold $M$ with empty boundary, and let $v$ be a smooth vector field on $M$. By Cartan's Formula for the Lie derivative,

$$\mathcal{L}_v\theta = i_v(d\theta) + d(i_v\theta), \tag{86}$$

where $i_v(\cdot)$ denotes the interior product. Since $\theta$ is a closed 1-form, $d\theta = 0$. So $\mathcal{L}_v\theta = d(i_v\theta)$. Since $\theta$ is a 1-form, $i_v\theta$ is a 0-form on $M$, i.e., a smooth function on $M$. Since $M$ is compact, $i_v\theta$ must attain a extrema at some point in $M$, which means that its differential $d(i_v\theta)$ must vanish where it achieves its maximum or minimum. This then implies that $\mathcal{L}_v\theta$ vanishes at this point.

## August 2020

**Problem 2020-A-II-1 (Complex Analysis).** How many roots (counted with multiplicity) does the function

$$g(z) = 6z^3 + e^z + 1$$

have in the unit disk $|z| < 1$?

Let $g(z) = 6z^3 + e^z + 1$, which is holomorphic. Let $f(z) = 6z^3$ and $h(z) = e^z + 1$. Then on the unit circle $|z| = 1$,

$$\begin{aligned} |h(z)| &\leq |e^z| + 1 \leq e^{|z|} + 1 \\ &\leq e + 1 \\ &< 6 = 6|z|^3 = |f(z)|. \end{aligned} \tag{87}$$

Hence, by Rouché's Formula, $g(z)$ has the same number of zeros as $f(z)$. Counting multiplicity, $f(z)$ has three solutions in the unit disk, which means that $g(z)$ also has three solutions in the unit disk.

**Problem 2020-A-II-4 (Geometry/Topology).** Let $M$ and $N$ be compact connected orientable smooth manifolds and let $f : M \to N$ be a smooth mapping. Recall the degree of $f$ is the integral

$$\deg(f) = \int_M f^* \omega$$

over $M$ of the pullback $f^*\omega$ of any top-degree smooth form $\omega$ on $N$ whose integral over $N$ is one. Recall the degree is an integer, denote it by $\deg(f)$. Now consider the map

$$f_\# : \pi_1(M) \to \pi_1(N)$$

on fundamental groups induced by $f$. Suppose that the image of $f_\#$ has finite index, $\mathrm{ind}(f)$. Prove that $\mathrm{ind}(f)$ divides $\deg(f)$.

Let $M, N$ be compact connected orientable smooth manifolds and let $f : M \to N$ be a smooth mapping. Suppose that $H := f_\#(\pi_1(M))$ is a subgroup of $\pi_1(N)$ of finite index $k$. This means there exists a $k$-sheeted covering $p : \widetilde{N} \to N$ so that $p_\#(\pi_1(\widetilde{N})) = H$. By the lifting criterion for coverings, $f$ lifts to a smooth map

$$\widetilde{f} : M \to \widetilde{N} \tag{88}$$

such that $f = p \circ \widetilde{f}$. Let $\omega$ be a top-degree smooth form on $N$ whose integral over $N$ is one. Since $p : \widetilde{N} \to N$ is a $k$-sheeted covering of orientable manifolds, we must have $\deg(p) = k$. Therefore,

$$\deg(f) = \deg(p \circ \widetilde{f}) = \deg(p)\deg(\widetilde{f}) = \mathrm{ind}(f) \cdot \deg(\widetilde{f}). \tag{89}$$

Since $\deg(\widetilde{f})$ is an integer, we conclude that $\mathrm{ind}(f) \mid \deg(f)$.

**Problem 2020-J-I-2 (Geometry/Topology).** Let $M$ and $N$ be smooth compact connected oriented $n$-manifolds without boundary. Suppose that $\pi_1(M)$ is finite, but that $\pi_1(M)$ is infinite. Prove that every smooth map $\Psi : M \to N$ has degree zero.

## January 2019

**Problem 2019-J-I-1 (Algebra).** Let $A$ and $B$ be $n \times n$ invertible matrices over complex numbers, satisfying
$$AB = \lambda BA \text{ for some } \lambda \in \mathbb{C}.$$
Prove that $A^n$ and $B$ commute.

Let $A$ and $B$ be $n \times n$ invertible matrices over complex numbers so that $AB = \lambda BA$ for some $\lambda \in \mathbb{C}$. Since $A$ is invertible, left-multiplying both sides by $A^{-1}$ yields,

$$B = \lambda A^{-1} BA. \tag{90}$$

So taking the determinant, we obtain:

$$\det B = \lambda^n \det A^{-1} \det B \det A = \lambda^n \det A^{-1} \det B \det A = \lambda^n \det B. \tag{91}$$

Since $B$ is invertible, $\det B \neq 0$, which means that $\lambda^n = 1$ (i.e., $\lambda$ is an $n^{\text{th}}$ root of unity). Now, we claim that for any $m \in \mathbb{N}$, $A^m B = \lambda^m BA^m$. By hypothesis, this claim is true for the base case $m = 1$. Suppose the claim is true for some $m \geq 1$. Then

$$A^{m+1}B = A(A^m B) = \lambda^m (ABA^m) = \lambda^m (\lambda BA)A^m = \lambda^{m+1} BA^{m+1}. \tag{92}$$

Therefore, the claim is true by induction. This implies that

$$A^n B = \lambda^n BA^n = BA^n, \tag{93}$$

so that $A^n$ and $B$ commute.

**Problem 2019-J-II-5.** Let $G$ be a finite group, and let $H$ be a non-normal subgroup of $G$ of index $n$. Show that if $|H|$ is divisible by a prime $p \geq n$, then $G$ is not simple.

Let $G$ be a finite group, $H$ a non-normal subgroup of $G$ of index $n$ such that $|H|$ is divisible by a prime $p \geq n$. Let $G$ act on the set of left cosets of $H$; this induces a group homomorphism $\varphi : G \to S_n$. Consider the kernel of this group action, $K = \ker \varphi$. *If* $K = G$, then for every $g \in G$, $gHg^{-1} = H$, which implies that $H$ is a normal subgroup of $G$ – a contradiction. Hence, $\ker \varphi$ is a proper normal subgroup of $G$. Likewise, $\ker \varphi \neq H$ since this equality also forces $H$ to be normal. All that remains is to show that $\ker \varphi$ is not trivial. Since $p \mid |H|$, let $P$ be a Sylow $p$-subgroup of $H$. [!! Complete Later !!]

## August 2018

**Problem 2018-A-II-3 (Analysis).** Suppose $E, F$ are two measurable subsets of the real numbers that both have positive measure. Prove that $E + F = \{x + y : x \in E, y \in F\}$ contains an interval.

## January 2017

**Problem 2017-J-I-1 (Geometry/Topology).** Let $\Sigma_1$ be a torus and let $\Sigma_2$ be a genus-2 surface. Show that there is no submersion from $\Sigma_2$ to $\Sigma_1$.

Let $\Sigma_1$ be a torus and $\Sigma_2$ be a genus-2 surface. We begin with a second modification to the Comps Lemma. Assume to the contrary that $F$ is a submersion from $\Sigma_2$ to $\Sigma_1$. By the second modification to the Comps Lemma, $F : \Sigma_2 \to \Sigma_1$ must be a $k$-sheeted covering map for some finite $k > 0$. This implies that $\chi(\Sigma_2) = k \cdot \chi(\Sigma_1)$, where $\chi(\cdot) = 2 - 2g$ denotes the Euler characteristic of a closed surface of genus $g$. But this is impossible since $\chi(\Sigma_2) = -2 < 0 = k \cdot 0 = k \cdot \chi(\Sigma_1)$. Hence, by contradiction, there cannot be any submersions from $\Sigma_2$ to $\Sigma_1$.

**Problem 2017-J-I-6 (Geometry/Topology).** Let $M$ be a smooth 4-manifold, let $\phi$ be a 3-form on $M$, and let $U \subset M$ be the open set of points where $\varphi \neq 0$. Show that $\varphi$ is closed if and only if, near any $p \in U$, one can find a smooth coordinate system $(x^1, x^2, x^3, x^4)$ in which

$$\varphi = dx^1 \wedge dx^2 \wedge dx^3.$$

Assume the hypotheses of the problem. Recall that $\varphi$ is closed if and only if $d\varphi$ is identically zero. Let $p \in U$ and suppose that we can find a smooth coordinate system $(x^1, x^2, x^3, x^4)$ in some neighborhood of $p$ in $U$ so that $\varphi = dx^1 \wedge dx^2 \wedge dx^3$. Then $d\varphi_p = d^2x^1 \wedge dx^2 \wedge dx^3 + \cdots + dx^1 \wedge dx^2 \wedge d^2x^3 = 0$. Since this is true for all $p \in U$, we conclude that $d\varphi$ is identically zero on $M$, and hence $\varphi$ is closed.
Now assume that $\varphi$ is closed, which means that $\varphi \wedge d\varphi$ is identically zero. At each point $p \in U$, define

$$D_p = \ker \varphi_p,$$

which is Frobenius integrable by our previous observation. In particular, $D_p$ is a 1-dimensional distribution. Since $L$ is integrable, we can find smooth coordinates $(x^1, \ldots, x^4)$ near $p$ such that $D_p = \operatorname{span}\{\partial_{x^4}\}$. Since $\varphi$ annihilates $\partial_{x^4}$, it must be a linear combination of $dx^1, dx^2$, and $dx^3$. Suppose $\varphi = f dx^1 \wedge dx^2 \wedge dx^3$. Then

$$0 = d\varphi = f_{x^1} dx^1 \wedge dx^1 \wedge \cdots \wedge dx^3 + f_{x^2} dx^2 \wedge dx^1 \wedge \cdots \wedge dx^3 + \cdots + f_{x^4} \wedge dx^1 \wedge \cdots \wedge dx^4. \tag{94}$$

The first three terms are all zero. The last term is zero iff $f_{x^4} = 0$, which means $f = f(x^1, x^2, x^3)$. [!! Complete Later !!]

**Problem 2017-J-II-1.** Let $f : M \to \mathbb{R}$ be a smooth function on a smooth manifold $M$. In an arbitrary smooth local coordinate chart $x : U \to \mathbb{R}^n$ of $M$, define

$$\mathcal{D} f \coloneqq \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

Does $\mathcal{D} f$ give a well-defined vector field on $M$?

We claim that $\mathcal{D} f$ does not give a well-defined vector field on $M$. Let $(U, (x^i))$ and $(V, (\widetilde{x}^i))$ denote two overlapping smooth local coordinate charts on $M$, and let $p \in U \cap V$. Then

$$\begin{aligned} \mathcal{D} f &= \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}\bigg|_p \\ &= \frac{\partial f}{\partial \widetilde{x}^j} \frac{\partial \widetilde{x}^j}{\partial x^i}\bigg|_p \frac{\partial \widetilde{x}^k}{\partial x^i}\bigg|_p \frac{\partial}{\partial \widetilde{x}^k}\bigg|_{\hat{p}}, \end{aligned} \tag{95}$$

which is identically not equal to $(\partial_{\widetilde{x}^k} f)\partial_{\widetilde{x}^k}$, which is the expression for $\mathcal{D} f$ in the smooth coordinate chart $(V, (\widetilde{x}^j))$.

**Problem 2017-J-II-2 (Real Analysis).** Suppose $f : [0,1] \to \mathbb{R}$ is measurable. Suppose further that for all $g \in L^2([0,1])$, we have that $fg \in L^2([0,1])$. Show that $f$ is in $L^\infty([0,1])$.

Let $f : [0,1] \to \mathbb{R}$ be measurable, and suppose that for all $g \in L^2([0,1])$, $fg \in L^2([0,1])$. Assume to the contrary that $f \notin L^\infty([0,1])$, which means that for every positive integer $n$, the set

$$E_n = \{x : |f_n(x)| \geq n\} \tag{96}$$

has positive measure. Consider the simple function

$$g = \sum_{1}^{\infty} \frac{1}{n\sqrt{m(E_n)}} \chi_{E_n} \tag{97}$$

so that

$$\|g\|_2^2 = \int_0^1 \sum_1^\infty \frac{1}{n^2 m(E_n)}\chi_{E_n} = \sum_1^\infty \frac{1}{n^2} < \infty. \tag{98}$$

On the other hand

$$\|fg\|_2^2 = \int_0^1 |f|^2 \sum_1^\infty \frac{1}{n^2 m(E_n)}\chi_{E_n} \geq \sum_1^\infty \int_{E_n} \frac{1}{m(E_n)} = \sum_1^\infty 1 > \infty, \tag{99}$$

which means $fg \notin L^2$. This is a contradiction. Hence, by contradiction, $f \in L^\infty([0,1])$.

## August 2017

**Problem 2017-A-I-1 (Geometry/Topology).** Let $M$ be a smooth compact connected $n$-manifold (without boundary), and let $F : M \to \mathbb{R}^n$ be a smooth map. Does $F$ necessarily have a critical point?

Let $M$ be a smooth compact connected $n$-manifold (without boundary), and let $F : M \to \mathbb{R}^n$ be a smooth map. Suppose $F$ has no critical points, which means that $dF_p$ is surjective at every $p \in M$. I.e., $\operatorname{rank} dF_p = n$ for every $p \in M$. Let $F = (f_1, \dots, f_n)$, where each $f_j : M \to \mathbb{R}$ is a component function of $F$. Fix some $f_j$; since $M$ is compact, $f_j$ must attain a maximum or minimum at some point $p \in M$. This means that $df_j(p) = 0$. But since $dF_p = (df_1(p), \dots, df_j(p), \dots, df_n(p))$, $\operatorname{rank} dF_p \neq n$, which is a contradiction. Hence, $F$ must have a critical point.

**Problem 2017-A-II-3 (Algebra).** Let $K$ denote the splitting field of $f(x) = x^4 + x^2 + 1$ over $\mathbb{Q}$. Compute the Galois group $\operatorname{Gal}(K/\mathbb{Q})$.

Let $f(x) = x^4 + x^2 + 1$; by the rational root test, $f(x)$ has no rational roots. However,

$$f(x) = x^4 + x + 1 = (x^2 + x + 1)(x^2 - x + 1), \tag{100}$$

where each quadratic factor is irreducible by the rational root test. The roots of these quadratic factors are

$$x = \pm\sqrt{\frac{-1 \pm \sqrt{-3}}{2}}. \tag{101}$$

Let $\alpha = \sqrt{\frac{-1+\sqrt{-3}}{2}}$ and $\beta = \sqrt{\frac{-1-\sqrt{-3}}{2}}$. We observe then that $\alpha^2\beta^2 = 1 \implies \beta = \pm\frac{1}{\alpha}$. On the other hand,

$$\alpha^2 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}, \tag{102}$$

so that $\alpha \in \mathbb{Q}(\sqrt{-3})$. Hence, we conclude that the splitting field of $f(x)$ over $\mathbb{Q}$ is $K = \mathbb{Q}(\sqrt{-3})$. Since the minimal polynomial of $\sqrt{-3}$ over $\mathbb{Q}$ has degree 2, $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$. Hence, the Galois group $\operatorname{Gal}(K/\mathbb{Q})$ has order 2, which means $\operatorname{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$.

## January 2013

**Problem 2013-J-II-6 (Geometry/Topology).** Let $M$ be a smooth compact manifold, and suppose that there is a smooth map $F : M \to S^1$ whose derivative is non-zero at every point. Prove that the de Rham cohomology space $H^1_{\text{dr}}(M)$ is non-zero.

Let $M$ be a smooth compact manifold, and $F : M \to S^1$ a smooth map whose derivative is non-zero at every point. Assume to the contrary that the de Rham cohomology space $H^1_{\text{dr}}(M) = 0$, which means that every closed 1-form on $M$ is exact. Since $H^1_{\text{dr}}(S^1) \cong \mathbb{R}$, there exists a nowhere vanishing closed 1-form $\omega$ on $S^1$ such that its equivalence class generates $H^1_{\text{dr}}(S^1)$. Then since $F$ is a smooth map, $F^*\omega$

is a closed 1-form on $M$. Since $H^1_{\mathrm{dr}}(M) = 0$, $F^*\omega$ is an exact form; i.e., there exists a smooth function: $f : M \to \mathbb{R}$ such that $F^*\omega = df$. Since $f$ is smooth and $M$ is compact, $f$ must have a maximum or minimum at some point $p \in M$, which implies that $df_p = 0$ at $p \in M$. Therefore, $0 = (F^*\omega)_p = \omega_{F(p)} \circ dF_p$. Since $\omega$ is nowhere vanishing, we conclude that $dF_p = 0$. But this contradicts our assumption that $dF$ is non-zero at every point. Hence, by contradiction, $H^1_{\mathrm{dr}}(M) \neq 0$.

## August 2013

**Problem 2013-A-II-4 (Geometry/Topology).** Let $\theta$ be a smooth 1-form on a manifold $M$ such that $\theta \neq 0$ everywhere. Let $D \subset TM$ be the vector subbundle defined by

$$D = \ker \theta = \{v \in TM : \theta(v) = 0\}.$$

Prove that $D$ is Frobenius integrable if and only if $\theta \wedge d\theta = 0$ everywhere.

Assume the hypotheses of the problem. We recall that $D$ is Frobenius integrable if and only if for any pair of smooth sections $X, Y$ of $D$, $[X, Y]$ is a smooth section of $D$. So let $X, Y$ be smooth sections of $D$, which means that $\theta(X) = \theta(Y) = 0$ everywhere. Suppose that $D$ is Frobenius integrable so that $\theta([X, Y]) = 0$. Since $\theta$ is not identically zero, for any $p \in M$, there exists a vector $R_p$ with $\theta_p(R_p) = 1$. This means that locally one can choose a smooth vector field $R$ with $\theta(R) = 1$. On this neighborhood, we have $T_pM = RR_p \oplus D_p$. Now, we note that

$$\theta \wedge d\theta(X, Y, R) = \theta(X)d\theta(Y, R) + \theta(Y)d\theta(R, X) + \theta(R)d\theta(X, Y). \tag{103}$$

The first two terms are identically zero by our hypothesis. For the latter, we note that

$$d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]), \tag{104}$$

which is identically zero. Hence, this means that $\theta \wedge d\theta(R, X, Y)$ is zero. This means that $(\theta \wedge d\theta)_p = 0$ for all $p \in M$. Hence, $\theta \wedge d\theta$ is identically zero. Now suppose $\theta \wedge d\theta$ is identically zero. Let $X, Y$ be smooth sections of $D$ and pick a local vector field $R$ such that $\theta(R) = 1$. We recover once again that

$$0 = \theta \wedge d\theta(X, Y, R) = -\theta(R)\theta([X, Y]) \implies \theta([X, Y]) = 0. \tag{105}$$

Hence, $[X, Y] \in \Gamma(D)$, which means that $D$ is Frobenius integrable.

## Textbook Problems

**Problem Lee-7-5.** Let $M$ be a smooth compact manifold. Show that there is no submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.

Let $M$ be a smooth compact manifold, and assume to the contrary that there exists a submersion $F : M \to \mathbb{R}^k$ for some $k > 0$. Since $M$ is compact, $F$ must attain either a maximum or minimum at some point $p \in M$, which means that $dF_p = 0$. But this is impossible since $F$ is a submersion, which means that $\operatorname{rank} dF_p = \dim \mathbb{R}^k = k > 0$. Hence, by contradiction, $F$ cannot be a submersion.

**Problem D&F-14.6.2.** Determine the Galois groups of the following polynomials:

(i) $x^3 - x^2 - 4$

(ii) $x^3 - 2x + 4$

(iii) $x^3 - x + 1$

(iv) $x^3 + x^2 - 2x - 1$.

(a) Let $f(x) = x^3 - x^2 - 4$. We note that $f$ has a rational root $x = 2$ since $2^3 - 2^2 - 4 = 8 - 4 - 4 = 0$. Using polynomial long division, we find that $f(x)$ is reducible over $\mathbb{Q}$ as the product

$$f(x) = (x-2)(x^2 + x + 2). \tag{106}$$

By the rational root test, the quadratic factor is irreducible and has complex roots

$$x_{1,2} = \frac{-1 \pm \sqrt{-7}}{2}. \tag{107}$$

Therefore, the splitting field of $f(x)$ is $\mathbb{Q}(\sqrt{-7})$, which has degree 2 since the minimal polynomial of $\sqrt{-7}$ is $x^2 + 7$. Therefore, the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{-7})/\mathbb{Q})$ has order 2; hence the Galois group is $\mathbb{Z}/2\mathbb{Z}$.

(b) Let $f(x) = x^3 - 2x + 4$. We note that $f(x)$ has a rational root $x = -2$ since $(-2)^3 - 2(-2) + 4 = -8 + 4 + 4 = 0$. Hence using polynomial long division,

$$f(x) = (x+2)(x^2 - 2x + 2). \tag{108}$$

By the rational root test, $x^2 - 2x + 2$ is irreducible over $\mathbb{Q}$ with complex roots $1 \pm i$. Therefore, the splitting field of $f(x)$ is $\mathbb{Q}(i)$, which has degree 2 since the minimal polynomial of $i$ is $x^2 + 1$. Therefore, the Galois group $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ has order 2; hence the Galois group is $\mathbb{Z}/2\mathbb{Z}$.

(c) Let $f(x) = x^3 - x + 1$; by the rational root test $f(x)$ is irreducible over $\mathbb{Q}$. However, since $f$ is already a depressed cubic, we note that its discrimant is $-4p^3 - 27q^2 = 4 - 27 = -23$. Since $-23$ is not a perfect square, we conclude that the Galois group is $S_3$. In fact, the splitting field for this cubic is $\mathbb{Q}(\alpha, \sqrt{-23})$, where $\alpha$ is a root of $x^3 - x + 1$.

(d) Let $f(x) = x^3 + x^2 - 2x - 1$; by the rational root test $f(x)$ is irreducible over $\mathbb{Q}$. Therefore, we will now depress the cubic. Let $x = y - 1/3$. Then

$$x^3 + x^2 - 2x - 1 = y^3 - \frac{7}{3}y - \frac{7}{27}. \tag{109}$$

The discriminant of the depressed cubic is,

$$D = -4p^3 - 27q^2 = 4\left(\frac{7^3}{27}\right) - 27\left(\frac{7^2}{27^2}\right) = \frac{7^2}{27}(4 \cdot 7 - 1) = 7^2. \tag{110}$$

Since the discriminant is a square, we see that the Galois group of the polynomial is $A_3$.

**Problem D&F-14.6.4.** Determine the Galois group of $x^4 - 25$.

Let $f(x) = x^4 - 25$. The roots of $f(x)$ are $\zeta_4^0\sqrt[4]{25}, \zeta_4^1\sqrt[4]{25}, \zeta_4^2\sqrt[4]{25}$, and $\zeta_4^3\sqrt[4]{25}$, where $\zeta_4$ is the primitive 4th root of unity. Here, we recall that the automorphisms in the Galois group of $f$ act transitively on the roots of $f(x)$. Hence, the Galois group of $f(x)$ must contain the automorphism that maps $\sqrt[4]{25} \mapsto -\sqrt[4]{25}$ (i.e., a reflection) and $\sqrt[4]{25} \mapsto \zeta_4^j\sqrt[4]{25}$ (i.e., a rotation). Hence, the Galois group is $D_8$.

**Problem D&F-14.6.5.** Determine the Galois group of $x^4 + 4$.

Let $f(x) = x^4 + 4$, which is irreducible over $\mathbb{Q}$. However, the four roots of $f(x)$ are $\pm 1 \pm i$. This means that the splitting field of $f(x)$ is $\mathbb{Q}(i)$, which is a degree 2 extension over $\mathbb{Q}$. Hence, the Galois group is of order 2, which implies that the Galois group is the cyclic group $\mathbb{Z}/2\mathbb{Z}$.

**Problem MAT532-F-4.** Suppose $E \subset \mathbb{R}^2$ is Lebesgue measurable. For a square $Q$, let $C_Q$ be the *white* squares of a $(8 \times 8)$ checkerboard fitted exactly in $Q$ (so a white square has sidelength 1/8 the sidelength of $Q$). Suppose that for almost any $x \in E$, and any square $Q_x$ with $x$ in its lower left corner, we have that $E \cap C_{Q_x} = \varnothing$, i.e., $E$ does not intersect the white squares of a checkerboard fitted to $Q_x$. Show $m(E) = 0$, where $m$ is Lebesgue measure.

Let $E \subset \mathbb{R}^2$ be Lebesgue measurable, and set $A = \{x \in E : E \cap C_{Q_x} = \varnothing \text{ for any square } Q_x\}$; by hypothesis, $A$ consists of almost every $x \in E$. Assume to the contrary that $m(E) \neq 0$ and pick $x \in A$. For this $x$, construct a family of sets $\{E_r\}_{r>0}$ as follows: for each $r$, let $E_r$ be a square of sidelength $r/\sqrt{2}$ with $x$ in its lower left corner. It is straightforward to see that for every $r > 0$, $E_r \subset B(x,r)$ and $m(E_r) = 2\pi^{-1}m(B(r,x))$. Hence, $\{E_r\}$ shrinks nicely to $x$. Now, by hypothesis, $m(E \cap E_r) \leq \frac{1}{2}m(E_r)$ for every $r$ since $E$ intersects at most half of $E_r$. This means that

$$\limsup_{r\to 0} \frac{m(E \cap E_r)}{m(E_r)} \leq \frac{1}{2}. \tag{111}$$

I.e., for almost every $x \in E$, the Lebesgue density is at most 1/2, which contradicts the Lebesgue Density Theorem. Therefore, by contradiction, $m(E) = 0$.

**Problem MAT532-7-4.** Suppose a set $E \subset \mathbb{R}^3$ satisfies that for every $x \in \mathbb{R}^3$ and $r > 0$, there exists a point $z \in B(x,r)$ such that $E \cap B(z, r/2) \cap B(x, 2r) = \varnothing$. Show that $m(E) = 0$, where $m$ is the Lebesgue measure on $\mathbb{R}^3$.

[!! Complete Later !!]

**Problem (Algebra-Classification-I).** Classify all groups of order 2026.

Let $G$ be a group of order $2026 = 2 \cdot 1013$. By Sylow's Theorem, $G$ must contain a normal Sylow 5-subgroup, which we denote by $H$. Let $K$ be a Sylow 2-subgroup of $G$; note $K \cong \mathbb{Z}_2$. By Lagrange's Theorem, $H$ and $K$ must intersect trivially. Moreover, $|HK| = |H||K|/|H \cap K| = |H||K| = |G|$, so that $G = HK$. Hence, by the recognition theorem for semidirect products, $G \cong H \rtimes_\varphi \mathbb{Z}_2$, where $\varphi \in \operatorname{Aut} H = \mathbb{Z}_{1013}^* \cong \mathbb{Z}_{1012}$. So we look for homomorphisms $\varphi : \mathbb{Z}_2 \to \mathbb{Z}_{1012}$; each homomorphism is completely determined by where the generator 1 is mapped to.

(i) Consider the map $1 \mapsto 0$, which corresponds to the trivial homomorphism. Then the semidirect product is just the direct product, and so $G \cong \mathbb{Z}_{1013} \times \mathbb{Z}_2$.

(ii) Consider the map $\varphi : 1 \mapsto 506$, where 506 is the unique element of $\mathbb{Z}_{1012}$ with order 2. This is a non-trivial homomorphism with kernel $\{0\}$. Hence, this gives a non-abelian group $\mathbb{Z}_{1013} \rtimes_\varphi \mathbb{Z}_2$.

Hence, up to isomorphism, there are only two groups of order 2026.

**Problem (Algebra-Classification-II).** Classify all groups of order 1969.

Let $G$ be a group of order $1969 = 11 \cdot 179$. By Sylow's Theorem, $G$ must contain a normal Sylow 179-subgroup, which we denote by $H$. Let $K$ be a Sylow 11-subgroup of $G$; note $K \cong \mathbb{Z}_{11}$. By Langrage's Theorem, $H$ and $K$ must intersect trivially and $G = HK$. Therefore, $G = H \rtimes_\varphi K$ for some $\varphi \in \operatorname{Aut} H = \mathbb{Z}_{179}^* \cong \mathbb{Z}_{178}$. So we look for homomorphisms $\varphi : \mathbb{Z}_{11} \to \mathbb{Z}_{178}$; each homomorphism is completely determine by where the generator 1 is mapped to.

(i) Consider the map $1 \mapsto 0$. This corresponds to the trivial homomorphism so that the semidirect product is just the direct product. Therefore, $G \cong \mathbb{Z}_{179} \times \mathbb{Z}_{11} \cong \mathbb{Z}_{1969}$ (by the Chinese Remainder Theorem).

(ii) Since 1 has order 11, 1 must map to some nonzero element of $\mathbb{Z}_{178}$ of order 11; but since 11 and 178 are relatively prime, there exists no such element.

Hence, we conclude that there is exactly one group of order 1969, which is precisely $\mathbb{Z}_{1969}$.

**Problem 2008-J-I-3 (Algebra).** Classify all groups of order 28.

Let $G$ be a group of rder $28 = 2^2 \cdot 7$. By Sylow's Theorem, $G$ contains a normal Sylow 7-subgroup, which we denote by $H$. Let $K$ be a Sylow 2-subgroup, which has order 4. By Lagrange's Theorem, $H$ and $K$ must intersect trivially and $G = HK$. Hence, by the recognition theorem for semidirect products, $G = H \rtimes_\varphi K$ for some $\varphi \in \operatorname{Aut}(H) = \mathbb{Z}_7^* \cong \mathbb{Z}_6$. So we look for homomorphisms $\varphi : K \to \mathbb{Z}_6$, where $K$ is a group of order 4. Up to isomorphism, there are precisely two groups of order 4: (1) $\mathbb{Z}_4$, and (2) $\mathbb{Z}_2 \times \mathbb{Z}_2$. We consider each case separately:

(I) Consider the case $K = \mathbb{Z}_4$, which has two generators: 1 and 3. Each homomorphism $\varphi : \mathbb{Z}_4 \to \mathbb{Z}_6$ is determined by where $\varphi$ sends a generator with the constrain that 1 may be sent to only those elements of $\mathbb{Z}_6$ whose order divides 4 (namely $0, 3$).

  (i) Suppose $\varphi_1 : 1 \mapsto 0$. Then since $\varphi(3) = 3 \cdot \varphi(1) = 0$, $\varphi$ is the trivial homomorphism. In this case, the semidirect product is the direct product and $G$ is isomorphic to the abelian group $\mathbb{Z}_7 \times \mathbb{Z}_4$.

  (ii) Suppose $\varphi_2 : 1 \mapsto 3$. Then this is a nontrivial homomorphism with image consisting of $\{0, 3\}$ and kernel consisting of $\{0, 2\}$. Hence, this produces a non-abelian group $\mathbb{Z}_7 \rtimes_{\varphi_2} \mathbb{Z}_4$.

(II) Now consider the case $K = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$. $\psi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_6$ is determined uniquely by $\psi(a)$ and $\psi(b)$ provided that its order divides 2. This means $\psi(a), \psi(b) \in \{0, 3\}$.

  (i) Suppose $\psi_1(a) = \psi_1(b) = 0$. The semidirect is then a direct product and so $G \cong \mathbb{Z}_7 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_{14} \times \mathbb{Z}_2$.

  (ii) Suppose $\psi_2(a) = 0$ and $\psi_2(b) = 3$. This is a nontrivial homomorphism so that $G \cong \mathbb{Z}_7 \rtimes_{\psi_2} \mathbb{Z}_2^2$ is non-abelian.

  (iii) Suppose $\psi_3(a) = 3$ and $\psi_3(b) = 0$. Then $\psi_3 = \psi_2 \circ \theta$ where $\theta$ is the automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by $\theta(a) = b$ and $\theta(b) = a$. Hence, this semidirect product gives the same group as in case (ii).

  (iv) Suppose $\psi_4(a) = \psi_4(b) = 3$. Then $\psi_4 = \psi_3 \circ \theta$ where $\theta$ is the automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by $\theta(a) = a$ and $\theta(b) = ab$. Hence, this semidirect product gives the same group as in case (iii).

Altogether, we conclude that there are exactly four isomorphism classes of groups of order 28, namely $\mathbb{Z}_7 \times \mathbb{Z}_4$, $\mathbb{Z}_7 \rtimes_{\varphi_2} \mathbb{Z}_4$, $\mathbb{Z}_{14} \times \mathbb{Z}_2$, and $\mathbb{Z}_7 \rtimes_{\psi_2} \mathbb{Z}_2^2$, of which exactly two are abelian.

**Problem 2010-J-II-5 (Algebra).** Classify (up to isomorphism) all groups of order 45.

Let $G$ be a group of order $45 = 3^2 \cdot 5$. By Sylow's Theorem, $G$ has a normal Sylow 5-subgroup, which we denote by $H$. Let $K$ denote a Sylow 3-subgroup of $G$, which has order 9. By Lagrange's Theorem, $H, K$ intersect trivially and $|G| = |H||K|$ so that $G = HK$. Hence, $G \cong H \rtimes_\varphi K$ for some $\varphi \in \operatorname{Aut}(H) \cong \mathbb{Z}_5^* \cong \mathbb{Z}_4$. Hence, we look at homomorphisms $\varphi : K \to \mathbb{Z}_4$. There are exactly two groups of order 9, up to isomorphism; namely, these are $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$. Hence, we consider each separately.

(I) Let $K = \mathbb{Z}_9$, which has generators 1, 2, 4, 5, 7, and 8. Each homomorphism $\varphi : K \to \mathbb{Z}_4$ is determined uniquely by where $\varphi$ sends a generator with the constraint that they may only be sent to those elements of $\mathbb{Z}_4$ whose order divides 9. There is only one such element, namely 0. Hence, the only group we get is the direct product $\mathbb{Z}_9 \times \mathbb{Z}_5 \cong \mathbb{Z}_{45}$, which is abelian.

(II) Let $K = \mathbb{Z}_3 \times \mathbb{Z}_3$. Each $\psi : \mathbb{Z}_3 \times \mathbb{Z}_3 = \langle a \rangle \times \langle b \rangle \to \mathbb{Z}_4$ is uniquely determined by $\psi(a)$ and $\psi(b)$ provided they divide 3. But there is only one such element in $\mathbb{Z}_4$, which is zero. Hence, we only get the direct product $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_3 \times \mathbb{Z}_{15}$, which is abelian.

Therefore, we find that (1) there are exactly two groups, up to isomorphism, of order 45; and (2) both groups are abelian.

**Problem 2003-J-I-6 (Algebra).**
(a) Prove that a group of order $p^2$, where $p$ is a prime number, is abelian.
(b) Classify groups of order $p^2$ up to isomorphism.

(a) Let $G$ be a group of order $p^2$, and let $Z(G)$ be its center. By Lagrange's Theorem, $|Z(G)| \in \{1, p, p^2\}$. If $|Z(G)| = p^2$ and so $G = Z(G)$, which means $G$ is abelian. $|Z(G)| \neq p$ since otherwise $|G/Z(G)| = p$ forcing $G/Z$ to be cyclic and $G$ to be abelian (which contradicts $Z(G)$ being a proper subgroup of $G$). Finally $|Z(G)|$ cannot be one, since the center of any $p$-group must necessarily be nontrivial (by the class equation). Hence, $Z(G) = G$, which means $G$ is abelian.

(b) Since every group of order $p^2$ must necessarily be abelian, up to isomorphism, there must be exactly two groups, namely $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$.

**Problem 2010-J-I-5 (Algebra).** Consider the following irreducible polynomial over $\mathbb{Q}$: $p(x) = x^4 - 3x^2 - 1$.
(a) Describe the splitting field of $p(x)$.
(b) Consider the Galois group of $p(x)$. Compute its order and determine if it is abelian.

(a) Let $p(x) = x^4 - 3x^2 - 1$. By the rational root test, $p(x)$ has no roots over $\mathbb{Q}$. Moreover, it is straightforward to check that $p(x)$ is not the product of irreducible quadratics with rational coefficients. Hence, $p(x)$ is irreducible over $\mathbb{Q}$. We start by finding the roots of $p(x)$; let $u = x^2$. Then

$$u^2 - 3u - 1 = 0 \implies u = \frac{3 \pm \sqrt{13}}{2} \implies x = \pm\sqrt{\frac{3 \pm \sqrt{13}}{2}}. \tag{112}$$

Let

$$\alpha = \sqrt{\frac{3 + \sqrt{13}}{2}}, \qquad \beta = \sqrt{\frac{3 - \sqrt{13}}{2}}. \tag{113}$$

Observe that $\alpha^2\beta^2 = -1$ so that $\beta = \pm\frac{i}{\alpha}$. Therefore, the splitting field of $p(x)$ is

$$\mathbb{Q}(\alpha, i). \tag{114}$$

Observe that the minimal polynomial of $i$ is $x^2+1$, which is irreducible over $\mathbb{Q}(\alpha)$ so that $[\mathbb{Q}(\alpha,i) : \mathbb{Q}(\alpha)] = 2$. On the other hand, the minimal polynomial of $\alpha$ is a degree 4 polynomial so that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$. Hence, by the tower law, $[\mathbb{Q}(\alpha, i) : \mathbb{Q}] = 8$.

(b) By the last work in (a), the order of the Galois group of $p(x)$ is 8. Now, we will determine the Galois group of $p(x)$. Recall that elements of $\mathrm{Gal}(\mathbb{Q}(\alpha,i)/\mathbb{Q})$ are automorphisms $\varphi$ of the field $\mathbb{Q}(\alpha, i)$ with the constraints that: (1) $\varphi$ fixes $\mathbb{Q}$, (2) $\varphi(\alpha)$ must be another root of the minimal polynomial of $\alpha$ over $\mathbb{Q}$, and (3) $\varphi(i)$ must be another root of $x^2 + 1$. We will explicitly work through each of the elements.

(i) $\sigma : i \mapsto -i, \alpha \mapsto \alpha$. This permutation has order 2 since $\sigma^2(\alpha) = \sigma(\alpha) = \alpha$ and $\sigma^2(i) = \sigma(-i) = i$.
(ii) $\tau : i \mapsto i, \alpha \mapsto -\alpha$. Once again, this permutation has order 2.

(iii) $\rho : i \mapsto -i, \alpha \mapsto \beta = \frac{i}{\alpha}$. To compute the order of this permutation, observe that

$$\rho^2(\alpha) = \rho(i\alpha^{-1}) = (-i) \cdot \frac{1}{i/\alpha} = -\alpha \implies \rho^4(\alpha) = \rho^2(-\alpha) = \alpha. \tag{115}$$

Likewise, $\rho^4(i) = \rho^2(i) = i$. Hence, $\rho$ has order 4.

Now, consider the three elements given above. We compute

$$\sigma\rho\sigma(i) = \sigma\rho(-i) = \sigma(i) = -i = \rho^{-1}(i). \tag{116}$$

Likewise,

$$\sigma\rho\sigma(\alpha) = \sigma\rho(\alpha) = \sigma(i)\sigma(\alpha)^{-1} = -\frac{i}{\alpha} = \rho^{-1}(\alpha). \tag{117}$$

Therefore, $\sigma\rho\sigma = \rho^{-1}$. Hence,

$$\mathrm{Gal}(\mathbb{Q}(\alpha, i)/\mathbb{Q}) = \{1, \sigma, \rho, \rho^2, \rho^3, \sigma\rho, \sigma\rho^2, \sigma\rho^3\} \cong D_8. \tag{118}$$

Since the dihedral group is not abelian, we conclude that the Galois group for $p(x)$ is non-abelian.

**Problem 2015-A-II-5 (Algebra).** Find the splitting field and the Galois group of the polynomial $x^4 - 5x^2 + 5$ over $\mathbb{Q}$.

Let $p(x) = x^4 - 5x^2 + 5$. By the rational root test, $p(x)$ has no rational roots. Moreover, it is straightforward to see that $p(x)$ is not expressible as the product of irreducible quadratics. Hence, $p(x)$ is irreducible over $\mathbb{Q}$. We find its four complex roots as follows. Let $u = x^2$. Then

$$u^2 - 5u + 5 = 0 \implies u = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{5 \pm \sqrt{5}}{2} \implies x = \pm\sqrt{\frac{5 \pm \sqrt{5}}{2}}. \tag{119}$$

Let

$$\alpha \coloneqq \sqrt{\frac{5 + \sqrt{5}}{2}}, \qquad \beta \coloneqq \sqrt{\frac{5 - \sqrt{5}}{2}}. \tag{120}$$

We observe that

$$\alpha^2 = \frac{5}{2} + \frac{\sqrt{5}}{2} \qquad \text{and} \qquad \alpha^2\beta^2 = 5 \implies \beta = \pm\frac{5}{\alpha}. \tag{121}$$

Therefore, the splitting field is $\mathbb{Q}(\sqrt{5}, \alpha)$. Since the minimal polynomial of $\sqrt{5}$ over $\mathbb{Q}$ is $x^2 - 5$, which has degree 2, $[\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2$. On the other hand, the minimal polynomial of $\alpha$ over $\mathbb{Q}(\sqrt{5})$ is

$$x^2 - \frac{5 + \sqrt{5}}{2}, \tag{122}$$

so that $[\mathbb{Q}(\sqrt{5}, \alpha) : \mathbb{Q}] = 2$. Hence, by the Tower Law, $[\mathbb{Q}(\sqrt{5}, \alpha) : \mathbb{Q}] = 4$, which means that the corresponding Galois group has order 4; there are two groups, up to isomorphism, of order 4 (namely $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$). The elements of $\mathrm{Gal}(\mathbb{Q}(\sqrt{5}, \alpha)/\mathbb{Q})$ are precisely the automorphisms on $\mathbb{Q}(\sqrt{5}, \alpha)$ that fix $\mathbb{Q}$ such that the automorphism group acts transitively on the roots. Consider the permutation $\rho : \alpha \mapsto -\beta = -\frac{5}{\alpha}$ and $\rho : \sqrt{5} \mapsto -\sqrt{5}$. We observe that

$$\begin{aligned}
\rho^2(\sqrt{5}) &= \rho(-\sqrt{5}) = \sqrt{5}. \\
\rho^2(\alpha) &= \rho\left(-\frac{5}{\alpha}\right) = -5\rho(\alpha)^{-1} = \alpha \\
\implies \rho^3(\alpha) &= \rho(\alpha) = -5\alpha^{-1} \\
\implies \rho^4(\alpha) &= -5\rho(\alpha)^{-1} = -5 \cdot \left(-\frac{\alpha}{5}\right) = \alpha.
\end{aligned} \tag{123}$$

I.e., $\rho$ is an element of order 4. Therefore, since only $\mathbb{Z}_4$ has an element of order 4, we conclude that $\mathrm{Gal}(\sqrt{5}, \alpha)/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$.

**Problem 2003-A-II-4 (Algebra).** Let $E$ be a splitting field of $f(x) = x^3 + x^2 - 2x - 1$ over the field of rational numbers $\mathbb{Q}$. Find the Galois group of $E/\mathbb{Q}$. (Hint: first prove that $f(x) : f(x^2 - 2)$.) This was the exact notation used in the problem...

Let $f(x) = x^3 + x^2 - 2x - 1$. By the rational root test, $f(x)$ has no rational roots and hence is irreducible over $\mathbb{Q}$ (being a polynomial of degree 3). Consider the substitution $x = y - 1/3$:

$$f(y) = y^3 - \frac{7}{3}y - \frac{7}{27}. \tag{124}$$

The discriminant of this depressed cubic is

$$D = -4p^3 - 27q^2 = 4\left(\frac{7^3}{27}\right) - 27\left(\frac{7^2}{27^2}\right) = 7^2\left(\frac{28}{27} - \frac{1}{27}\right) = 7^2. \tag{125}$$

Since the discriminant is a perfect square, we conclude that the Galois group is $A_3$.

**Problem 2014-J-I-5 (Algebra).** Let $K$ denote the splitting field for $(x^5-1)(x^3-2)$ over the rational numbers $\mathbb{Q}$. Compute the cardinality of the Galois group $G$ for the extension $\mathbb{Q} \subset K$, and show that $G$ is not abelian.

Let $K$ denote the splitting field for $(x^5-1)(x^3-2)$. We note that the splitting field for $x^3-2$ is $\mathbb{Q}(\sqrt[3]{(2)}, \zeta_3)$, where $\zeta_3$ is the primitive 3rd root of unity, and the splitting field for $x^5 - 1$ is $\mathbb{Q}(\zeta_5)$, where $\zeta_5$ is the primitive 5th root of unity. Now, since 3 and 5 are relatively prime, the 3rd primitive roots of unity cannot be expressed as a linear combination of 5th roots of unity. Likewise, $\sqrt[3]{2} \notin \mathbb{Q}(\zeta_5)$. Hence, $\mathbb{Q}(\sqrt[3]{2}, \zeta_3) \cap \mathbb{Q}(\zeta_5) = \mathbb{Q}$, which means that

$$\mathrm{Gal}(K/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}). \tag{126}$$

From this, we see that the order of $G$ is 24. Now consider $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q})$. The corresponding minimal polynomial is $x^3 - 2$, which is a depressed cubic. Since its discriminant is $-108$, which is not a square, we conclude that $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}) \cong S_3$, which is not abelian. Hence, we conclude that $G$ is not abelian.

**Problem 2003-J-I-5 (Algebra).** Let $f(x) = x^5 - 2$. Find generators and relations for the Galois group $G \coloneqq \mathrm{Gal}(F/\mathbb{Q})$ of the splitting field $F$ of $f(x)$ over the rational numbers $\mathbb{Q}$.

Let $f(x) = x^5 - 2$, which has no roots in $\mathbb{Q}$ by the rational root test. It is also straightforward to check that $x^5 - 2$ cannot be written as the product of an irreducible cubic and irreducible quadratic so that $f(x)$ is indeed irreducible over $\mathbb{Q}$. The roots of this polynomial are $\sqrt[5]{2}, \zeta_5\sqrt[5]{2}, \ldots, \zeta_5^4\sqrt[5]{2}$, where $\zeta_5$ is the primitive 5th root of unity. Therefore, the splitting field $F$ of $f(x)$ must contain the field $\mathbb{Q}(\sqrt[5]{2}, \zeta_5)$. On the other hand, each of the roots mentioned above lie in this field so that $F = \mathbb{Q}(\sqrt[5]{2}, \zeta_5)$. Moreover, it follows that $[F : \mathbb{Q}] = 5 \cdot 4 = 20$ so that $G$ is a group of order 20. [!! Complete Later !!]

**Problem 2014-J-II-2 (Algebra).** Let $H$ denote a normal subgroup of the finite group $G$. If $P$ denotes a Sylow $p$-subgroup of $H$, then prove that $G = N(P;G)H$ (where $N(P;G)$ denotes the normalizer of $P$ in $G$).

Let $H$ denote a normal subgroup of the finite group $G$, and let $P$ be a Sylow $p$-subgroup of $H$. Let $N_G(P) \coloneqq N(P;G)$ denote the normalizer of $P$ in $G$. Since $H$ is normal, $KH \leq G$ for any $K \leq G$. In particular, $N_G(P)H \leq G$. Hence, it suffices to show that $G \leq N_G(P)H$. Since $P \leq H$ and $H$ is a normal subgroup of $G$, for any $g \in G$, $gPg^{-1} \leq gHg^{-1} = H$ so that $gPg^{-1}$ is another Sylow $p$-subgroup of $H$. On the other hand, we also know that all Sylow $p$-subgroups of $H$ are conjugate by elements of $H$. Hence, for each $g \in G$, there exists a corresponing $h \in H$ such that

$$hPh^{-1} = gPg^{-1} \implies (g^{-1}h)P(g^{-1}h)^{-1} = P. \tag{127}$$

I.e, $g^{-1}h \in N_G(P)$, which means $g \in HN_G(P) = N_G(P)H$, where the equality stems from $N_G(P)H$ being a subgroup of $G$. Hence, since $g$ was arbitrary, $G \subseteq N_G(P)H$, which concludes the proof.

## Other Qualifying Exams

**Problem RUT-2023-A-I-1 (Algebra).** Classify the groups of order $2023 = 7 \cdot 17^2$ up to isomorphism. (You may use without proof the well-known result that if $p$ is a prime, then every group of order $p^2$ is abelian.)

Let $G$ be a group of order $2023 = 7 \cdot 17^2$. By Sylow's Theorem, $G$ contains a normal Sylow 7-subgroup and a normal Sylow 17-subgroup. Let $H \cong \mathbb{Z}_7$ denote the Sylow 7-subgroup and $K$ denote the Sylow 17-subgroup; note that either $K \cong \mathbb{Z}_{17^2}$ or $K \cong \mathbb{Z}_{17} \times \mathbb{Z}_{17}$. Hence, $G \cong H \rtimes_\varphi K$, where $\varphi \in \mathrm{Aut}(H) \cong \mathbb{Z}_7^\times \cong \mathbb{Z}_6$. We consider various cases.

(I) Suppose $K = \mathbb{Z}_{17^2}$, which has a single generator, 1. Each homomorphism $\varphi : K \to \mathbb{Z}_6$ is uniquely determined by where the generator 1 is mapped to, with the constraint that $\varphi(1)$ is an element that divides the order of 1, namely $17^2$. Since the only such element is 0, $\varphi$ is the trivial homomorphism, which means that the semidirect product is just the direct product, and so $G \cong \mathbb{Z}_7 \times \mathbb{Z}_{17^2} \cong \mathbb{Z}_{2023}$; this is an abelian group.

(II) Suppose $K = \mathbb{Z}_{17} \times \mathbb{Z}_{17} = \langle a \rangle \times \langle b \rangle$. Each homomorphism $\psi : \mathbb{Z}_{17} \times \mathbb{Z}_{17} \to \mathbb{Z}_6$ is uniquely determined by $\psi(a)$ and $\psi(b)$ with the constraint that these elements divide the order of $a$ and $b$ in $\mathbb{Z}_{17}$, which is 17. Since there is only one such element, namely 0, we find that the semidirect is just the direct product, and $G \cong \mathbb{Z}_7 \times \mathbb{Z}_{17} \times \mathbb{Z}_{17} \cong \mathbb{Z}_{289} \times \mathbb{Z}_7$; which is abelian.

Hence, up to isomorphism, there are exactly two groups of order 2023, both of which are abelian.

## Classification of Finite Groups

Some facts we will use to classify groups are:

- Every group of order $p^2$, where $p$ is abelian, is abelian.
- Every group of order $p$, where $p$ is prime, is isomorphic to $\mathbb{Z}_p$.

(1) $|G| = 1$: This is the trivial group $\{1\}$.

(2) $|G| = 2$: There is exactly one group, up to ismorphism, which is $\mathbb{Z}/2\mathbb{Z}$. This follows from Cauchy's Theorem which states that if $p$ divides $|G|$, where $p$ is prime, then $G$ contains an element of order $p$.

(3) $|G| = 3$: There is exactly one group, up to isomorphism, which is $\mathbb{Z}/3\mathbb{Z}$. This follows from Cauchy's Theorem, which states that if $p$ divides $|G|$, where $p$ is prime, then $G$ contains an element of order $p$.

(4) $|G| = 4 = 2^2$: $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are both groups of order 4. Now let $G$ be an arbitrary group of order 4; by Lagrange's Theorem, each element of $G$ can have order 1, 2, or 4. Suppose $G$ contains an element $x$ of order 4. Then $G = \langle g \rangle$; let $\varphi : \mathbb{Z}_4 \to G$ be the map given by $\varphi(n) \mapsto g^n$; this is easily seen to be a group isomorphism. Now suppose $G$ has no element of order 4. Since the only element of $G$ with order 1 is the identity (by uniqueness of group identities), the three nontrivial elements of $G$ must have order 2. Consider the map $\varphi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to G$ defined as follows:

$$\begin{aligned} \varphi(0,0) &= 1_G, \quad & \varphi(1,0) &= a, \\ \varphi(0,1) &= b, \quad & \varphi(1,1) &= c, \end{aligned} \tag{128}$$

where $a, b$, and $c$ are the three nonidentity elements of $G$; $\varphi$ is easily seen to be an isomorphism. Hence, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(5) $|G| = 5$: There is exactly one group, up to isomorphism, which is $\mathbb{Z}/5\mathbb{Z}$.

(6) $|G| = 6 = 2 \cdot 3$. By Sylow's Theorem, there exists a normal Sylow 3-subgroup, which we denote by $H$. Let $K$ be a Sylow 2-subgroup. By Lagrange's Theorem, $H$ and $K$ intersect trivially and $|HK| = |H||K|/|H \cap K| = |H||K| = 6 = |G|$ so that $G = HK$. Hence, by the recognition theorem for semidirect products, $G \cong H \rtimes_\varphi K$, where $\varphi \in \mathrm{Aut}(H) \cong \mathbb{Z}_3^* \cong \mathbb{Z}_2$. Hence, we look for homomorphisms $\varphi : K \to \mathbb{Z}_2$. Since $K$ is a group of order 2, $K \cong \mathbb{Z}_2$. Homomorphisms $\varphi : \mathbb{Z}_2 \to \mathbb{Z}_2$ are determined uniquely by where the generator $1$ is sent to with the constraint that $\varphi(1)$ divides the order of $1$, which is $2$. Hence, either $\varphi_1(1) = 0$ (in which case, the homomorphism is trivial, the semidirect is just the direct product, and $G$ is the abelian group $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$), or $\varphi_2(1) = 1$ (in which case, the homomorphism is *non*trivial, and $G$ is the nonabelian group $\mathbb{Z}_3 \rtimes_{\varphi_2} \mathbb{Z}_2$). Hence, up to isomorphism, there are exactly two groups of order 6, one abelian and the other nonabelian.

(7)

## Essential Review Notes

### Topological Vector Spaces

- **Def. (Topological Vector Space)** A vector space $\mathcal{X}$ over a field $K$ such that vector addition in $\mathcal{X}$ and scalar multiplication are continuous maps from $\mathcal{X} \times \mathcal{X}$ and $K \times \mathcal{X}$, respectively, to $\mathcal{X}$.
- **Def. (Weak Convergence)** A sequence $\{x_n\}$ in a normed linear space $\mathcal{X}$ *converges weakly* to $x \in X$ if the sequence of scalars $\{f(x_n)\}$ converges to $f(x)$ for all $f \in \mathcal{X}^*$.
- **Def. (Weak$^*$ Convergence)** Let $\mathcal{X}$ be a normed linear space. A sequence $\{f_n\} \subseteq \mathcal{X}^*$ is *weak$^*$ convergent* to $f \in \mathcal{X}^*$ if $\{f_n(x)\}$ converges to $f(x)$ for all $x \in \mathcal{X}$. Note, all this really says that the sequence of scalars $\{\hat{x}(f_n)\} = \{f_n(x)\}$ converges to $\hat{x}(f) = f(x)$ for all $\hat{x} \in \mathcal{X}^{**}$ (read $x \in \mathcal{X}$).