

Geometry Crash Course

Aniruddha Madhava

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1 Smooth Manifolds

1.1 Topological Manifolds

- **Def. (Topological Manifold)** A topological space M with the following properties:
 1. M is Hausdorff;
 2. M is second countable (i.e., has a countable basis for its topology);
 3. M is locally Euclidean of dimension n (i.e., for each $p \in M$, there exists a neighborhood $U \subset M$, an open set $\tilde{U} \subset \mathbb{R}^n$, and a homeomorphism $\varphi : U \rightarrow \tilde{U}$).

Exercise 1.1. Show that equivalent definitions of locally Euclidean spaces are obtained if instead of requiring U to be homeomorphic to an open subset of \mathbb{R}^n , we require it to be homeomorphic to an open ball in \mathbb{R}^n , or to \mathbb{R}^n itself.

Let M be a topological space that satisfies conditions (1) and (2). (\Leftarrow) Suppose that for each $p \in M$, there exists a neighborhood U of p that is homeomorphic to an open ball in \mathbb{R}^n , or to \mathbb{R}^n itself. Since each of these are open subsets of \mathbb{R}^n , it follows that M satisfies condition (3). (\Rightarrow) Suppose that M satisfies conditions (1) - (3). Suppose that for some $p \in U \subset M$, $U \cong_{\varphi} \tilde{U} \subseteq \mathbb{R}^n$. Since every open subset of \mathbb{R}^n is the countable union of open balls in \mathbb{R}^n , suppose that $\tilde{U} = \bigcup_1^{\infty} B_j$. Pick some ball B_{j_0} containing $\varphi(p)$. Then $V = \varphi^{-1}(B_{j_0})$ is an open neighborhood of p in M that is homeomorphic, under the map $\varphi|_V : V \rightarrow B_{j_0}$, to the open ball B_{j_0} .

- **Def. (Coordinate Chart)** A pair (U, φ) , where U is an open subset of M and $\varphi : U \rightarrow \tilde{U}$ is a homeomorphism from U to an open subset $\tilde{U} = \varphi(U) \subseteq \mathbb{R}^n$.
- **Def. (Precompact Subset)** Let X be a topological space. A subset $K \subset X$ is said to be *precompact* (or *relatively compact*) in X if its closure in X is compact. E.g., the subsets $(-1, 1)$, $(2, 3]$, $(4, 5) \cup \{6\}$ are all precompact in \mathbb{R} , but the subset $(-1, \infty)$ is not.
- **Lem 1.6. (Topological Manifolds have Precompact Basis)** Every topological manifold has a countable basis of precompact coordinate balls.

Let M be a topological n -manifold. Suppose $\varphi : M \rightarrow \tilde{U} \subset \mathbb{R}^n$ is a global coordinate map. Let \mathcal{B} be the collection of all open balls $B_r(x) \subset \mathbb{R}^n$ such that (1) r is rational, (2) x has rational coordinates, and (3) $\overline{B_r(x)} \subset \tilde{U}$. By definition, each such ball is precompact in \tilde{U} and \mathcal{B} is a countable basis for the topology of \tilde{U} . Since φ is a homeomorphism, the collection $\mathcal{B}^{-1} = \{\varphi^{-1}(B) : B \in \mathcal{B}\}$ is a countable basis for the topology of M . Moreover, each of the sets in this collection is precompact in M : for each $B \in \mathcal{B}$, $\varphi^{-1}(\overline{B}) = \overline{\varphi^{-1}(B)} \subset M$; since φ^{-1} is continuous, $\varphi^{-1}(\overline{B})$ is compact in M . The restrictions of φ are the coordinate maps. In this case, we assumed that M had a global coordinate map, which might not necessarily be true in general.

So, let M be an arbitrary topological n -manifold. By definition, every point of M is contained in the domain of a chart. Since every open cover of a second countable space has a countable subcover, M is covered by countably many charts $\{(U_i, \varphi_i)\}$. By the preceding argument, for each i , U_i has a countable basis of precompact coordinate balls, and the union of all these balls is a countable basis for the topology on M . Suppose $V \subset U_i$ is one of these precompact balls. Since the closure of V in U_i is compact, the closure must be closed in M . Hence, the closure of V in M is the same as the closure of V in U_i , so that V is precompact in M .

1.2 Smooth Manifolds

- **Def. (Transition Map between Charts)** Let M be a topological n -manifold. Let $(U, \varphi), (V, \psi)$ be two charts such that $U \cap V \neq \emptyset$. The composite map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$ is called the *transition map* from φ to ψ .

- **Def. (Smoothly Compatible Charts)** Two charts (U, φ) and (V, ψ) are said to be *smoothly compatible* if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism.
- **Def. ((Smooth) Atlases)** Let M be a topological n -manifold. (1) An *atlas* \mathcal{A} for M is a collection of charts whose domains cover M ; (2) \mathcal{A} is said to be a *smooth atlas* if any two charts in \mathcal{A} are smoothly compatible.
- **Def. (Maximal Atlas):** A smooth atlas \mathcal{A} on M is said to be *maximal* iff it is not contained in any strictly larger smooth atlas. I.e., any chart that is smoothly compatible with every chart in \mathcal{A} is already contained in \mathcal{A} . A *smooth structure* on M is a maximal atlas.
- **Lem 1.10. (Smooth Atlases)** Let M be a topological manifold.
 - (a) Every smooth atlas for M is contained in a unique maximal smooth atlas.
 - (b) Two smooth atlases for M determine the same maximal smooth atlas if and only if their union is a smooth atlas.

The proof of (b) was left as an exercise (see below). The proof of (a) is given. Let \mathcal{A} be a smooth atlas for M , and let $\bar{\mathcal{A}}$ denote the set of all charts that are smoothly compatible with every chart in \mathcal{A} . To show that $\bar{\mathcal{A}}$ is a smooth atlas, we need to show that any two charts of $\bar{\mathcal{A}}$ are smoothly compatible with each other, which is to say that for any $(U, \varphi), (V, \psi) \in \bar{\mathcal{A}}$, $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$ is smooth.

Let $x = \varphi(p) = \varphi(U \cap V)$ be arbitrary. Because the domains of the charts in \mathcal{A} cover M , there is some chart (W, θ) in \mathcal{A} such that $p \in W$. Since every chart in $\bar{\mathcal{A}}$ is smoothly compatible with (W, θ) , both of the maps $\theta \circ \varphi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth where they are defined. Since $p \in U \cap V \cap W$, it follows that

$$\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1}) \quad (1)$$

is smooth on a neighborhood of x . Hence, $\psi \circ \varphi^{-1}$ is smooth in a neighborhood of each point in $\varphi(U \cap V)$. This concludes that $\bar{\mathcal{A}}$ is a smooth atlas. Now, we need to show that $\bar{\mathcal{A}}$ is maximal. But this is straightforward to see: any chart that is smoothly compatible with every chart contained in $\bar{\mathcal{A}}$ must be smoothly compatible with every chart contained in \mathcal{A} , and hence, must be contained in $\bar{\mathcal{A}}$. Therefore, $\bar{\mathcal{A}}$ is maximal. Uniqueness also follows in a straightforward way: suppose \mathcal{B} is another maximal atlas containing \mathcal{A} . Then since every chart in \mathcal{B} is smoothly compatible with every chart in \mathcal{A} , it follows that $\mathcal{B} \subset \bar{\mathcal{A}}$. Hence by maximality of \mathcal{B} , $\mathcal{B} = \bar{\mathcal{A}}$.

Exercise 1.4. Prove Lemma 1.10(b).

Let M be a topological n -manifold, $\mathcal{A}_1, \mathcal{A}_2$ be two smooth atlas on M , and $\bar{\mathcal{A}}_1, \bar{\mathcal{A}}_2$ the maximal smooth atlases determined by the two smooth atlases, respectively. This means that among all the smooth atlases that contain $\mathcal{A}_{1,2}$, $\bar{\mathcal{A}}_{1,2}$ are maximal, respectively. (\Rightarrow) Suppose that $\bar{\mathcal{A}}_1 = \bar{\mathcal{A}}_2$. This means that every chart contained in $\bar{\mathcal{A}}_2$ is smoothly compatible with every chart in \mathcal{A}_1 ; since $\mathcal{A}_2 \subset \bar{\mathcal{A}}_2$, this implies that every chart of \mathcal{A}_2 is smoothly compatible with every chart of \mathcal{A}_1 . Likewise, since every chart in $\bar{\mathcal{A}}_1$ is smoothly compatible with every chart in \mathcal{A}_2 , and $\mathcal{A}_1 \subset \bar{\mathcal{A}}_1$, it follows that every chart in \mathcal{A}_1 is smoothly compatible with every chart in \mathcal{A}_2 . Hence, it follows that every pair of charts in $\mathcal{A}_1 \cup \mathcal{A}_2$ is smoothly compatible, showing that the union is a smooth atlas. (\Leftarrow) Suppose that $\mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas. This implies that every chart in \mathcal{A}_2 is smoothly compatible with every chart in \mathcal{A}_1 , and thus, $\mathcal{A}_2 \subset \bar{\mathcal{A}}_1$; by maximality of $\bar{\mathcal{A}}_2$, $\bar{\mathcal{A}}_1 \subset \bar{\mathcal{A}}_2$. Likewise, we can show that $\bar{\mathcal{A}}_1 \subset \bar{\mathcal{A}}_2$; by maximality of $\bar{\mathcal{A}}_2$, $\bar{\mathcal{A}}_2 \subset \bar{\mathcal{A}}_1$. Therefore, $\bar{\mathcal{A}}_1 = \bar{\mathcal{A}}_2$.

2 Smooth Maps

2.1 Smooth Functions and Smooth Maps

- **Def. (Smooth Function)** Let M be a smooth n -manifold. A function $f : M \rightarrow \mathbb{R}^k$ is *smooth* if for every $p \in M$, there exists a smooth chart (U, φ) for M whose domain contains p and such that the composite function $f \circ \varphi^{-1}$ is smooth on the open subset $\tilde{U} = \varphi(U) \subset \mathbb{R}^n$.

Exercise 2.3. Suppose M is a smooth manifold and $f : M \rightarrow \mathbb{R}^k$ is a smooth function. Show that $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{R}^k$ is smooth for every smooth chart (U, φ) for M .

Suppose M is a smooth manifold and $f : M \rightarrow \mathbb{R}^k$ is a smooth function. Let (U, φ) be a smooth chart for M . By definition of a smooth function, for every $p \in U$, there exists a smooth chart (V_p, ψ_p) for M containing p in its domain such that $f \circ \psi_p^{-1} : \psi_p(V_p) \rightarrow \mathbb{R}^k$ is smooth. Since

$$U = \bigcup_{p \in U} (U \cap V_p) \implies \varphi(U) = \varphi \left(\bigcup_{p \in U} (U \cap V_p) \right) = \bigcup_{p \in U} \varphi(U \cap V_p), \quad (2)$$

it suffices to show that $f \circ \varphi^{-1}$ is smooth on $\varphi(U \cap V_p)$ for each p . Indeed, since (V_p, ψ_p) and (U, φ) are smoothly compatible for all p , $\psi_p \circ \varphi^{-1} : \varphi(U \cap V_p) \rightarrow \psi_p(U \cap V_p)$ is smooth. Since $f \circ \psi_p^{-1}$ is smooth on $\psi_p(V_p)$, it must be smooth on the subset $\psi_p(U \cap V_p)$. Therefore,

$$f \circ \varphi^{-1} = (f \circ \psi_p^{-1}) \circ (\psi_p \circ \varphi^{-1}) : \varphi(U \cap V_p) \rightarrow \mathbb{R}^k \quad (3)$$

is smooth for all p . Thus, we conclude that $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{R}^k$ is smooth.

- **Def. (Coordinate Representation)** Given a function $f : M \rightarrow \mathbb{R}^k$ and a chart (U, φ) for M , the function $\hat{f} : \varphi(U) \rightarrow \mathbb{R}^k$ defined by $\hat{f}(x) = f \circ \varphi^{-1}(x)$ is called the *coordinate representation* of f . By definition, f is smooth iff its coordinate representation is *smooth* in some smooth chart of M ; but by the preceding exercise, the coordinate representation of f is smooth in every smooth chart of M .
- **Def. (Smooth Map between Manifolds)** Let M, N be smooth manifolds, and let $F : M \rightarrow N$ be any map. F is a *smooth map* if for every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and the composite map $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.

Exercise 2.4. Let M and N be smooth manifolds, and let $F : M \rightarrow N$ be a map. If every point $p \in M$ has a neighborhood U such that the restriction $F|_U$ is smooth, show that F is smooth. Conversely, if F is smooth, show that its restriction to any open subset is smooth.

Let M and N be smooth manifolds, and let $F : M \rightarrow N$ be a map.

- Let $p \in M$, and let W be a neighborhood of p such that $F|_W$ is smooth. This means that there exist smooth charts (U, φ) , where $p \in U \subset W$, and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and the composite function $\psi \circ (F|_W) \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$ is smooth. Since $U \subset W$, it follows that $(F|_W)|_U = F|_U$. This means that $\psi \circ F|_U \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$ is smooth. Hence, since p was arbitrary, we conclude that F is smooth.
- Now assume that F is smooth, and let W be an arbitrary open subset of M . By definition of smoothness, for each $p \in W$, there exist smooth charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$ such that $F(U) \subset V$ and the composite function $\psi \circ F \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$ is smooth. Since $\varphi(U \cap W)$ is an open subset of $\varphi(U)$, it follows that $\psi \circ F \circ \varphi^{-1}$ is smooth on $\varphi(U \cap W)$; that is, $F|_{(U \cap W)}$ is smooth. Hence, we have shown that for every $p \in W$, there exists a neighborhood of p such that the restriction of F to this neighborhood is smooth. Therefore, we conclude that $F|_W$ is smooth.

- **Lem 2.1. (Constructing Smooth Maps)** Let M and N be smooth manifolds, and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M . Suppose that for each $\alpha \in A$, we are given a smooth map $F_\alpha : U_\alpha \rightarrow N$ such that the maps agree on overlaps $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all α and β . Then there exists a unique smooth map $F : M \rightarrow N$ such that $F|_{U_\alpha} = F_\alpha$ for each $\alpha \in A$.

- **Lem. 2.2 (Smoothness Implies Continuity)** Every smooth map between smooth manifolds is continuous.

Suppose $F : M \rightarrow N$ is smooth. By definition of smoothness, for each $p \in M$, we can choose smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subset V$ and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$ is a smooth map, and hence continuous. Since $\varphi : U \rightarrow \varphi(U)$ and $\psi : V \rightarrow \psi(V)$ are homeomorphisms, this implies in turn that

$$F|_U = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi : U \rightarrow V, \quad (4)$$

which is a composition of continuous maps, is continuous. Hence, since F is continuous in a neighborhood of each point, it is continuous on M .

- **Def. (Coordinate Representation)** Let $F : M \rightarrow N$ be a smooth map, and (U, φ) , (V, ψ) be any smooth charts for M and N , respectively. Then we call $\hat{F} = \psi \circ F \circ \varphi^{-1}$ the coordinate representation of F with respect to the given coordinates.

Exercise 2.6. Suppose $F : M \rightarrow N$ is a smooth map between smooth manifolds. Show that the coordinate representation of F with respect to any pair of smooth charts for M and N is smooth.

Let $F : M \rightarrow N$ be a smooth map between smooth manifolds, and let (U, φ) , (V, ψ) be any pair of smooth charts for M and N . Without loss of generality, assume that $F(U) \subset V$. Our task is to show that $\psi \circ F \circ \varphi^{-1}$ is smooth. Let $p \in U$. Since F is smooth, there exist smooth charts (W, θ) and (R, ϑ) containing p and $F(p)$, respectively, such that $F(W) \subset V \cap R$ and the composite function $\vartheta \circ F \circ \theta^{-1} : \theta(W) \rightarrow \vartheta(R)$ is smooth. Since $U \cap W$ is nonempty and the corresponding charts are smoothly compatible, the transition map $\theta \circ \varphi^{-1} : \varphi(U \cap W) \rightarrow \theta(U \cap W)$ is smooth. Likewise, the transition map $\psi \circ \vartheta^{-1}$ is smooth. Hence, the composite function:

$$\psi \circ F \circ \varphi^{-1} = (\psi \circ \vartheta^{-1}) \circ (\vartheta \circ F \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1}) \quad (5)$$

is smooth on $\varphi(U \cap W)$. By locality of smoothness, since for each $p \in U$, there exists a neighborhood on which $\psi \circ F \circ \varphi^{-1}$ is smooth, we conclude that the coordinate representation of F with respect to the given coordinates is smooth.

2.2 Smooth Covering Maps

- **Def. (Covering Map)** A surjective continuous map $\pi : \widetilde{M} \rightarrow M$ between connected, locally path connected spaces with the property that for every $p \in M$, there exists a neighborhood U that is *evenly covered* (i.e., U is connected, and each component of $\pi^{-1}(U)$ is mapped homeomorphically onto U by π).
- **Def. (Smooth Covering Map)** Let \widetilde{M} and M be connected smooth manifolds. A smooth covering map $\pi : \widetilde{M} \rightarrow M$ is a smooth surjective map with the property that every $p \in M$ has a connected neighborhood U such that each component of $\pi^{-1}(U)$ is mapped *diffeomorphically* onto U by π . In this instance also, we say that U is evenly covered.
- **Prop 2.9. (Properties of Smooth Coverings)**
 - (a) Any smooth covering map is a local diffeomorphism and an open map.
 - (b) An injective smooth covering map is a diffeomorphism.
 - (c) A topological covering map is a smooth covering map if and only if it is a local diffeomorphism.

Exercise 2.12. If $\pi_1 : \widetilde{M}_1 \rightarrow M_1$ and $\pi_2 : \widetilde{M}_2 \rightarrow M_2$ are smooth covering maps, show that $\pi_1 \times \pi_2 : \widetilde{M}_1 \times \widetilde{M}_2 \rightarrow M_1 \times M_2$ is a smooth covering map.

Since $\widetilde{M}_{1,2}$ and $M_{1,2}$ are all connected smooth manifolds, $\widetilde{M}_1 \times \widetilde{M}_2$ and $M_1 \times M_2$ are all connected smooth manifolds. Now let $(p, q) \in M_1 \times M_2$. Since π_1 is surjective, there exists $\tilde{p} \in \widetilde{M}_1$ such that $\pi_1(\tilde{p}) = p$; likewise, there exists $\tilde{q} \in \widetilde{M}_2$ such that $\pi_2(\tilde{q}) = q$. Hence, $\pi_1 \times \pi_2 : (\tilde{p}, \tilde{q}) \mapsto (p, q)$, which shows that $\pi_1 \times \pi_2$ is surjective. Likewise, since π_1, π_2 are smooth, $\pi_1 \times \pi_2$ is smooth. Now we need to verify the evenly covered property for $\pi_1 \times \pi_2$.

Let $(p, q) \in M_1 \times M_2$. By the definition of smooth covering maps, there exist connected neighborhoods $U \subset M_1$ and $V \subset M_2$ such that each component of $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$ is mapped diffeomorphically onto U and V by π_1 and π_2 , respectively. Since the product of connected open sets is connected, $U \times V$ is a connected neighborhood of (p, q) . Then since $(\pi_1 \times \pi_2)^{-1}(U \times V) = \pi_1^{-1}(U) \times \pi_2^{-1}(V)$, the components of $(\pi_1 \times \pi_2)^{-1}(U \times V)$ are just the products of the components of $\pi_1^{-1}(U)$ with the components of $\pi_2^{-1}(V)$. Hence, since π_1 (π_2) maps each component of $\pi_1^{-1}(U)$ ($\pi_2^{-1}(V)$) diffeomorphically onto U (V), it follows that $\pi_1 \times \pi_2$ maps each component of $\pi_1^{-1}(U) \times \pi_2^{-1}(V)$ diffeomorphically onto $U \times V$. Therefore, $\pi_1 \times \pi_2$ is a smooth covering map.

- **Def. (Section of a Continuous Map)** If $\pi : \widetilde{M} \rightarrow M$ is any continuous map, a *section* of π is a continuous map $\sigma : M \rightarrow \widetilde{M}$ such that $\pi \circ \sigma = \text{Id}_M$:

$$\begin{array}{c} \widetilde{M} \\ \downarrow \pi \quad \nearrow \sigma \\ M \end{array}$$

Figure 1: Section of π .

- **Def. (Local Section of a Continuous Map)** A continuous map $\sigma : U \subset M \rightarrow \widetilde{M}$ such that $\pi \circ \sigma = \text{Id}_U$.

2.3 Proper Maps

- **Def. (Proper Maps)** Let M, N be topological spaces. $F : M \rightarrow N$ is *proper* if for every compact set $K \subset N$, $F^{-1}(K)$ is compact.

- **Lem. 2.14 (Sufficient Condition for Proper Map I)** Suppose M is a compact space and N is Hausdorff space. Then every continuous map $F : M \rightarrow N$ is proper.

Let $K \subset N$ be compact; since N is Hausdorff, K is closed. Then by continuity of F , $F^{-1}(K)$ is closed in M . Since M is compact, $F^{-1}(K)$ must be compact in K .

- **Def. (Saturated Subset)** A subset $A \subset M$ is said to be saturated with respect to a map $F : M \rightarrow N$ if $A = F^{-1}(F(A))$.

- **Lem. 2.15. (Sufficient Condition for Proper Map II)** Suppose $F : M \rightarrow N$ is a proper map between topological spaces, and $A \subset M$ is any subset that is saturated with respect to F . Then $F|_A : A \rightarrow F(A)$ is proper.

Let $K \subset F(A)$ be compact. Since A is saturated, $(F|_A)^{-1}(K) = F^{-1}(K)$, which is compact since F is proper.

3 Tangent Vectors

- **Def. (Derivation at a Point)** Let $a \in \mathbb{R}^n$. A linear map $X : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$ is called a *derivation at a* iff it satisfies the following product rule:

$$X(fg) = f(a)Xg + g(a)Xf. \quad (6)$$

- **Lem 3.1. (Properties of Derivations)** Suppose $a \in \mathbb{R}^n$ and $X \in T_a(\mathbb{R}^n)$.
 - (a) If f is a constant function, then $Xf = 0$.
 - (b) If $f(a) = g(a) = 0$, then $X(fg) = 0$.

(a) It suffices to show that if $f \equiv 1$, then $Xf = 0$. Indeed,

$$Xf = X(ff) = f(a)Xf + f(a)Xf = 2f(a)Xf = 2Xf, \quad (7)$$

whence $Xf = 0$.

(b) From the product rule, $X(fg) = f(a)Xg + g(a)Xf = 0 + 0 = 0$.

3.1 Tangent Vectors on a Manifold

- **Def. (Derivations on Manifolds)** Let M be a smooth manifold and $p \in M$. A linear map $X : C^\infty(M) \rightarrow \mathbb{R}$ is called a *derivation at p* if it satisfies

$$X(fg) = f(p)Xg + g(p)Xf \quad (8)$$

for all $f, g \in C^\infty(M)$. The set of all derivations at p is called the *tangent space* to M at p , and is denoted by T_pM .

- **Lem 3.4. (Properties of Tangent Vectors on Manifolds)** Let M be a smooth manifold, and suppose $p \in M$ and $X \in T_pM$.
 - (a) If f is a constant function, then $Xf = 0$.
 - (b) If $f(p) = g(p) = 0$, then $X(fg) = 0$.

3.2 Pushforwards

- **Def. (Pushforward associated with a Map)** Let M, N be smooth manifolds and $F : M \rightarrow N$ a smooth map. For each $p \in M$, we define a map $F_* : T_pM \rightarrow T_{F(p)}(N)$, called the *pushforward* associated with F as follows:

$$(F_*X)(f) = X(f \circ F). \quad (9)$$

It is straightforward to see that the pushforward is linear. It is also a derivation at p :

$$\begin{aligned} (F_*X)(fg) &= X(fg \circ F) = X((f \circ F)(g \circ F)) \\ &= (f \circ F)(p)X(g \circ F) + (g \circ F)(p)X(f \circ F) \\ &= (f \circ F)(p)(F_*X)(g) + (g \circ F)(p)(F_*X)(f). \end{aligned} \quad (10)$$

- **Lem 3.5. (Properties of Pushforwards)** Let $F : M \rightarrow N$ and $G : N \rightarrow P$ be smooth maps, and let $p \in M$.
 - (a) $F_* : T_pM \rightarrow T_{F(p)}N$ is linear.
 - (b) $(G \circ F)_* = G_* \circ F_* : T_pM \rightarrow T_{(G \circ F)(p)}P$.
 - (c) $(\text{Id}_M)_* = \text{Id}_{T_pM} : T_pM \rightarrow T_pM$.
 - (d) If F is a diffeomorphism, then $F_* : T_pM \rightarrow T_{F(p)}N$ is an isomorphism.

Exercise 3.2. Prove Lemma 3.5.

(a) Let $f \in C^\infty(N)$, $X, Y \in T_p(M)$, $c_{1,2} \in \mathbb{R}^n$. Then

$$\begin{aligned} (F_*(c_1X + c_2Y))(f) &= (c_1X + c_2Y)(f \circ F) \\ &= c_1X(f \circ F) + c_2Y(f \circ F) = c_1F_*(X)(f) + c_2F_*(Y)(f). \end{aligned} \quad (11)$$

(b) Let $f \in C^\infty(N)$, and $X \in T_p(M)$. Then

$$\begin{aligned} ((G \circ F)_*X)(f) &= X(f \circ (G \circ F)) = X((f \circ G) \circ F) \\ &= (F_*X)(f \circ G) \\ &= (G_*(F_*X))(f) = ((G_* \circ F_*)X)(f). \end{aligned} \quad (12)$$

(c) Let $f \in C^\infty(N)$, and $X \in T_pM$. Then

$$(\text{Id}_M_*X)(f) = X(f \circ \text{Id}_M) = X(f). \quad (13)$$

- **Prop. 3.6. (Tangent Space is Local)** Suppose M is a smooth manifold, $p \in M$, and $X \in T_pM$. If f and g are smooth functions in M that agree on some neighborhood of p , then $Xf = Xg$.

Let $h = f - g$. It suffices to show that $Xh = 0$ by linearity of X whenever h vanishes in a neighborhood of p . Let $\psi \in C^\infty(M)$ be a smooth function that is identically 1 on the support of h and supported in $M \setminus \{p\}$. Because $\psi \equiv 1$ where h is nonzero, the product ψh is identically equal to h . Since $h(p) = \psi(p) = 0$, Lemma 3.4(b) implies that $Xh = X(\psi h) = 0$.

3.3 Computation in Coordinates

- **Def. (Basis for T_pM in Coordinates)** Let (U, φ) be a smooth coordinate chart on M ; in particular, $\varphi : U \rightarrow \tilde{U} \subset \mathbb{R}^n$ is a diffeomorphism. This implies that $\varphi_* : T_pM \rightarrow T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism. We've seen that $T_{\varphi(p)}\mathbb{R}^n$ has as a basis consisting of all the derivations $\partial_{x^i}|_{\varphi(p)}$, $i = 1, \dots, n$. Therefore, the pushforward of these vectors under $(\varphi^{-1})_*$ form a basis for T_pM . We use the following notation:

$$\frac{\partial}{\partial x^i} \Big|_p = (\varphi^{-1})_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)}. \quad (14)$$

Indeed, if $f : U \rightarrow \mathbb{R}$ is smooth, then

$$\frac{\partial}{\partial x^i} \Big|_p f = \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} (f \circ \varphi^{-1}) = \frac{\partial \hat{f}}{\partial x^i}(\hat{p}), \quad (15)$$

where \hat{f} is the coordinate representation of f , and $\hat{p} = (p^1, \dots, p^n) = \varphi(p)$ is the coordinate representation of p .

- **Def. (Pushforward in Coordinates I)** Consider a smooth map $F : U \rightarrow V$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open subsets of Euclidean spaces. Let $p \in U$. We will use (x^1, \dots, x^n) to denote the coordinates in the domain and (y^1, \dots, y^m) to denote the coordinates in the range. Then using the chain rule,

$$\begin{aligned} \left(F_* \frac{\partial}{\partial x^i} \Big|_p \right) f &= \frac{\partial}{\partial x^i} \Big|_p (f \circ F) \\ &= \frac{\partial f}{\partial y^j}(F(p)) \frac{\partial F^j}{\partial x^i}(p) \\ &= \left(\frac{\partial F^j}{\partial x^i} \frac{\partial}{\partial y^j} \Big|_{F(p)} \right) f. \end{aligned} \quad (16)$$

Since f was arbitrary, we conclude that

$$F_* \frac{\partial}{\partial x^i} \Big|_p = \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j} \Big|_{F(p)}. \quad (17)$$

In other words, the matrix of F_* in terms of the standard coordinate basis is given by

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}. \quad (18)$$

This is precisely the Jacobian matrix of F .

- **Def. (Pushforward in Coordinates II)** Let $F : M \rightarrow N$ be an arbitrary smooth map. Choosing smooth coordinate charts (U, φ) for M near p and (V, ψ) for N near $F(p)$, we obtain the coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \rightarrow \psi(V)$. Now we apply the chain rule:

$$\begin{aligned} F_* \frac{\partial}{\partial x^i} \Big|_p &= F_* \left((\varphi^{-1})_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) = (F \circ \varphi^{-1})_* \left(\frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) \\ &= (\psi^{-1})_* \left(\hat{F}_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) = (\psi^{-1})_* \left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j} \Big|_{\hat{F}(\varphi(p))} \right) = \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j} \Big|_{F(p)}. \end{aligned} \quad (19)$$

In other words, the pushforward of F is precisely the Jacobian matrix of its coordinate representation.

- **Obs. (Transformation of Vectors)** Suppose (U, φ) and (V, ψ) are two smooth charts on M , and let $p \in U \cap V$. We have two bases for the tangent space at p , namely $\{\partial/\partial x^i|_p\}$, where the coordinate functions of φ are (x^i) , and $\{\partial/\partial \tilde{x}^i|_p\}$, where the coordinate functions of ψ are (\tilde{x}^i) . By **Def. (Pushforward in Coordinates I)**, we have

$$(\psi \circ \varphi^{-1})_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} = \frac{\partial \tilde{x}^j(x)}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j} \Big|_{\psi(p)}. \quad (20)$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial x^i} \Big|_p &= (\varphi^{-1})_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} = (\psi^{-1} \circ \psi \circ \varphi^{-1})_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \\ &= (\psi^{-1})_* \left((\psi \circ \varphi^{-1})_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) \\ &= (\psi^{-1})_* \left(\frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j} \Big|_{\psi(p)} \right) = \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j} \Big|_p. \end{aligned} \quad (21)$$

In particular, for any $X \in T_p M$, if

$$X = X^i \frac{\partial}{\partial x^i} \Big|_p = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i} \Big|_p, \quad (22)$$

then by the above result,

$$\tilde{X}^i = X^i \frac{\partial \tilde{x}^i}{\partial x^i}(\varphi(p)) = X^i \frac{\partial \tilde{x}^i}{\partial x^i}(\hat{p}), \quad (23)$$

where $\hat{p} = \varphi(p)$ is the representation of p in x^i -coordinates.

4 Vector Fields

- **Def. (Tangent Bundle)** Let M be a smooth manifold. Then the *tangent bundle* of M is the disjoint union of the tangent spaces at all points of M :

$$TM := \coprod_{p \in M} T_p M. \quad (24)$$

A typical element of the tangent bundle is of the form (p, X) , where $p \in M$ and $X \in T_p M$.

- **Lem. 4.1: (Tangent Bundle is a Manifold)** For any smooth n -manifold M , the tangent bundle TM has a natural topology and smooth structure that make it into a $2n$ -dimensional smooth manifold. With this structure, the canonical projection map $\pi : TM \rightarrow M$, defined as the map $\pi : (p, X) \mapsto p$, is a smooth map.

We start by defining the smooth charts that will give TM its smooth structure. For some given smooth chart (U, φ) for M , let (x^1, \dots, x^n) denote the coordinate functions of φ , and define the map $\tilde{\varphi} : \pi^{-1}(U) \rightarrow \mathbb{R}^{2n}$ by

$$\tilde{\varphi} \left(v^i \frac{\partial}{\partial x^i} \Big|_p \right) = (x^1(p), \dots, x^n(p), v^1, \dots, v^n). \quad (25)$$

More precisely, the image set of $\tilde{\varphi}$ is the set $\varphi(U) \times \mathbb{R}^n$, which is an open subset of \mathbb{R}^{2n} . [!! Complete Later !!]

Exercise 4.2. Suppose $F : M \rightarrow N$ is a smooth map. By examining the local expression (3.6) for F_* in coordinates, show that $F_* : TM \rightarrow TN$ is a smooth map.

Let $F : M \rightarrow N$ be a smooth map, and consider its pushforward $F_* : TM \rightarrow TN$. Our goal is to show that F_* is a smooth map. Let $p \in M$; by smoothness there exist smooth charts (U, φ) containing p in its domain, and (V, ψ) containing $F(p)$ in its domain such that $F(U) \subset V$ and the composite function $\psi \circ F \circ \varphi^{-1}$ is smooth. Let (x^i) denote the coordinate functions of φ and (y^j) denote the coordinate functions of ψ . Let $(\pi^{-1}(U), \tilde{\varphi})$ and $(\pi^{-1}(V), \tilde{\psi})$ be the corresponding smooth charts for TM and TN , respectively, where π is the canonical projection map from the tangent bundle of a manifold onto the manifold. These charts are equipped with the standard coordinates (x^i, v^i) and (y^j, w^j) , respectively. Then in coordinates, the local expression for F_* is given by,

$$F_* \frac{\partial}{\partial x^i} \Big|_p = \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j} \Big|_{F(p)}. \quad (26)$$

This implies that

$$F_* : (p, v) \mapsto \left(F^j(p), v^i \frac{\partial F^j}{\partial x^i}(p) \right). \quad (27)$$

Since F is smooth, each coordinate function F^j must be smooth. Likewise, each $\partial F^j / \partial x^i(p)$. Since the map $(x^i, v^i) \mapsto v^i \frac{\partial F^j}{\partial x^i}(x)$ is the finite sum of smooth functions, it must be smooth as well. Hence, we conclude that F_* is smooth.

- **Def. (Vector Field)** A section of the map $\pi : TM \rightarrow M$; i.e., a vector field is a continuous map $Y : M \rightarrow TM$, usually written $p \mapsto Y_p$, with the property that

$$\pi \circ Y = \text{Id}_M. \quad (28)$$

- **Def. (Smooth Vector Field)** A smooth vector field.

- **Lem. 4.2 (Smoothness Criterion for Vector Fields)** Let M be a smooth manifold, and let $Y : M \rightarrow TM$ be a rough vector field. If $(U, (x^i))$ is *any* smooth coordinate chart on M , then Y is smooth on U if and only if its component functions with respect to this chart are smooth.

Let (x^i, v^i) be the standard coordinates on $\pi^{-1}(U) \subset TM$ associated with the chart $(U, (x^i))$. By definition of the standard coordinate representation of Y ,

$$\hat{Y}(x) = (x^1, \dots, x^n, Y^1(x), \dots, Y^n(x)), \quad (29)$$

where Y^i is the i th component function of Y in x^i -coordinates. Hence, smoothness of Y is equivalent to smoothness of the component functions.

- **Lem 4.5. (Extending a Tangent Vector)** Let M be a smooth manifold. If $p \in M$ and $X \in T_p M$, there is a smooth vector field \tilde{X} on M such that $\tilde{X}_p = X$.

Let (x^i) be smooth coordinates on a neighborhood U of p , and let $X^i \partial / \partial x^i|_p$ be the coordinate expression for X . Let ψ be a smooth bump function supported in U and with $\psi(p) = 1$. Then the vector field \tilde{X} defined by

$$\tilde{X}_q = \begin{cases} \psi(q) X^i \frac{\partial}{\partial x^i} \Big|_q, & q \in U, \\ 0, & q \notin \text{supp } \psi \end{cases} \quad (30)$$

is a smooth vector field whose value at p is equal to X .

- **Def. (Set of all Smooth Vector Fields)** Let $\mathcal{T}(M)$ denote the set of all smooth vector fields on M ; $\mathcal{T}(M)$ is a vector space under pointwise addition and scalar multiplication:

$$(aY + bZ)_p = aY_p + bZ_p. \quad (31)$$

If $f \in C^\infty(M)$ and $Y \in \mathcal{T}(M)$, we define $fY : M \rightarrow TM$ by

$$(fY)_p = f(p)Y_p. \quad (32)$$

Exercise 4.3. If Y and Z are smooth vector fields on M and $f, g \in C^\infty(M)$, show that $fY + gZ$ is a smooth vector field.

Let $(U, (x^i))$ be a smooth coordinate chart on M . Then in these coordinates,

$$Y = Y^i \frac{\partial}{\partial x^i}, \quad Z = Z^i \frac{\partial}{\partial x^i}. \quad (33)$$

Then

$$fY + gZ = (fY^i + gZ^i) \frac{\partial}{\partial x^i}. \quad (34)$$

Since f, g, Y^i, Z^i are all smooth, and the product/sum of smooth functions is smooth, $fY^i + gZ^i$ is smooth for all i . Hence, since the component functions of $fY + gZ$ are smooth on any smooth coordinate chart on M , it follows that $fY + gZ$ is a smooth vector field on M .

- **Def. (Action of Vector Field on Functions)** If $Y \in \mathcal{T}(M)$ and f is a smooth real-valued function defined on an open set $U \subset M$, we obtain a new function $Yf : U \rightarrow \mathbb{R}$ defined by

$$Yf(p) = Y_p f. \quad (35)$$

5 Cotangent Bundle

5.1 Covectors

- **Def. (Covector)** Let V be a finite-dimensional vector space. A *covector* on V is a real-valued linear functional on V ; i.e., a linear map $\omega : V \rightarrow \mathbb{R}$. The vector space of all covectors on V is denoted by V^* and called the *dual* space to V .
- **Prop. 6.1. (Dual Basis)** Let V be a finite-dimensional vector space. If (E_1, \dots, E_n) is any basis for V , then the covectors $(\varepsilon^1, \dots, \varepsilon^n)$, defined by

$$\varepsilon^i(E_j) = \delta_j^i = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad (36)$$

form a basis for V^* , called the dual basis to (E_i) . Therefore, $\dim V^* = \dim V$.

Exercise 6.1. Prove Proposition 6.1.

Assume the given hypotheses of Proposition 6.1. We must show that (ε^i) is a linearly independent collection of covectors that spans V^* ; we start by showing linear independence. Suppose $a_1, \dots, a_n \in \mathbb{R}$ are scalars such that

$$a_1 \varepsilon^1 + \dots + a_n \varepsilon^n = 0. \quad (37)$$

Then allowing the left side to act on the V -basis vector E_j , for some $j \in \{1, \dots, n\}$,

$$0 = (a_1 \varepsilon^1 + \dots + a_n \varepsilon^n)(E_j) = \sum_1^n a_i \varepsilon^i(E_j) = a_j. \quad (38)$$

Since this is true for all $j \in \{1, \dots, n\}$, we conclude that each $a_j = 0$. Therefore, (ε^i) is linearly independent. Now let $\omega \in V^*$. For each $i = 1, \dots, n$, let $\omega(E_i) = a_i \in \mathbb{R}$. Then we claim that $\omega = a_i \varepsilon^i$ (where, we follow Einstein Summation Convention as usual). Indeed,

$$\begin{aligned} \omega(v) &= \omega(v^i E_i) = a_i v^i, \\ a_i \varepsilon^i(v) &= a_i \varepsilon^i(v^j E_j) = a_i v^j \varepsilon^i(E_j) = a_i v^j \delta_j^i = a_i v^i. \end{aligned} \quad (39)$$

Hence, it follows that (ε^i) spans V^* . Altogether, we have shown that this collection forms a basis for the dual space.

- **Def. (Dual Map)** Suppose V and W are vector spaces, and $A : V \rightarrow W$ is a linear map. Define a linear map $A^* : W^* \rightarrow V^*$, called the *dual map* of A by,

$$(A^* \omega)(X) = \omega(AX) \quad \text{for } \omega \in W^*, X \in V. \quad (40)$$

Exercise 6.2. Show that $A^* \omega$ is actually a linear functional on V , and that A^* is a linear map.

Let V, W be vector spaces, $A : V \rightarrow W$ a linear map, and $A^* : W^* \rightarrow V^*$ the dual map of A .

- (i) Let $\omega \in W^*$ be a fixed covector, and let $X, Y \in V$, $a_1, a_2 \in \mathbb{R}$. Then

$$\begin{aligned} (A^* \omega)(a_1 X + a_2 Y) &= \omega(A(a_1 X + a_2 Y)) \\ &= \omega(a_1 AX + a_2 AY) \\ &= \omega(a_1 AX) + \omega(a_2 AY) \\ &= a_1 \omega(AX) + a_2 \omega(AY) \\ &= a_1 (A^* \omega)(X) + a_2 (A^* \omega)(Y), \end{aligned} \quad (41)$$

where the second inequality follows from linearity of A , and the third and fourth inequalities follow from linearity of ω . Hence, $A^* \omega$ is a linear functional for each $\omega \in W^*$.

(ii) Now let $X \in V$ be fixed, and let $\omega_1, \omega_2 \in V^*$, $a_1, a_2 \in \mathbb{R}$. Then

$$\begin{aligned} (A^*(a_1\omega_1 + a_2\omega_2))(X) &= (a_1\omega_1 + a_2\omega_2)(AX) \\ &= a_1\omega_1(AX) + a_2\omega_2(AX) \\ &= (a_1(A^*\omega_1) + a_2(A^*\omega_2))(X). \end{aligned} \quad (42)$$

Since X was arbitrary, it follows that A^* is a linear map.

• **Prop. 6.2. (Properties of Dual Maps)** The dual map satisfies the following properties:

- (a) $(A \circ B)^* = B^* \circ A^*$.
- (b) $(\text{Id}_V)^* : V^* \rightarrow V^*$ is the identity map of V^* .

Exercise 6.3. Prove the preceding proposition.

(a) Let $B : V \rightarrow W$ and $A : W \rightarrow Y$ be linear maps, and A^*, B^* their corresponding dual maps. Let $\omega \in Y^*$ and $X \in V$. Then

$$\begin{aligned} ((B^* \circ A^*)\omega)(X) &= B^*(A^*\omega)(X) \\ &= A^*\omega(BX) = \omega(ABX) = \omega((A \circ B)X) \\ &= ((A \circ B)^*\omega)(X). \end{aligned} \quad (43)$$

Since X, ω were arbitrary, $(A \circ B)^* = B^* \circ A^*$.

(b) Let $\omega \in V^*$, and $X \in V$. Then

$$((\text{Id}_V)^*\omega)(X) = \omega(\text{Id}_V X) = \omega(X). \quad (44)$$

Since X was arbitrary, we conclude that $(\text{Id}_V)^*\omega = \omega$ for all $\omega \in V^*$.

• **Def. (Natural Basis-Independent Map)** For each vector space V , there is a natural, basis-independent map $\xi : V \rightarrow V^{**}$, defined as follows: for each vector $X \in V$, define a linear functional $\xi(X) : V^* \rightarrow \mathbb{R}$ by

$$\xi(X)(\omega) = \omega(X), \quad \text{for } \omega \in V^*. \quad (45)$$

Exercise 6.4. Let V be a vector space.

- (a) For any $X \in V$, show that $\xi(X)(\omega)$ depends linearly on ω , so that $\xi(X) \in V^{**}$.
- (b) Show that the map $\xi : V \rightarrow V^{**}$ is linear.

Let V be a vector space.

(a) Fix $X \in V$, and let $\omega_1, \omega_2 \in V^*$, $a_1, a_2 \in \mathbb{R}$. Then

$$\begin{aligned} \xi(X)(a_1\omega_1 + a_2\omega_2) &= (a_1\omega_1 + a_2\omega_2)(X) = a_1\omega_1(X) + a_2\omega_2(X) \\ &= a_1\xi(X)(\omega_1) + a_2\xi(X)(\omega_2). \end{aligned} \quad (46)$$

Hence, $\xi(X) \in V^{**}$.

(b) Fix $\omega \in V^*$, and let $X_1, X_2 \in V$, $a_1, a_2 \in \mathbb{R}$. Then

$$\begin{aligned} \xi(a_1X_1 + a_2X_2)(\omega) &= \omega(a_1X_1 + a_2X_2) = a_1\omega(X_1) + a_2\omega(X_2) \\ &= a_1\xi(X_1)(\omega) + a_2\xi(X_2)(\omega). \end{aligned} \quad (47)$$

Hence, since $\omega \in V^*$ was arbitrary, we conclude that $\xi : V \rightarrow V^{**}$ is linear.

• **Prop. 6.4 (Dual Dual Space is Isomorphic)** Let V be a finite-dimensional vector space. The map $\xi : V \rightarrow V^{**}$ is an isomorphism.

Since V and V^{**} have the same dimension, it suffices to check that ξ is injective. Suppose $X \in V \setminus \{0\}$. Extend X to a basis $(X = E_1, \dots, E_n)$, and let $(\varepsilon^1, \dots, \varepsilon^n)$ be the corresponding dual basis. Then

$$\xi(X)(\varepsilon^1) = \varepsilon^1(X) = \varepsilon^1(E_1) = 1 \neq 0, \quad (48)$$

so that $\xi(X) \neq 0$. Hence, the kernel is trivial, which proves injectivity.

5.2 Tangent Covectors on Manifolds

- **Def. (Cotangent Space)** Let M be a smooth manifold. For each $p \in M$, define the *cotangent space* at p , denoted by T_p^*M , to be the dual space to T_pM : $T_p^*M = (T_pM)^*$.
- **Obs. (Transformation Law for Covectors)** Suppose (U, φ) and (V, ψ) are two smooth charts on M , and let $p \in U \cap V$. As we saw before, we have two bases for the tangent space at p , namely $\{\partial/\partial x^i|_p\}$, where the coordinate functions of φ are (x^i) , and $\{\partial/\partial \tilde{x}^i|_p\}$, where the coordinate functions of ψ are (\tilde{x}^i) . Let (dx^i) and $(d\tilde{x}^i)$ be the corresponding dual bases for the cotangent space at p . In particular, we have

$$\omega = \omega_i dx^i = \tilde{\omega}_j d\tilde{x}^j \iff \omega \left(\frac{\partial}{\partial x^i} \Big|_p \right) dx^i = \tilde{\omega}_j d\tilde{x}^j. \quad (49)$$

Using the transformation law for vectors,

$$\omega \left(\frac{\partial}{\partial x^i} \Big|_p \right) dx^i = \omega \left(\frac{\partial \tilde{x}^j}{\partial x^i}(p) \frac{\partial}{\partial \tilde{x}^j} \Big|_p \right) dx^i = \tilde{\omega}_j \frac{\partial \tilde{x}^j}{\partial x^i}(p) dx^i = \tilde{\omega}_j d\tilde{x}^j. \quad (50)$$

Therefore, we conclude that

$$d\tilde{x}^j = \frac{\partial \tilde{x}^j}{\partial x^i} dx^i. \quad (51)$$

Contrast this with the transformation law for vectors.

5.3 The Cotangent Bundle

- **Def. (Cotangent Bundle)** The disjoint union

$$T^*M = \coprod_{p \in M} T_p^*M. \quad (52)$$

- **Lem. 6.6. (Smoothness Criteria for Covector Fields)** Let M be a smooth manifold, and let $\omega : M \rightarrow T^*M$ be a rough section.
 - If $\omega = \omega_i \lambda^i$ is the coordinate representation for ω in any smooth chart $(U, (x^i))$ for M , then ω is smooth on U if and only if its component functions ω_i are smooth
 - ω is smooth if and only if for every smooth vector field X on an open subset $U \subset M$, the function $\langle \omega, X \rangle : U \rightarrow \mathbb{R}$ defined by

$$\langle \omega, X \rangle(p) = \langle \omega_p, X_p \rangle = \omega_p(X_p) \quad (53)$$

is smooth.

Exercise 6.5. Prove Lemma 6.6.

5.4 The Differential of a Function

Exercise 6.6. Let $f(x, y) = x^2$ on \mathbb{R}^2 , and let X be the vector field

$$X = \text{grad}(f) = 2x \frac{\partial}{\partial x}. \quad (54)$$

Compute the coordinate expression of X in polar coordinates (on some open set on which they

are defined) using (6.4) and show that it is *not* equal to

$$\frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta}. \quad (55)$$

Recall that (6.4) stated the following:

$$\left. \frac{\partial}{\partial x^i} \right|_p = \frac{\partial \tilde{x}^j}{\partial x^i}(p) \left. \frac{\partial}{\partial \tilde{x}^j} \right|_p. \quad (56)$$

Remember that polar coordinates are given by $(x, y) = (r \cos(\theta), r \sin(\theta))$. In particular, by (6.4),

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta}. \quad (57)$$

Using the polar coordinates,

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}(\sqrt{x^2 + y^2}) \\ &= \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{r} = \cos(\theta). \\ \frac{\partial \theta}{\partial x} &= \frac{\partial}{\partial x}(\arctan(x^{-1}y)) \\ &= -\frac{y}{x^2 + y^2} = -\frac{r \sin(\theta)}{r^2} = -\frac{\sin(\theta)}{r}. \end{aligned} \quad (58)$$

Hence, this implies that

$$X = 2r \cos^2(\theta) \frac{\partial}{\partial r} - 2 \sin(\theta) \cos(\theta) \frac{\partial}{\partial \theta}. \quad (59)$$

On the other hand,

$$\frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} = 2r \cos^2(\theta) \frac{\partial}{\partial r} - 2r^2 \sin(\theta) \cos(\theta) \frac{\partial}{\partial \theta} \neq X. \quad (60)$$

- **Def. (Differential of a Function)** Let f be a smooth real-valued function on a smooth manifold M . Define the covector field df , called the *differential* of f , by

$$df_p(X_p) = X_p f \quad \text{for } X_p \in T_p M. \quad (61)$$

- **Lem. 6.7. (Differential is Smooth Covector Field)** The differential of a smooth function is a smooth covector field.
- **Obs. (Differential in Coordinates)** Let (x^i) be smooth coordinates on an open subset $U \subset M$, and let (λ_i) be the corresponding coframe on U . Suppose that in coordinates, $df_p = A_i(p) \lambda^i|_p$ for some functions $A_i : U \rightarrow \mathbb{R}$. This implies the following:

$$\begin{aligned} A_i(p) &= A_i(p) \lambda^i|_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = df_p \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) \\ &= \frac{\partial f}{\partial x^i}(p). \end{aligned} \quad (62)$$

This implies that

$$df_p = \frac{\partial f}{\partial x^i}(p) \lambda^i|_p. \quad (63)$$

Taking f to be $x^j : U \rightarrow \mathbb{R}$, we obtain

$$dx^j|_p = \frac{\partial x^j}{\partial x^i}(p) \lambda^i|_p = \lambda^j|_p. \quad (64)$$

Therefore, this proves that

$$df|_p = \frac{\partial f}{\partial x^i}(p) dx^i|_p \iff df = \frac{\partial f}{\partial x^i} dx^i. \quad (65)$$

- **Prop. 6.9 (Properties of Differentials)** Let M be a smooth manifold, and let $f, g \in C^\infty(M)$.
 - (a) For any constants a, b , $d(af + bg) = a df + b dg$.
 - (b) $d(fg) = f dg + g df$.
 - (c) $d(f/g) = (g df - f dg)/g^2$ on the set where $g \neq 0$.
 - (d) If $J \subset \mathbb{R}$ is an interval containing the image of f , and $h : J \rightarrow \mathbb{R}$ is a smooth function, then $d(h \circ f) = (h' \circ f) df$.
 - (e) If f is constant, then $df = 0$.

Exercise 6.7. Prove Proposition 6.9.

Let $a, b \in \mathbb{R}$, and $f, g \in C^\infty(M)$.

- (a) Let $U \subseteq M$ be open, (x^i) smooth coordinates on U , and (dx^i) the corresponding coordinate coframe. Then

$$\begin{aligned} d(af + bg) &= \frac{\partial(af + bg)}{\partial x^i} dx^i = \left[\frac{\partial(af)}{\partial x^i} + \frac{\partial(bg)}{\partial x^i} \right] dx^i \\ &= \left[a \frac{\partial f}{\partial x^i} + b \frac{\partial g}{\partial x^i} \right] dx^i = a df + b dg. \end{aligned} \quad (66)$$

- (b) We will work in coordinates as in (a). Then we observe that

$$\begin{aligned} d(fg) &= \frac{\partial(fg)}{\partial x^i} dx^i = \left[g \frac{\partial f}{\partial x^i} + f \frac{\partial g}{\partial x^i} \right] dx^i \\ &= g df + f dg. \end{aligned} \quad (67)$$

- (c) Let $E = \{x \in M : g(x) \neq 0\}$. Then let $U \subseteq E$ be an open subset, (x^i) be smooth coordinates, and (dx^i) the corresponding coframe. Then

$$\begin{aligned} d(f/g) &= \frac{\partial(f/g)}{\partial x^i} dx^i = \frac{g \frac{\partial f}{\partial x^i} - f \frac{\partial g}{\partial x^i}}{g^2} dx^i \\ &= \frac{g df - f dg}{g^2}. \end{aligned} \quad (68)$$

- (d) Let $J \subset \mathbb{R}$ be an interval containing the image of f , and $h : J \rightarrow \mathbb{R}$ be smooth. Let $U \subseteq M$ be open, (x^i) smooth coordinates on U , and (dx^i) the corresponding coordinate coframe. Then

$$d(h \circ f) = \frac{\partial(h \circ f)}{\partial x^i} dx^i = (h' \circ f) \cdot \frac{\partial f}{\partial x^i} dx^i = (h' \circ f) df, \quad (69)$$

where the second equality follows from the chain rule.

- (e) It suffices to show that if $f = 1$, then $df = 0$. Indeed,

$$df = d(ff) = f df + f df = 2f df = 2 df. \quad (70)$$

Hence, this proves that $df = 0$.

- **Prop. 6.10. (Functions with Vanishing Differentials)** If f is a smooth real-valued function on a smooth manifold M , then $df = 0$ if and only if f is constant on each component of M .

It suffices to assume that M is connected and to show that $df = 0$ if and only if f is constant. Indeed, assume f is constant. Then by Prop. 6.9(e), $df = 0$. Now suppose $df = 0$, $p \in M$, and let $\mathcal{C} = \{q \in M : f(p) = f(q)\}$. If q is any point in \mathcal{C} , then let U be a smooth coordinate ball centered at q . By virtue of the differential being zero, we must have $\partial f / \partial x^i = 0$ in U for each i . This implies that f is constant on U . Hence, \mathcal{C} is open. On the other hand, by continuity of f , \mathcal{C} is closed. Since the only open and closed sets in a connected set are the empty set and M , it follows that $\mathcal{C} = M$; i.e., f is constant on M .

5.5 Pullbacks

- **Def. (Pullback of a Smooth Map)** Let $F : M \rightarrow N$ be a smooth map, and $F_* : T_p M \rightarrow T_{F(p)} N$ its pushforward. Then the pushforward induces a dual linear map $F^* : T_{F(p)}^* N \rightarrow T_p^* M$ defined by

$$(F^* \omega)(X_p) = \omega(F_* X), \quad \text{for } \omega \in T_{F(p)}^* N, X \in T_p M. \quad (71)$$

- **Obs. (Pullback in Coordinates)** Let $p \in M$ be arbitrary, and choose smooth coordinates (x^i) for M near p and (y^j) for N near $G(p)$. Then

$$G^* \omega = G^*(\omega_j dy^j) = (\omega_j \circ G) dG^j. \quad (72)$$

- **Ex. 6.14. (Example of Pullback)** Let $G : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the map defined by

$$(u, v) = G(x, y, z) = (x^2 y, y \sin(z)), \quad (73)$$

and let $\omega \in \mathcal{T}^*(\mathbb{R}^2)$ be the covector field

$$\omega = u \, dv + v \, du. \quad (74)$$

First, we shall compute the differentials.

$$\begin{aligned} du &= \frac{\partial u}{\partial x^i} dx^i = 2xy \, dx + x^2 \, dy. \\ dv &= \frac{\partial v}{\partial x^i} dx^i = \sin(z) \, dy + y \cos(z) \, dz. \end{aligned} \quad (75)$$

Therefore,

$$\begin{aligned} G^* \omega &= x^2 y [\sin(z) \, dy + y \cos(z) \, dz] + y \sin(z) [2xy \, dx + x^2 \, dy] \\ &= 2xy^2 \sin(z) \, dx + 2xy^2 \sin(z) \, dy + x^2 y^2 \cos(z) \, dz. \end{aligned} \quad (76)$$

5.6 Line Integrals

- **Prop. 6.16 (Diffeomorphism Invariance of the Integral)** Let ω be a smooth covector field on the compact interval $[a, b] \subset \mathbb{R}$. If $\varphi : [c, d] \rightarrow [a, b]$ is an increasing diffeomorphism (meaning that $t_1 < t_2$ implies $\varphi(t_1) < \varphi(t_2)$), then

$$\int_{[c,d]} \varphi^* \omega = \int_{[a,b]} \omega. \quad (77)$$

Let s be the standard coordinates on $[c, d]$ and t be the standard coordinates on $[a, b]$. We may write $\omega_t = f(t) \, dt$ for some smooth function $f : [a, b] \rightarrow \mathbb{R}$. Then using the pullback expression in local coordinates,

$$(\varphi^* \omega)_s = f(\varphi(s)) \varphi'(s) \, ds. \quad (78)$$

Therefore,

$$\int_{[c,d]} \varphi^* \omega = \int_c^d f(\varphi(s)) \varphi'(s) \, ds \stackrel{t:=\varphi(s)}{=} \int_a^b f(t) \, dt = \int_{[a,b]} \omega. \quad (79)$$

Exercise 6.8. If $\varphi : [c, d] \rightarrow [a, b]$ is a decreasing diffeomorphism, show that $\int_{[c,d]} \varphi^* \omega = - \int_{[a,b]} \omega$.

The proof follows almost nearly identically to the proof from above. Suppose that s is the standard coordinate on $[c, d]$, and let t be the standard coordinate on $[a, b]$. We may assume that $\omega_t = f(t) dt$ for some smooth function $f : [a, b] \rightarrow \mathbb{R}$. Note that because of the decreasing property, $\varphi(c) = b$ and $\varphi(d) = a$. Hence,

$$\int_{[c,d]} \varphi^* \omega = \int_c^d f(\varphi(s)) \varphi'(s) ds = \int_b^a f(t) dt = - \int_{[a,b]} f(t) dt = - \int_{[a,b]} \omega. \quad (80)$$

- **Def. (Curve Segment)** Let M be a smooth manifold. A *curve segment* is a continuous curve $\gamma : [a, b] \rightarrow M$ whose domain is a compact interval. It is a *smooth curve segment* if it has a smooth extension to an open interval containing $[a, b]$. A *piecewise smooth curve segment* is a piecewise smooth curve segment.
- **Def. (Line Integral)** Let $\gamma : [a, b] \rightarrow M$ be a smooth curve segment and ω a smooth covector field on M . The *line integral* of ω over γ is defined to be the real number

$$\int_{\gamma} \omega = \int_{[a,b]} \gamma^* \omega. \quad (81)$$

If γ is *piecewise smooth*, then

$$\int_{\gamma} \omega = \sum_{i=1}^n \int_{[a_{i-1}, a_i]} \gamma^* \omega, \quad (82)$$

where $\{a_i\}_0^n$ is a partition of $[a, b]$.

- **Prop. 6.18. (Properties of Line Integrals)** Let M be a smooth manifold. Suppose $\gamma : [a, b] \rightarrow M$ is a piecewise smooth curve segment and $\omega, \omega_1, \omega_2 \in \mathcal{T}^*(M)$.

(a) For any $c_1, c_2 \in \mathbb{R}$,

$$\int_{\gamma} (c_1 \omega_1 + c_2 \omega_2) = c_1 \int_{\gamma} \omega_1 + c_2 \int_{\gamma} \omega_2. \quad (83)$$

(b) If γ is a constant map, then $\int_{\gamma} \omega = 0$.

(c) If $a < c < b$, then

$$\int_{\gamma} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega, \quad (84)$$

where $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$.

Exercise 6.9. Prove Proposition 6.18

Assume all of the hypotheses given in the statement of the proposition. Let $a = a_0 < a_1 < \dots < a_n = b$ be a partition of $[a, b]$ such that γ is smooth on each subinterval.

(a) By linearity of pullbacks,

$$\begin{aligned} \int_{\gamma} (c_1 \omega_1 + c_2 \omega_2) &= \sum_{i=1}^n \int_{[a_{i-1}, a_i]} \gamma^* (c_1 \omega_1 + c_2 \omega_2) = \sum_{i=1}^n \int_{[a,b]} [\gamma^* (c_1 \omega_1) + \gamma^* (c_2 \omega_2)] \\ &= \sum_{i=1}^n \left[\int_{[a_{i-1}, a_i]} \gamma^* (c_1 \omega_1) + \int_{[a_{i-1}, a_i]} \gamma^* (c_2 \omega_2) \right] \\ &= c_1 \int_{\gamma} \omega_1 + c_2 \int_{\gamma} \omega_2. \end{aligned} \quad (85)$$

(c) Let $a < c < b$; let $a = a'_0 < a'_1 < \dots < a'_n = c$ and $c = a'_{n+1} < \dots < a'_m = b$ be partitions of $[a, c]$ and $[c, b]$, respectively. Clearly $\{a'_i\}_{i=0}^m$ is also a partition of $[a, b]$. Then

$$\int_{\gamma} \omega = \sum_{i=1}^m \int_{[a'_{i-1}, a'_i]} \gamma^* \omega = \sum_{i=1}^n \int_{[a'_{i-1}, a'_i]} \gamma^* \omega + \sum_{i=n+1}^m \int_{[a'_{i-1}, a'_i]} \gamma^* \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega. \quad (86)$$

- (b) It suffices to assume that γ is a smooth curve segment. Let s be the standard coordinates on $[a, b]$. Then in local coordinates, $\gamma^*\omega = \omega(\gamma(s))\gamma'(s) ds = 0$ since $\gamma'(s) = 0$ for all s . Hence,

$$\int_{\gamma} \omega = \int_{[a,b]} \gamma^*\omega = \int_{[a,b]} 0 ds = 0. \quad (87)$$

5.7 Conservative Vector Fields

- **Def. (Exact Smooth Covector Field)** Let ω be a smooth covector field on a smooth manifold M . ω is *exact* if it is the differential of some $f \in C^\infty(M)$. The function f is called a *potential* for ω .
- **Def. (Conservative Covector Field)** A smooth covector field ω is *conservative* if the line integral of ω over *any* closed piecewise smooth curve segment is zero.
- **Lem. 6.23. (Conservative Covector Field Criterion I)** A smooth covector field ω is conservative if and only if the line integral of ω depends only on the endpoints of the curve, i.e., $\int_{\gamma} \omega = \int_{\tilde{\gamma}} \omega$ whenever γ and $\tilde{\gamma}$ are piecewise smooth curves are piecewise smooth curve segments with the same starting and ending points.

Exercise 6.10. Prove Lemma 6.23. [Observe that this would be much harder to prove if we defined conservative fields in terms of smooth curves instead of piecewise smooth ones.]

6 Submersions, Immersions, and Embeddings

6.1 Maps of Constant Rank

- **Def. (Rank of a Smooth Map)** Let M and N be smooth manifolds, and $F : M \rightarrow N$ a smooth map. The *rank* of F at $p \in M$ is the rank of the linear map $F_* : T_p M \rightarrow T_{F(p)} N$; this is equivalent to the rank of the matrix of partial derivatives of F in any smooth chart, or to the dimension of $\text{Im } F_* \subset T_{F(p)} N$. I.e., the rank is equivalent to the maximum number of linearly independent rows/columns of the corresponding matrix.
- **Def. (Submersion)** A smooth map $F : M \rightarrow N$ such that F_* is surjective at each point, which is to say that $\text{rank } F = \dim N$.
- **Def. (Immersion)** A smooth map $F : M \rightarrow N$ such that F_* is injective at each point; equivalently $\text{rank } F = \dim M$.
- **Def. (Smooth Embedding)** An immersion $F : M \rightarrow N$ such that $F : M \rightarrow F(M) \subset N$ is a homeomorphism.

Exercise 7.2. Show that a composition of submersions is a submersion, a composition of immersions is an immersion, and a composition of smooth embeddings is a smooth embedding.

- (i) Let $F : M \rightarrow N$ and $G : N \rightarrow P$ be submersions, where M is a smooth m -manifold, N is a smooth n -manifold, and P is a smooth p -manifold. Let U, V, T be smooth coordinate charts for M, N , and P , respectively, such that (wlog) $F(U) \subset V$ and $G(V) \subset T$. Then since $(G \circ F)_* = G_* \circ F_*$, in local coordinates, $(G \circ F)_*$ corresponds to the matrix product of an $p \times n$ matrix with an $n \times m$ matrix (the $p \times n$ matrix representing G_* , and the $n \times m$ matrix representing F_*); the rank of the $p \times n$ matrix is p , while the rank of the $n \times m$ matrix is n . Hence, by the properties of the rank of a matrix product, the matrix representation of $G_* \circ F_*$ has rank p , which proves that $(G \circ F)$ is a submersion.
- (ii) Let $F : M \rightarrow N$ and $G : N \rightarrow P$ be immersions. Since F_* is injective and G_* is injective, and the composition of injective functions is injective,

$$(G \circ F)_* = G_* \circ F_* \quad (88)$$

is injective. Hence, $G \circ F$ is an immersion.

- (iii) Let $F : M \rightarrow N$ and $G : N \rightarrow P$ be smooth embeddings. From (ii), $G \circ F$ is an immersion. Finally, since the composition of homeomorphisms is always another homeomorphism, we conclude that $G \circ F$ is a smooth embedding.

- **Prop. 7.4. (Smooth Embedding Criteria)** Suppose $F : M \rightarrow N$ is an injective immersion. If either of the following condition holds, then F is a smooth embedding with closed image:
 - (a) M is compact.
 - (b) F is a proper map.

7 Tensors

7.1 The Algebra of Tensors

- **Def. (Multilinear Function)** Suppose V_1, \dots, V_n and W are vector spaces. A map $F : V_1 \times \dots \times V_n \rightarrow W$ is said to be *multilinear* if it is linear as a function of each variable separately:

$$F(v_1, \dots, av_i + a'v'_i, \dots, v_k) = aF(v_1, \dots, v_i, \dots, v_k) + a'F(v_1, \dots, v'_i, \dots, v_k). \quad (89)$$

- **Def. (Covariant k -Tensor)** Let V be a finite-dimensional real vector space, and let k be a natural number. A *covariant k -tensor* on V is a real-valued multilinear function of k elements of V :

$$T : \underbrace{V \times \dots \times V}_{k \text{ copies}} \rightarrow \mathbb{R}. \quad (90)$$

The number k is called the *rank* of T .

- **Def. (Tensor Product)** We can build up covariant tensors of larger ranks as follows: let V be a finite-dimensional real vector space and let $S \in T^k(V)$, $T \in T^l(V)$. Define a map $S \otimes T : \underbrace{V \times \dots \times V}_{k+l \text{ copies}} \rightarrow \mathbb{R}$ by

$$S \otimes T(X_1, \dots, X_{k+l}) = S(X_1, \dots, X_k)T(X_{k+1}, \dots, X_{k+l}). \quad (91)$$

We will check multilinearity. WLOG, assume $i \leq k$. Then

$$\begin{aligned} S \otimes T(X_1, \dots, aX_i + a'X'_i, \dots, X_k, \dots, X_{k+l}) &= S(X_1, \dots, aX_i + a'X'_i, \dots, X_k)T(X_{k+1}, \dots, X_{k+l}) \\ &= [aS(X_1, \dots, X_i, \dots, X_k) + a'S(X_1, \dots, X'_i, \dots, X_k)]T(X_{k+1}, \dots, X_{k+l}) \\ &= a(S \otimes T)(X_1, \dots, X_i, \dots, X_{k+l}) + a'(S \otimes T)(X_1, \dots, X'_i, \dots, X_{k+l}). \end{aligned} \quad (92)$$

Hence, $S \otimes T$ is a covariant $(k+l)$ -tensor.

Exercise 11.1. Show that the tensor product operation is bilinear and associative. More precisely, show that $S \otimes T$ depends linearly on each of the tensors S and T , and that $(R \otimes S) \otimes T = R \otimes (S \otimes T)$.

Let P, R, S, T be k, k, l, l -tensors, respectively. Then

$$\begin{aligned} (a_1P + a_2R) \otimes (a_3S + a_4T)(X_1, \dots, X_{k+l}) &= (a_1P + a_2R)(X_1, \dots, X_k) \cdot (a_3S + a_4T)(X_{k+1}, \dots, X_{k+l}) \\ &= a_1P(X_1, \dots, X_k) \cdot (a_3S + a_4T)(X_{k+1}, \dots, X_{k+l}) \\ &\quad + a_2R(X_1, \dots, X_k)(a_3S + a_4T)(X_{k+1}, \dots, X_{k+l}) \\ &= a_1a_3P(X_1, \dots, X_k)S(X_{k+1}, \dots, X_{k+l}) \\ &\quad + a_1a_4P(X_1, \dots, X_k)T(X_{k+1}, \dots, X_{k+l}) \\ &\quad + a_2a_3P(X_1, \dots, X_k)S(X_{k+1}, \dots, X_{k+l}) \\ &\quad + a_2a_4P(X_1, \dots, X_k)T(X_{k+1}, \dots, X_{k+l}). \end{aligned} \quad (93)$$

Using the definition of the tensor products, we can simplify the final expressions to see that the tensor product is, indeed, linear in each of the tensor terms.

- **Prop. 11.2. (Basis for $T^k V$)** Let V be a real vector space of dimension n , let (E_i) be any basis for V , and let (ε^i) be the dual basis. The set of all k -tensors of the form $\varepsilon^{i_1} \otimes \dots \otimes \varepsilon^{i_k}$ for $1 \leq i_1 \leq \dots \leq i_k \leq n$ is a basis for $T^k V$, which therefore has dimension n^k .

Let \mathcal{B} denote the set $\{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$. It suffices to show that \mathcal{B} is linearly independent and spans $T^k V$. Let $T \in T^k(V)$. For any k -tuple of integers (i_1, \dots, i_k) , where $1 \leq i_j \leq n$ for all $j = 1, \dots, k$, define the number $T_{i_1 \dots i_k}$ as follows:

$$T_{i_1 \dots i_k} = T(E_{i_1}, \dots, E_{i_k}). \quad (94)$$

We will show that $T = T_{i_1 \dots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}$. Indeed,

$$\begin{aligned} T_{i_1 \dots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} (E_{j_1}, \dots, E_{j_k}) &= T_{i_1 \dots i_k} \varepsilon^{i_1}(E_{j_1}) \cdots \varepsilon^{i_k}(E_{j_k}) \\ &= T_{j_1 \dots j_k} \\ &= T(E_{j_1}, \dots, E_{j_k}). \end{aligned} \quad (95)$$

By multilinearity, since a tensor is completely determined by its action on sequences of basis vectors, this proves the claim that \mathcal{B} spans $T^k V$. Now, we must prove linear independence. But this is straightforward to show by letting $T_{i_1 \dots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} = 0$ act on a sequence of basis vectors.

- **Def. (Free Vector Space)** Let S be a set. The *free vector space on S* , denoted by $\mathbb{R}\langle S \rangle$, is the set of all finite formal linear combinations of S with real coefficients. More precisely, a finite formal linear combination is a function $\mathcal{F} : S \rightarrow \mathbb{R}$ such that $\mathcal{F}(s) = 0$ for all but finitely many $s \in S$.

Exercise 11.2 (Characteristic Property of Free Vector Spaces). Let S be a set and W a vector space. Show that any map $F : S \rightarrow W$ has a unique extension to a linear map $\bar{F} : \mathbb{R}\langle S \rangle \rightarrow W$.

Let S be a set, W a vector space, and $F : S \rightarrow W$ an arbitrary map. Define the map $\bar{F} : \mathbb{R}\langle S \rangle \rightarrow W$ as follows: given a formal sum $\sum_{s \in S} \alpha_s s$, where $\alpha_s = 0$ for all but finitely many elements $s \in S$, let

$$\bar{F} \left(\sum_{s \in S} \alpha_s s \right) = \sum_{s \in S} \alpha_s F(s). \quad (96)$$

Since each $\alpha_s \in \mathbb{R}$ and $F(s) \in W$, it follows that $\sum_{s \in S} \alpha_s F(s) \in W$. First, we must show that \bar{F} is a linear map. Let $\sum_{s \in S} \alpha_s s, \sum_{s \in S} \beta_s s \in \mathbb{R}\langle S \rangle$, where α_s, β_s are zero for all but finitely elements (not necessarily the same) of S . Then

$$\begin{aligned} \bar{F} \left(\sum_{s \in S} \alpha_s s + \sum_{s \in S} \beta_s s \right) &= \bar{F} \left(\sum_{s \in S} (\alpha_s + \beta_s) s \right) \\ &= \sum_{s \in S} (\alpha_s + \beta_s) F(s) \\ &= \sum_{s \in S} \alpha_s F(s) + \sum_{s \in S} \beta_s F(s) \\ &= \bar{F} \left(\sum_{s \in S} \alpha_s s \right) + \bar{F} \left(\sum_{s \in S} \beta_s s \right). \end{aligned} \quad (97)$$

Hence, \bar{F} is linear. The proof of uniqueness follows as proceeds: if F extends to two linear maps \bar{F}_1 and \bar{F}_2 , let these linear maps act on each element of S . By construction of these maps, it follows that $\bar{F}_1(s) = \bar{F}_2(s)$ for all $s \in S$. Since S is a basis for $\mathbb{R}\langle S \rangle$ and $\bar{F}_{1,2}$ are completely determined by their actions on the basis elements, we conclude that $\bar{F}_1 = \bar{F}_2$, and so this extension is unique.

- **Def. (Tensor Product of Vector Spaces)** Let V and W be finite-dimensional real vector spaces, and let \mathcal{R} be the subspace of the free vector space $\mathbb{R}\langle V \times W \rangle$ spanned by all elements of the

following forms:

$$\begin{aligned}
 & \alpha(v, w) - (\alpha v, w), \\
 & \alpha(v, w) - (v, \alpha w), \\
 & (v, w) + (v', w) - (v + v', w) \\
 & (v, w) + (v, w') - (v, w + w'),
 \end{aligned} \tag{98}$$

for $\alpha \in \mathbb{R}$, $v, v' \in V$, and $w, w' \in W$. Define the *tensor product of V and W* , denoted $V \otimes W$ to be the quotient space $\mathbb{R}\langle V \times W \rangle / \mathcal{R}$. The equivalence class of an element $(v, w) \in V \otimes W$ is denoted by $v \otimes w$, and is called the *tensor product of v and w* .

- **Prop. 11.3. (Characteristic Property of Tensor Products)** Let V and W be finite dimensional real vector spaces. IF $A : V \times W \rightarrow X$ is a bilinear map into any vector space X , there is a unique linear map $\tilde{A} : V \otimes W \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc}
 V \times W & \xrightarrow{A} & X \\
 \downarrow \pi & \nearrow \tilde{A} & \\
 V \otimes W & &
 \end{array}$$

where $\pi(v, w) = v \otimes w$.

- **Prop. 11.4. (Other Properties of Tensor Products)** Let V, W , and X be finite-dimensional real vector spaces.
 - (a) The tensor product $V^* \otimes W^*$ is canonically isomorphic to the space $B(V, W)$ of bilinear maps from $V \times W$ into \mathbb{R} .
 - (b) If (E_i) is a basis for V and (F_j) is a basis for W , then the set of all elements of the form $E_i \otimes F_j$ is a basis for $V \otimes W$, which therefore has dimension $(\dim V)(\dim W)$.
 - (c) There is a unique isomorphism $V \otimes (W \otimes X) \rightarrow (V \otimes W) \otimes X$ sending $v \otimes (w \otimes x)$ to $(v \otimes w) \otimes x$.
- **Cor. 11.5. (Space of Covariant k -Tensors and Tensor Products)** If V is a finite-dimensional real vector space, the space $T^k(V)$ of covariant k -tensors on V is canonically isomorphic to the k -fold tensor product $V^* \otimes \cdots \otimes V^*$.
- **Def. (Space of Contravariant k -Tensors)** Let V be a finite-dimensional real vector space, and define the space of all *contravariant k -tensors* on V to be the space

$$T_k(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}}. \tag{99}$$

Because of the canonical identification $V = V^{**}$, one may think of an element of $T_k V$ as a multilinear function from $V^* \times \cdots \times V^*$ into \mathbb{R} .

7.2 Tensors and Tensor Fields on Manifolds

- **Def. (Various Tensor Bundles)** Let M be a smooth manifold. Define the following:
 - (a) *Bundle of covariant k -tensors on M :*

$$T^k M = \coprod_{p \in M} T^k(T_p M). \tag{100}$$

- (b) *Bundle of contravariant l -tensors on M :*

$$T_l M = \coprod_{p \in M} T_l(T_p M). \tag{101}$$

(c) *Bundle of mixed tensors of type $\binom{k}{l}$ on M :*

$$T_l^k M = \coprod_{p \in M} T_l^k(T_p M). \quad (102)$$

- **Def. (Smooth Tensor Fields)** A *smooth tensor field* is a smooth section of the above tensor bundles.
- **Obs. (Smooth Tensor Fields in Coordinates)** Given any smooth local coordinates (x^i) on M , sections of the above bundles can be written as:

$$\sigma = \begin{cases} \sigma_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}, & \sigma \in \mathcal{T}^k(M); \\ \sigma^{j_1 \dots j_l} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}}, & \sigma \in \mathcal{T}_l(M). \\ \sigma_{i_1 \dots i_k}^{j_1 \dots j_l} dx^{i_1} \otimes \dots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}}, & \sigma \in \mathcal{T}_l^k(M). \end{cases} \quad (103)$$

- **Lem. 11.6. (Equivalent Conditions for Smooth Tensor Fields)** Let M be a smooth manifold, and let $\sigma : M \rightarrow T^k M$ be a rough section. The following are equivalent:
 - (a) σ is smooth.
 - (b) In any smooth coordinate chart, the composition functions of σ are smooth.
 - (c) If X_1, \dots, X_k are smooth vector fields defined on an open subset $U \subset M$, then the function $\sigma(X_1, \dots, X_k) : U \rightarrow \mathbb{R}$, defined by

$$\sigma(X_1, \dots, X_k)(p) = \sigma_p(X_1|_p, \dots, X_k|_p) \quad (104)$$

is smooth.

7.3 Pullbacks of Smooth Tensor Fields

- **Def. (Pullback of a Smooth Map in Relation to Tensor Fields)** If $F : M \rightarrow N$ is a smooth map, for each integer $k \geq 0$ and each $p \in M$, we obtain a map $F_* : T^k(T_{F(p)}N) \rightarrow T^k(T_p M)$ called the pullback by

$$(F^*S)(X_1, \dots, X_k) = S(F_*X_1, \dots, F_*X_k). \quad (105)$$

- **Prop. 11.8. (Properties of Tensor Pullbacks)** Suppose $F : M \rightarrow N$ and $G : N \rightarrow P$ are smooth maps, $p \in M$, $S \in T^k(T_{F(p)}N)$, and $T \in T^l(T_{G(F(p))}P)$.
 - (a)

8 Homotopy and the Fundamental Group

8.1 Homotopy

- **Def. (Homotopy of Maps)** Let X and Y be *topological spaces*, and $f, g \in C(X, Y)$. Then a *homotopy* from f to g is a continuous map $H : X \times I \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.
- **Def. (Homotopy Relative to a Subset)** Let X and Y be topological spaces, and $A \subset X$ an arbitrary subspace. A homotopy H between maps $f, g : X \rightarrow Y$ is called a *homotopy relative to A* if

$$H(x, t) = f(x), \quad \text{for all } x \in A, t \in I. \quad (106)$$
- **Def. (Path Homotopy)** Given two paths f, g on X , a *path homotopy* from f to g is a homotopy between the paths relative to the subset $\{0, 1\} \subset I$.
- **Def. (Fundamental Group)** The *fundamental group* of X based at $q \in X$, denoted by $\pi_1(X, q)$ is the set of all path classes of loops based at q , with operation defined by concatenation.
- **Def. (Simply Connected Topological Space)** Let X be a topological space. If X is path connected and $\pi_1(X)$ is trivial, then X is said to be *simply connected*.

Exercise 7.2. Let X be a topological space.

- Let $f, g : I \rightarrow X$ be two paths from p to q . Show that $f \sim g$ if and only if $f \cdot g^{-1} \sim c_p$.
- Show that X is simply connected if and only if any two paths in X with the same initial and terminal points are path homotopic.

Let X be a topological space.

- Let $f, g : I \rightarrow X$ be two paths from p to q . Suppose $f \sim g$. Then since $[f] = [g]$,

$$[g] \cdot [g^{-1}] = [c_p] \implies [f] \cdot [g^{-1}] = [c_p] \implies f \cdot g^{-1} \sim c_p. \quad (107)$$

On the other hand, if $f \cdot g^{-1} \sim c_p$, then

$$[f] \cdot [g^{-1}] = [c_p] = [g] \cdot [g^{-1}] \implies [f] = [g] \implies f \sim g. \quad (108)$$

- Suppose that X is a simply connected space, and let $f, g : I \rightarrow X$ be two paths from p to q . Then the product $f \cdot g^{-1}$ is well-defined and is a loop based at p . By simple connectedness, $f \cdot g^{-1} \sim c_p$. Hence, by part (a), we conclude that $f \sim g$. Now suppose that X is path connected and that any two paths in X that have the same initial and terminal points are path homotopic. Let γ be an arbitrary loop based at $p \in X$. Then by hypothesis, $\gamma \sim c_p$. Hence, $\pi_1(X, p)$ is trivial. By path connectedness, $\pi_1(X)$ is trivial, and so X is simply connected.

8.2 Homomorphisms Induced by Continuous Maps

- **Lem. 7.14. (Path Homotopy is Preserved by Composition with Continuous Maps)** The path homotopy relation is preserved by composition with continuous maps. That is, if $f_0, f_1 : I \rightarrow X$ are path homotopic and $\varphi : X \rightarrow Y$ is continuous, then $\varphi \circ f_0$ and $\varphi \circ f_1$ are path homotopic.

Exercise 7.5. Prove Lemma 7.14.

Suppose that $f_0, f_1 : I \rightarrow X$ are path homotopic, and that $\varphi : X \rightarrow Y$ is continuous. Let $H : I \times I \rightarrow X$ be the path homotopy from f_0 to f_1 , and consider the map $\varphi \circ H : I \times I \rightarrow Y$. Since H and φ are continuous on their respective domains, it follows that $\varphi \circ H$ is continuous. Moreover, for any $s \in I$,

$$(\varphi \circ H)(s, 0) = (\varphi \circ f_0)(s), \quad \text{and} \quad (\varphi \circ H)(s, 1) = (\varphi \circ f_1)(s). \quad (109)$$

Hence, $\varphi \circ H$ is a path homotopy from $\varphi \circ f_0$ to $\varphi \circ f_1$.

- **Def. (Homomorphism Induced by a Continuous Map)** Let X and Y be topological spaces, and $\varphi : X \rightarrow Y$ a continuous map. The map $\varphi_* : \pi_1(X, q) \rightarrow \pi_1(X, \varphi(q))$ defined by $\varphi_*([f]) = [\varphi \circ f]$ is a group homomorphism, and is called the *homomorphism induced by φ* .

- **Prop. 7.16 (Properties of the Induced Homomorphism)**

- Let $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ be continuous maps. Then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.
- If $\text{Id}_X : X \rightarrow X$ denotes the identity map of X , then for any $q \in X$, $(\text{Id}_X)_*$ is the identity map of $\pi_1(X, q)$.

- Let $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ be continuous maps, $p \in X$, and $[f] \in \pi_1(X, p)$. Then

$$(\psi_* \circ \varphi_*)([f]) = \psi_*([\varphi \circ f]) = [(\psi \circ \varphi) \circ f] = (\psi \circ \varphi)_*([f]). \quad (110)$$

Since this is true for all $[f] \in \pi_1(X, p)$, $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

- Let $[f] \in \pi_1(X, q)$. Then

$$(\text{Id}_X)_*([f]) = [\text{Id}_X \circ f] = [f]. \quad (111)$$

Since this is true for all $[f] \in \pi_1(X, q)$, we conclude that $(\text{Id}_X)_*$ is the identity map of $\pi_1(X, q)$.

- **Cor. 7.17 (Induced Isomorphism)** Homeomorphic spaces have isomorphic fundamental groups; namely, if $\varphi : X \rightarrow Y$ is a homeomorphism, then $\varphi_* : \pi_1(X, q) \rightarrow \pi_1(X, \varphi(q))$ is an isomorphism.

- **Def. (Retraction of a Space)** Let X be a topological space, and A a subspace of X . A continuous map $r : X \rightarrow A$ is called a *retraction* if $r|_A = \text{Id}_A$. Equivalently, r is a retraction if $r \circ \iota_A = \text{Id}_A$, where $\iota_A : A \hookrightarrow X$ is the inclusion map. If there exists a retraction from X to A , then we say that A is a *retract* of X .

- **Prop. 7.18. (Injective Induced Homomorphism)** Suppose A is a retract of X . If $r : X \rightarrow A$ is any retraction, then for any $q \in A$, $(\iota_A)_* : \pi_1(A, q) \rightarrow \pi_1(X, q)$ is injective and $r_* : \pi_1(X, q) \rightarrow \pi_1(A, q)$ is surjective.

Since $r \circ \iota_A = \text{Id}_A$, $r_* \circ (\iota_A)_*$ is the identity on $\pi_1(A, q)$, from which it follows that $(\iota_A)_*$ is injective and r_* is surjective.

8.3 Homotopy Equivalence

- **Def. (Homotopy Equivalences)** Let $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow X$.

- ψ is a *homotopy inverse* for φ if $\psi \circ \varphi \simeq \text{Id}_X$ and $\varphi \circ \psi \simeq \text{Id}_Y$.

- If φ has a homotopy inverse ψ , then φ is called a *homotopy equivalence*, and we say that X is *homotopically equivalent* to Y , or that X has the same *homotopy type* as Y . We denote $X \simeq Y$.

- **Def. (Deformation Retract)** A subspace $A \subset X$ is said to be a *deformation retract* if there exists a retraction $r : X \rightarrow A$ such that the identity of X is homotopic to $\iota_A \circ r$; the homotopy $H : \text{Id}_X \simeq \iota_A \circ r$ is called a *deformation retraction*. Intuitively, this means that points in A end up at the same position they started at. A deformation retraction is *strong* iff $\text{Id}_X \simeq_A (r \circ \iota_A)$, which is to say that the points of A remain *fixed* throughout the retraction.¹

- **Def. (Contractible Space)** Let X be any topological space. X is said to be *contractible* iff the identity map of X is homotopic to a constant map (i.e., if Id_X is nullhomotopic).

¹See https://encycla.com/Deformation_retraction for a gif of a (strong) deformation retraction.

9 Problems

9.1 Smooth Maps

Problem 2-5. Let M be a nonempty smooth manifold of dimension $n \geq 1$. Show that $C^\infty(M)$ is infinite dimensional.

Let M be a nonempty smooth manifold of dimension $n \geq 1$. Let (U, φ) be a smooth chart for M , and let x_1, \dots, x_k be k distinct points contained in U . For each $j = 1, \dots, k$, define the smooth function real-valued function f_j with compact support inside $\varphi(U)$ as follows: $f_j(x_m) = \delta_{mj}$. Then for each j , define the function $g_j : M \rightarrow \mathbb{R}$ as follows:

$$g_j(x) = \begin{cases} f_j(\varphi(x)), & x \in U. \\ 0, & x \in M \setminus U. \end{cases} \quad (112)$$

Then since U is open, it follows that g_j is smooth for each j . Hence, we have obtained a linearly independent subset of $C^\infty(M)$ consisting of k vectors. Since k was arbitrary, we conclude that $C^\infty(M)$ is infinite dimensional.

Problem 2-6. For any topological space M , let $C(M)$ denote the algebra of continuous functions $f : M \rightarrow \mathbb{R}$. If $F : M \rightarrow N$ is a continuous map, define $F^* : C(N) \rightarrow C(M)$ by $F^* = f \circ F$,

- (a) Show that F^* is a linear map.
- (b) If M and N are smooth manifolds, show that F is smooth if and only if $F^*(C^\infty(N)) \subset C^\infty(M)$.
- (c) If $F : M \rightarrow N$ is a homeomorphism between smooth manifolds, show that it is a diffeomorphism if and only if F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

- (a) Let $a, b \in \mathbb{R}$ and $f, g \in C(N)$. Then

$$F^*(af + bg) = (af + bg) \circ F = a(f \circ F) + b(g \circ F) = aF^*(f) + bF^*(g). \quad (113)$$

- (b) Let M and N be smooth manifolds. Assume that F is smooth, and let $f \in C^\infty(N)$. Then

$$F^*(f) = (f \circ F) : M \rightarrow \mathbb{R} \quad (114)$$

is smooth since it is the composition of smooth functions. Hence, $F^*(C^\infty(N)) \subset C^\infty(M)$. Now we need to show the converse. Suppose $F^*(C^\infty(N)) \subset C^\infty(M)$. Let (U, φ) and (V, ψ) be smooth charts for M and N , respectively such that $F(U) \subset V$. Let $\psi = (\psi^i)$, where each coordinate function $\psi^i : V \rightarrow \mathbb{R}$ is smooth. Note we can extend ψ^i to a smooth function on N by means of a smooth bump function. By our hypothesis, $F^*(\psi^i) = \psi^i \circ F$ is smooth. Then since $\varphi^{-1} : \varphi(U) \rightarrow M$ is smooth,

$$\psi^i \circ F \circ \varphi^{-1} \quad (115)$$

is smooth for each i , which means that $\psi \circ F \circ \varphi^{-1}$ is smooth. Hence, we conclude that F is smooth on M .

- (c) Let $F : M \rightarrow N$ be a homeomorphism between smooth manifolds. Suppose F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$. Then since $F^*(C^\infty(N)) \subset C^\infty(M)$, F is a smooth map. Let G be the inverse function of F . Then G^* is the inverse function of F^* , and so G^* restricts to an isomorphism from $C^\infty(M)$ to $C^\infty(N)$. In particular, this implies that G is a smooth map. Hence, F is a diffeomorphism. Now assume that F is a diffeomorphism. Let G be its inverse map. Since $G : N \rightarrow M$ is smooth, $G^*(C^\infty(M)) \subset C^\infty(N)$. Let $g \in C^\infty(M)$ and let $C^\infty(N) \ni f = G^*(g)$. Then

$$F^*(f) = f \circ F = g \circ G \circ F = g, \quad (116)$$

so that F^* is surjective. Now we show injectivity of F^* . Suppose $F^*(f) = F^*(g) \iff f \circ F = g \circ F$. Then $(f \circ F) \circ G = (g \circ F) \circ G \iff f = g$. Hence, F^* is injective. Using (a), we conclude that F^* restricts to an isomorphism from $C^\infty(N)$ to $C^\infty(M)$.

9.2 Tangent Vectors

Problem 3-1. Suppose M and N are smooth manifolds with M connected, and $F : M \rightarrow N$ is a smooth map such that $F_* : T_p M \rightarrow T_{F(p)} N$ is the zero map for each $p \in M$. Show that F is a constant map.

Let M and N be smooth manifolds with M connected, and $F : M \rightarrow N$ be a smooth map such that F_* is the zero map for each $p \in M$. Let $p \in M$, and define the subset

$$\mathcal{C} = \{q \in M : F(q) = F(p)\}. \quad (117)$$

Clearly this subset is nonempty since it at least contains $p \in M$. If $q \in \mathcal{C}$, let U be a smooth coordinate chart containing q . By hypothesis, for all $r \in U$, F_* is the zero map; in local coordinates, this is possible iff all of the partial derivatives of the coordinate representation of F is zero at each $r \in U$. But this means that F is constant on U . Hence, $U \subset \mathcal{C}$, which means that \mathcal{C} is an open subset of M . By continuity of F , \mathcal{C} is also a closed subset of M . Since M is connected and \mathcal{C} is nonempty, it then follows that $\mathcal{C} = M$. Hence, F is a constant map.

Problem 3-3. If a nonempty smooth n -manifold is diffeomorphic to an m -manifold, show that $n = m$.

Let M be a nonempty m -manifold and N a nonempty n -manifold; let $F : M \rightarrow N$ be a diffeomorphism. Then since F is a local diffeomorphism, for each $p \in M$, $F_* : T_p M \rightarrow T_{F(p)} N$ is an isomorphism. Since $\dim T_p M = m$ and $\dim T_{F(p)} N = n$ for every $p \in M$, it then follows that $m = n$.

Problem 3-4. Let $C \subset \mathbb{R}^2$ be the unit circle, and let $S \subset \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin:

$$S = \{(x, y) : \max(|x|, |y|) = 1\}.$$

Show that there is a homeomorphism $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F(C) = S$, but there is no *diffeomorphism* with the same property. [Hint: Consider what F does to the tangent vector to a suitable curve in C .]

Let $C \subset \mathbb{R}^2$ be the unit circle and $S \subset \mathbb{R}^2$ the boundary of the square of side 2 centered at the origin. Consider the map $G : S \rightarrow C$ defined as follows

$$G(x, y) = \frac{(x, y)}{\sqrt{x^2 + y^2}} \in S. \quad (118)$$

First, we show that G is injective. Suppose $G(x_1, y_1) = G(x_2, y_2)$. Since $\sqrt{x_1^2 + y_1^2}, \sqrt{x_2^2 + y_2^2}$ are nonzero, multiplying both sides by $\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$, we get $(x_1, y_1) = (x_2, y_2)$. Hence, G is injective. Now we show that G is surjective. Let $(\tilde{x}, \tilde{y}) \in C$. We give a rough sketch for surjectivity, but the idea is clear. Consider the ray connecting the origin $(0, 0)$ to the point (\tilde{x}, \tilde{y}) . Extend this ray indefinitely. Then this ray must intersect S at some point (x_0, y_0) . Since G radially projects all of the points in S inwards onto C , it follows that $G(x_0, y_0) = (\tilde{x}, \tilde{y})$. Hence, G is bijective. By calculus, G is continuous. Since continuous bijections from compact spaces onto Hausdorff spaces is a homeomorphism, G is a homeomorphism. Notably, its inverse F must also be a homeomorphism, proving the claim. However, there can be no diffeomorphism between C and S . Suppose F was such a diffeomorphism, and let a be one of the corners of the square, and $p = F^{-1}(a)$. Since F is a diffeomorphism, $T_p C \cong T_a S$ under the isomorphism F_* . As we showed before $T_p C$ is 1-dimensional. On the other hand, $T_a S$ is not well-defined. But this is a contradiction. Therefore, C and S are not diffeomorphic.

9.3 The Cotangent Bundle

Problem 6-1.

- (a) If V and W are finite-dimensional vector spaces and $A : V \rightarrow W$ is any linear map, show that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{A} & W \\ \downarrow \xi_V & & \downarrow \xi_W \\ V^{**} & \xrightarrow{(A^*)^*} & W^{**}, \end{array}$$

where ξ_V and ξ_W denote the isomorphisms defined by (6.3) for V and W , respectively.

- (a) Assume all of the given hypotheses. Let $X \in V$ and let $\omega \in W^*$. Then

$$\xi_W(AX)(\omega) = \omega(AX). \quad (119)$$

On the other hand, since $A^*\omega \in V^*$, $\xi_V(X)(A^*\omega) = A^*\omega(X)$.

Problem 6-2.

- (a) If $F : M \rightarrow N$ is a smooth map, show that $F^* : T^*N \rightarrow T^*M$ is a smooth bundle map.
 (b) Show that the assignment $M \mapsto T^*M$, $F \mapsto F^*$ defines a contravariant functor from the category of smooth manifolds to the category of smooth vector bundles.

9.4 Comps

Problem 2017-J-II-1. Let $f : M \rightarrow \mathbb{R}$ be a smooth function on a smooth manifold M . In an arbitrary smooth local coordinate chart $x : U \rightarrow \mathbb{R}^n$ of M , define

$$\mathcal{D}f := \sum_{i=1}^n \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}. \quad (120)$$

Does $\mathcal{D}f$ give a well-defined vector field on M ?

Let $f : M \rightarrow \mathbb{R}$ be a smooth function on a smooth manifold M , and define $\mathcal{D}f$ as prescribed above. For any given smooth local coordinate chart $(U, (x^i))$, by smoothness of f , all of the partial derivatives $\partial f / \partial x^i$ are smooth so that the component functions of $\mathcal{D}f$ are all smooth; hence $\mathcal{D}f$ is smooth in each smooth local coordinate chart. However, we need to check if $\mathcal{D}f$ transforms like a vector field. Suppose $p \in (U, (x^i)) \cap (V, (\tilde{x}^i))$. Then

$$\begin{aligned} \mathcal{D}f &= \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i} \Big|_p \\ &= \frac{\partial f}{\partial \tilde{x}^j}(p) \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) \cdot \frac{\partial \tilde{x}^k}{\partial x^i}(\hat{p}) \frac{\partial}{\partial \tilde{x}^k} \Big|_p \\ &\neq \frac{\partial f}{\partial \tilde{x}^k}(p) \frac{\partial}{\partial \tilde{x}^k} \Big|_p, \end{aligned} \quad (121)$$

where \hat{p} is the coordinate representation of p in the (x^i) coordinates, and we used the contravariant vector transformation law in the second line. Therefore, since $\mathcal{D}f$ does not transform as a vector field on M , it cannot be a well-defined vector field on M .

Problem 2023-J-II-4. Prove that $S^2 \times S^2$ is not diffeomorphic to $M_1 \times M_2 \times M_3$, where M_1, M_2, M_3 are smooth manifolds of nonzero dimension.

Assume to the contrary that $S^2 \times S^2$ is diffeomorphic to $M_1 \times M_2 \times M_3$; since diffeomorphisms preserve dimensions, $\dim(S^2 \times S^2) = \dim S^2 + \dim S^2 = 4$, and $\dim M_{1,2,3} \neq 0$, without loss of generality, we must have $\dim M_1 = \dim M_2 = 1$ and $\dim M_3 = 2$. Additionally, since S^2 is compact and connected, $\times_{j=1}^3 M_j$ must be compact and connected, which then implies that each M_j must be compact and connected. Moreover, since diffeomorphisms induce isomorphisms between fundamental groups, we must have

$$\pi_1(S^2 \times S^2) \cong \pi_1\left(\bigtimes_{j=1}^3 M_j\right). \quad (122)$$

On the left side, since S^2 is simply connected, $\pi_1(S^2 \times S^2)$ is trivial. On the right side, since the only compact, connected, smooth 1-manifold, up to diffeomorphism, is S^1 , and $\pi_1(S^1) \cong \mathbb{Z}$,

$$\pi_1\left(\bigcup_{j=1}^3 M_j\right) = \mathbb{Z} \times \mathbb{Z} \times \pi_1(M_3), \quad (123)$$

which is clearly not isomorphic to the trivial group. But this is a contradiction. Hence, by contradiction, $S^2 \times S^2$ cannot be diffeomorphic to $M_1 \times M_2 \times M_3$.

Problem 2021-A-II-3. Let X be the space obtained by gluing the boundary circle of a Möbius band to a great circle in S^2 . Compute the fundamental group of X .

Let X be the space obtained by gluing the boundary circle of a Möbius band to a great circle in S^2 .