

# Comps Practice

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## **Some Helpful Resources**

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- (2) [https://encyclopediaofmath.org/wiki/Main\\_Page](https://encyclopediaofmath.org/wiki/Main_Page)
- (3) <https://www.math3ma.com/>
- (4) <https://mathworld.wolfram.com/>
- (5) <https://math.stackexchange.com/>
- (6) *Folland, Real Analysis*
- (7) *Dummit & Foote, Abstract Algebra*
- (8) Comps solutions for 2014-2018 by Mads Bach Villadsen

## Comps Lemma

**Problem Comps Lemma.** Let  $M, N$  be smooth, connected,  $n$ -manifolds, and  $f : M \rightarrow N$  a (smooth) immersion. If  $M$  is compact and nonempty, then  $N$  is compact and  $f$  is a (smooth) covering map.

Let  $M, N$  be smooth, connected  $n$ -manifolds and  $f : M \rightarrow N$  an immersion. Assume that  $M$  is compact and nonempty. Since  $\dim N = n$  and  $f$  is an immersion,  $\text{rank } df_p = n$  at every  $p \in M$ . Hence, by the Inverse Function Theorem,  $f$  is a local diffeomorphism. Since local diffeomorphisms are open maps,  $f(M)$  is open in  $N$ . On the other hand, since the continuous image of compact sets is compact,  $f(M)$  is compact in  $N$ . Since  $N$  is Hausdorff,  $f(M)$  is closed in  $N$ . Since  $N$  is connected,  $f(M) = N$ . Therefore,  $N$  is compact.

Now, let  $q \in N$ , and consider  $f^{-1}(q) \subset M$ . For each  $x \in f^{-1}(q)$ , let  $U_x$  be an open neighborhood of  $M$  containing  $x$ . Since  $M$  is Hausdorff, we can shrink each  $U_x$  so that these neighborhoods are pairwise disjoint. This means that each  $x \in f^{-1}(q)$  is isolated, and hence  $f^{-1}(q)$  is discrete. Since  $M$  is compact, we conclude that  $f^{-1}(q)$  must be finite; let  $f^{-1}(q) = \{x_1, \dots, x_s\}$ . As noted above, for each  $j = 1, \dots, s$ , let  $U_j$  be a neighborhood of  $x_j$  such that  $f|_{U_j} : U_j \rightarrow V_j \subset N$  is a diffeomorphism. Then by the Hausdorff condition on  $M$ , shrink each  $U_j$  so that  $U_i \cap U_j = \emptyset$  for all  $i \neq j$ ;  $f$  remains a diffeomorphism on these shrunken neighborhoods. Setting  $V = \bigcap_1^s f(U_j)$  and taking  $\tilde{U}_j = f^{-1}(V) \cap U_j$  gives us an evenly covered neighborhood of  $q$  in  $N$ .

**Problem (Comps Lemma - Local Homeomorphisms).** Let  $M, N$  be smooth, connected  $n$ -manifolds and  $f : M \rightarrow N$  a local homeomorphism. If  $M$  is compact and nonempty, then  $N$  is compact and  $f$  is a covering map.

**Problem (Comps Lemma - Submersions).** Let  $M, N$  be smooth, connected  $n$ -manifolds and  $F : M \rightarrow N$  a submersion. If  $M$  is compact and nonempty, then  $N$  is compact and  $F$  is a covering map.

Let  $M, N$  be smooth, connected  $n$ -manifolds and  $F : M \rightarrow N$  a submersion. Also assume  $M$  is compact and nonempty. Since submersions are open maps,  $f(M)$  is open in  $N$ . On the other hand, since  $F$  is continuous, continuous images of compact sets are compact, and compact subsets of Hausdorff spaces are closed,  $F(M)$  is closed in  $N$ . Hence, since  $N$  is connected and  $F(M)$  is nonempty,  $F(M) = N$ . This proves that  $N$  is compact. We also claim that  $F$  is a local diffeomorphism. Since  $F$  is a submersion, at every  $p \in M$ ,  $dF_p : T_p M \rightarrow T_{f(p)} N$  is surjective. Since  $\dim M = \dim N = n$ , it follows that  $dF_p$  is bijective. Hence, by the Inverse Function Theorem,  $F$  is a local diffeomorphism.

All that remains to be seen is that  $F$  is a covering map. Let  $q \in N$  and consider the closed subset  $F^{-1}(q) \subset M$ . Since  $F$  is a local diffeomorphism, for each  $x \in F^{-1}(q)$ , there exists a neighborhood  $U_x$  such that  $F|_{U_x}$  is a local diffeomorphism. Since  $M$  is Hausdorff, we may shrink these neighborhoods so that they are pairwise disjoint. This means that each  $x \in F^{-1}(q)$  is isolated, and hence,  $f^{-1}(q)$  is discrete. Since  $M$  is compact,  $f^{-1}(q)$  is finite; let  $f^{-1}(q) = \{x_1, \dots, x_s\}$ . For each  $j = 1, \dots, s$ , let  $U_j$  be a neighborhood of  $x_j$  such that  $F|_{U_j}$  is a diffeomorphism. Since  $M$  is Hausdorff, we shrink these neighborhoods such that they are pairwise disjoint;  $F$  remains a diffeomorphism on each shrunken  $U_j$ . Set  $V = \bigcap_1^s f(U_j)$ , and let  $\tilde{U}_j = f^{-1}(V) \cap U_j$ . Hence,  $V$  is an evenly covered neighborhood of  $q \in N$ , which concludes the proof that  $F$  is a covering map.

## Steinhaus Theorem

**Problem (Steinhaus Theorem).** Let  $E$  be a Lebesgue measurable subset of  $\mathbb{R}^n$  such that  $m^n(E) > 0$ , and let  $v_1, \dots, v_N$  be a finite collection of vectors in  $\mathbb{R}^n$ . Then there exists  $R > 0$ , depending on  $E$ , and  $M = \max\{|v_1|, \dots, |v_N|\}$  such that for all  $0 < r < R$ , there exists  $p \in S$  so that the  $(N + 1)$ -points,  $p, p + rv_1, \dots, p + rv_1 + \dots + rv_n \in S$ .

Let  $E$  be a measurable subset of  $\mathbb{R}^n$  with positive Lebesgue measure. We recall that the Lebesgue measure is *regular* (which means it is both *inner* and *outer* regular). By inner regularity, there exists

a compact set  $K_1 \subset E$  such that  $m^n(K_1) > 0$ . Let  $\beta < (2^N - 1)^{-1}$ ; by outer regularity, there exists an open set  $U$  containing  $K_1$  such that

$$m^n(U) \leq (1 + \beta)m^n(K_1). \quad (1)$$

Since  $K_1$  is compact,  $d_1 = d(K_1, U^c) > 0$ . Let  $R = d_1/M$ , and choose an arbitrary  $r$  such that  $0 < r < R$ . First, observe that the set  $K_1 + rv_1$  is contained in  $U$ , since otherwise,

$$d(K_1, U^c) \leq |rv_1| \leq rM < d_1. \quad (2)$$

Therefore,  $K_1 \cup (K_1 + rv_1) \subset U$ , and so

$$m^n(U) \geq m^n(K_1 \cup (K_1 + rv_1)) = m^n(K_1) + m^n(K_1 + rv_1) - m^n(K_1 \cap (K_1 + rv_1)). \quad (3)$$

Since the Lebesgue measure is translation invariant,

$$m^n(K_1 \cap (K_1 + rv_1)) \geq 2m^n(K_1) - m^n(U) \geq 2m^n(K_1) - m^n(K_1) - \beta m^n(K_1) = (1 - \beta)m^n(K_1). \quad (4)$$

Since  $\beta < 1$ , it follows that  $m^n(K_1 \cap (K_1 + rv_1)) > 0$ , and so  $K_1 \cap (K_1 + rv_1) \neq \emptyset$ . Now we proceed by induction. For each  $i = 1, \dots, N$ , let  $K_{i+1} = K_i \cap (K_i + rv_i)$ . Each  $K_i + rv_i$  must be contained in  $U$  (by a generalization of the argument made above) and each  $K_{i+1} \subset K_i \subset U$ . We claim that for each  $i$ ,  $m^n(K_{i+1}) \geq (1 - (2^i - 1)\beta)m^n(K_1)$ . We have already proven the base case  $i = 1$ . So assume the result holds for some  $1 \leq m < N$ . Then

$$m^n(U) \geq m^n(K_i \cup (K_i + rv_i)) = m^n(K_i) + m^n(K_i + rv_i) - m^n(K_i \cap (K_i + rv_i)). \quad (5)$$

By translation invariance of the Lebesgue measure,

$$\begin{aligned} m^n(K_{i+1}) &= m^n(K_i + rv_i) \geq 2m^n(K_i) - m^n(U) \geq 2(1 - (2^i - 1)\beta)m^n(K_1) - (1 + \beta)m^n(K_1) \\ &= m^n(K_1) - 2^{i+1}\beta m^n(K_1) + 2\beta m^n(K_1) - \beta m^n(K_1) \\ &= (1 - (2^{i+1} - 1)\beta)m^n(K_1). \end{aligned} \quad (6)$$

Hence, since  $\beta < (2^N - 1)^{-1}$ , we obtain a nested sequence of compact subsets  $\emptyset \neq K_{N+1} \subset K_N \subset \dots \subset K_1 \subset U$ . Let  $q \in K_{N+1}$  be arbitrary. Since  $K_{N+1} = K_N \cap (K_N + rv_N)$ , the point  $q - rv_N$  is contained in  $K_N$ . Then since  $K_N = K_{N-1} \cap (K_{N-1} \cap rv_{N-1})$ ,  $q - rv_N - rv_{N-1} \in K_{N-1}$ . Proceeding inductively, we obtain the sequence  $\{q, q - rv_N, q - rv_N - rv_{N-1}, \dots, q - rv_N - \dots - rv_1\} \subset K_1 \subset E$ . Hence, the proof concludes.

## Fat Cantor Set

**Problem (Fat Cantor Set).** There exists a closed nowhere dense subset  $E \subset \mathbb{R}$  of positive Lebesgue measure.

Consider the interval  $[0, 1] \subset \mathbb{R}$ . Delete the open set fourth  $(\frac{3}{8}, \frac{5}{8})$ , which leaves the two line segments

$$\left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right]. \quad (7)$$

From each of these segments, remove the corresponding open middle fourths again, yielding the set

$$\left[0, \frac{5}{32}\right] \cup \left[\frac{7}{32}, \frac{3}{8}\right] \cup \left[\frac{5}{8}, \frac{25}{32}\right] \cup \left[\frac{27}{32}, 1\right]. \quad (8)$$

We repeat this procedure inductively, removing an interval of width  $4^n$  from each of the remaining  $2^{n-1}$  intervalsOverall, we remove intervals of total length

$$\sum_{n=0}^{\infty} \frac{2^n}{2^{2n+2}} = \frac{1}{2}, \quad (9)$$

which means that the Lebesgue measure of the overall set is  $1/2 > 0$ . Moreover, the set is the intersection of a sequence of closed sets so that it is closed. Finally, the set does not contain any intervals so that it has empty interior. Therefore, the fat Cantor set is a closed nowhere dense subset of  $\mathbb{R}$  of positive Lebesgue measure.

## Fundamental Group of the Projective Planes

**Problem (Fundamental Group of the Projective Planes).** Compute the fundamental group of  $\mathbb{RP}^n$  for all  $n \geq 1$ .

We recall the definition of  $\mathbb{RP}^n$  for  $n \geq 1$ .

**(Real Projective Space)**  $\mathbb{RP}^n$  is the quotient space obtained from  $S^n$  by identifying each point  $x$  of  $S^n$  with its antipodal point  $-x$ .

Now we prove the following:

**(Theorem)** For all  $n \geq 1$ ,  $\mathbb{RP}^n$  is compact, and the quotient map  $p : S^n \rightarrow \mathbb{RP}^n$  is a covering map.

*Proof.* First, we will show that  $\mathbb{RP}^n$  is compact. Let  $U \subset S^n$  be open. The antipodal map  $a : S^n \rightarrow S^n$  given by  $a(x) = -x$  is a homeomorphism of  $S^n$ , which means that  $a(U)$  is open in  $S^n$ . Then since  $p$  is a quotient map,

$$p^{-1}(p(U)) = U \cup a(U), \quad (10)$$

the set is also open in  $S^n$ . Therefore,  $p(U)$  is open in  $\mathbb{RP}^n$ . Hence,  $p$  is an open map. Likewise, we can show that  $p$  is a closed map. Since  $\mathbb{RP}^n$  is connected,  $S^n$  is compact, and  $p$  is surjective, we conclude that  $p(S^n) = \mathbb{RP}^n$ , and so  $\mathbb{RP}^n$  is compact.

Now let  $y \in \mathbb{RP}^n$ , and choose  $x \in p^{-1}(y)$ . Then choose an  $\varepsilon$ -neighborhood  $U$  of  $x$  in  $S^n$  for some  $\varepsilon < 1$ , using the euclidean metric of  $\mathbb{R}^n$ . Then  $U$  contains no pair  $\{z, a(z)\}$  of antipodal points of  $S^n$ , since  $d(z, a(z)) = 2$ . As a result, the map

$$p : U \rightarrow p(U)$$

is bijective. Since  $p$  is continuous and open,  $p$  is a homeomorphism. Likewise,  $p : a(U) \rightarrow p(a(U)) = p(U)$  is a homeomorphism. The set  $p^{-1}(p(U))$  is thus the union of the two disjoint open sets  $U$  and  $a(U)$ , each of which is homeomorphically mapped by  $p$  onto  $p(U)$ . Hence,  $p(U)$  is a neighborhood of  $p(x) = y$  that is evenly covered by  $p$ .  $\square$

Now we have the following theorem

**(Theorem)** Let  $p : E \rightarrow B$  be a covering map; let  $p(e_0) = b_0$ . If  $E$  is path connected, then the lifting correspondence

$$\phi : \pi_1(B, b_0) \rightarrow p^{-1}(b_0)$$

is surjective. If  $E$  is simply connected, it is bijective.

*Proof.* Let  $E$  be path connected. Then given  $e_1 \in p^{-1}(b_0)$ , there is a path  $\tilde{f}$  in  $E$  from  $e_0$  to  $e_1$ . Then  $f = p \circ \tilde{f}$  is a loop in  $B$  at  $b_0$ , and  $\phi([f]) = e_1$  by definition. Therefore,  $\phi$  is surjective.

Now suppose  $E$  is simply connected. Let  $[f], [g]$  be two elements of  $\pi_1(B, b_0)$  such that  $\phi([f]) = \phi([g])$ . Let  $\tilde{f}$  and  $\tilde{g}$  be the liftings of  $f$  and  $g$ , respectively, to paths in  $E$  that begin at  $e_0$ ; then  $\tilde{f}(1) = \tilde{g}(1)$ . Since  $E$  is simply connected, there is a path homotopy  $\tilde{F}$  in  $E$  between  $\tilde{f}$  and  $\tilde{g}$ . Hence,  $F = p \circ \tilde{F}$  is a path homotopy between  $f$  and  $g$ , which proves injectivity.  $\square$

Now, suppose  $n \geq 2$ . Then since  $S^n$  is simply connected,  $\phi : \pi_1(\mathbb{RP}^n, b_0) \rightarrow p^{-1}(b_0)$  is bijective. But since  $|p^{-1}(b_0)| = 2$ ,  $|\pi_1(\mathbb{RP}^n, b_0)| = 2$  for all  $b_0 \in \mathbb{RP}^n$ . Therefore,  $\pi_1(\mathbb{RP}^n, b_0) \cong \mathbb{Z}/2\mathbb{Z}$ . Now suppose  $n = 1$ . Since  $\mathbb{RP}^1$  is homeomorphic to  $S^1$ , their fundamental groups must be isomorphic. Therefore,  $\pi_1(\mathbb{RP}^1, b_0) \cong \pi_1(S^1, p_0) \cong \mathbb{Z}$ . Hence, we conclude that

$$\pi_1(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}, & n = 1, \\ \mathbb{Z}/2\mathbb{Z}, & n \geq 2. \end{cases} \quad (11)$$

**Problem (Lifting Criterion).** Let  $f : (Y, y_0) \rightarrow (X, x_0)$  be a map with  $Y$  path-connected and locally path-connected. Then a lift of  $f$  exists if and only if

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

## January 2025

**Problem 2025-J-I-1 (Algebra).** Let  $R$  be a UFD (unique factorization domain). Let  $F$  be its quotient field. Let  $p(x) = x^n + b_{n-1}x^{n-1} + \dots + b_0 \in F[x]$  be a monic polynomial with coefficients in  $R$  admitting a root  $a \in F$ . Prove that  $a \in R$ .

Let  $R$  be a UFD, and  $F$  its quotient field. Let  $p(x) = x^n + b_{n-1}x^{n-1} + \dots + b_0 \in F[x]$  be a monic polynomial with coefficients in  $R$  admitting a root  $a \in F$ . Let  $a = c/d$ , where  $c, d \in R \setminus \{0\}$  so that  $\gcd(c, d) = 1$ . By definition of a root, we must have

$$0 = p(a) = \left(\frac{c}{d}\right)^n + b_{n-1}\left(\frac{c}{d}\right)^{n-1} + \dots + b_0. \quad (12)$$

Multiplying both sides by  $d^n$ ,

$$c^n + d(b_{n-1}c^{n-1} + b_{n-2}c^{n-2}d + \dots + b_0d^{n-1}) = 0 \implies c^n = -d(b_{n-1}c^{n-1} + \dots + b_0d^{n-1}). \quad (13)$$

From this, we observe that  $d \mid c^n$ . If  $d$  is not a unit in  $R$ , then every nonidentity irreducible divisor of  $d$  is an irreducible divisor of  $c^n$ , and hence an irreducible divisor of  $c$ . But this contradicts our hypothesis that  $\gcd(c, d) = 1$ . Hence,  $d$  has to be a unit of  $R$ . If  $v \in R \setminus \{0\}$  such that  $dv = vd = 1$ , then

$$a = \frac{c}{d} = \frac{c}{d} \cdot \frac{v}{v} = cv \in R. \quad (14)$$

Hence, this concludes the proof.

**Problem 2025-J-I-2 (Real Analysis).** Let  $\{f_n\}_{n \geq 1}$  be a sequence of Lebesgue-measurable functions on  $[0, 1]$ . Suppose that

$$\int_0^1 f_n^2 dm \leq \frac{1}{n^2} \quad \text{for all } n \geq 1.$$

Prove that  $f_n$  converges to 0 a.e. on  $[0, 1]$ .

Let  $\{f_n\}_{n \geq 1}$  be a sequence of Lebesgue-measurable functions on  $[0, 1]$  so that

$$\int_0^1 f_n^2 dm \leq \frac{1}{n^2} \quad \text{for all } n \geq 1. \quad (15)$$

Consider the sequence  $\{\sum_1^m f_n^2\}$ , which is increasing and converges a.e. to  $\sum_1^\infty f_n^2$ . Hence, by the Monotone Convergence Theorem,

$$\sum_1^\infty \int_0^1 f_n^2 dm = \lim_{m \rightarrow \infty} \sum_1^m \int_0^1 f_n^2 dm = \lim_{m \rightarrow \infty} \int_0^1 \sum_1^m f_n^2 dm = \int_0^1 \sum_1^\infty f_n^2 dm \leq \int_0^1 \sum_1^\infty \frac{1}{n^2} dm < \infty. \quad (16)$$

Therefore,  $\sum_1^\infty f_n^2 \in L^1(\mathbb{R})$ , which means that  $\sum_1^\infty f_n^2 < \infty$  a.e. on  $[0, 1]$ . Hence,  $\sum_1^\infty f_n^2$  converges a.e. on  $[0, 1]$ . This implies that  $f_n^2 \rightarrow 0$  a.e. on  $[0, 1]$ , and hence  $f_n \rightarrow 0$  a.e. on  $[0, 1]$ .

**Problem 2025-J-I-3 (Geometry/Topology).** Let  $M$  be an orientable, connected, and compact smooth  $n$ -manifold with boundary. Show that there is no (smooth) retraction to the boundary, that is, there does not exist a smooth map  $f : M \rightarrow \partial M$  such that  $f(x) = x$  when  $x \in \partial M$ .

Let  $M$  be an orientable, connected, and compact smooth  $n$ -manifold with boundary. Assume to the contrary that there exists a smooth map  $f : M \rightarrow \partial M$  such that  $f(x) = x$  when  $x \in \partial M$ . Let  $\omega \in \Omega^{n-1}(\partial M)$  be a volume form for the boundary of  $M$ . Since volume forms are closed (hence,  $\omega$  is closed), we have by Stokes's theorem

$$0 = \int_M f^* d\omega = \int_M d(f^* \omega) = \int_{\partial M} f^* \omega = \int_{\partial M} \omega > 0, \quad (17)$$

which is a contradiction. Hence, by contradiction, there cannot exist a smooth retraction to the boundary.

**Problem 2025-J-II-3 (Algebra).** Let  $V$  be a vector space of dimension  $n$  over  $\mathbb{Q}$ . Let  $T : V \rightarrow V$  be a linear transformation with minimal polynomial  $x^4 - x^2 - 2$  over  $\mathbb{Q}$ . Show that  $n$  must be even.

Consider  $V$  as a module over the ring  $\mathbb{Q}[x]$  by letting a polynomial  $f(x) \in \mathbb{Q}[x]$  act as the linear operator  $f(T)$ . Since  $\dim V = n$ , this module is finitely generated. By the structure theorem for finitely generated modules over principal ideal domains,  $V$  is isomorphic to a direct sum of modules of the form  $\mathbb{Q}[x]/(p(x))^e$ , where  $p(x) \in \mathbb{Q}[x]$  is irreducible. Moreover, each  $p(x)$  must divide the minimal polynomial of  $T$ . We note that over  $\mathbb{Q}$ ,

$$x^4 - x^2 - 2 = (x^2 - 2)(x^2 + 1), \quad (18)$$

where both factors are irreducible over  $\mathbb{Q}$ . Therefore, the only choices for  $p(x)$  are  $x^2 - 2$  and  $x^2 + 1$ . Therefore,  $\mathbb{Q}[x]/(p(x))^e$  has dimension  $\deg p \cdot e = 2e$  for each choice of  $p$ . Since 2 divides these dimensions, we conclude that 2 must divide  $n$ . Hence,  $n$  is even.

**Problem 2025-J-II-4 (Topology).** Let  $\Sigma_2$  be a compact oriented surface of genus 2. Is there a submersion  $f : \Sigma_2 \rightarrow S^1 \times S^1$ , where  $S^1$  denotes the unit circle?

Assume to the contrary that there exists a submersion  $f : \Sigma_2 \rightarrow S^1 \times S^1$ , where  $S^1$  denotes the unit circle. Since  $\dim \Sigma_2 = \dim S^1 \times S^1 = 2$ ,  $df_p$  must have constant rank 2 at every  $p \in \Sigma_2$ . Hence,  $f$  is a local diffeomorphism. Since  $f$  is a local diffeomorphism,  $f(\Sigma_2)$  is compact in  $S^1 \times S^1$ ; since  $S^1 \times S^1$  is Hausdorff,  $f(\Sigma_2)$  must be closed in  $S^1 \times S^1$ . On the other hand, since local diffeomorphisms are open maps,  $f(\Sigma_2)$  is open in  $S^1 \times S^1$ . Therefore, since  $S^1 \times S^1$  is connected,  $f(\Sigma_2) = S^1 \times S^1$ ; i.e.,  $f$  is surjective. Therefore,  $f$  is a covering map. This means that the induced homomorphism,  $f_* : \pi_1(\Sigma_2) \rightarrow \pi_1(S^1 \times S^1)$  is injective, and so  $f_*(\pi_1(\Sigma_2)) \cong \text{img } f_* \leq \pi_1(S^1 \times S^1)$ . However,  $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$  is an abelian group and cannot have any nonabelian subgroups, whereas  $\pi_1(\Sigma_2)$  is nonabelian. Hence, by contradiction,  $f$  cannot be a submersion.

**Problem 2025-J-II-5 (Analysis).** Let  $V$  be a topological vector space whose topology is Hausdorff. Let  $X_1$  and  $X_2$  be two Banach spaces, and assume there exist continuous linear bijections  $F_1 : X_1 \rightarrow V$  and  $F_2 : X_2 \rightarrow V$ . Show that there is a continuous linear bijection  $G : X_1 \rightarrow X_2$ .

Assume the given hypotheses. Let  $G = F_2^{-1} \circ F_1$ . Since  $F_1, F_2$  are bijections, we conclude that  $G$  is a bijection. Likewise, since  $F_1, F_2$  are linear,  $G$  must also be linear. It suffices to prove that  $G$  is continuous. By the Closed Graph Theorem, continuity of  $G$  is equivalent to the graph of  $G$  being a closed subspace of  $X_1 \times X_2$ . Let  $\{x_n\} \subset X_1$  be a sequence in  $X_1$  such that  $x_n \rightarrow x$  and  $y_n = Gx_n \rightarrow y$ . We need to show that  $y = Gx$ . By continuity of  $F_1$ ,  $F_1 x_n \rightarrow F_1 x$ . By continuity of  $F_2$ ,

$$F_2 y = \lim F_2 y_n = \lim F_2 G x_n = \lim F_1 x_n = F_1 x. \quad (19)$$

Since  $F_2$  is bijective,  $y = F_2^{-1} F_1 x = Gx$ . Hence, the graph of  $G$  is closed, which implies that  $G$  is continuous.

## August 2025

**Problem 2025-A-I-1 (Geometry/Topology).** Let  $S$  be a closed orientable surface of genus 4 and  $C$  be an embedded circle that partitions  $S$  into two subsurfaces of genus 2. Does  $S$  retract to  $C$ ?

We claim that the answer is no; assume to the contrary that there exists a retraction  $r : S \rightarrow C$ . Let  $i : C \hookrightarrow S$  be the inclusion map so that  $r \circ i = \text{id}_C$ . Now since  $C$  is an embedded circle,  $H_1(C)$  (i.e., the first homology) is isomorphic to  $H_1(S^1) = \mathbb{Z}$ . On the other hand, since  $C$  is separating in  $S$ , its homology class in  $H_1(S)$  is the zero element. Hence, the induced map  $i_* : H_1(C) \rightarrow H_1(S)$  is the zero map. But this is impossible since if  $i_*$  is the zero map,

$$0 = r_* \circ i_* = (r \circ i)_* = \text{id}_{H_1}(C), \quad (20)$$

which is a contradiction. Hence, no such retraction can exist.

**Problem 2025-A-II-1 (Geometry/Topology).** For  $n \geq 2$ ,

- Calculate the fundamental group of the real  $n$ -dimensional projective space  $\mathbb{P}^n$ ;
- Show that any continuous map  $f : \mathbb{P}^n \rightarrow \mathbb{T}^n$  from  $\mathbb{P}^n$  to the  $n$ -dimensional torus is homotopic to a constant map.

Let  $n \geq 2$ .

- Let  $\sim$  be an equivalence relation on  $S^n$  that identifies each  $x \in S^n$  with its antipodal point  $-x$ , and let  $\mathbb{RP}^n = S^n / \sim$  with quotient map  $p : S^n \rightarrow \mathbb{RP}^n$ . We begin by proving that  $p : S^n \rightarrow \mathbb{RP}^n$  is a covering map.

Let  $y \in \mathbb{RP}^n$ . For each  $x \in p^{-1}(y)$ , choose an  $\varepsilon$ -neighborhood  $U$  in  $S^n$  for  $\varepsilon < 1$  using the Euclidean metric on  $\mathbb{R}^n$ . Let  $a : S^n \rightarrow S^n$  be the antipodal map given by  $a(x) = -x$ , which is clearly seen to be a homeomorphism of  $S^n$ ; this means that  $a(U)$  is open in  $S^n$ . By construction of  $U$ ,  $U$  does not contain a pair  $\{z, a(z)\}$  of antipodal points of  $S^n$  since  $d(z, a(z)) = 2$ . Therefore, the map  $p : U \rightarrow p(U)$  is bijective. Since  $p$  is continuous and open,  $p$  is a homeomorphism. Likewise,  $p : a(U) \rightarrow p(a(U)) = p(U)$  is a homeomorphism. The set  $p^{-1}(p(U)) = U \sqcup a(U)$ , each of which is homeomorphically mapped by  $p$  onto  $p(U)$ . Therefore,  $p(U)$  is a neighborhood of  $p(x) = y$  that is evenly covered by  $p$ . Hence,  $p$  is a covering map.

Now we recall that if  $p : E \rightarrow B$  is a covering map,  $b_0 \in B$ ,  $e_0 \in p^{-1}(b_0)$ , and  $E$  is simply connected, then the map  $\phi : \pi_1(B, b_0) \rightarrow p^{-1}(b_0)$  defined by  $\phi([f]) = \tilde{f}(1)$ , where  $\tilde{f}$  is a lift of  $f$  to a path in  $E$  beginning at  $e_0$ , is bijective. Since  $p^{-1}(b_0)$  contains only two elements for every  $b_0 \in \mathbb{RP}^n$  and  $S^n$  is simply connected,  $\pi_1(\mathbb{RP}^n, b_0)$  is a group of two elements. Therefore, the fundamental group of  $\mathbb{RP}^n$  is simply  $\mathbb{Z}_2$ .

- Now let  $f : \mathbb{RP}^n \rightarrow \mathbb{T}^n$  be a continuous map;  $f$  induces a group homomorphism  $f_* : \pi_1(\mathbb{RP}^n) \rightarrow \pi_1(\mathbb{T}^n)$ . We observe that  $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$ , while  $\pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$ . Since  $\mathbb{Z}^n$  has no subgroup of order 2, we conclude that  $f_*$  is the trivial homomorphism. By the lifting criterion, this then implies that  $\tilde{f}$  lifts to a path  $\tilde{f} : \mathbb{RP}^n \rightarrow \mathbb{R}^n$ , where  $\mathbb{R}^n$  is the universal cover of  $S^n$ . But since  $\mathbb{R}^n$  is contractible,  $\tilde{f}$  is homotopic to a constant map. Hence, projecting down, we conclude that  $f$  is homotopic to a constant map.

**Problem 2025-A-I-6 (Algebra).** Let  $f(x)$  be an irreducible polynomial of degree  $n$  over a field  $F$ , and let  $g(x)$  be any polynomial in  $F[x]$ . Prove that every irreducible factor of the composition  $f(g(x))$  has degree divisible by  $n$ .

Let  $h(x)$  be an irreducible factor of  $f(g(x))$  in  $F[x]$  and let  $\alpha$  be the root of  $h(x)$  in some algebraic closure of  $F$ . Since  $h$  is irreducible and  $\alpha$  is a root, the minimum polynomial of  $\alpha$  over  $F$  is  $h$ . Therefore,

$$\deg h = [F(\alpha) : F]. \quad (21)$$

Now since  $\alpha$  is a root of  $h(x) = f(g(x))$ ,  $f(g(\alpha)) = 0$ . In particular,  $g(\alpha)$  is a root of  $f$ . Since  $f$  is irreducible of degree  $n$  over  $F$ , the minimal polynomial of  $g(\alpha)$  over  $\alpha$  is  $f$ . Hence,

$$[F(g(\alpha)) : F] = n. \quad (22)$$

Since  $F \subset F(g(\alpha)) \subset F(\alpha)$ , by the Tower Law,

$$\deg h = [F : (\alpha) : F] = [F(\alpha) : F(g(\alpha))] \cdot [F(g(\alpha)) : F] = n[F(\alpha) : F(g(\alpha))], \quad (23)$$

so that  $n \mid \deg h$ . Hence, this concludes the proof.

**Problem 2025-A-II-2 (Geometry/Topology).** Consider the plane distribution in  $\mathbb{R}^3$  spanned by two vector fields

$$V = \partial_x + 2xy\partial_z, \quad W = x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z. \quad (24)$$

- Show that this distribution is integrable.
- Does the pair of vector fields  $V$  and  $W$  generate a coordinate system on integral surfaces? If not, find a pair that can play this role for the local integral surfaces passing through points

$(0, 0, z_0)$ .

- (i) Let  $D$  be the plane distribution in  $\mathbb{R}^3$  spanned by the two vector fields  $V$  and  $W$  given above. Then by the Frobenius Theorem,  $D$  is integrable if and only if  $D$  is involutive, which is true if and only if the Lie Bracket of  $V$  and  $W$  is a smooth section of  $D$  at each  $p \in \mathbb{R}^3$ . We observe that:

$$\begin{aligned} V(W) &= (\partial_x + 2xy\partial_z)(x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z) \\ &= \partial_x + (4xy + 2x)\partial_z. \\ W(V) &= (x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z)(\partial_x + 2xy\partial_z) \\ &= 2xy\partial_z + 2x\partial_z. \end{aligned} \tag{25}$$

Therefore, for any  $p \in \mathbb{R}^3$ ,

$$[V, W] = V(W) - W(V) = \partial_x + 2xy\partial_z = V. \tag{26}$$

Since  $V$  is a smooth section of  $D$ , we conclude that  $D$  is involutive, and hence integrable.

- (ii) Let  $S$  be an integral surface, and assume there are coordinates  $(u, v)$  on  $S$  such that  $V|_S = \partial_u$  and  $W|_S = \partial_v$ . Then we observe that  $[V|_S, W|_S] = \partial_u(\partial_v) - \partial_v(\partial_u) = 0$ . On the other hand,

$$[V|_S, W|_S] = ([V, W])|_S = V|_S \neq 0, \tag{27}$$

which is a contradiction. Therefore,  $V$  and  $W$  cannot generate a coordinate system on integral surfaces. However, consider the fields  $\tilde{V} = V$  and  $\tilde{W} = W - xV$  on  $\mathbb{R}^3$ . Then since

$$[\tilde{V}, \tilde{W}] = V(W - xV) - (W - xV)(V) = VW - xVV - W(V) + xVV = 0, \tag{28}$$

and so this pair generates a coordinate system on all integral surfaces.

## January 2024

**Problem 2024-J-I-1 (Algebra).** For distinct odd primes  $p$  and  $q$ , prove that every finite group of order  $2pq$  is a semidirect product of a normal subgroup of order  $pq$  and a subgroup of order 2.

Let  $G$  be a group of order  $2pq$ , where  $p, q$  are distinct odd primes. Without loss of generality, assume  $q > p$ . By Sylow's Theorem,

$$n_q \in \{1, 2, p, 2p\} \cap \{1, q+1, \dots\} = 1, \tag{29}$$

since  $q > 2$  and  $q > p$ . Therefore,  $G$  has a unique, normal, Sylow  $q$ -subgroup, which we denote as  $Q$ . Let  $P$  be a Sylow  $p$ -subgroup of  $G$ . By the Second Isomorphism Theorem, we conclude that  $N = PQ$  is a subgroup of  $G$  of order  $|P||Q| = pq$ . Since  $|G : N| = 2pq/(pq) = 2$ , where 2 is the smallest prime dividing  $|G|$ , we conclude that  $N$  is a normal subgroup of  $G$ . Next, by Cauchy's Theorem,  $G$  contains an element of order 2. Let  $M$  be the subgroup generated by this element, which also must have order 2. By Lagrange's Theorem,  $N \cap M = \{e\}$ . Next,

$$|NM| = \frac{|N||M|}{|N \cap M|} = |N||M| = 2pq = |G|, \tag{30}$$

so that  $G = NM$ . Therefore, we conclude that  $G = N \rtimes M$ .

**Problem 2024-J-I-2 (Geometry/Topology).** Let  $p : E \rightarrow B$  be a covering space map, with  $B$  and  $E$  path connected. Choose a point  $e_0 \in E$  and  $b_0 \in B$  such that  $p(e_0) = b_0$ . This gives us a subgroup  $H = p_*\pi_1(E, e_0)$  of the fundamental group  $G = \pi_1(B, b_0)$ .

Construct a bijection between the fiber  $p^{-1}(b_0)$  and the set of right cosets of  $H$  and prove that this is indeed a bijection. Prove that the number of sheets of  $p$  equals the index  $(G : H)$ .

Assume all of the given hypotheses. Let  $\phi : \pi_1(B, b_0) \rightarrow p^{-1}(b_0)$  be the lifting correspondence induced by  $p$  defined by  $\phi([f]) = \tilde{f}(1)$ , where  $\tilde{f}$  is the lift of  $f$ , and let  $\Phi : \pi_1(B, b_0)/H \rightarrow p^{-1}(b_0)$  be the map induced by  $\phi$ . It suffices to prove that  $\Phi$  is a bijection.

- (i) Since  $E$  is path connected and  $p : E \rightarrow B$  is a covering map, the lifting correspondence  $\phi$  must be surjective. Hence, since  $\Phi$  is induced by  $\phi$ , it follows that  $\Phi$  is also surjective.
- (ii) Now we will show that  $\Phi$  is injective. Let  $f$  and  $g$  be two paths in  $B$ , and  $\tilde{f}, \tilde{g}$  their liftings to paths in  $E$  that begin at  $e_0$ . We must show that  $\tilde{f}(1) = \tilde{g}(1)$  iff  $[f] \in H * [g]$ .
  - ( $\Leftarrow$ ) Suppose  $[f] = [h * g]$ , where  $h = p \circ \tilde{h}$  for some loop  $\tilde{h}$  in  $E$  based at  $e_0$ . Since  $\tilde{g}$  is a path in  $E$  that begins at  $e_0$ , the product  $\tilde{h} * \tilde{g}$  is well-defined. Since  $[f] = [h * g]$ , it follows that  $\tilde{f}$  and  $\tilde{h} * \tilde{g}$  must end at the same point. Hence,  $\tilde{f}$  and  $\tilde{g}$  end at the same point. Therefore,  $\phi([f]) = \phi([g])$ .
  - ( $\Rightarrow$ ) Suppose  $\phi([f]) = \phi([g])$ , which means that  $\tilde{f}(1) = \tilde{g}(1)$ . Then the product of  $\tilde{f}$  with the reverse of  $\tilde{g}$  is well-defined and is a loop  $\tilde{h}$  in  $E$  based at  $e_0$ . By direct computation,  $[\tilde{h} * \tilde{g}] = [\tilde{f}]$ . If  $\tilde{F}$  is a path homotopy between  $\tilde{h} * \tilde{g}$  and  $\tilde{f}$ , then  $p \circ \tilde{F}$  is a path homotopy between  $h * g$  and  $f$ , which means that  $[f] \in H * [g]$ . Hence, this concludes the proof that  $\Phi$  is injective.

Hence,  $|p^{-1}(b_0)| = |G/H| = (G : H)$ .

**Problem 2024-J-I-3 (Complex Analysis).** Suppose  $f$  is continuous on the plane and holomorphic on  $\mathbb{C} \setminus \mathbb{R}$ . Prove that  $f$  is holomorphic on the whole plane.

Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be continuous on  $\mathbb{C}$  and holomorphic on  $\mathbb{C} \setminus \mathbb{R}$ . We show that  $f$  is holomorphic on all of  $\mathbb{C}$ .

By Morera's Theorem, it suffices to prove that

$$\oint_{\gamma} f(z) dz = 0$$

for every closed piecewise  $C^1$  curve  $\gamma \subset \mathbb{C}$ .

If  $\gamma$  lies entirely in the upper or lower half-plane, then  $f$  is holomorphic on a neighborhood of  $\gamma$ , and by the Cauchy–Goursat theorem,

$$\oint_{\gamma} f(z) dz = 0.$$

Now suppose that  $\gamma$  intersects the real axis. For  $\varepsilon > 0$ , construct a closed piecewise  $C^1$  curve  $\gamma_\varepsilon$  by modifying  $\gamma$  so that it avoids the real axis by small detours of height  $\pm\varepsilon$ . Then  $\gamma_\varepsilon \subset \mathbb{C} \setminus \mathbb{R}$ , so  $f$  is holomorphic on a neighborhood of  $\gamma_\varepsilon$ , and hence

$$\oint_{\gamma_\varepsilon} f(z) dz = 0.$$

Since  $f$  is continuous on  $\mathbb{C}$ , it is uniformly continuous on compact sets, and the total length of the detours tends to 0 as  $\varepsilon \rightarrow 0$ . Therefore,

$$\lim_{\varepsilon \rightarrow 0} \oint_{\gamma_\varepsilon} f(z) dz = \oint_{\gamma} f(z) dz.$$

Thus  $\oint_{\gamma} f(z) dz = 0$ .

Since this holds for every closed piecewise  $C^1$  curve in  $\mathbb{C}$ , Morera's Theorem implies that  $f$  is holomorphic on all of  $\mathbb{C}$ .

**Problem 2024-J-I-4 (Algebra).** For each field  $K$ , prove that the polynomial ring  $K[x, y]$  in two variables is not a principal ideal domain.

Let  $K$  be a field, and consider the polynomial ring  $K[x, y]$ . Let  $(x, y)$  be the proper ideal of  $K[x, y]$  generated by the monomials  $x$  and  $y$ . Assume to the contrary that  $(x, y) = (f(x, y))$  where  $f(x, y) \in K[x, y]$  is not a unit of the polynomial ring. Since  $x \in (f(x, y))$ ,  $f(x, y) \mid x$ . By our assumption that  $f$  is not a unit, it follows that  $f(x, y)$  is an associate of  $x$ . Likewise,  $f(x, y)$  must be an associate of  $y$ . But this is impossible since  $x$  and  $y$  are not associates of each other. This forces  $f(x, y)$  to be a unit, which means that  $(f(x, y)) = K[x, y]$ . But this contradicts the fact that  $(x, y) = (f(x, y))$  is a proper ideal. Hence, by contradiction,  $(x, y)$  is not a principal ideal, and so  $K[x, y]$  is not a principal ideal domain.

**Problem 2024-J-I-5 (Geometry/Topology).** Let  $\alpha$  be a closed 1-form on  $\mathbb{RP}^n$ ,  $n > 1$ . Show that if  $f : [0, 1] \rightarrow \mathbb{RP}^n$  is a smooth function such that  $f(0) = f(1)$ , then

$$\int_{[0,1]} f^* \alpha = 0.$$

Include all calculations that are relevant to your solution.

We recall that  $H^k(\mathbb{RP}^n) = 0$  for all  $0 < k < n$  so that  $H^1(\mathbb{RP}^n) = 0$  if  $n > 1$ . In particular, this means that  $\alpha$  is also an exact 1-form on  $\mathbb{RP}^n$ . Let  $g$  be a smooth function on  $\mathbb{RP}^n$  so that  $\alpha = dg$ . Then

$$\int_0^1 f^* \alpha = \int_0^1 f^* dg = \int_0^1 d(f^* g) = g(f(1)) - g(f(0)) = 0, \quad (31)$$

where the last equality follows from the fact that  $f(1) = f(0)$ . Hence, the proof concludes.

**Problem 2024-J-I-6 (Real Analysis).** Let  $f$  and  $g$  be Lebesgue-measurable functions on  $\mathbb{R}$ . Define the convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(x - y)g(y) dy$$

for all  $x$  such that the integral exists. Prove that if  $f \in L^p(\mathbb{R})$  and  $g \in L^q(\mathbb{R})$  with  $p, q \in (1, \infty)$  satisfying  $\frac{1}{p} + \frac{1}{q} = 1$ , then  $f * g$  is a bounded continuous function on  $\mathbb{R}$ .

Assume the given hypotheses. Then by Hölder's inequality, for any  $x \in \mathbb{R}$ ,

$$|(f * g)(x)| \leq \int_{\mathbb{R}} |f(x - y)g(y)| dy \leq \|f(x - \cdot)\|_p \|g\|_q. \quad (32)$$

Since  $L^p$  norms are translation invariant,  $\|f(x - \cdot)\|_p = \|f\|_p$ . Hence,  $|(f * g)(x)| \leq \|f\|_p \|g\|_q = M < \infty$  for all  $x \in \mathbb{R}$ . Hence, we conclude that  $f * g$  is a bounded function on  $\mathbb{R}$ . Next, let  $\tau_z$  be the translation operator defined by  $\tau_z f = f(x - z)$ . Since translation operators are continuous in the  $L^p$  norms,  $\|\tau_z f - f\| \rightarrow 0$  as  $z \rightarrow 0$ , which implies that

$$\|\tau_z(f * g) - (f * g)\|_{\infty} = \|(\tau_z f - f) * g\|_{\infty} \quad (33)$$

$$\leq \|\tau_z f - f\|_p \|g\|_q \rightarrow 0 \text{ as } z \rightarrow 0. \quad (34)$$

Hence,  $f * g$  is uniformly continuous, and therefore continuous on  $\mathbb{R}$ . Note that the inequality used in the second line of the above equation comes from *Young's convolution inequality*, which states the following:

**(Young's Convolution Inequality)** Let  $f \in L^p$ ,  $g \in L^q$ , and  $p^{-1} + q^{-1} = r^{-1} + 1$ . Then  $\|f * g\|_r \leq \|f\|_p \|g\|_q$ .

In our case, we had  $r = \infty$  so that  $r^{-1} = 0$ .

**Problem 2024-J-II-2.** Suppose  $E \subset \mathbb{R}^2$  is a set of positive Lebesgue measure. Show that there are points  $a, b, c$  in  $E$  such that their connecting segments form a right angle, i.e.,  $a - b$  is perpendicular to  $c - b$  (as vectors in  $\mathbb{R}^2$ ).

Let  $E \subset \mathbb{R}^2$  be a set of positive Lebesgue measure; let  $m^2$  denote the Lebesgue measure on  $\mathbb{R}^2$ . Let  $\{v_1, v_2, v_3\}$  be a collection of vectors in  $\mathbb{R}^2$  such that  $v_1 \perp v_2$ , and  $v_3 = -v_1$ . Without loss of generality, assume that  $\|v_j\| = 1$  for all  $j = 1, \dots, 3$ . By inner regularity of the Lebesgue measure, there exists a compact subset  $K_1 \subset E$  such that  $m^2(K_1) > 0$ . Taking  $\beta < 1/7$ , by outer regularity of the Lebesgue measure, there exists an open set  $U$  containing  $K_1$  such that  $m^2(U) \leq (1 + \beta)m^2(K_1)$ .

Since  $K_1$  is compact,  $d_1 = d(K_1, U^c) > 0$ . Hence, let  $R = d_1$ . Fix some  $r \in (0, R)$  and consider the set  $K_1 + rv_1$ . We claim that  $K_1 + rv_1 \subset U$  since if otherwise,

$$d(K_1, U^c) \leq |rv_1| = r < d_1, \text{ which is a contradiction.} \quad (35)$$

Hence,  $K_1 \cup (K_1 + rv_1) \subset U$ , which means that

$$m^2(U) \geq m^2(K_1 \cup (K_1 + rv_1)) = m^2(K_1) + m^2(K_1 + rv_1) - m^2(K_1 \cap (K_1 + rv_1)). \quad (36)$$

By translation invariance of the Lebesgue measure,  $m^2(K_1) + m^2(K_1 + rv_1) = 2m^2(K_1)$  so that

$$m^2(K_1 \cap (K_1 + rv_1)) = 2m^2(K_1) - m^2(U) \geq (1 - \beta)m^2(K_1). \quad (37)$$

Since  $\beta < 1$ ,  $m^2(K_1 \cap (K_1 + rv_1)) > 0$  so that the set is nonempty. For  $i = 1, \dots, 3$ , define  $K_{i+1} = K_i \cap (K_i + rv_i)$ . Generalizing the argument from above shows that each  $K_{i+1} \subset U$ . We claim that  $m^2(K_{i+1}) \geq (1 - (2^i - 1)\beta)m^2(K_1)$  for each  $i$ ; the above work establishes the result for  $i = 1$ . Now assume the result holds for some  $1 \leq j < 3$ . Then

$$m^2(U) \geq m^2(K_j \cup (K_j + rv_j)) = m^2(K_j) + m^2(K_j + rv_j) - m^2(K_j \cap (K_j + rv_j)) = 2m^2(K_j) - m^2(K_j \cap (K_j + rv_j)). \quad (38)$$

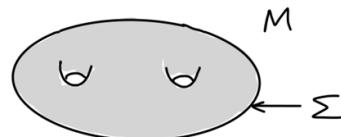
Therefore,

$$\begin{aligned} m^2(K_j \cap (K_j + rv_j)) &= 2m^2(K_j) - m^2(U) \\ &\geq 2m^2(K_1) - 2^{j+1}\beta m^2(K_1) + 2\beta m^2(K_1) - m^2(K_1) - \beta m^2(K_1) \\ &= (1 - (2^{j+1} - 1)\beta)m^2(K_1). \end{aligned} \quad (39)$$

Since  $\beta < (2^3 - 1)^{-1} = 7^{-1}$ , we conclude that each  $K_i$  is nonempty. Hence, we obtain a nested sequence  $\emptyset \neq K_4 \subset \dots \subset K_1 \subset E$ . Let  $q \in K_4$ ; since  $K_4 = K_3 \cap (K_3 + rv_3)$ ,  $q - rv_3 \in K_3$ . Following inductively, we obtain a sequence of points  $\{p, p + rv_1, p + rv_1 + rv_2, p + rv_1 + rv_2 + rv_3\} \subset E$ , with  $p \in K_1$ , and  $p + rv_j \in K_j$  for  $j = 1, 2, 3$  (note we have renamed  $q - rv_1 - \dots - rv_3 = p$ , and so on). Let  $a = p$ ,  $b = p + rv_1$ , and  $c = p + rv_1 + rv_2$ . Then  $a - b = -rv_1$  and  $c - b = rv_2$ . By hypothesis on  $v_1$  and  $v_2$ ,  $a - b$  is orthogonal to  $c - b$ .

**Problem 2024-J-II-3 (Geometry/Topology).** Let  $\Sigma$  be a genus 2 surface embedded in  $\mathbb{R}^3$  as shown in the picture. Let  $M$  be the closure of the *unbounded* component of  $\mathbb{R}^3 \setminus \Sigma$ ; in other words,  $M$  is the part of  $\mathbb{R}^3$  which is *not* enclosed by  $\Sigma$ .

- (a) Compute  $\pi_1(M)$ .
- (b) Is  $\Sigma$  a retract of  $M$ ?



(a)

**Problem 2024-J-II-5 (Real Analysis).** Let  $P$  be the vector space over  $\mathbb{R}$  of (finite degree) polynomials in the variable  $x \in (-\infty, \infty)$ . Show that  $P$  cannot be a Banach space with respect to any norm, that is, if  $\|\cdot\|$  is some norm on  $P$ , then  $P$  is not complete under this norm. Hint: You may use the Baire Category Theorem.

We recall the Baire Category Theorem:

**(Baire Category Theorem)** Let  $X$  be a complete metric space.

- (a) If  $\{U_n\}_1^\infty$  is a sequence of open dense subsets of  $X$ , then  $\bigcap_1^\infty U_n$  is dense in  $X$ .
- (b)  $X$  is not a countable union of nowhere dense sets.

For each positive integer  $n$ , let  $P_n$  be the vector space of all polynomials of degree  $\leq n$  so that  $P = \bigcup_{n \in \mathbb{N}} P_n$ . Let  $\|\cdot\|$  be a norm on  $P$  and assume to the contrary that  $P$  is complete under this norm; this means that  $P$  is a complete metric space. Since  $X$  cannot be the countable union of nowhere dense sets, it follows that there exists some positive integer  $n_0$  so that  $P_{n_0}$  is not nowhere dense; i.e., the closure of  $P_{n_0}$  has nonempty interior. Since any finite dimensional vector subspace of a normed vector space is closed, it follows that  $P_{n_0}$  is closed in  $P$ ; i.e.,  $\bar{P}_{n_0} = P_{n_0}$ . Hence, by our hypothesis,  $P_{n_0}$  has nonempty interior. Let  $p \in P_{n_0}$  and  $B(r, p)$  a ball of radius  $r > 0$  centered at  $p$  that is contained entirely within  $P_{n_0}$ . Let  $u \in P \setminus \{0\}$  be arbitrary, and set

$$u' = p + \frac{r \cdot u}{2\|u\|} \implies u' \in B(r, p) \subset P_{n_0}. \quad (40)$$

But since  $P_{n_0}$  is a vector space, this implies that  $u \in P_{n_0}$ . Since  $u$  was arbitrary in  $P$ , this means that  $P_{n_0} = P$ , which is a contradiction. Hence, every  $P_n$  must have empty interior, which then contradicts the Baire Category Theorem. Hence,  $P$  cannot be a Banach space with respect to any norm.

**Problem 2024-J-II-6 (Geometry/Topology).** Let  $M$  be a smooth  $n$ -manifold, and let  $\varphi$  be a differential  $k$ -form on  $M$  which is closed, in the sense that  $d\varphi = 0$ . At each point  $p \in M$ , define

$$D_p = \{v \in T_p M : v \lrcorner \varphi = 0\}, \quad (41)$$

where  $\lrcorner$  denotes the interior product. Assume  $\ell := \dim D_p$ , so that  $D \subset TM$  is a rank- $\ell$  vector subbundle of the tangent bundle of  $M$ . Prove that  $D$  is an integrable distribution of  $\ell$ -planes, in the sense of the Frobenius Theorem.

By the Frobenius Theorem, it suffices to prove that  $D$  is involutive, which is to say that if  $X, Y$  are smooth sections of  $D$ , then  $[X, Y]$  is also a smooth section of  $D$ . Indeed, let  $X, Y$  be smooth sections of  $D$ , which means that  $X \lrcorner \varphi, Y \lrcorner \varphi = 0$ . Observe that,

$$[X, Y] \lrcorner \varphi = \mathcal{L}_X(Y \lrcorner \varphi) - Y \lrcorner (\mathcal{L}_X \varphi). \quad (42)$$

By hypothesis,  $Y \lrcorner \varphi = 0$  so that  $\mathcal{L}_X(Y \lrcorner \varphi) = 0$ . On the other hand, by Cartan's Formula,

$$\mathcal{L}_X \varphi = d(X \lrcorner \varphi) + X \lrcorner d\varphi = 0, \quad (43)$$

by the hypotheses. Hence, this shows that  $[X, Y] \lrcorner \varphi = 0$ , and so  $[X, Y]$  is a smooth section of  $D$ . Therefore,  $D$  is involutive, which means that it is Frobenius integrable.

**Problem 2024-J-II-4 (Algebra).** Let  $\alpha = \sqrt{2 + \sqrt{3}} \in \mathbb{C}$ . Let  $K$  be the smallest Galois extension of  $\mathbb{Q}$  which contains  $\alpha$ . Describe the Galois group  $\text{Gal}(K/\mathbb{Q})$ .

Let  $\alpha = \sqrt{2 + \sqrt{3}} \in \mathbb{C}$ , and  $K$  the smallest Galois extension of  $\mathbb{Q}$  that contains  $\alpha$ . We start by finding the minimal polynomial of  $\alpha$ . We observe that

$$\alpha^2 = 2 + \sqrt{3} \implies (\alpha^2 - 2)^2 - 3 = 0. \quad (44)$$

Simplifying,

$$\alpha^4 - 4\alpha^2 + 1 = 0. \quad (45)$$

I.e., the polynomial  $x^4 - 4x^2 + 1$  is the minimal polynomial of  $\alpha$ . Solving this polynomial over an algebraic closure of  $\mathbb{Q}$ , we obtain the four roots,  $\pm\sqrt{2 + \sqrt{3}}, \pm\sqrt{2 - \sqrt{3}}$ . Hence, the elements of the Galois group  $\text{Gal}(K/\mathbb{Q})$  are the identity permutation, the permutation  $\sigma$  that fixes  $\pm\sqrt{2 - \sqrt{3}}$  and permutes  $\pm\sqrt{2 + \sqrt{3}}$ , the permutation  $\tau$  that fixes  $\pm\sqrt{2 + \sqrt{3}}$  and permutes  $\pm\sqrt{2 - \sqrt{3}}$ , and the permutation  $\sigma\tau$ . Labeling these roots as  $\alpha_1, \dots, \alpha_4$ , we see that  $\text{Gal}(K/\mathbb{Q}) \cong \{1, (1 2), (3 4), (1 2)(3 4)\} \cong V \subset S_4$ , where  $V$  is the Klein-4 subgroup.

## August 2024

**Problem 2024-A-I-1 (Geometry/Topology).** Let  $M$  be a smooth compact manifold without boundary, and let  $\varphi$  be a smooth closed 1-form on  $M$  that has the property that  $\varphi \neq 0$  at every point of  $M$ . Prove that the first de Rham cohomology  $H_{\text{dr}}^1(M)$  of the given manifold is non-zero.

Let  $M$  be a smooth compact manifold without boundary and let  $\varphi$  be a smooth closed 1-form on  $M$  that has the property that  $\varphi \neq 0$  at every point of  $M$ . Suppose that  $\varphi$  is exact; i.e., assume there exists a smooth function  $f$  on  $M$  such that  $\varphi = df$ . By the Extreme Value Theorem, since  $M$  is compact,  $f$  must have either a maximum or minimum value at some point  $p \in M$ . Since all of the first-order partial derivatives of  $f$  must vanish at the point  $p$  where  $f$  attains its maximum/minimum value,  $df|_p = 0$ . This means that  $\varphi$  must also vanish at  $p$ , which contradicts our hypothesis that  $\varphi$  is nowhere vanishing. Hence, by contradiction,  $\varphi$  cannot be an exact form. Since  $H_{\text{dr}}^1(M) := \{\text{closed 1-forms on } M\}/\{\text{exact 1-forms on } M\}$  and we have shown the existence of a closed 1-form that is *not* an exact 1-form, we conclude that  $H_{\text{dr}}^1(M)$  is non-zero.

**Problem 2024-A-I-2 (Geometry/Topology).** Suppose that  $f : \Sigma_2 \rightarrow \Sigma_1$  is a continuous map between a genus 2 closed orientable surface  $\Sigma_2$  and a torus  $\Sigma_1$ . Prove that  $f$  is not a local homeomorphism. In other words, show that there exists a point  $x \in \Sigma_2$  which does not have an open neighborhood  $U \subset \Sigma_2$  on which the restriction  $f|_U$  is a homeomorphism between  $U$  and  $f(U)$ .

Before presenting our argument, we will state and prove a quick technical lemma.

**(Modified Comps Lemma)** Let  $M$  and  $N$  be smooth connected manifolds, and  $f : M \rightarrow N$  a local homeomorphism. If  $M$  is compact and nonempty, then  $N$  is compact and  $f$  is a covering map.

*Proof.* Let  $M$  and  $N$  be smooth connected manifolds, and  $f : M \rightarrow N$  a local homeomorphism. Since  $f$  is an open map,  $f(M)$  is open in  $N$ . Next since the continuous image of a compact set is compact and a compact subset of a Hausdorff space is closed,  $f(M)$  is closed in  $N$ . Hence, since  $N$  is connected,  $f(M) = N$ , which means  $N$  is connected and  $f$  is surjective.

Now let  $q \in N$ , and consider the closed subset  $f^{-1}(q) \subset M$ . For each  $x \in f^{-1}(q)$ , there exists a neighborhood  $U_x$  such that  $f|_{U_x}$  is a homeomorphism. Since  $M$  is Hausdorff, we may shrink these neighborhoods so that they are pairwise disjoint. Hence, each  $x \in f^{-1}(q)$  is isolated, which means  $f^{-1}(q)$  is discrete. Since discrete subspaces of compact spaces is necessarily finite,  $f^{-1}(q)$  is finite; let  $\{x_1, \dots, x_s\} = f^{-1}(q)$ . As stated above, for each  $j = 1, \dots, s$ , we may find a neighborhood  $U'_j$  such that  $f|_{U'_j}$  is a homeomorphism. Using Hausdorff-ness of  $M$ , we may shrink these neighborhoods to obtain the collection  $\{\tilde{U}_j\}_1^s$  of pairwise disjoint open neighborhoods. Set  $V = \bigcap_1^s U_j$ , which is then an evenly covered neighborhood of  $q$ . Therefore,  $f$  is a covering map.  $\square$

Now assume to the contrary that  $f : \Sigma_2 \rightarrow \Sigma_1$  is a local homeomorphism; by the modified Comps Lemma,  $f$  is a covering map. Moreover,  $\Sigma_2$  must be a  $k$ -sheeted covering space for some finite positive integer  $k$ , which means that  $\chi(\Sigma_2) = k \cdot \chi(\Sigma_1)$ . However, this is impossible since  $\chi(\Sigma_1) = 0$ , while  $\chi(\Sigma_2) = 2 - 2(2) = 2 - 4 = -2$ . Therefore,  $f$  cannot be a local homeomorphism.

**Problem 2024-A-I-5 (Algebra).** Determine whether or not the complex number  $i = \sqrt{-1}$  is in the field  $\mathbb{Q}(\alpha)$ , where  $\alpha$  is any complex number subject to the relation  $\alpha^3 + \alpha + 1 = 0$ . Justify your answer.

The polynomial  $x^3 + x + 1$  has no roots in  $\mathbb{Q}$  (by the rational root test), and so is irreducible (since it is a cubic). This means that  $\mathbb{Q}(\alpha)$  is an extension of degree 3 over  $\mathbb{Q}$ . Therefore, it cannot contain the field  $\mathbb{Q}(i)$ , which has degree 2 over  $\mathbb{Q}$  (since the minimal polynomial of  $i$  is  $x^2 + 1$ ) since  $2 \nmid 3$ .

**Problem 2024-A-II-1 (Geometry/Topology).** Recall that  $S^n$  denotes the unit sphere in  $\mathbb{R}^{n+1}$ . Also recall that a smooth map is called a smooth submersion if its differential is everywhere surjective. Prove or disprove each of the following statements:

- (a) There is a smooth submersion  $F : S^3 \rightarrow S^1$ .
- (b) There is a smooth submersion  $F : S^3 \rightarrow S^2$ .

(a) [!! Complete Later !!]

**Problem 2024-A-II-2 (Geometry/Topology).** On  $\mathbb{R}^5$ , equipped with standard coordinates  $(v, w, x, y, z)$ , consider the 1-form

$$\theta = dz + v \, dx + w \, dy.$$

Are there two smooth functions  $f, g : \mathbb{R}^5 \rightarrow \mathbb{R}$  such that  $\theta = f \, dg$ ? Justify your answer by means of concrete solutions.

We claim that there do *not* exist smooth functions  $f, g : \mathbb{R}^5 \rightarrow \mathbb{R}$  such that  $\theta = f \, dg$ . Assume to the contrary. First, we observe that if  $\theta = f \, dg$ , then

$$d\theta = d(f \, dg) = df \wedge dg \implies \theta \wedge d\theta = f \, dg \wedge df \wedge dg = 0. \quad (46)$$

I.e., if  $\theta = f \, dg$ , then  $\theta \wedge d\theta$  must be identically zero. However, since  $\theta = dz + v \, dx + w \, dy$ , we note that

$$d\theta = d^2z + d(v \, dx) + d(w \, dy) = dv \wedge dx + dw \wedge dy \implies \theta \wedge d\theta = dz \wedge dv \wedge dx + dz \wedge dw \wedge dy + v \, dx \wedge dw \wedge dy + w \, dy \wedge dv \wedge dx, \quad (47)$$

which is nowhere vanishing on  $\mathbb{R}^5$ . Hence, by contradiction, there cannot exist two smooth functions  $f, g : \mathbb{R}^5 \rightarrow \mathbb{R}$  such that  $\theta = f \, dg$ .

## January 2023

**Problem 2023-J-I-3 (Geometry/Topology).** Show that if  $M$  is a closed manifold that has an even dimensional sphere  $S^{2n}$  as its universal cover, then its fundamental group is either trivial or  $\mathbb{Z}_2$ .

Let  $M$  be a closed manifold that has an even dimensional sphere  $S^{2n}$  as its universal cover. Then  $M = S^{2n}/G$ , where  $G$  is the group of deck transformations. Since  $S^{2n}$  is path connected, each nonidentity deck transformation  $g$  is a homeomorphism  $S^{2n} \rightarrow S^{2n}$  with no fixed points. Then

$$\deg g = (-1)^{2n+1} = -1. \quad (48)$$

If  $G$  contains two nontrivial elements  $f$  and  $g$ , this implies  $\deg(f \circ g) = \deg f \deg g = 1$ . Hence, by our observation  $f \circ g = \text{id}_{S^{2n}}$ , and so  $f = g^{-1}$ . Likewise,  $g^2 = \text{id}_{S^{2n}}$  so that  $f = g$ . Since  $S^{2n}$  is simply connected, we have  $\pi_1(M) \cong G$ , and therefore  $\pi_1(M)$  is either trivial or  $\mathbb{Z}_2$ .

**Problem 2023-J-II-4 (Geometry/Topology).** Prove that  $S^2 \times S^2$  is not diffeomorphic to  $M_1 \times M_2 \times M_3$ , where  $M_1, M_2, M_3$  are smooth manifolds of nonzero dimension.

We begin with a technical lemma, that we will use to prove the desired result.

**(Comps Lemma)** Let  $M, N$  be smooth, connected  $n$ -manifolds and  $f : M \rightarrow N$  a (smooth) immersion. If  $M$  is compact and nonempty, then  $N$  is compact and  $f$  is a (smooth) covering map.

*Proof.* Let  $M, N$  be smooth connected  $n$ -manifolds,  $f : M \rightarrow N$  an immersion, and  $M$  compact and nonempty. Since  $\dim N = n$  everywhere and  $f$  is an immersion,  $d_f : T_p M \rightarrow T_{f(p)} N$  has constant rank  $n$  everywhere. Hence, by the Inverse Function Theorem,  $f$  is a local diffeomorphism. Since local diffeomorphisms are open maps,  $f(M)$  is open in  $N$ . Next since the continuous image of compact sets is compact,  $f(M)$  is compact in  $N$ . Since  $N$  is Hausdorff,  $f(M)$  must be closed in  $N$ . Therefore, since  $N$  is connected, we conclude that  $f(M) = N$ . This means that  $N$  is compact and  $f$  is surjective. All that remains is to show that  $f$  is a covering map.

Let  $q \in N$ , and consider  $f^{-1}(q)$ , which is closed in  $M$ . For each  $x \in f^{-1}(q)$ , there exists a neighborhood  $U_x$  of  $x$  such that  $f|_{U_x}$  is a diffeomorphism. Since  $M$  is Hausdorff, we may shrink these neighborhoods so that they are pairwise disjoint. This means that each  $x \in f^{-1}(q)$  is isolated. Hence,  $f^{-1}(q)$  is discrete in  $M$ . Since discrete subspaces of compact spaces must be finite, it follows that  $f^{-1}(q)$  is finite; let  $f^{-1}(q) = \{x_1, \dots, x_s\}$ . As stated above, for each  $j = 1, \dots, s$ , we can find a neighborhood  $U_j$  of  $x_j$  such that  $f|_{U_j} : U_j \rightarrow V_j \subset N$  is a diffeomorphism. Since  $M$  is Hausdorff, we may shrink these neighborhoods so that  $U_i \cap U_j = \emptyset$  for all  $i \neq j$ ;  $f$  restricted to each of these new  $U_j$ 's remains a diffeomorphism. Set  $V = \bigcap_1^s f(U_j)$ , and define  $\tilde{U}_j = f^{-1}(V) \cap U_j$ . For each  $j$ ,  $f : \tilde{U}_j \rightarrow V$  is a diffeomorphism and  $V = \bigsqcup_1^s f(U_j)$ . Hence,  $V$  is an evenly covered neighborhood of  $q$ , so that  $f$  is a covering map.  $\square$

Now, assume to the contrary that  $f : S^2 \times S^2 \rightarrow M_1 \times M_2 \times M_3$  is a diffeomorphism; since diffeomorphisms preserve dimensions and  $M_1, M_2, M_3$  have nonzero dimensions, it follows, without loss of generality, that  $M_1, M_2$  are 1-dimensional and  $M_3$  is 2-dimensional. Since diffeomorphisms of manifolds are immersions, by the Comps Lemma,  $M_1 \times M_2 \times M_3$  must be compact and connected; by projecting onto each manifold,  $M_1, M_2, M_3$  must be compact and connected. Moreover, the induced group homomorphism  $f_* : \pi_1(S^2 \times S^2) \rightarrow \pi_1(M_1 \times M_2 \times M_3) = \pi_1(M_1) \times \pi_1(M_2) \times \pi_1(M_3)$  must be an isomorphism. Since  $S^2$  is simply connected,

$$\pi_1(S^2 \times S^2) = \pi_1(S^2) \times \pi_1(S^2) = \{0\}. \quad (49)$$

On the other hand, since the only compact connected 1-manifold, up to diffeomorphism, is the unit circle  $S^1$ , and  $\pi_1(S^1) \cong \mathbb{Z}$  is not trivial,  $\pi_1(M_1 \times M_2 \times M_3)$  is not trivial. But this contradicts our claim that  $f_*$  is an isomorphism. Hence, by contradiction,  $f$  cannot be a diffeomorphism.

**Problem 2023-J-II-3 (Geometry/Topology).** Consider the form  $\omega = (x^2 + x + y)dy \wedge dz$  on  $\mathbb{R}^3$ . Let  $S^2 \subset \mathbb{R}^3$  be the unit sphere, and  $i : S^2 \rightarrow \mathbb{R}^3$  be the inclusion map.

- (a) Calculate  $\int_{S^2} i^* \omega$ .
- (b) Construct a closed form  $\alpha$  on  $\mathbb{R}^3$  such that  $i^* \alpha = i^* \omega$ , or show that such a form  $\alpha$  does not exist.

- (a) **(Method 1)** Consider the form  $\omega = (x^2 + x + y)dy \wedge dz$  on  $\mathbb{R}^3$ , and let  $i : S^2 \hookrightarrow \mathbb{R}^3$  be the inclusion map. Let  $D = [0, \pi] \times [0, 2\pi]$ , and  $F : D \rightarrow S^2$  be the coordinate map defined by

$$F(\phi, \theta) = (\sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), \cos(\phi)). \quad (50)$$

Taking  $D_1 = [0, \pi] \times [0, \pi]$  and  $D_2 = [0, \pi] \times [\pi, 2\pi]$ , and letting  $F_1 = F|_{D_1}$  and  $F_2 = F|_{D_2}$ , we observe that

$$\int_{S^2} i^* \omega = \int_{D_1} F_1^* i^* \omega + \int_{D_2} F_2^* \omega = \int_{D_1} (i \circ F_1)^* \omega + \int_{D_2} (i \circ F_2^*) \omega = \int_D F^* \omega, \quad (51)$$

where the last equality follows from the fact that  $i \circ F_{1,2} = F_{1,2}$ . We observe that

$$F^* dy = \cos(\varphi) \sin(\theta) d\varphi + \sin(\varphi) \cos(\theta) d\theta \quad \text{and} \quad F^* dz = -\sin(\varphi) d\varphi. \quad (52)$$

Therefore,

$$F^* \omega = [\sin^2(\varphi) \cos^2(\theta) + \sin(\varphi) \cos(\theta) + \sin(\varphi) \sin(\theta)] \sin^2(\varphi) \cos(\theta) d\varphi \wedge d\theta. \quad (53)$$

From this, we conclude that

$$\int_{S^2} i^* \omega = \int_0^{2\pi} \int_0^\pi [\sin^2(\varphi) \cos^2(\theta) + \sin(\varphi) \cos(\theta) + \sin(\varphi) \sin(\theta)] \sin^2(\varphi) \cos(\theta) d\varphi d\theta = \frac{4\pi}{3}. \quad (54)$$

**(Method 2)** Using Stokes Theorem,

$$\int_{S^2} i^* \omega = \int_{B^3} d\omega, \quad (55)$$

where  $B^3$  indicates the 3-ball (recall that  $S^1 = \partial B^3$ ). We compute,  $d\omega = (2x+1)dx \wedge dy \wedge dz$  so that

$$\int_{S^2} i^* \omega = \int_{B^3} d\omega = \int_{B^3} 2xdxdydz + \int_{B^3} dx dy dz = \int_{B^3} dx dy dz = \frac{4\pi}{3}, \quad (56)$$

where the first integral after the second inequality is zero due to symmetry.

(b) Suppose there exists a closed form  $\alpha$  on  $\mathbb{R}^3$  such that  $i^* \alpha = i^* \omega$ . Since  $\alpha$  is closed,  $d\alpha = 0$ . Hence,

$$\int_{S^2} i^* \alpha = \int_{B^3} d(i^* \alpha) = \int_{B^3} i^* d\alpha = 0 \neq \frac{4\pi}{3} = \int_{S^2} i^* \omega, \quad (57)$$

which is a contradiction. Hence, such a closed form cannot exist.

**Problem 2023-J-I-5 (Algebra).** Consider the following irreducible polynomial over  $\mathbb{Q}$ :  $p(x) = x^4 - 3x^2 - 1$ .

- (a) Describe the splitting field of  $p(x)$ .
- (b) Consider the Galois group of  $p(x)$ . Compute its order and determine if it is abelian.

(a) Let  $p(x) = x^4 - 3x^2 - 1$ . By the rational root test,  $p(x)$  has no roots over  $\mathbb{Q}$ . Moreover, it is straightforward to check that  $p(x)$  is not the product of irreducible quadratics with rational coefficients. Hence,  $p(x)$  is irreducible over  $\mathbb{Q}$ . We start by finding the roots of  $p(x)$ ; let  $u = x^2$ . Then

$$u^2 - 3u - 1 = 0 \implies u = \frac{3 \pm \sqrt{13}}{2} \implies x = \pm \sqrt{\frac{3 \pm \sqrt{13}}{2}}. \quad (58)$$

Let

$$\alpha = \sqrt{\frac{3 + \sqrt{13}}{2}}, \quad \beta = \sqrt{\frac{3 - \sqrt{13}}{2}}. \quad (59)$$

Observe that  $\alpha^2 \beta^2 = -1$  so that  $\beta = \pm \frac{i}{\alpha}$ . Therefore, the splitting field of  $p(x)$  is

$$\mathbb{Q}(\alpha, i). \quad (60)$$

Observe that the minimal polynomial of  $i$  is  $x^2 + 1$ , which is irreducible over  $\mathbb{Q}(\alpha)$  so that  $[\mathbb{Q}(\alpha, i) : \mathbb{Q}(\alpha)] = 2$ . On the other hand, the minimal polynomial of  $\alpha$  is a degree 4 polynomial so that  $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ . Hence, by the tower law,  $[\mathbb{Q}(\alpha, i) : \mathbb{Q}] = 8$ .

(b) By the last work in (a), the order of the Galois group of  $p(x)$  is 8. Now, we will determine the Galois group of  $p(x)$ . Recall that elements of  $\text{Gal}(\mathbb{Q}(\alpha, i)/\mathbb{Q})$  are automorphisms  $\varphi$  of the field  $\mathbb{Q}(\alpha, i)$  with the constraints that: (1)  $\varphi$  fixes  $\mathbb{Q}$ , (2)  $\varphi(\alpha)$  must be another root of the minimal polynomial of  $\alpha$  over  $\mathbb{Q}$ , and (3)  $\varphi(i)$  must be another root of  $x^2 + 1$ . We will explicitly work through each of the elements.

- (i)  $\sigma : i \mapsto -i, \alpha \mapsto \alpha$ . This permutation has order 2 since  $\sigma^2(\alpha) = \sigma(\alpha) = \alpha$  and  $\sigma^2(i) = \sigma(-i) = i$ .
- (ii)  $\tau : i \mapsto i, \alpha \mapsto -\alpha$ . Once again, this permutation has order 2.
- (iii)  $\rho : i \mapsto -i, \alpha \mapsto \beta = \frac{i}{\alpha}$ . To compute the order of this permutation, observe that

$$\rho^2(\alpha) = \rho(i\alpha^{-1}) = (-i) \cdot \frac{1}{i/\alpha} = -\alpha \implies \rho^4(\alpha) = \rho^2(-\alpha) = \alpha. \quad (61)$$

Likewise,  $\rho^4(i) = \rho^2(i) = i$ . Hence,  $\rho$  has order 4.

Now, consider the three elements given above. We compute

$$\sigma\rho\sigma(i) = \sigma\rho(-i) = \sigma(i) = -i = \rho^{-1}(i). \quad (62)$$

Likewise,

$$\sigma\rho\sigma(\alpha) = \sigma\rho(\alpha) = \sigma(i)\sigma(\alpha)^{-1} = -\frac{i}{\alpha} = \rho^{-1}(\alpha). \quad (63)$$

Therefore,  $\sigma\rho\sigma = \rho^{-1}$ . Hence,

$$\text{Gal}(\mathbb{Q}(\alpha, i)/\mathbb{Q}) = \{1, \sigma, \rho, \rho^2, \rho^3, \sigma\rho, \sigma\rho^2, \sigma\rho^3\} \cong D_8. \quad (64)$$

Since the dihedral group is not abelian, we conclude that the Galois group for  $p(x)$  is non-abelian.

**Problem 2023-J-I-5 (Algebra I).** Determine the Galois group of  $x^3 - x^2 - 4$ .

Let  $p(x) = x^3 - x^2 - 4$ . We start by finding the roots of  $p(x)$  over some algebraic closure of  $\mathbb{Q}$ . Observe that 2 is a solution. Using polynomial long division,

$$p(x) = (x - 2)(x^2 + x + 2) \implies x = 2, \frac{-1 \pm \sqrt{-7}}{2}. \quad (65)$$

Hence, the splitting field of  $p(x)$  is  $\mathbb{Q}(\sqrt{7}i)$ . Now since  $\text{Gal}(\mathbb{Q}(\sqrt{7}i)/\mathbb{Q})$  is the group of automorphisms of the splitting field  $\mathbb{Q}(\sqrt{7}i)$  that preserve  $\mathbb{Q}$ . Since there are exactly two automorphisms (namely, the identity permutation fixing  $\sqrt{7}i$  and the conjugation map  $\sqrt{7}i \mapsto -\sqrt{7}i$ ), we conclude that  $\text{Gal}(\mathbb{Q}(\sqrt{7}i)/\mathbb{Q}) \cong \mathbb{Z}_2$ .

**Problem 2023-J-I-5 (Algebra II).** Determine the Galois group of  $x^3 - 2x + 4$ .

Let  $p(x) = x^3 - 2x + 4$ . We start by finding the roots of  $p(x)$  over some algebraic closure of  $\mathbb{Q}$ . Clearly  $-2$  is a root of  $p(x)$ . Using polynomial long division,

$$p(x) = (x + 2)(x^2 - 2x + 2) \implies x = -2, 1 \pm \sqrt{-1}. \quad (66)$$

Hence, the splitting field of  $p(x)$  is  $\mathbb{Q}(i)$ , which is a quadratic extension of  $\mathbb{Q}$ . Now since  $\text{Gal}(\mathbb{Q}(i)/\mathbb{Q})$  is the group of automorphisms of the splitting field  $\mathbb{Q}(i)$  that preserve  $\mathbb{Q}$ , and there exactly two such automorphisms (namely, the identity fixing  $i$ , and the conjugation map  $i \mapsto -i$ ), we conclude that  $\text{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ .

**Problem 2023-J-I-5 (Algebra III).** Determine the Galois group of  $x^3 - x + 1$ .

Let  $p(x) = x^3 - x + 1$ . We start by finding the roots of  $x$  over some algebraic closure of  $\mathbb{Q}$ . Since the only possible rational roots of  $p$  over  $\mathbb{Q}$  are  $\pm 1$  by the Rational Root Test, and neither of these are actually roots of  $p$ , we conclude that  $p$  is irreducible. Hence, a root of  $f(x)$  generates an extension of degree 3 so that the degree of the splitting field of  $F$  is divisible by 3. Since the Galois group is a subgroup of  $S_3$ , either  $\text{Gal}(\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)/\mathbb{Q}) \cong A_3$  or  $\text{Gal}(\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)/\mathbb{Q}) \cong S_3$ . Since  $p$  is already a depressed cubic, we calculate its discriminant to be  $-4(-1)^3 - 27(1)^2 = -23$ . Since the discriminant is not a perfect square in  $\mathbb{Q}$ , we conclude that  $\text{Gal}(\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)/\mathbb{Q}) \cong S_3$ .

**Problem 2023-J-I-4 (Geometry/Topology).** Let  $\omega$  be a smooth nowhere vanishing 1-form on a smooth 3-manifold  $M^3$ .

- (a) Show that the distribution defined at each point  $p \in M$  by

$$\ker \omega_p = \{v \in T_p M^3 : \omega_p(v) = 0\} \quad (67)$$

is integrable if and only if  $\omega \wedge d\omega = 0$ .

- (b) Give an example of a codimension one distribution on  $\mathbb{R}^3$  that is not integrable.

- (a) We recall that a distribution  $D$  is Frobenius integrable if and only if given two smooth sections  $X, Y$  of  $D$ , the Lie Bracket  $[X, Y]$  is also a smooth section of  $D$ . Therefore, let  $X, Y$  be smooth sections of  $D$ , which means that  $\omega(X), \omega(Y) = 0$  by definition of  $D$ . We recall that

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) = -\omega([X, Y]), \quad (68)$$

where the first two terms are identically zero by our hypothesis. Therefore,  $D$  is integrable if and only if  $[X, Y]$  is a smooth section of  $D$  if and only if  $\omega([X, Y]) = 0$ . Now, if  $D$  were integrable, then for any field  $Z$  on  $\mathbb{R}^3$ ,

$$\omega \wedge d\omega(X, Y, Z) = \omega(Z)d\omega(X, Y) = 0, \quad (69)$$

where the other terms vanish by assumption on  $X$  and  $Y$ . Hence, since  $X, Y \in \ker \omega$  were arbitrary and  $Z$  was arbitrary,  $\omega \wedge d\omega = 0$ . On the other hand, if  $\omega \wedge d\omega = 0$ , let  $p \in M$ ,  $Z_p \in T_p M$  with  $\omega_p(Z_p) \neq 0$  and  $X_p, Y_p \in \ker \omega_p$ . Then

$$0 = (\omega \wedge d\omega)_p(X_p, Y_p, Z_p) = \omega_p(Z_p)d\omega_p(X_p, Y_p). \quad (70)$$

Hece,  $d\omega_p(X_p, Y_p) = 0$ . This means that for smooth sections  $X, Y$  of  $\ker \omega$ ,  $d\omega(X, Y) = 0$ , and so  $D$  is integrable.

- (b) Consider the smooth nowhere vanishing 1-form  $\omega = ydx + dy + dz$  on  $\mathbb{R}^3$ , and let  $D$  be the distribution on  $\mathbb{R}^3$  defined at each point  $p \in M$  by  $D_p = \ker \omega_p$ . By the rank-nullity theorem,  $\dim D = \dim T_p \mathbb{R}^3 - \text{rank } \omega = 3 - 1 = 2$ . Hence,  $\text{codim } D = 3 - 2 = 1$ . Next, we observe that  $d\omega = dy \wedge dx$ , which is identically not zero. Then  $\omega \wedge d\omega = dz \wedge dy \wedge dx$ , which is also not identically zero. Hence, by the conclusion in (a),  $D$  is not integrable.

**Problem 2023-J-I-1 (Real Analysis).** Give (with proof) an example of a Banach space  $X$  and a norm closed set  $E \subset X$  that is not weakly closed.

Let  $X = \mathcal{H}$ , where  $\mathcal{H}$  is any infinite-dimensional Hilbert space, and let  $E = \{v_n\}$  be an infinite orthonormal set in  $\mathcal{H}$ . Since  $\|x_i - x_j\| \geq 1$  for any two distinct vectors  $x_i, x_j$ , it follows that if  $\|x_n - x\| \rightarrow 0$ , then the sequence must eventually be constant, which means  $x \in E$ . However,  $E$  is not weakly closed: fix an arbitrary element  $y \in \mathcal{H}$ . Then by Bessel's inequality,

$$\sum_1^\infty |\langle y, v_n \rangle|^2 \leq \|y\|^2, \quad (71)$$

so that the sequence of inner products  $a_n = \langle y, v_n \rangle$  is square summable, so  $a_n \rightarrow 0$ . This means that  $v_n \rightarrow 0$  weakly. However,  $0 \notin E$  since  $\|0\| = 0 \neq 1$ . Hence,  $E$  is not weakly closed.

**Problem 2023-J-I-2 (Complex Analysis).** Set  $\mathbb{D} = \{z \in \mathbb{C} : \|z\| < 1\}$ ,  $f : \mathbb{D} \rightarrow \{w \in \mathbb{C} : e^{-\pi/2} < |w| < e^{\pi/2}\}$  be a holomorphic map satisfying  $f(0) = 1$ . Show that  $|f'(0)| \leq 2$ .

[!! Complete Later]

**Problem 2023-J-II-1 (Real Analysis).** Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous. Show that  $f^{-1}(y) = \{x \in \mathbb{R} : f(x) = y\}$  has Lebesgue measure zero for Lebesgue almost all  $y$ .

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous, and define the following:

$$\begin{aligned}\Gamma &:= \{(x, y) \in \mathbb{R}^2 : f(x) = y\}. \\ \Gamma_x &:= \{y \in \mathbb{R} : f(x) = y\} = \{f(x)\}. \\ \Gamma^y &:= \{x \in \mathbb{R} : f(x) = y\} = f^{-1}(y).\end{aligned}\tag{72}$$

$g : \mathbb{R} \rightarrow [0, \infty)$  defined by  $g(y) = m(f^{-1}(y))$ .

Since  $\Gamma_x$  is a singleton for every  $x \in \mathbb{R}$ , we conclude that  $m(\Gamma_x) = 0$  for all  $x$ . By Fubini-Tonelli, we recall that

$$m^2(\Gamma) = \int_{\mathbb{R}^2} \chi_{\Gamma} dx dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \chi_{\Gamma} dx \right) dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \chi_{\Gamma} dy \right) dx.\tag{73}$$

However,

$$\int_{\mathbb{R}} \chi_{\Gamma} dy = m(\Gamma_x) = 0.\tag{74}$$

This means that

$$\int_{\mathbb{R}} \left( \int_{\mathbb{R}} \chi_{\Gamma} dx \right) dy = 0 \implies \int_{\mathbb{R}} \chi_{\Gamma} dx = 0 \text{ a.e.}\tag{75}$$

Now, we observe that

$$\int_{\mathbb{R}} \chi_{\Gamma} dx = g(y) = m(f^{-1}(y)).\tag{76}$$

Hence, this means that  $m(f^{-1}(y)) = 0$  a.e., which implies that  $f^{-1}(y)$  has Lebesgue measure zero for Lebesgue almost all  $y$ .

**Problem 2023-J-II-2 (Real Analysis).** Suppose that  $f$  is continuous on  $[0, 1]$  and  $\int_0^1 f(x)x^k dx = 0$  for  $k = 0, \dots, n$ . Prove that either  $f$  is identically zero, or  $f$  must change sign at least  $n + 1$  times. We say that  $f$  changes sign  $n$  times if there are points  $x_1 < \dots < x_{n+1}$  so that  $f(x_j)f(x_{j+1}) < 0$  for  $j = 1, \dots, n$ .

Suppose that  $f$  is continuous on  $[0, 1]$  and  $\int_0^1 f(x)x^k dx = 0$  for  $k = 0, \dots, n$ . If  $f$  is identically zero, then the claim is trivial and we are done. So assume that  $f \neq 0$ . Suppose  $f$  changes sign only  $n$  times. By the definition provided above, we can find  $n$  points  $x_1, \dots, x_n$  such that  $f(x_j) = 0$  for each  $j = 1, \dots, n$ . Consider the function  $g(x) = \pm f(x) \cdot \prod_{j=1}^n (x - x_j)$ , which must be continuous on  $[0, 1]$  since it is the product of finitely many continuous functions. For some choice of  $\pm$ ,  $g(x) \geq 0$  for all  $x \in [0, 1]$ . Since  $\prod_{j=1}^n (x - x_j)$  is a polynomial of degree  $n$ , we conclude by the hypothesis that

$$\int_0^1 g(x) dx = 0.\tag{77}$$

Since  $g(x) \geq 0$ , this forces  $g(x) = 0$  and so  $f$  has to be identically zero, which contradicts our hypothesis. Hence, by contradiction,  $f$  has to change at least  $n + 1$  times.

**Problem 2023-A-I-1 (Algebra).** Let  $V$  be an  $n$ -dimensional vector space over a field  $F$ . An element  $A \in \text{End } V$  is called *nilpotent* if  $A^k = 0$  for some  $k > 1$ . Prove that  $A$  is nilpotent if and only if

$$\text{Tr}(\Lambda^i A) = 0, \quad i = 1, \dots, n$$

where  $\Lambda^i A$  denotes the induced action of  $A$  on the wedge product  $\Lambda^i V$  for each  $i$ .

**Problem 2023-A-I-5 (Geometry/Topology).** Let  $T$  be the 2-torus  $S^1 \times S^1$  with an open 2-disk removed:



Show that there is no continuous retraction  $r$  onto its boundary (i.e., no continuous map  $r : T \rightarrow \partial T$  satisfying  $r^2 = r$ ).

Let  $T$  be the 2-torus  $S^1 \times S^1$  with an open 2-disk removed,  $\iota : \partial T \rightarrow T$  the inclusion map, and assume to the contrary that  $r : T \rightarrow \partial T$  is a continuous retraction. Then the composition  $r_* \circ \iota_* : \pi_1(\partial T) \rightarrow \pi_1(\partial T)$  must be the identity map. Since  $\partial T \cong S^1$ ,  $\pi_1(\partial T) = \mathbb{Z}$ , and is generated by the element 1. By a direct computation, since  $\partial_1(T) = \mathbb{Z} * \mathbb{Z}$  is the free product on two generators  $a$  and  $b$   $\iota_*$  maps 1 to the element  $aba^{-1}b^{-1}$ . But then  $r_*$  maps the commutator into the abelian group  $\mathbb{Z}$ , where the commutator must be zero. This contradicts our claim that  $r_* \circ \iota_*$  is the identity map. Hence, by contradiction, there cannot be any continuous retraction of  $T$  onto its boundary.

**Problem 2023-A-I-6 (Complex Analysis).** Let  $\mathbb{D} \subset \mathbb{C}$  be the open unit disk. Is there a holomorphic function  $f$  with  $f(\mathbb{D}) = \mathbb{D}$ ,  $f(0) = f'(0) = 2/3$ ? If so, give a formula. If not, prove that it cannot exist.

The problem lends itself nicely to an application of the Schwarz-Pick Theorem:

**(Schwarz-Pick Theorem)** Let  $f : \mathbb{D} \rightarrow \mathbb{D}$  be holomorphic. If  $|f(z)| \leq 1$  for all  $z$ , and  $f(a) = b$  for some  $a, b \in \mathbb{D}$ , then

$$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}.$$

Now assume that a holomorphic function  $f$  with  $f(\mathbb{D}) = \mathbb{D}$ ,  $f(0) = f'(0) = 2/3$  exists. Then by the Schwarz-Pick Lemma,

$$\frac{2}{3} \leq \frac{1 - 4/9}{1 - 0} = \frac{5}{9} < \frac{2}{3}, \tag{78}$$

which is a contradiction. Hence, no such holomorphic function can exist.

**Problem 2023-A-I-2 (Geometry/Topology).** Let  $f : T^2 \rightarrow S^2$  be a smooth map from the 2-torus to the 2-sphere. Can  $f$  be an immersion? If the answer is yes, give an explicit example. If the answer is no, then give a proof.

We begin by stating and proving a technical lemma, which we will then use in our argument.

**(Comps Lemma)** Let  $M$  and  $N$  be smooth connected  $n$ -manifolds, and  $f : M \rightarrow N$  a (smooth) immersion. If  $M$  is compact and nonempty, then  $N$  is compact and  $f$  is a (smooth) covering map.

*Proof.* Let  $M$  and  $N$  be smooth connected  $n$ -manifolds, and  $f : M \rightarrow N$  an immersion. Since  $\dim M = \dim N = n$ , and  $f$  is an immersion, the map  $df_p : T_p M \rightarrow T_{f(p)} N$  has constant rank

$n$  at every  $p \in M$ . Hence, by the Inverse Function Theorem,  $f$  is a local diffeomorphism. Since local diffeomorphisms are open maps,  $f(M)$  is open in  $N$ . On the other hand, since continuous images of compact sets are compact,  $f(M)$  is compact in  $N$ ; since  $N$  is Hausdorff,  $f(M)$  is closed in  $N$ . Since  $N$  is connected, it follows that  $f(M) = N$ . Therefore,  $N$  is compact. All that remains is to show that  $f$  is a covering map.

Let  $q \in N$ ; by continuity of  $f$ ,  $f^{-1}(q)$  is a closed subset of  $M$ . For each  $x \in f^{-1}(q)$ , there exists an open neighborhood  $U_x$  of  $x$  such that  $f|_{U_x}$  is a diffeomorphism. Since  $M$  is Hausdorff, we can shrink these neighborhoods so that they are pairwise disjoint. This means that each  $x \in f^{-1}(q)$  is isolated, implying that  $f^{-1}(q)$  is discrete. Since  $M$  is compact, it follows that  $f^{-1}(q)$  is finite; let  $f^{-1}(q) = \{x_1, \dots, x_s\}$ . As stated above, for each  $j = 1, \dots, s$ , we may find an open neighborhood  $U'_j$  so that  $f|_{U'_j}$  is a diffeomorphism. Moreover, we can shrink these neighborhoods to obtain a pairwise disjoint collection  $\{\tilde{U}_j\}_1^s$  of neighborhoods. Set  $V = \bigcap_1^s f(\tilde{U}_j)$ . Then taking  $U_j = f^{-1}(V) \cap \tilde{U}_j$ ,  $V$  is an evenly covered neighborhood of  $p$ , so that  $f$  is a covering map.  $\square$

Now assume to the contrary that there exists an immersion  $f : T^2 \rightarrow S^2$ . By the Comps Lemma,  $f$  must be a covering map. Hence, the induced homomorphism of groups  $f_* : \pi_1(T^2) \rightarrow \pi_1(S^2)$  must be injective. Since  $S^2$  is simply connected,  $\pi_1(S^2) \cong \{0\}$ . However,  $\pi_1(T^2)$  is not a trivial group (in fact,  $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$ ). This means that  $f_*$  cannot be injective. Therefore, by contradiction,  $f$  cannot be an immersion. Hence, there exist no immersions from  $T^2$  to  $S^2$ .

**Problem 2023-A-II-1 (Algebra).** A field extension  $K/L$  is called algebraic, if every element in  $K$  satisfies a polynomial equation with coefficients in  $L$ . Let  $F, K, L$  be fields such that  $F \supset K \supset L$ , and  $F/K$  and  $K/L$  are algebraic extensions. Prove that  $F/L$  is also an algebraic extension.

Since subfields of subfields is a subfield,  $L$  is a subfield of  $F$ . Hence, it suffices to show that every element in  $F$  satisfies a polynomial equation with coefficients in  $L$ . Let  $a \in F$ , and let

$$k(x) = k_n x^n + k_{n-1} x^{n-1} + \dots + k_0 \in K[x] \quad (79)$$

such that  $k(a) = 0$ ; this follows since  $F/K$  is an algebraic extension. Each  $k_j \in K$ , and hence is algebraic over  $L$ . Therefore,  $L' = L(k_0, \dots, k_n)$  is a finite extension of  $L$ . Since  $k(a) = 0$  and  $k(x)$  now has its coefficients in  $L'$ , it follows that  $a$  is algebraic over  $L'$  so that  $L'(a)$  is a finite extension of  $L$ . Then since

$$[L(a) : L] = [L(a) : L'][L' : L], \quad (80)$$

it follows that  $L(a)$  is a finite extension of  $L$ . Therefore,  $a$  is algebraic over  $L$ . Since  $a$  was arbitrary,  $F/L$  is an algebraic extension.

**Problem 2023-A-I-2 (Geometry/Topology).** Let  $f : T^2 \rightarrow S^2$  be a smooth map from the 2-torus to the 2-sphere. Can  $f$  be an immersion? If the answer is yes, give an explicit example. If the answer is no, then give a proof.

There cannot be an immersion  $f : T^2 \rightarrow S^2$ . To prove our answer, we will state and proof a technical lemma.

**(Comps Lemma)** Let  $M, N$  be smooth, connected,  $n$ -manifolds and  $f : M \rightarrow N$  a (smooth) immersion. If  $M$  is compact and nonempty, then  $f$  is a (smooth) covering map.

*Proof.* Let  $M, N$  be smooth connected  $n$ -manifolds,  $M$  compact, and  $f : M \rightarrow N$  an immersion. Since  $\dim N = n$  everywhere and  $f$  is an immersion,  $df_p : T_p M \rightarrow T_{f(p)} N$  has constant rank  $n$  everywhere. Hence, by the Inverse Function Theorem,  $f$  is a local diffeomorphism. Let  $q \in N$  so that  $f^{-1}(q) \subset M$  is closed. For each  $x \in f^{-1}(q)$ , there exists a neighborhood  $U_x$  such that  $f|_{U_x} : U_x \rightarrow V_x \subset N$  is a diffeomorphism. Since  $M$  is Hausdorff, we can shrink these neighborhoods so that they are pairwise disjoint. Since every  $x \in f^{-1}(q)$  is now isolated, it

follows that  $f^{-1}(q)$  is discrete. Since  $M$  is compact, we conclude that  $f^{-1}(q)$  must be finite; let  $f^{-1}(q) = \{x_1, \dots, x_s\}$ . As stated above, for each  $j = 1, \dots, s$ , we can find a neighborhood  $U_j$  of  $x_j$  so that  $f|_{U_j} : U_j \rightarrow V_j \subset N$  is a diffeomorphism. Again, since  $M$  is Hausdorff, we can shrink these neighborhoods so that  $U_i \cap U_j = \emptyset$  for all  $i \neq j$ ;  $f$  restricted to each of these shrunken neighborhoods remains a diffeomorphism. Now set  $V = \bigcap_{j=1}^s f(U_j)$ , and define  $\tilde{U}_j \subset M$  by  $\tilde{U}_j = f^{-1}(V) \cap U_j$  for each  $j = 1, \dots, s$ . Hence,  $V$  is an evenly covered neighborhood of  $q \in N$ , which means  $f$  is a covering map. That  $f$  is surjective comes from recognizing that  $f(M) = N$  due to connectedness of  $N$ .  $\square$

Now, assume  $f : T^2 \rightarrow S^2$  is an immersion. Since  $T^2, S^2$  are smooth, connected 2-manifolds, and  $T^2$  is compact and nonempty, by the Comps Lemma,  $f$  is a covering map. Hence, the induced homomorphism  $f_* : \pi_1(T^2) \rightarrow \pi_1(S^2)$  is injective. Since  $S^2$  is simply connected,  $\pi_1(S^2) \cong \{0\}$ . On the other hand,  $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$ . Since the order of  $\pi_1(T^2)$  is more than one,  $f_*$  cannot be injective. Hence,  $f$  cannot be an immersion.

**Problem 2023-A-II-5 (Geometry/Topology).** Let  $(t, x, y, z)$  be the standard coordinate system on  $\mathbb{R}^4$ , and let  $\phi$  be the non-zero smooth 1-form on  $\mathbb{R}^4$  defined by

$$\phi = dt + ydx + zd\gamma.$$

Let  $D$  be the 3-plane field on  $\mathbb{R}^4$  that consists of tangent vectors  $V$  such that  $\phi(V) = 0$ . Is  $D$  Frobenius integrable? Support your answer with a proof.

Let  $D$  be the 3-plane field on  $\mathbb{R}^4$  defined as follows: for each  $p \in \mathbb{R}^4$ ,

$$D_p = \{v \in T_p \mathbb{R}^4 : \phi(v) = 0\} = \ker \phi_p. \quad (81)$$

Hence, by the Frobenius Theorem,  $D$  is Frobenius integrable if and only if  $\phi \wedge d\phi = 0$ . We compute:

$$d\phi = d(dt + ydx + zd\gamma) = d^2t + dy \wedge dx + dz \wedge dy = dy \wedge dx + dz \wedge dy. \quad (82)$$

Therefore,

$$\phi \wedge d\phi = dt \wedge dy \wedge dx + dt \wedge dz \wedge dy + ydx \wedge dz \wedge dy. \quad (83)$$

Since  $\phi \wedge d\phi$  is nowhere vanishing on  $\mathbb{R}^4$ ,  $D$  is not Frobenius integrable.

**Problem 2023-A-I-1 (Algebra).** Let  $V$  be a  $n$ -dimensional vector space over a field  $F$ . An element  $A \in \text{End } V$  is called *nilpotent*, if  $A^k = 0$  for some  $k > 1$ . Prove that  $A$  is nilpotent if and only if

$$\text{Tr}(\Lambda^i A) = 0, \quad i = 1, \dots, n, \quad (84)$$

where  $\Lambda^i A$  denotes the induced action of  $A$  on the wedge product  $\Lambda^i V$  for each  $i$ .

Let  $V$  be a  $n$ -dimensional vector space over a field  $F$ , and let  $A \in \text{End } V$ . Recall that  $\Lambda^i A$ , the induced action of  $A$  on the wedge product  $\Lambda^i V$ , is defined to be

$$(\Lambda^i A)(v_1 \wedge \dots \wedge v_i) = Av_1 \wedge \dots \wedge Av_i, \quad v_j \in V \text{ for all } j = 1, \dots, i. \quad (85)$$

Over an algebraic closure of  $F$ ,  $A$  has eigenvalues  $\lambda_1, \dots, \lambda_n$ . Suppose  $A$  is diagonalizable, with the set of eigenvectors given by  $\{v_1, \dots, v_n\}$ . Then for each  $i = 1, \dots, n$ , since the collection

$$\{v_{j_1} \wedge \dots \wedge v_{j_i} : 1 \leq j_1 < \dots < j_i \leq n\}$$

is a basis of  $\Lambda^i V$ , and for each  $i$ -tuple,  $\Lambda^i A(v_{j_1} \wedge \dots \wedge v_{j_i}) = Av_{j_1} \wedge \dots \wedge Av_{j_i} = (\lambda_{j_1} \cdots \lambda_{j_i})(v_{j_1} \wedge \dots \wedge v_{j_i})$ , it follows that the eigenvalues of  $\Lambda^i A$  are the set of all products of the form  $\lambda_{j_1} \cdots \lambda_{j_i}$  for  $1 \leq j_1 < \dots < j_i \leq n$ , counting for multiplicity. Hence,

$$\text{Tr}(\Lambda^i A) = \sum_{1 \leq j_1 < \dots < j_i \leq n} \lambda_{j_1} \cdots \lambda_{j_i}. \quad (86)$$

If  $A$  is not diagonalizable, since the eigenvalues of  $\Lambda^i A$  depend only on the eigenvalues of  $A$ , we may assume  $A$  is in Jordan normal form. Indeed, if  $A = PJP^{-1}$ , then

$$\Lambda^i(A) = \Lambda^i(PJP^{-1}) = \Lambda^i(P)\Lambda^i(J)\Lambda^i(P^{-1}), \quad (87)$$

so  $\Lambda^i A$  and  $\Lambda^i J$  are similar and therefore have the same eigenvalues. Thus it suffices to compute the eigenvalues of  $\Lambda^i J$ , which are exactly the products  $\lambda_{j_1} \cdots \lambda_{j_i}$  of the eigenvalues of  $A$ .

If  $A$  is nilpotent so that  $A^k = 0$  for some  $k > 1$ , then since  $0 = A^k v = \lambda^k v$  for all eigenvectors  $v$  of  $A$ , it follows that every eigenvalue of  $A$  is zero. Therefore, the above expression implies that  $\text{Tr}(\Lambda^i A) = 0$  for all  $i = 1, \dots, n$ . On the other hand, expanding the characteristic polynomial for  $A$  is given by:

$$p_A(t) = \det(tI - A) = t^n - \text{Tr}(\Lambda^1 A)t^{n-1} + \cdots + (-1)^n \text{Tr}(\Lambda^n A). \quad (88)$$

If  $\text{Tr}(\Lambda^i A) = 0$  for all  $i = 1, \dots, n$ , then we conclude that the characteristic polynomial of  $A$  is precisely  $t^n$ . Therefore,  $A$ 's eigenvalues are all zero. Hence, the minimal polynomial of  $A$  is of the form  $t^k$  for some  $k \leq n$ . This implies that  $A^k = 0$ , and so  $A$  is nilpotent.

**Problem 2023-A-II-6 (Complex Analysis).** Find the number of solutions (counting multiplicity) to  $z^8 - 5z^6 + 2z^3 - z - 1 = 0$  that lie inside the unit disk.

Recall Rouché's Formula, which states that

For any two complex-valued functions  $f$  and  $g$  holomorphic inside some region  $K$  with closed and simple contour  $\partial K$ , if  $|g(z)| < |f(z)|$  on  $\partial K$ , then  $f$  and  $f+g$  have the same number of zeros inside  $K$ , where each zero is counted as many times as its multiplicity.

Pick  $f(z) = 5z^6$  and set  $h(z) = z^8 + 2z^3 - z - 1$  so that  $p(z) = z^8 - 5z^6 + 2z^3 - z - 1 = h(z) - f(z)$ . On the unit disk  $\partial S^1$ , we observe that

$$\begin{aligned} |f(z)| &= |5z^6| = 5 \\ &= 1 + 2 + 1 + 1 \\ &= |z^8| + 2|z^3| + |z| + |1| \\ &\geq |h(z)|. \end{aligned} \quad (89)$$

Hence,  $p(z) = h(z) - f(z)$  has the same number of zeros, counting multiplicity, as  $f(z)$ . Since  $f(z)$  has six zeros in the unit disk, we conclude that  $p(z)$  must also have six zeros inside the unit disk.

**Problem 2023-A-II-4 (Real Analysis).** Let  $\mu$  be a (positive) Borel probability measure on  $[0, 1]$ , such that for all  $t \in [0, 1]$  we have  $\mu(\{t\}) = 0$ . Let  $\mu_n$  be a (positive) Borel probability measure on  $[0, 1]$  for  $n = 1, 2, \dots$ . Suppose  $\mu_n \rightarrow \mu$  in the weak\* topology. Let  $F(t) = \mu([0, t])$  and  $F_n(t) = \mu_n([0, t])$ . Prove that  $F_n \rightarrow F$  uniformly.

## January 2022

**Problem 2022-J-I-1 (Complex Analysis).** Suppose  $f : \mathbb{D} \rightarrow \mathbb{C}$  is a holomorphic function on the unit disk so that

$$f\left(\frac{1}{n}\right) = \frac{5}{n^2}, \quad \forall n \in \mathbb{Z}, n \geq 2.$$

Find  $f''(0)$ .

This problem requires the notion of an analytic continuation:

**(Analytic Continuation)** Let  $f$  be an analytic function defined on a non-empty open subset  $U$  of the complex plane  $\mathbb{C}$ . If  $V$  is a larger open subset of  $\mathbb{C}$  containing  $U$ , and  $F$  is an analytic function defined on  $V$  such that  $F(z) = f(z)$  for all  $z \in U$ , then  $F$  is called an *analytic continuation* of  $f$ . I.e.,  $F|_U = f$ .

For our problem, since the function  $F(z) = 5z^2$  agrees with  $f(z)$  at a sequence of points converging in  $\mathbb{D}$  and  $\mathbb{D}$  is connected,  $f(z) = 5z^2$ . Hence,  $f''(0) = 10$ .

**Problem 2022-J-I-3 (Algebra).** Show that a group of order 1,000,000 contains a proper normal subgroup (i.e., is not simple).

Let  $G$  be a group of order  $1,000,000 = 10^6 = 2^6 \cdot 5^6$ . By Sylow's Theorem,

$$\begin{aligned} n_2 &\in \{1, 5, 5^2, 5^3, 5^4, 5^5, 5^6\} \cap \{2k + 1 : k \in \mathbb{N}\}, \\ n_5 &\in \{1, 2, 4, 8, 16, 32, 64\} \cap \{5k + 1 : k \in \mathbb{N}\} = \{1, 16\}. \end{aligned} \tag{90}$$

If  $n_5 = 1$ , then we are done since the unique Sylow 5-subgroup must necessarily be normal. So suppose  $n_5 = 16$ , and let  $G$  act on  $Syl_5(G)$  by conjugation. This induces a homomorphism  $\varphi : G \rightarrow \varphi(G) \leq S_{16}$ . However,  $|G| = 10^6 = 16! = |S_{16}|$ . This means that  $\varphi$  cannot be an injective homomorphism since if otherwise,  $|\varphi(G)| = |G|$ , but this is impossible since  $|G| \neq |S_{16}|$ . Therefore,  $\ker \varphi$  is a nontrivial normal subgroup of  $G$ . If  $\ker \varphi = G$ , then every Sylow 5-subgroup of  $G$  is normal and is, in fact, unique, which contradicts our hypothesis that  $n_5 = 16$ . Hence,  $\ker \varphi$  is a proper nontrivial normal subgroup of  $G$ , which means that  $G$  cannot be simple.

**Problem 2022-J-II-3 (Real Analysis).** Prove or give a counterexample: if  $E \subset \mathbb{R}$  is a Lebesgue measurable subset of positive Lebesgue measure, then some countable union of translates of  $E$  covers  $\mathbb{R}$ .

The statement is not necessarily true. Consider the *fat Cantor* set, which is a subset of  $\mathbb{R}$  that is nowhere dense and has positive Lebesgue measure  $1/2$ ; call this set  $E$ . Assume to the contrary that  $\mathbb{R}$  is the countable union of translates of  $E$ . Since  $E$  is nowhere dense, each translate of  $E$  must also be nowhere dense. Then  $\mathbb{R}$  is the countable union of nowhere dense sets, which violates the Baire Category Theorem. Hence, the statement is not necessarily true.

**Problem 2022-J-II-4 (Complex Analysis).** Let  $U \subset \mathbb{C}$  be an open subset. Suppose  $f_i : U \rightarrow \mathbb{C}$  is a sequence of holomorphic functions converging uniformly on compact subsets to a function  $f : U \rightarrow \mathbb{C}$ . Show that  $f$  is also holomorphic. Justify each step clearly.

It is sufficient to show that  $f$  is holomorphic on any disk  $\mathbb{D} \subset \mathbb{C}$  with  $\overline{\mathbb{D}} \subset \mathbb{C}$ . Since  $f_i \rightarrow f$  uniformly on  $\mathbb{D}$ ,  $f$  is continuous on  $\mathbb{D}$ . We will use Morera's Theorem, which states the following:

**(Morera's Theorem)** A continuous, complex-valued function  $f$  defined on an open set  $D$  in the complex plane that satisfies

$$\oint_{\gamma} f(z) dz = 0$$

for every closed piecewise  $C^1$  curve  $\gamma$  in  $D$  must be holomorphic on  $D$ .

Let  $\gamma$  be a closed piecewise  $C^1$  curve in  $\mathbb{D}$ . Then

$$\oint_{\gamma} f(z) dz = \oint_{\gamma} \lim_{i \rightarrow \infty} f_i(z) dz = \lim_{i \rightarrow \infty} \oint_{\gamma} f_i(z) dz = \lim_{i \rightarrow \infty} (0) = 0. \tag{91}$$

The first equality follows from the hypothesis that  $f_i \rightarrow f$ ; the second equality follows from the fact that this convergence is uniform; and the third equality follows from the fact that each  $f_i$  is holomorphic. Therefore,  $f$  is also holomorphic.

**Problem 2022-A-I-1 (Geometry/Topology).** Let  $Y \subset \mathbb{R}^3$  be a surface of genus two.



Let  $Z$  be the closure of the bounded component in  $\mathbb{R}^3 \setminus Y$ . (Thus,  $Z$  is the compact domain-with-boundary whose interior is the region inside of  $Y$ .) Prove that the inclusion homomorphism  $\pi_1(Y, \text{pt}) \rightarrow \pi_1(Z, \text{pt})$  is surjective. Then use this to show that the fundamental group  $\pi_1(Y, \text{pt})$  of the surface  $Y$  is non-abelian.

Let  $X \subset Y$  be a figure-eight along the top of the surface. Then  $Z$  is a deformation retract of  $Z$  and  $X \hookrightarrow Z$  induces an isomorphism of fundamental groups. Hence, the image of  $\pi_1(X)$  in  $\pi_1(Y)$  maps isomorphically to  $\pi_1(Z)$ . In particular,  $\pi_1(X) \rightarrow \pi_1(Z)$  is surjective. But since  $\pi_1(Z) \cong \pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$  is non-abelian. Since any homomorphic image of an abelian group is abelian, it follows that  $\pi_1(Y)$  cannot be abelian.

**Problem 2022-A-I-3 (Real Analysis).** Show that  $L^4(X, \mu) \neq L^5(X, \mu)$  if and only if there are subsets of arbitrarily small measure.

( $\Rightarrow$ ) Suppose  $X$  contains subsets of arbitrarily small positive measure. This means that for each positive integer  $n$ , there exists a measurable subset  $\tilde{E}_n$  of  $X$  such that  $\mu(\tilde{E}_n) = 2^{-n}$ . From this collection, we can obtain a sequence of disjoint sets  $\{E_n\}$  such that for each  $n$ ,  $0 < \mu(E_n) < 2^{-n}$ . Define the function

$$f = \sum_1^\infty \mu(E_n)^{-1/5} \chi_{E_n}. \quad (92)$$

We claim that  $f \in L^4(X, \mu)$ :

$$\begin{aligned} \|f\|_4^4 &= \int_X |f|^4 = \int_X \sum_1^\infty |\mu(E_n)|^{-4/5} \chi_{E_n} \\ &= \sum_1^\infty \mu(E_n)^{-4/5} \int_X \chi_{E_n} \\ &= \sum_1^\infty \mu(E_n)^{1/5} = \sum_1^\infty 2^{-n/5} = \frac{2^{4/5}}{2^{4/5} - 2} < \infty. \end{aligned} \quad (93)$$

On the other hand, we claim that  $f \notin L^5(X, \mu)$ :

$$\begin{aligned} \|f\|_5^5 &= \int_X |f|^5 = \int_X \sum_1^\infty \mu(E_n)^{-1} \chi_{E_n} \\ &= \sum_1^\infty \mu(E_n)^{-1} \int_X \chi_{E_n} \\ &= \sum_1^\infty 1 = \infty. \end{aligned} \quad (94)$$

Hence,  $L^4(X, \mu) \neq L^5(X, \mu)$ .

( $\Leftarrow$ ) Now suppose that  $L^4(X, \mu) \neq L^5(X, \mu)$ . Let  $f \in L^4 \setminus L^5$ , and for each positive integer  $n$ , define  $E_n = \{x \in X : |f(x)| > n\}$ . We observe that

$$\infty > \|f\|_4^4 = \int_X |f|^4 \geq \int_{E_n} |f|^4 \geq \int_{E_n} n^4 = n^4 \mu(E_n) \implies \mu(E_n) \leq \frac{\|f\|_4^4}{n^4} \quad (95)$$

I.e., we observe that  $\mu(E_n) \rightarrow 0$  as  $n \rightarrow \infty$ . It suffices to show that each  $E_j$  has positive measure (combined with the previous observation, this yields the result that  $X$  has arbitrarily small positively

measured subsets). Assume to the contrary. Suppose there exists some  $j_0$  for which  $\mu(E_{j_0}) = 0$ . Let  $F_{j_0} = E_{j_0}^c$ . Then we observe that since  $|f| \leq n$  on  $F_{j_0}$ ,

$$\begin{aligned}\|f\|_q^q &= \int |f|^q = \int |f|^q \chi_{F_{j_0}} \\ &= \int |f|^{q-p} |f|^p \chi_{F_{j_0}} \leq n^{q-p} \int |f|^p \chi_{F_{j_0}} \\ &=: n^{q-p} \|f\|_p^p < \infty.\end{aligned}\tag{96}$$

This shows that  $f \in L^p$ , which contradicts our hypothesis that  $f \in L^p \setminus L^q$ . Therefore, by contradiction,  $\mu(E_n) \rightarrow 0$  and  $\mu(E_n) > 0$  for all  $n$ .

**Problem 2022-A-I-4 (Geometry/Topology).** Find the fundamental group of the space of unordered pairs of distinct points of  $S^n$ .

Let  $X$  be the space of unordered pairs of distinct points of  $S^n$ . We claim that  $X$  is homotopy equivalent to the real projective space  $\mathbb{RP}^n$ . Indeed the projective space of dimension  $n$  can be identified to the space of unordered pairs of antipodal points of  $S^n$ , and there is a deformation retraction of the whole space to this subspace: draw through the pair of distinct points a line and take the parallel line through the center of  $S^n$ . This means that  $\pi_1(X) \cong \pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$  for  $n \geq 2$ ,  $\cong \mathbb{Z}$  for  $n = 1$ , and  $\cong \{0\}$  for  $n = 0$ .

**Problem 2022-A-I-5 (Complex Analysis).** If  $\Omega \subset \mathbb{C}$  is simply connected and  $f : \Omega \rightarrow \mathbb{C} \setminus \{0\}$  is a holomorphic function, is there a holomorphic function  $g : \Omega \rightarrow \mathbb{C}$  such that  $f = \exp(g)$  on  $\Omega$ ? Prove or give a counterexample.

Yes, there exists a function  $g : \Omega \rightarrow \mathbb{C}$  such that  $f = \exp(g)$ . Consider the function  $f'/f : \Omega \rightarrow \mathbb{C}$ , which is holomorphic since  $f$  is zero-free. Fix  $z_0 \in \Omega$ , and define

$$h(z) = \int_{\gamma|z} \frac{f'(\zeta)}{f(\zeta)} d\zeta, \quad z \in \Omega, \tag{97}$$

where  $\gamma|z$  is a smooth curve connecting  $z_0$  to  $z$ ; since  $\Omega$  is simply connected, Cauchy's Theorem tells us that  $h(z)$  is independent of the choice of curve  $\gamma|z$ . Then, we note that

$$h'(z) = \frac{f'(z)}{f(z)}. \tag{98}$$

So consider the function  $f \exp(-h)$ . Then

$$\frac{d}{dz} f \exp(-h) = \exp(-h)(f' - h'f) = 0, \tag{99}$$

so that  $f \exp(-h)$  is a non-zero constant  $c$ . So we must have  $f = c \exp(h)$ . Let  $g = h + C$  where  $e^C = c$ . Then

$$\exp(g) = \exp(h + C) = e^C \exp(h) = c \exp(h) = f. \tag{100}$$

The proof concludes. The main ideas are simple connectedness and Cauchy's Theorem, and the fact that for every holomorphic function  $f$ , there exists a primitive holomorphic function  $g$  such that  $g'(z) = f(z)$ . But the existence of a primitive function holds only on simply connected regions.

**Problem 2022-A-II-I (Real Analysis).** Suppose  $E \subset \mathbb{R}^2$  has positive Lebesgue area. Show that  $E$  contains 3 points that form the vertices of an equilateral triangle.

Let  $E \subset \mathbb{R}^2$  be a set of positive Lebesgue measure (we will denote by  $m^2$  the Lebesgue measure on  $\mathbb{R}^2$ ). Let  $\{v_1, v_2\}$  be a collection of unit vectors in  $\mathbb{R}^2$  so that the angle between  $v_1$  and  $v_2$  is  $120^\circ$ , and let  $\beta < 1/3$ . By inner regularity of the Lebesgue measure, there exists a compact set  $K_1 \subset E$  so that

$m^2(K_1) > 0$ . Then by outer regularity of the Lebesgue measure, there exists an open set  $U$  containing  $K_1$  such that  $m^2(U) \leq (1 + \beta)m^2(K_1)$ .

Since  $K_1$  is compact,  $d_1 = d(K_1, U^c)$  is positive; so let  $R = d_1$ , pick an arbitrary  $r \in (0, R)$ , and consider the set  $K_1 + rv_1$ .  $K_1 + rv_1$  has to be contained within  $U$  since otherwise,

$$d(K_1, U^c) < |rv_1| = r < d_1, \text{ which is a contradiction.} \quad (101)$$

Hence,  $K_1 \cup (K_1 + rv_1) \subset U$ , which means

$$m^2(U) \geq m^2(K_1 \cup (K_1 + rv_1)) = m^2(K_1) + m^2(K_1 + rv_1) - m^2(K_1 \cap (K_1 + rv_1)) = 2m^2(K_1) - m^2(K_1 \cap (K_1 + rv_1)), \quad (102)$$

where the last equality follows from translation invariance of the Lebesgue measure. Hence,  $m^2(K_1 \cap (K_1 + rv_1)) = 2m^2(K_1) - m^2(U) \geq (1 - \beta)m^2(K_1) > 0$ . Therefore,  $K_2 := K_1 \cap (K_1 + rv_1)$  is nonempty. Now define  $K_3 = K_2 \cap (K_2 + rv_2)$ . Using the same reasoning as above, we observe that  $K_3 \neq \emptyset$  and  $K_3 \subset K_2$ . Hence, we obtain a nested sequence of sets  $\emptyset \neq K_3 \subset K_2 \subset K_1 \subset E$ . Let  $M \in K_3$ . Since  $K_3 = K_2 \cap (K_2 + rv_1)$ ,  $N = q - rv_2 \in K_2$ . Likewise,  $O = q - rv_2 - rv_1 \in K_1$ . These three points form the vertices of a triangle. Then since

$$\|M - N\| = r, \quad \|N - O\| = r, \quad \|M - O\| = \|r(v_2 + v_1)\| = r \|v_2 + v_1\| = r. \quad (103)$$

**Problem 2022-A-II-4 (Algebra).** Let  $G$  be a finite group in which  $(ab)^p = a^p b^p$  for every  $a, b \in G$ , where  $p$  is a prime dividing  $|G|$ . Prove that the Sylow  $p$ -subgroup of  $G$  is normal in  $G$  (and is in fact unique).

Let  $G$  be a finite group in which  $(ab)^p = a^p b^p$  for every  $a, b \in G$ , where  $p$  is a prime dividing  $|G|$ . Consider the map  $\varphi : G \rightarrow G$  defined by  $\varphi(g) = g^p$ . This map is a homomorphism since for any  $g, h \in G$ ,

$$\varphi(gh) = (gh)^p = g^p h^p = \varphi(g)\varphi(h), \quad (104)$$

where the second equality follows from the hypothesis. Consider the map

$$\varphi^k := \underbrace{\varphi \circ \cdots \circ \varphi}_{k \text{ copies}}, \quad (105)$$

which must also be a homomorphism since the composition of homomorphisms is a homomorphism. The kernel of  $\varphi^k$  consists exactly of those elements  $x \in G$  whose order is a power of  $p$  (i.e.,  $x^{p^r} = 1$  for some positive integer  $r$ ) since

$$\varphi^k(x) = x^{p^k} = x^{p^{r+(k-r)}} = \left(x^{p^r}\right)^{p^{k-r}} = 1^{p^{k-r}} = 1. \quad (106)$$

Hence, since every element with order equal to some power of  $p$  belongs in a Sylow  $p$ -subgroup of  $G$ ,

$$\ker \varphi^k = \bigcup_{P \in \text{Syl}_p(G)} P. \quad (107)$$

Moreover,  $\ker \varphi^k$  must be a  $p$ -subgroup of  $G$  since if not, there exists a prime  $p' \neq p$  dividing  $|\ker \varphi^k|$ , which means by Cauchy's Theorem that  $\ker \varphi^k$  contains an element of order  $p'$  (which is impossible). Hence, since  $\ker \varphi^k$  is a  $p$ -subgroup of  $G$  containing a Sylow  $p$ -subgroup, by maximality of Sylow  $p$ -subgroups,  $\ker \varphi^k$  must be a Sylow  $p$ -subgroup of  $G$ . Hence,  $G$  has a unique Sylow  $p$ -subgroup. And since kernels of homomorphisms are normal subgroups, this Sylow  $p$ -subgroup must be normal.

**Problem 2022-A-II-5.** If  $f : [-1, 2] \rightarrow \mathbb{R}$  is continuous and increasing, show that the set of  $x \in [0, 1]$  where

$$\int_0^1 \frac{f(x+t) - f(x-t)}{t} dt = \infty,$$

has Lebesgue zero measure.

Since  $f$  is continuous and increasing on  $[0, 1]$ , we must have

$$f(x) = a + \int_0^x d\mu(t), \quad (108)$$

where  $\mu$  is a non-atomic measure. Therefore,  $f'(x)$  exists and is finite for Lebesgue almost every  $x$ . Therefore, for almost every  $x$ , there exists a finite  $M < \infty$  so that

$$|f(x+t) - f(x)| \leq M|t|, \quad (109)$$

which means

$$\left| \int_0^1 \frac{f(x+t) - f(x-t)}{t} dt \right| \leq \int_0^1 \frac{|f(x+t) - f(x-t)|}{t} dt \leq \int_0^1 M dt < \infty \quad (110)$$

for Lebesgue almost  $x$ .

**Problem 2021-A-II-6 (Real Analysis).** Suppose  $\mu$  is a finite positive measure of compact support and

$$\int_{\mathbb{R}} x^n d\mu(x) = 0$$

for every  $n \in \{0, 1, 2, \dots\}$ . Show that  $\mu$  is the zero measure.

Suppose  $\mu$  is a finite positive measure of compact support and that

$$\int_{\mathbb{R}} x^n d\mu(x) = 0, \quad \forall n \in \{0, 1, 2, \dots\}. \quad (111)$$

Let  $E$  be the support of  $\mu$ . Our strategy is to show that  $\int_{\mathbb{R}} f d\mu(x) = 0$  for all continuous functions  $f$  on  $\mathbb{R}$ , for which we shall use the Stone-Weierstraß theorem:

**(Stone-Weierstraß Theorem)** Let  $X$  be a compact Hausdorff space,  $C(X, \mathbb{R})$  the space of all continuous functions on  $X$ . Suppose  $\mathcal{B}$  is a subalgebra of  $C(X, \mathbb{R})$  that separates points. If there exists  $x_0 \in X$  such that  $f(x_0) = 0$  for all  $f \in \mathcal{B}$ , then  $\mathcal{B}$  is dense in  $\{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$ . Otherwise,  $\mathcal{B}$  is dense in  $C(X, \mathbb{R})$ .

Since  $\mathbb{R}$  is Hausdorff,  $E$  is Hausdorff; by hypothesis,  $E$  is compact. Hence, the Stone-Weierstrass theorem is applicable for our case. Let  $\mathcal{B}$  be the subalgebra of  $C(E, \mathbb{R})$  that separates points. Then by the theorem,  $\mathcal{B}$  is dense in  $C(E, \mathbb{R})$ , which means that any continuous function can be uniformly approximated by a sequence of polynomials in  $\mathcal{B}$ . Let  $f \in C(E, \mathbb{R})$  be arbitrary, and consider a sequence  $\{p_j(x)\}_1^\infty$  that uniformly converges to  $f$ . Then

$$\int_{\mathbb{R}} f(x) d\mu(x) = \int_E f(x) d\mu(x) = \int_E \lim_{j \rightarrow \infty} p_j(x) d\mu(x) = \lim_{j \rightarrow \infty} \int_E p_j(x) d\mu(x) = \lim_{j \rightarrow \infty} \left[ \sum_{k=1}^{\deg p_j} a_k \int_E x^k d\mu(x) \right] = 0, \quad (112)$$

where the last equality follows from the hypothesis. Hence, since for every continuous function  $f$ , the integral over  $\mathbb{R}$  with respect to  $\mu$  is zero, we conclude that  $\mu$  has to be the zero measure.

## January 2021

**Problem 2021-J-I-2 (Geometry/Topology).** Define an equivalence relation  $\sim$  on the 2-dimensional torus  $S^1 \times S^1$  as follows:  $(x, y) \sim (u, v)$  iff either  $(x, y) = (u, v)$  or  $(x, y) = (v, u)$ . Compute the fundamental group for the quotient space  $S^1 \times S^1 / \sim$  and describe the universal cover for this space.

[!! Complete Later !!]

**Problem 2021-J-1-3 (Real Analysis).** If  $E \subset \mathbb{R}$  is Lebesgue measurable and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz, then show that  $f(E)$  is also Lebesgue measurable.

## August 2021

**Problem 2021-A-I-2 (Complex Analysis).** Let  $P(z) = z^n + a_1 z^{n-1} + \dots + a_n$ . Prove that either there is  $|z| = 1$  such that  $|P(z)| > 1$  or  $P(z) = z^n$ .

This problem requires the Cauchy Integral Formula, which states the following:

**(Cauchy's Integral Formula)** Let  $U$  be an open subset of the complex plane  $\mathbb{C}$ , and suppose the closed disk  $\mathbb{D}$  defined as

$$\mathbb{D} = \{z \in \mathbb{C} : |z - z_0| \leq r\} \quad (113)$$

is completely contained in  $U$ . Let  $f : U \rightarrow \mathbb{C}$  be a holomorphic function. Then for every  $a \in \text{int } \mathbb{D}$ ,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\partial \mathbb{D}} \frac{f(z)}{(z - a)^{n+1}} dz. \quad (114)$$

Returning to our original problem, assume that  $|P(z)| \leq 1$  for all  $|z| = 1$ . Then by the Cauchy Integral Formula,

$$n! = P^{(n)}(0) = \frac{n!}{2\pi i} \oint_{|z|=1} \frac{P(z)}{z^{n+1}} dz = \quad (115)$$

[!! Complete Later !!]

**Problem 2021-A-I-6 (Geometry/Topology).** What connected spaces can be finitely-sheeted covering spaces of a sphere with three handles?

We claim that the finitely-sheeted covering spaces of a sphere with three handles are exactly the closed orientable connected surfaces of genus of the form  $2k + 1$  for some positive integer  $k$ . Let  $M$  be a  $k$ -sheeted covering space of a sphere with three handles. If  $M$  were nonorientable, then since covering maps are local diffeomorphisms and local diffeomorphisms preserve orientability, the sphere with three handles must also be nonorientable, which is a contradiction. Hence,  $M$  has to be orientable. Next, since  $M$  is a  $k$ -sheeted covering space of the sphere with three handles, which has Euler characteristic  $2 - 2(3) = -4$ , we must have

$$2 - 2g_M = \chi(M) = -4k \implies g_M - 1 = 2k \implies g_M = 2k + 1. \quad (116)$$

**Problem 2021-A-II-1 (Geometry/Topology).** Let  $M$  be a compact manifold (without boundary) and  $\pi : M \rightarrow S^1$  a submersion onto the circle. Show that the de Rham group  $H_{\text{dr}}^1(M) \neq 0$ .

Let  $M$  be a compact manifold (without boundary) and  $\pi : M \rightarrow S^1$  a submersion onto the circle. Assume to the contrary that  $H_{\text{dr}}^1(M) = 0$  which means that every closed form on  $M$  is an exact form. Since  $H_{\text{dr}}^1(S^1) \cong \mathbb{R}$ , let  $[\omega]$  be a generator of this cohomology group, where  $\omega$  is a nowhere vanishing closed 1-form on  $S^1$ . Since  $\pi$  is a submersion, the 1-form  $\pi^* \omega$  must also be a nowhere vanishing closed form on  $M$ . By our hypothesis on the de Rham cohomology group in degree one of  $M$ ,  $\pi^* \omega$  is exact, which means there exists a smooth function  $f$  such that  $\pi^* \omega = df$ . Since  $M$  is compact and  $f$  is smooth,  $f$  must attain either a maximum or minimum value at some  $p_0 \in M$ . This means that  $df_{p_0} = 0$ . But this contradicts our claim that  $\pi^* \omega$  is nowhere vanishing. Hence, by contradiction,  $H_{\text{dr}}^1(M) \neq 0$ .

## January 2020

**Problem 2020-J-I-1 (Algebra).** Let  $G$  be a finite non-abelian group, and let  $Z(G)$  denote its center. Prove that  $|Z(G)| \leq \frac{1}{4}|G|$ , and then give an example where equality holds.

Let  $G$  be a finite non-abelian group, and let  $Z(G)$  denote its center. Assume to the contrary that  $|Z(G)| > \frac{1}{4}|G| \implies |G|/|Z(G)| < 4$ . Since  $|Z(G)| \mid |G|$ ,  $|G|/|Z(G)|$  is a positive integer. Therefore, one of the three must necessarily be true: (1)  $|G|/|Z(G)| = 1$ , (2)  $|G|/|Z(G)| = 2$ , (3)  $|G|/|Z(G)| = 3$ . If (1) were true, then since  $|Z(G)| = |G|$ ,  $G$  has to be abelian, which contradicts our hypothesis. If (2) were true, then  $G/Z \cong \mathbb{Z}/2\mathbb{Z}$  which is cyclic. Hence,  $G$  would have to be abelian, which is a contradiction. Finally, if (3) were true, then  $G/Z \cong \mathbb{Z}/3\mathbb{Z}$  which is cyclic. Hence,  $G$  would have to be abelian, which is a contradiction. Hence,  $|Z(G)| \not> \frac{1}{4}|G|$ , which means  $|Z(G)| \leq \frac{1}{4}|G|$ .

**Problem 2020-J-I-4 (Geometry/Topology).** Let  $\theta$  be a closed smooth 1-form on a compact  $C^\infty$  manifold  $M$  with empty boundary, and let  $v$  be a smooth vector field on  $M$ . Prove that the Lie derivative  $\mathcal{L}_v\theta$  vanishes at some point of  $M$ .

Let  $\theta$  be a closed smooth 1-form on a compact  $C^\infty$  manifold  $M$  with empty boundary, and let  $v$  be a smooth vector field on  $M$ . By Cartan's Formula for the Lie derivative,

$$\mathcal{L}_v\theta = i_v(d\theta) + d(i_v\theta), \quad (117)$$

where  $i_v(\cdot)$  denotes the interior product. Since  $\theta$  is a closed 1-form,  $d\theta = 0$ . So  $\mathcal{L}_v\theta = d(i_v\theta)$ . Since  $\theta$  is a 1-form,  $i_v\theta$  is a 0-form on  $M$ , i.e., a smooth function on  $M$ . Since  $M$  is compact,  $i_v\theta$  must attain a extrema at some point in  $M$ , which means that its differential  $d(i_v\theta)$  must vanish where it achieves its maximum or minimum. This then implies that  $\mathcal{L}_v\theta$  vanishes at this point.

## August 2020

**Problem 2020-A-II-1 (Complex Analysis).** How many roots (counted with multiplicity) does the function

$$g(z) = 6z^3 + e^z + 1$$

have in the unit disk  $|z| < 1$ ?

Let  $g(z) = 6z^3 + e^z + 1$ , which is holomorphic. Let  $f(z) = 6z^3$  and  $h(z) = e^z + 1$ . Then on the unit circle  $|z| = 1$ ,

$$\begin{aligned} |h(z)| &\leq |e^z| + 1 \leq e^{|z|} + 1 \\ &\leq e + 1 \\ &< 6 = 6|z|^3 = |f(z)|. \end{aligned} \quad (118)$$

Hence, by Rouché's Formula,  $g(z)$  has the same number of zeros as  $f(z)$ . Counting multiplicity,  $f(z)$  has three solutions in the unit disk, which means that  $g(z)$  also has three solutions in the unit disk.

**Problem 2020-A-II-4 (Geometry/Topology).** Let  $M$  and  $N$  be compact connected orientable smooth manifolds and let  $f : M \rightarrow N$  be a smooth mapping. Recall the degree of  $f$  is the integral

$$\deg(f) = \int_M f^*\omega$$

over  $M$  of the pullback  $f^*\omega$  of any top-degree smooth form  $\omega$  on  $N$  whose integral over  $N$  is one. Recall the degree is an integer, denote it by  $\deg(f)$ . Now consider the map

$$f_\# : \pi_1(M) \rightarrow \pi_1(N)$$

on fundamental groups induced by  $f$ . Suppose that the image of  $f_*$  has finite index,  $\text{ind}(f)$ . Prove that  $\text{ind}(f)$  divides  $\deg(f)$ .

Let  $M, N$  be compact connected orientable smooth manifolds and let  $f : M \rightarrow N$  be a smooth mapping. Suppose that  $H := f_*(\pi_1(M))$  is a subgroup of  $\pi_1(N)$  of finite index  $k$ . This means there exists a  $k$ -sheeted covering  $p : \tilde{N} \rightarrow N$  so that  $p_*(\pi_1(\tilde{N})) = H$ . By the lifting criterion for coverings,  $f$  lifts to a smooth map

$$\tilde{f} : M \rightarrow \tilde{N} \quad (119)$$

such that  $f = p \circ \tilde{f}$ . Let  $\omega$  be a top-degree smooth form on  $N$  whose integral over  $N$  is one. Since  $p : \tilde{N} \rightarrow N$  is a  $k$ -sheeted covering of orientable manifolds, we must have  $\deg(p) = k$ . Therefore,

$$\deg(f) = \deg(p \circ \tilde{f}) = \deg(p) \deg(\tilde{f}) = \text{ind}(f) \cdot \deg(\tilde{f}). \quad (120)$$

Since  $\deg(\tilde{f})$  is an integer, we conclude that  $\text{ind}(f) \mid \deg(f)$ .

**Problem 2020-J-I-2 (Geometry/Topology).** Let  $M$  and  $N$  be smooth compact connected oriented  $n$ -manifolds without boundary. Suppose that  $\pi_1(M)$  is finite, but that  $\pi_1(M)$  is infinite. Prove that every smooth map  $\Psi : M \rightarrow N$  has degree zero.

**Problem 2020-A-II-6 (Real Analysis).** Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a strictly decreasing function.

(a) Prove there is no continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(f(x)) = g(x), \quad \text{for all } x \in \mathbb{R}.$$

(b) Show that there exists a function  $f : [-1, 1] \rightarrow [-1, 1]$  with finitely many points of discontinuity such that

$$f(f(x)) = -x \quad \text{for all } x \in [-1, 1].$$

(a) Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a strictly decreasing function, and assume to the contrary that there exists a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(f(x)) = g(x)$  for all  $x \in \mathbb{R}$ . Then  $f(f(x))$  must be strictly decreasing as well. Suppose  $f$  is strictly decreasing. Then for any  $x_1 < x_2$ ,  $f(x_1) > f(x_2) \implies f(f(x_1)) < f(f(x_2))$  so that  $f \circ f$  is strictly increasing. Suppose  $f$  is strictly increasing. Then for any  $x_1 < x_2$ ,  $f(x_1) < f(x_2)$  so that  $f(f(x_1)) < f(f(x_2))$ , so that  $f$  is again strictly increasing. Hence, we run into a contradiction, which means that such a continuous function  $f$  cannot exist.

(b) We explicitly construct such a function:

$$f(x) = \begin{cases} x - 0.5, & x \in [-0.5, 0] \cup [0.5, 1] \\ x + 0.5, & x \in [-1, -0.5] \cup [0, 0.5]. \end{cases} \quad (121)$$

It is easily checked that this function satisfies the desired properties.

## January 2019

**Problem 2019-J-I-1 (Algebra).** Let  $A$  and  $B$  be  $n \times n$  invertible matrices over complex numbers, satisfying

$$AB = \lambda BA \text{ for some } \lambda \in \mathbb{C}.$$

Prove that  $A^n$  and  $B$  commute.

Let  $A$  and  $B$  be  $n \times n$  invertible matrices over complex numbers so that  $AB = \lambda BA$  for some  $\lambda \in \mathbb{C}$ . Since  $A$  is invertible, left-multiplying both sides by  $A^{-1}$  yields,

$$B = \lambda A^{-1}BA. \quad (122)$$

So taking the determinant, we obtain:

$$\det B = \lambda^n \det A^{-1} \det B \det A = \lambda^n \det A^{-1} \det B \det A = \lambda^n \det B. \quad (123)$$

Since  $B$  is invertible,  $\det B \neq 0$ , which means that  $\lambda^n = 1$  (i.e.,  $\lambda$  is an  $n^{\text{th}}$  root of unity). Now, we claim that for any  $m \in \mathbb{N}$ ,  $A^m B = \lambda^m B A^m$ . By hypothesis, this claim is true for the base case  $m = 1$ . Suppose the claim is true for some  $m \geq 1$ . Then

$$A^{m+1}B = A(A^m B) = \lambda^m(ABA^m) = \lambda^m(\lambda BA)A^m = \lambda^{m+1}BA^{m+1}. \quad (124)$$

Therefore, the claim is true by induction. This implies that

$$A^n B = \lambda^n B A^n = B A^n, \quad (125)$$

so that  $A^n$  and  $B$  commute.

**Problem 2019-J-II-5.** Let  $G$  be a finite group, and let  $H$  be a non-normal subgroup of  $G$  of index  $n$ . Show that if  $|H|$  is divisible by a prime  $p \geq n$ , then  $G$  is not simple.

Let  $G$  be a finite group,  $H$  a non-normal subgroup of  $G$  of index  $n$  such that  $|H|$  is divisible by a prime  $p \geq n$ . Let  $G$  act on the set of left cosets of  $H$ ; this induces a group homomorphism  $\varphi : G \rightarrow S_n$ . Consider the kernel of this group action,  $K = \ker \varphi$ . If  $K = G$ , then for every  $g \in G$ ,  $gHg^{-1} = H$ , which implies that  $H$  is a normal subgroup of  $G$  – a contradiction. Hence,  $\ker \varphi$  is a proper normal subgroup of  $G$ . Likewise,  $\ker \varphi \neq H$  since this equality also forces  $H$  to be normal. All that remains is to show that  $\ker \varphi$  is not trivial. Since  $p \mid |H|$ , let  $P$  be a Sylow  $p$ -subgroup of  $H$ . [!! Complete Later !!]

**Problem 2019-A-I-3 (Geometry/Topology).** Let  $M^n$  be a compact, connected, oriented  $n$ -manifold, and assume that  $n$  is odd. Compute the Euler characteristic  $\chi(M^n \setminus \{\text{pt}\})$ .

Let  $M^n$  be a compact, connected, oriented  $n$ -manifold, where  $n$  is odd. We recall the following two facts:

**(Euler Characteristic)** The Euler characteristic is defined to be the constant:

$$\chi(M^n) := \sum_1^n (-1)^n \text{rank}(H_i(M^n)). \quad (126)$$

**(Euler Characteristic of Disjoint Union)** The Euler characteristic is additive over disjoint unions.

Let  $U = M^n \setminus \{\text{pt}\}$  and  $V$  be an open ball around  $p$  so that  $U \cap V$  is homeomorphic to  $\mathbb{R}^n \setminus \{\text{pt}\}$ . Hence, we obtain

$$\chi(U) + \chi(V) = \chi(M) + \chi(\mathbb{R}^n \setminus \{\text{pt}\}). \quad (127)$$

Since  $V$  is contractible,  $\chi(V) = 0$ ; since  $M$  is a compact, odd-dimensional manifold,  $\chi(M) = 0$ ; since  $\mathbb{R}^n \setminus \{\text{pt}\}$  deformation retracts onto  $S^{n-1}$ , which is an even-dimensional sphere,  $\chi(\mathbb{R}^n \setminus \{\text{pt}\}) = 2$ . Hence,

$$\chi(U) = 2 - 1 = 1. \quad (128)$$

## August 2019

**Problem 2019-A-I-3 (Real Analysis).** For every positive integer  $n$ , let  $T_n : \mathbb{X} \rightarrow \mathbb{Y}$  be a linear transformation with  $\mathbb{X}, \mathbb{Y}$  normed vector spaces. Suppose that  $\|T_n\| \leq 1$ . Let  $\{x_\alpha\}$  be a dense subset of  $\mathbb{X}$ . Suppose that for any  $\alpha$ ,

$$\lim_n T_n(x_\alpha) = y_\alpha$$

for some  $y_\alpha \in \mathbb{Y}$ . Suppose further that if  $x = \lim_i x_{\alpha_i}$ , then  $y_{\alpha_i}$  converges to some  $y_x \in \mathbb{Y}$ . Show that

$$\lim_n T_n(x) = y_x.$$

Assume all of the given hypotheses of the problem. Then

$$\begin{aligned} \|T_n(x) - y_x\| &\leq \|T_n(x) - T_n(x_{\alpha_i})\| + \|T_n(x_{\alpha_i}) - y_{\alpha_i}\| + \|y_{\alpha_i} - y_x\| \\ &\leq \|T_n\| \|x - x_{\alpha_i}\| + \|T_n(x_{\alpha_i}) - y_{\alpha_i}\| + \|y_{\alpha_i} - y_x\| \\ &\leq \|x - x_{\alpha_i}\| + \|T_n(x_{\alpha_i}) - y_{\alpha_i}\| + \|y_{\alpha_i} - y_x\|. \end{aligned} \quad (129)$$

Let  $\varepsilon > 0$ . Choose  $I$  large enough so that for all  $i \geq I$ ,  $\|x - x_{\alpha_i}\|, \|y_x - y_{\alpha_i}\| < \varepsilon/3$ . Now choose  $N$  large enough so that for all  $n \geq N$ ,  $\|T_n(x_{\alpha_i}) - y_{\alpha_i}\| < \varepsilon/3$ . Hence, it follows that  $T_n(x) \rightarrow y_x$ .

## August 2018

**Problem 2018-A-I-2 (Geometry/Topology).** Let  $M^n$  be a compact, connected, oriented  $n$ -manifold, and assume that  $n$  is odd. Compute the Euler characteristic  $\chi(M \setminus \{\text{pt}\})$ .

Let  $U = M^n \setminus \{\text{pt}\}$  and  $V$  be an open ball around  $p$  so that  $U \cap V$  is homeomorphic to  $\mathbb{R}^n \setminus \{\text{pt}\}$ . Hence, we obtain

$$\chi(U) + \chi(V) = \chi(M) + \chi(\mathbb{R}^n \setminus \{\text{pt}\}). \quad (130)$$

Since  $V$  is contractible,  $\chi(V) = 0$ ; since  $M$  is a compact, odd-dimensional manifold,  $\chi(M) = 0$ ; since  $\mathbb{R}^n \setminus \{\text{pt}\}$  deformation retracts onto  $S^{n-1}$ , which is an even-dimensional sphere,  $\chi(\mathbb{R}^n \setminus \{\text{pt}\}) = 2$ . Hence,

$$\chi(U) = 2 - 1 = 1. \quad (131)$$

**Problem 2018-A-II-3 (Analysis).** Suppose  $E, F$  are two measurable subsets of the real numbers that both have positive measure. Prove that  $E + F = \{x + y : x \in E, y \in F\}$  contains an interval.

**Problem 2018-A-I-3 (Complex Analysis).** Show that if  $c > 1$ , then the function

$$f(z) = ze^{c-z} - 1$$

has precisely one root in  $\Delta = \{|z| < 1\}$ , and this root is real and positive.

Let  $c > 1$ , and  $f(z) = ze^{c-z} - 1$ . Let  $g(z) = ze^{c-z}$  and  $h(z) = 1$ . On  $\partial\Delta$ ,

$$|h(z)| = 1 < e^{c-1} = |g(z)|, \quad (132)$$

so that by Rouché's Theorem,  $f(z)$  has the same number of roots as  $g(z)$ . Since the exponential has no roots but the function  $z$  has one root inside the unit disk, we conclude that  $f(z)$  has precisely one root in  $\Delta$ . Now consider the real-valued function  $\tilde{f}(x) = xe^{c-x} - 1$  obtained by restricting  $f(z)$  to the real line. We observe that

$$f(0) = -1 < 0 \quad \text{and} \quad f(1) = e^{c-1} - 1 > 0. \quad (133)$$

Since  $f(z)$  is continuous, it follows from the intermediate value theorem that  $f(x)$  must have a root inside the interval  $(0, 1)$ . Such a root must necessarily be real and positive. Hence, the proof concludes.

**Problem 2018-A-I-5 (Real Analysis).** If  $\{\mu_n\}$  are finite positive measures on  $[0, 1]$  that converge in the weak\* topology to  $\mu$ , prove  $\{\mu_n \times \mu_n\}$  converges in the weak\*-topology to  $\mu \times \mu$ .

Recall that if  $\mu_n \rightarrow \mu$  in the weak\* topology, then for all  $f \in C([0, 1])$ ,

$$\int_{[0,1]} f \, d\mu_n(x) \rightarrow \int_{[0,1]} f \, d\mu(x). \quad (134)$$

Therefore, our goal is to show that for any  $f \in C([0, 1]^2)$ ,

$$\int_{[0,1]^2} f \, d(\mu_n \times \mu_n) \rightarrow \int_{[0,1]^2} f \, d(\mu \times \mu). \quad (135)$$

By the Stone-Weierstrass, functions of the form  $g(x)h(y)$ , where  $g(x), h(y) \in C([0, 1])$  are dense in  $C([0, 1]^2)$ . Hence it suffices to prove that the claim holds for these functions. Since continuous functions on compact spaces are in  $L^1$ , by Fubini-Tonelli,

$$\begin{aligned} \int g(x)h(y) \, d(\mu_n \times \mu_n) &= \left( \int g(x) \, d\mu_n(x) \right) \left( \int h(y) \, d\mu_n(y) \right) \\ &\rightarrow \left( \int g(x) \, d\mu(x) \right) \left( \int h(y) \, d\mu(y) \right) = \int g(x)h(y) \, d(\mu \times \mu). \end{aligned} \quad (136)$$

Hence,  $\mu_n \times \mu_n \rightarrow \mu \times \mu$  in the weak\*-topology.

## January 2017

**Problem 2017-J-I-1 (Geometry/Topology).** Let  $\Sigma_1$  be a torus and let  $\Sigma_2$  be a genus-2 surface. Show that there is no submersion from  $\Sigma_2$  to  $\Sigma_1$ .

Let  $\Sigma_1$  be a torus and  $\Sigma_2$  be a genus-2 surface. We begin with a second modification to the Comps Lemma. Assume to the contrary that  $F$  is a submersion from  $\Sigma_2$  to  $\Sigma_1$ . By the second modification to the Comps Lemma,  $F : \Sigma_2 \rightarrow \Sigma_1$  must be a  $k$ -sheeted covering map for some finite  $k > 0$ . This implies that  $\chi(\Sigma_2) = k \cdot \chi(\Sigma_1)$ , where  $\chi(\cdot) = 2 - 2g$  denotes the Euler characteristic of a closed surface of genus  $g$ . But this is impossible since  $\chi(\Sigma_2) = -2 < 0 = k \cdot 0 = k \cdot \chi(\Sigma_1)$ . Hence, by contradiction, there cannot be any submersions from  $\Sigma_2$  to  $\Sigma_1$ .

**Problem 2017-J-I-6 (Geometry/Topology).** Let  $M$  be a smooth 4-manifold, let  $\phi$  be a 3-form on  $M$ , and let  $U \subset M$  be the open set of points where  $\phi \neq 0$ . Show that  $\phi$  is closed if and only if, near any  $p \in U$ , one can find a smooth coordinate system  $(x^1, x^2, x^3, x^4)$  in which

$$\phi = dx^1 \wedge dx^2 \wedge dx^3.$$

Assume the hypotheses of the problem. Recall that  $\phi$  is closed if and only if  $d\phi$  is identically zero. Let  $p \in U$  and suppose that we can find a smooth coordinate system  $(x^1, x^2, x^3, x^4)$  in some neighborhood of  $p$  in  $U$  so that  $\phi = dx^1 \wedge dx^2 \wedge dx^3$ . Then  $d\phi_p = d^2x^1 \wedge dx^2 \wedge dx^3 + \dots + dx^1 \wedge dx^2 \wedge d^2x^3 = 0$ . Since this is true for all  $p \in U$ , we conclude that  $d\phi$  is identically zero on  $M$ , and hence  $\phi$  is closed.

Now assume that  $\phi$  is closed, which means that  $\phi \wedge d\phi$  is identically zero. At each point  $p \in U$ , define

$$D_p = \ker \phi_p,$$

which is Frobenius integrable by our previous observation. In particular,  $D_p$  is a 1-dimensional distribution. Since  $L$  is integrable, we can find smooth coordinates  $(x^1, \dots, x^4)$  near  $p$  such that  $D_p = \text{span}\{\partial_{x^4}\}$ . Since  $\phi$  annihilates  $\partial_{x^4}$ , it must be a linear combination of  $dx^1, dx^2$ , and  $dx^3$ . Suppose  $\phi = f dx^1 \wedge dx^2 \wedge dx^3$ . Then

$$0 = d\phi = f_{x^1} dx^1 \wedge dx^1 \wedge \dots \wedge dx^3 + f_{x^2} dx^2 \wedge dx^1 \wedge \dots \wedge dx^3 + \dots + f_{x^4} dx^1 \wedge \dots \wedge dx^4. \quad (137)$$

The first three terms are all zero. The last term is zero iff  $f_{x^4} = 0$ , which means  $f = f(x^1, x^2, x^3)$ . [!! Complete Later !!]

**Problem 2017-J-II-1.** Let  $f : M \rightarrow \mathbb{R}$  be a smooth function on a smooth manifold  $M$ . In an arbitrary smooth local coordinate chart  $x : U \rightarrow \mathbb{R}^n$  of  $M$ , define

$$\mathcal{D}f := \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

Does  $\mathcal{D}f$  give a well-defined vector field on  $M$ ?

We claim that  $\mathcal{D}f$  does not give a well-defined vector field on  $M$ . Let  $(U, (x^i))$  and  $(V, (\tilde{x}^i))$  denote two overlapping smooth local coordinate charts on  $M$ , and let  $p \in U \cap V$ . Then

$$\begin{aligned} \mathcal{D}f &= \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i} \Big|_p \\ &= \frac{\partial f}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^j}{\partial x^i} \Big|_p \frac{\partial}{\partial \tilde{x}^k} \Big|_p \frac{\partial}{\partial \tilde{x}^k} \Big|_p, \end{aligned} \tag{138}$$

which is identically not equal to  $(\partial_{\tilde{x}^k} f) \partial_{\tilde{x}^k}$ , which is the expression for  $\mathcal{D}f$  in the smooth coordinate chart  $(V, (\tilde{x}^i))$ .

**Problem 2017-J-II-2 (Real Analysis).** Suppose  $f : [0, 1] \rightarrow \mathbb{R}$  is measurable. Suppose further that for all  $g \in L^2([0, 1])$ , we have that  $fg \in L^2([0, 1])$ . Show that  $f$  is in  $L^\infty([0, 1])$ .

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be measurable, and suppose that for all  $g \in L^2([0, 1])$ ,  $fg \in L^2([0, 1])$ . Assume to the contrary that  $f \notin L^\infty([0, 1])$ , which means that for every positive integer  $n$ , the set

$$E_n = \{x : |f_n(x)| \geq n\} \tag{139}$$

has positive measure. Consider the simple function

$$g = \sum_1^\infty \frac{1}{n\sqrt{m(E_n)}} \chi_{E_n} \tag{140}$$

so that

$$\|g\|_2^2 = \int_0^1 \sum_1^\infty \frac{1}{n^2 m(E_n)} \chi_{E_n} = \sum_1^\infty \frac{1}{n^2} < \infty. \tag{141}$$

On the other hand

$$\|fg\|_2^2 = \int_0^1 |f|^2 \sum_1^\infty \frac{1}{n^2 m(E_n)} \chi_{E_n} \geq \sum_1^\infty \int_{E_n} \frac{1}{m(E_n)} = \sum_1^\infty 1 > \infty, \tag{142}$$

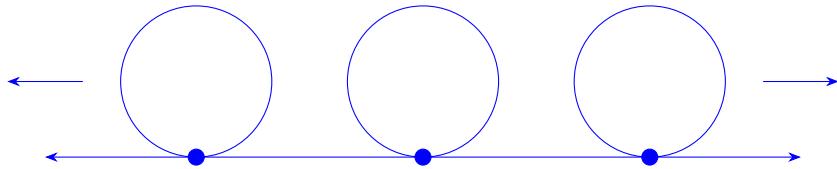
which means  $fg \notin L^2$ . This is a contradiction. Hence, by contradiction,  $f \in L^\infty([0, 1])$ .

**Problem 2017-J-II-4 (Geometry/Topology).** Let  $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$  be the “wedge” of two real projective planes, meaning the quotient space obtained from the disjoint union  $\mathbb{RP}^2 \sqcup \mathbb{RP}^2$  by identifying a single point  $p$  in one copy of  $\mathbb{RP}^2$  with a single point  $\tilde{p}$  in the other. (1) What is the fundamental group of  $X$ ? (2) Give a concrete description of the universal cover  $\tilde{X}$  of  $X$ , accompanied by a drawing of  $X$ .

- (1) Let  $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$ . Since the fundamental group of the wedge product of two manifolds corresponds to the free product of the fundamental groups of each manifold, and the fundamental group of  $\mathbb{RP}^2$  is  $\mathbb{Z}/2\mathbb{Z}$ , we conclude that

$$\pi_1(X) \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}). \tag{143}$$

(2) Recall that  $S^2$  is the universal covering of  $\mathbb{RP}^2$ ; a base point in  $\mathbb{RP}^2$  has two preimages in  $S^2$ . To obtain  $\tilde{X}$ , wedge another sphere to each of the two preimages of the base point. Then wedge another sphere to each of the new preimages of the base point, and keep repeating.  $\tilde{X}$  resembles the following diagram:



## August 2017

**Problem 2017-A-I-1 (Geometry/Topology).** Let  $M$  be a smooth compact connected  $n$ -manifold (without boundary), and let  $F : M \rightarrow \mathbb{R}^n$  be a smooth map. Does  $F$  necessarily have a critical point?

Let  $M$  be a smooth compact connected  $n$ -manifold (without boundary), and let  $F : M \rightarrow \mathbb{R}^n$  be a smooth map. Suppose  $F$  has no critical points, which means that  $dF_p$  is surjective at every  $p \in M$ . I.e.,  $\text{rank } dF_p = n$  for every  $p \in M$ . Let  $F = (f_1, \dots, f_n)$ , where each  $f_j : M \rightarrow \mathbb{R}$  is a component function of  $F$ . Fix some  $f_j$ ; since  $M$  is compact,  $f_j$  must attain a maximum or minimum at some point  $p \in M$ . This means that  $df_j(p) = 0$ . But since  $dF_p = (df_1(p), \dots, df_j(p), \dots, df_n(p))$ ,  $\text{rank } dF_p \neq n$ , which is a contradiction. Hence,  $F$  must have a critical point.

**Problem 2017-A-II-1 (Real Analysis).** Suppose that  $f \in C([0, 1])$  is a continuous real-valued function on  $[0, 1]$  for which

$$\int_0^1 x^n f(x) dx = 0$$

for all non-negative integers  $n$ . Does it follow that  $f(x) = 0$  for all  $x \in [0, 1]$ .

It is necessarily true that  $f(x) = 0$  for all  $x \in [0, 1]$ . By the Stone-Weierstraß Theorem, one can show that the hypothesis forces

$$\int_0^1 f(x) g(x) dx = 0, \quad (144)$$

for all continuous functions  $g(x)$  on  $[0, 1]$ . Taking  $g(x) = f(x)$  so that  $f(x)g(x) = f(x)^2$ , we observe that

$$\int_0^1 f^2(x) dx = 0 \implies f(x)^2 = 0 \implies f(x) = 0. \quad (145)$$

Hence, the proof concludes.

**Problem 2017-A-II-3 (Algebra).** Let  $K$  denote the splitting field of  $f(x) = x^4 + x^2 + 1$  over  $\mathbb{Q}$ . Compute the Galois group  $\text{Gal}(K/\mathbb{Q})$ .

Let  $f(x) = x^4 + x^2 + 1$ ; by the rational root test,  $f(x)$  has no rational roots. However,

$$f(x) = x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1), \quad (146)$$

where each quadratic factor is irreducible by the rational root test. The roots of these quadratic factors are

$$x = \pm \sqrt{\frac{-1 \pm \sqrt{-3}}{2}}. \quad (147)$$

Let  $\alpha = \sqrt{\frac{-1+\sqrt{-3}}{2}}$  and  $\beta = \sqrt{\frac{-1-\sqrt{-3}}{2}}$ . We observe then that  $\alpha^2\beta^2 = 1 \implies \beta = \pm \frac{1}{\alpha}$ . On the other hand,

$$\alpha^2 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}, \quad (148)$$

so that  $\alpha \in \mathbb{Q}(\sqrt{-3})$ . Hence, we conclude that the splitting field of  $f(x)$  over  $\mathbb{Q}$  is  $K = \mathbb{Q}(\sqrt{-3})$ . Since the minimal polynomial of  $\sqrt{-3}$  over  $\mathbb{Q}$  has degree 2,  $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$ . Hence, the Galois group  $\text{Gal}(K/\mathbb{Q})$  has order 2, which means  $\text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ .

## January 2013

**Problem 2013-J-II-6 (Geometry/Topology).** Let  $M$  be a smooth compact manifold, and suppose that there is a smooth map  $F : M \rightarrow S^1$  whose derivative is non-zero at every point. Prove that the de Rham cohomology space  $H_{\text{dr}}^1(M)$  is non-zero.

Let  $M$  be a smooth compact manifold, and  $F : M \rightarrow S^1$  a smooth map whose derivative is non-zero at every point. Assume to the contrary that the de Rham cohomology space  $H_{\text{dr}}^1(M) = 0$ , which means that every closed 1-form on  $M$  is exact. Since  $H_{\text{dr}}^1(S^1) \cong \mathbb{R}$ , there exists a nowhere vanishing closed 1-form  $\omega$  on  $S^1$  such that its equivalence class generates  $H_{\text{dr}}^1(S^1)$ . Then since  $F$  is a smooth map,  $F^*\omega$  is a closed 1-form on  $M$ . Since  $H_{\text{dr}}^1(M) = 0$ ,  $F^*\omega$  is an exact form; i.e., there exists a smooth function:  $f : M \rightarrow \mathbb{R}$  such that  $F^*\omega = df$ . Since  $f$  is smooth and  $M$  is compact,  $f$  must have a maximum or minimum at some point  $p \in M$ , which implies that  $df_p = 0$  at  $p \in M$ . Therefore,  $0 = (F^*\omega)_p = \omega_{F(p)} \circ dF_p$ . Since  $\omega$  is nowhere vanishing, we conclude that  $dF_p = 0$ . But this contradicts our assumption that  $dF$  is non-zero at every point. Hence, by contradiction,  $H_{\text{dr}}^1(M) \neq 0$ .

## August 2013

**Problem 2013-A-II-4 (Geometry/Topology).** Let  $\theta$  be a smooth 1-form on a manifold  $M$  such that  $\theta \neq 0$  everywhere. Let  $D \subset TM$  be the vector subbundle defined by

$$D = \ker \theta = \{v \in TM : \theta(v) = 0\}.$$

Prove that  $D$  is Frobenius integrable if and only if  $\theta \wedge d\theta = 0$  everywhere.

Assume the hypotheses of the problem. We recall that  $D$  is Frobenius integrable if and only if for any pair of smooth sections  $X, Y$  of  $D$ ,  $[X, Y]$  is a smooth section of  $D$ . So let  $X, Y$  be smooth sections of  $D$ , which means that  $\theta(X) = \theta(Y) = 0$  everywhere. Suppose that  $D$  is Frobenius integrable so that  $\theta([X, Y]) = 0$ . Since  $\theta$  is not identically zero, for any  $p \in M$ , there exists a vector  $R_p$  with  $\theta_p(R_p) = 1$ . This means that locally one can choose a smooth vector field  $R$  with  $\theta(R) = 1$ . On this neighborhood, we have  $T_p M = RR_p \oplus D_p$ . Now, we note that

$$\theta \wedge d\theta(X, Y, R) = \theta(X)d\theta(Y, R) + \theta(Y)d\theta(R, X) + \theta(R)d\theta(X, Y). \quad (149)$$

The first two terms are identically zero by our hypothesis. For the latter, we note that

$$d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]), \quad (150)$$

which is identically zero. Hence, this means that  $\theta \wedge d\theta(R, X, Y)$  is zero. This means that  $(\theta \wedge d\theta)_p = 0$  for all  $p \in M$ . Hence,  $\theta \wedge d\theta$  is identically zero. Now suppose  $\theta \wedge d\theta$  is identically zero. Let  $X, Y$  be smooth sections of  $D$  and pick a local vector field  $R$  such that  $\theta(R) = 1$ . We recover once again that

$$0 = \theta \wedge d\theta(X, Y, R) = -\theta(R)\theta([X, Y]) \implies \theta([X, Y]) = 0. \quad (151)$$

Hence,  $[X, Y] \in \Gamma(D)$ , which means that  $D$  is Frobenius integrable.

**Problem 2013-J-II-5 (Real Analysis).** Let  $E \subset [0, 1]$  be a measurable set. Assume  $E$  has positive Lebesgue measure. Show that there are  $\alpha$  and  $\beta$  such that all three numbers  $\alpha, \alpha + \beta, \alpha + 2\beta \in E$ .

Let  $E \subset [0, 1]$  be measurable set with positive Lebesgue measure, and  $\varepsilon < 1/3$ . By inner regularity of the Lebesgue measure, there exists a compact set  $K_1 \subset E$  so that  $m(K_1) > 0$ . By outer regularity of the Lebesgue measure, there exists an open set  $U \supset K_1$  so that

$$m(U) \leq (1 + \varepsilon)m(K_1). \quad (152)$$

Since  $K_1$  is compact, the quantity  $D = d(K_1, U^c) > 0$ . So let  $R = D/2$ , and pick an arbitrary  $\beta \in (0, D/2)$ . We first claim that  $K_1 + \beta \subset U$ , since if not, then

$$d(K_1, U^c) < \beta = \frac{D}{2} < D, \quad (153)$$

which is a contradiction. In particular, this means that  $K_1 \cup (K_1 + \beta) \subset U$  so that

$$m(U) \geq m(K_1 \cup (K_1 + \beta)) = m(K_1) + m(K_1 + \beta) - m(K_1 \cap (K_1 + \beta)). \quad (154)$$

By translation invariance of the Lebesgue measure,  $m(K_1) = m(K_1 + \beta)$  so that

$$m(K_1 \cap (K_1 + \beta)) \geq 2m(K_1) - m(U) \geq 2m(K_1) - (1 + \varepsilon)m(K_1) = (1 - \varepsilon)m(K_1). \quad (155)$$

Since  $\varepsilon < 1$ , we conclude that  $m(K_1 \cap (K_1 + \beta)) > 0$  so that  $K_1 \cap (K_1 + \beta) \neq \emptyset$ . Now for  $j = 1, 2$ , define  $K_{j+1} = K_j \cap (K_j + \beta)$ . Generalizing the arguments from above, we see that  $K_j + \beta \subset U$  for  $j = 1, 2$  and  $m(K_{j+1}) \geq (1 - \varepsilon(2^j - 1))m(K_1) > 0$  so that  $K_1, K_2, K_3$  are nonempty. Hence, this produces a nested sequence of nonempty sets  $\emptyset \neq K_3 \subset K_2 \subset K_1 \subset E$ . Let  $q \in K_3$  be arbitrary; since  $K_3 = K_2 \cap (K_2 + \beta)$ ,  $q - \beta \in K_2$ . And since  $K_2 = K_1 \cap (K_1 + \beta)$ ,  $q - \beta - \beta = q - 2\beta \in K_1$ . Let  $\alpha = q - 2\beta$ . This proves that  $\{\alpha, \alpha + \beta, \alpha + 2\beta\} \subset E$ , concluding the proof.

## Textbook Problems

**Problem Lee-7-5.** Let  $M$  be a smooth compact manifold. Show that there is no submersion  $F : M \rightarrow \mathbb{R}^k$  for any  $k > 0$ .

Let  $M$  be a smooth compact manifold, and assume to the contrary that there exists a submersion  $F : M \rightarrow \mathbb{R}^k$  for some  $k > 0$ . Since  $M$  is compact,  $F$  must attain either a maximum or minimum at some point  $p \in M$ , which means that  $dF_p = 0$ . But this is impossible since  $F$  is a submersion, which means that  $\text{rank } dF_p = \dim \mathbb{R}^k = k > 0$ . Hence, by contradiction,  $F$  cannot be a submersion.

**Problem D&F-14.6.2.** Determine the Galois groups of the following polynomials:

- (i)  $x^3 - x^2 - 4$
- (ii)  $x^3 - 2x + 4$
- (iii)  $x^3 - x + 1$
- (iv)  $x^3 + x^2 - 2x - 1$ .

(a) Let  $f(x) = x^3 - x^2 - 4$ . We note that  $f$  has a rational root  $x = 2$  since  $2^3 - 2^2 - 4 = 8 - 4 - 4 = 0$ . Using polynomial long division, we find that  $f(x)$  is reducible over  $\mathbb{Q}$  as the product

$$f(x) = (x - 2)(x^2 + x + 2). \quad (156)$$

By the rational root test, the quadratic factor is irreducible and has complex roots

$$x_{1,2} = \frac{-1 \pm \sqrt{-7}}{2}. \quad (157)$$

Therefore, the splitting field of  $f(x)$  is  $\mathbb{Q}(\sqrt{-7})$ , which has degree 2 since the minimal polynomial of  $\sqrt{-7}$  is  $x^2 + 7$ . Therefore, the Galois group  $\text{Gal}(\mathbb{Q}(\sqrt{-7})/\mathbb{Q})$  has order 2; hence the Galois group is  $\mathbb{Z}/2\mathbb{Z}$ .

(b) Let  $f(x) = x^3 - 2x + 4$ . We note that  $f(x)$  has a rational root  $x = -2$  since  $(-2)^3 - 2(-2) + 4 = -8 + 4 + 4 = 0$ . Hence using polynomial long division,

$$f(x) = (x + 2)(x^2 - 2x + 2). \quad (158)$$

By the rational root test,  $x^2 - 2x + 2$  is irreducible over  $\mathbb{Q}$  with complex roots  $1 \pm i$ . Therefore, the splitting field of  $f(x)$  is  $\mathbb{Q}(i)$ , which has degree 2 since the minimal polynomial of  $i$  is  $x^2 + 1$ . Therefore, the Galois group  $\text{Gal}(\mathbb{Q}(i)/\mathbb{Q})$  has order 2; hence the Galois group is  $\mathbb{Z}/2\mathbb{Z}$ .

- (c) Let  $f(x) = x^3 - x + 1$ ; by the rational root test  $f(x)$  is irreducible over  $\mathbb{Q}$ . However, since  $f$  is already a depressed cubic, we note that its discriminant is  $-4p^3 - 27q^2 = 4 - 27 = -23$ . Since  $-23$  is not a perfect square, we conclude that the Galois group is  $S_3$ . In fact, the splitting field for this cubic is  $\mathbb{Q}(\alpha, \sqrt{-23})$ , where  $\alpha$  is a root of  $x^3 - x + 1$ .
- (d) Let  $f(x) = x^3 + x^2 - 2x - 1$ ; by the rational root test  $f(x)$  is irreducible over  $\mathbb{Q}$ . Therefore, we will now depress the cubic. Let  $x = y - 1/3$ . Then

$$x^3 + x^2 - 2x - 1 = y^3 - \frac{7}{3}y - \frac{7}{27}. \quad (159)$$

The discriminant of the depressed cubic is,

$$D = -4p^3 - 27q^2 = 4\left(\frac{7^3}{27}\right) - 27\left(\frac{7^2}{27^2}\right) = \frac{7^2}{27}(4 \cdot 7 - 1) = 7^2. \quad (160)$$

Since the discriminant is a square, we see that the Galois group of the polynomial is  $A_3$ .

**Problem D&F-14.6.4.** Determine the Galois group of  $x^4 - 25$ .

Let  $f(x) = x^4 - 25$ . The roots of  $f(x)$  are  $\zeta_4^0 \sqrt[4]{25}$ ,  $\zeta_4^1 \sqrt[4]{25}$ ,  $\zeta_4^2 \sqrt[4]{25}$ , and  $\zeta_4^3 \sqrt[4]{25}$ , where  $\zeta_4$  is the primitive 4th root of unity. Here, we recall that the automorphisms in the Galois group of  $f$  act transitively on the roots of  $f(x)$ . Hence, the Galois group of  $f(x)$  must contain the automorphism that maps  $\sqrt[4]{25} \mapsto -\sqrt[4]{25}$  (i.e., a reflection) and  $\sqrt[4]{25} \mapsto \zeta_4^j \sqrt[4]{25}$  (i.e., a rotation). Hence, the Galois group is  $D_8$ .

**Problem D&F-14.6.5.** Determine the Galois group of  $x^4 + 4$ .

Let  $f(x) = x^4 + 4$ , which is irreducible over  $\mathbb{Q}$ . However, the four roots of  $f(x)$  are  $\pm 1 \pm i$ . This means that the splitting field of  $f(x)$  is  $\mathbb{Q}(i)$ , which is a degree 2 extension over  $\mathbb{Q}$ . Hence, the Galois group is of order 2, which implies that the Galois group is the cyclic group  $\mathbb{Z}/2\mathbb{Z}$ .

**Problem MAT532-F-4.** Suppose  $E \subset \mathbb{R}^2$  is Lebesgue measurable. For a square  $Q$ , let  $C_Q$  be the white squares of a  $(8 \times 8)$  checkerboard fitted exactly in  $Q$  (so a white square has sidelength  $1/8$  the sidelength of  $Q$ ). Suppose that for almost any  $x \in E$ , and any square  $Q_x$  with  $x$  in its lower left corner, we have that  $E \cap C_{Q_x} = \emptyset$ , i.e.,  $E$  does not intersect the white squares of a checkerboard fitted to  $Q_x$ . Show  $m(E) = 0$ , where  $m$  is Lebesgue measure.

Let  $E \subset \mathbb{R}^2$  be Lebesgue measurable, and set  $A = \{x \in E : E \cap C_{Q_x} = \emptyset \text{ for any square } Q_x\}$ ; by hypothesis,  $A$  consists of almost every  $x \in E$ . Assume to the contrary that  $m(E) \neq 0$  and pick  $x \in A$ . For this  $x$ , construct a family of sets  $\{E_r\}_{r>0}$  as follows: for each  $r$ , let  $E_r$  be a square of sidelength  $r/\sqrt{2}$  with  $x$  in its lower left corner. It is straightforward to see that for every  $r > 0$ ,  $E_r \subset B(x, r)$  and  $m(E_r) = 2\pi^{-1}m(B(r, x))$ . Hence,  $\{E_r\}$  shrinks nicely to  $x$ . Now, by hypothesis,  $m(E \cap E_r) \leq \frac{1}{2}m(E_r)$  for every  $r$  since  $E$  intersects at most half of  $E_r$ . This means that

$$\limsup_{r \rightarrow 0} \frac{m(E \cap E_r)}{m(E_r)} \leq \frac{1}{2}. \quad (161)$$

I.e., for almost every  $x \in E$ , the Lebesgue density is at most  $1/2$ , which contradicts the Lebesgue Density Theorem. Therefore, by contradiction,  $m(E) = 0$ .

**Problem MAT532-7-4.** Suppose a set  $E \subset \mathbb{R}^3$  satisfies that for every  $x \in \mathbb{R}^3$  and  $r > 0$ , there exists a point  $z \in B(x, r)$  such that  $E \cap B(z, r/2) \cap B(x, 2r) = \emptyset$ . Show that  $m(E) = 0$ , where  $m$  is the Lebesgue measure on  $\mathbb{R}^3$ .

[!! Complete Later !!]

**Problem (Algebra-Classification-I).** Classify all groups of order 2026.

Let  $G$  be a group of order  $2026 = 2 \cdot 1013$ . By Sylow's Theorem,  $G$  must contain a normal Sylow 5-subgroup, which we denote by  $H$ . Let  $K$  be a Sylow 2-subgroup of  $G$ ; note  $K \cong \mathbb{Z}_2$ . By Lagrange's Theorem,  $H$  and  $K$  must intersect trivially. Moreover,  $|HK| = |H||K|/|H \cap K| = |H||K| = |G|$ , so that  $G = HK$ . Hence, by the recognition theorem for semidirect products,  $G \cong H \rtimes_{\phi} \mathbb{Z}_2$ , where  $\phi \in \text{Aut } H = \mathbb{Z}_{1013}^* \cong \mathbb{Z}_{1012}$ . So we look for homomorphisms  $\phi : \mathbb{Z}_2 \rightarrow \mathbb{Z}_{1012}$ ; each homomorphism is completely determined by where the generator 1 is mapped to.

- (i) Consider the map  $1 \mapsto 0$ , which corresponds to the trivial homomorphism. Then the semidirect product is just the direct product, and so  $G \cong \mathbb{Z}_{1013} \times \mathbb{Z}_2$ .
- (ii) Consider the map  $\phi : 1 \mapsto 506$ , where 506 is the unique element of  $\mathbb{Z}_{1012}$  with order 2. This is a non-trivial homomorphism with kernel  $\{0\}$ . Hence, this gives a non-abelian group  $\mathbb{Z}_{1013} \rtimes_{\phi} \mathbb{Z}_2$ .

Hence, up to isomorphism, there are only two groups of order 2026.

**Problem (Algebra-Classification-II).** Classify all groups of order 1969.

Let  $G$  be a group of order  $1969 = 11 \cdot 179$ . By Sylow's Theorem,  $G$  must contain a normal Sylow 179-subgroup, which we denote by  $H$ . Let  $K$  be a Sylow 11-subgroup of  $G$ ; note  $K \cong \mathbb{Z}_{11}$ . By Langrange's Theorem,  $H$  and  $K$  must intersect trivially and  $G = HK$ . Therefore,  $G = H \rtimes_{\varphi} K$  for some  $\varphi \in \text{Aut } H = \mathbb{Z}_{179}^* \cong \mathbb{Z}_{178}$ . So we look for homomorphisms  $\varphi : \mathbb{Z}_{11} \rightarrow \mathbb{Z}_{178}$ ; each homomorphism is completely determined by where the generator 1 is mapped to.

- (i) Consider the map  $1 \mapsto 0$ . This corresponds to the trivial homomorphism so that the semidirect product is just the direct product. Therefore,  $G \cong \mathbb{Z}_{179} \times \mathbb{Z}_{11} \cong \mathbb{Z}_{1969}$  (by the Chinese Remainder Theorem).
- (ii) Since 1 has order 11, 1 must map to some nonzero element of  $\mathbb{Z}_{178}$  of order 11; but since 11 and 178 are relatively prime, there exists no such element.

Hence, we conclude that there is exactly one group of order 1969, which is precisely  $\mathbb{Z}_{1969}$ .

**Problem 2008-J-I-3 (Algebra).** Classify all groups of order 28.

Let  $G$  be a group of order  $28 = 2^2 \cdot 7$ . By Sylow's Theorem,  $G$  contains a normal Sylow 7-subgroup, which we denote by  $H$ . Let  $K$  be a Sylow 2-subgroup, which has order 4. By Lagrange's Theorem,  $H$  and  $K$  must intersect trivially and  $G = HK$ . Hence, by the recognition theorem for semidirect products,  $G = H \rtimes_{\varphi} K$  for some  $\varphi \in \text{Aut}(H) = \mathbb{Z}_7^* \cong \mathbb{Z}_6$ . So we look for homomorphisms  $\varphi : K \rightarrow \mathbb{Z}_6$ , where  $K$  is a group of order 4. Up to isomorphism, there are precisely two groups of order 4: (1)  $\mathbb{Z}_4$ , and (2)  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . We consider each case separately:

- (I) Consider the case  $K = \mathbb{Z}_4$ , which has two generators: 1 and 3. Each homomorphism  $\varphi : \mathbb{Z}_4 \rightarrow \mathbb{Z}_6$  is determined by where  $\varphi$  sends a generator with the constraint that 1 may be sent to only those elements of  $\mathbb{Z}_6$  whose order divides 4 (namely 0, 3).
  - (i) Suppose  $\varphi_1 : 1 \mapsto 0$ . Then since  $\varphi(3) = 3 \cdot \varphi(1) = 0$ ,  $\varphi$  is the trivial homomorphism. In this case, the semidirect product is the direct product and  $G$  is isomorphic to the abelian group  $\mathbb{Z}_7 \times \mathbb{Z}_4$ .
  - (ii) Suppose  $\varphi_2 : 1 \mapsto 3$ . Then this is a nontrivial homomorphism with image consisting of {0, 3} and kernel consisting of {0, 2}. Hence, this produces a non-abelian group  $\mathbb{Z}_7 \rtimes_{\varphi_2} \mathbb{Z}_4$ .
- (II) Now consider the case  $K = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$ .  $\psi : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_6$  is determined uniquely by  $\psi(a)$  and  $\psi(b)$  provided that its order divides 2. This means  $\psi(a), \psi(b) \in \{0, 3\}$ .
  - (i) Suppose  $\psi_1(a) = \psi_1(b) = 0$ . The semidirect product is then a direct product and so  $G \cong \mathbb{Z}_7 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_{14} \times \mathbb{Z}_2$ .
  - (ii) Suppose  $\psi_2(a) = 0$  and  $\psi_2(b) = 3$ . This is a nontrivial homomorphism so that  $G \cong \mathbb{Z}_7 \rtimes_{\psi_2} \mathbb{Z}_2^2$  is non-abelian.
  - (iii) Suppose  $\psi_3(a) = 3$  and  $\psi_3(b) = 0$ . Then  $\psi_3 = \psi_2 \circ \theta$  where  $\theta$  is the automorphism of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  given by  $\theta(a) = b$  and  $\theta(b) = a$ . Hence, this semidirect product gives the same group as in case (ii).
  - (iv) Suppose  $\psi_4(a) = \psi_4(b) = 3$ . Then  $\psi_4 = \psi_3 \circ \theta$  where  $\theta$  is the automorphism of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  given by  $\theta(a) = a$  and  $\theta(b) = ab$ . Hence, this semidirect product gives the same group as in case (iii).

Altogether, we conclude that there are exactly four isomorphism classes of groups of order 28, namely  $\mathbb{Z}_7 \times \mathbb{Z}_4$ ,  $\mathbb{Z}_7 \rtimes_{\varphi_2} \mathbb{Z}_4$ ,  $\mathbb{Z}_{14} \times \mathbb{Z}_2$ , and  $\mathbb{Z}_7 \rtimes_{\psi_2} \mathbb{Z}_2^2$ , of which exactly two are abelian.

**Problem 2010-J-II-5 (Algebra).** Classify (up to isomorphism) all groups of order 45.

Let  $G$  be a group of order  $45 = 3^2 \cdot 5$ . By Sylow's Theorem,  $G$  has a normal Sylow 5-subgroup, which we denote by  $H$ . Let  $K$  denote a Sylow 3-subgroup of  $G$ , which has order 9. By Lagrange's Theorem,  $H, K$  intersect trivially and  $|G| = |H||K|$  so that  $G = HK$ . Hence,  $G \cong H \rtimes_{\varphi} K$  for some  $\varphi \in \text{Aut}(H) \cong \mathbb{Z}_5^* \cong \mathbb{Z}_4$ . Hence, we look at homomorphisms  $\varphi : K \rightarrow \mathbb{Z}_4$ . There are exactly two groups of order 9, up to isomorphism; namely, these are  $\mathbb{Z}_9$  and  $\mathbb{Z}_3 \times \mathbb{Z}_3$ . Hence, we consider each separately.

- (I) Let  $K = \mathbb{Z}_9$ , which has generators 1, 2, 4, 5, 7, and 8. Each homomorphism  $\varphi : K \rightarrow \mathbb{Z}_4$  is determined uniquely by where  $\varphi$  sends a generator with the constraint that they may only be sent to those elements of  $\mathbb{Z}_4$  whose order divides 9. There is only one such element, namely 0. Hence, the only group we get is the direct product  $\mathbb{Z}_9 \times \mathbb{Z}_5 \cong \mathbb{Z}_{45}$ , which is abelian.
- (II) Let  $K = \mathbb{Z}_3 \times \mathbb{Z}_3$ . Each  $\psi : \mathbb{Z}_3 \times \mathbb{Z}_3 = \langle a \rangle \times \langle b \rangle \rightarrow \mathbb{Z}_4$  is uniquely determined by  $\psi(a)$  and  $\psi(b)$  provided they divide 3. But there is only one such element in  $\mathbb{Z}_4$ , which is zero. Hence, we only get the direct product  $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_3 \times \mathbb{Z}_{15}$ , which is abelian.

Therefore, we find that (1) there are exactly two groups, up to isomorphism, of order 45; and (2) both groups are abelian.

**Problem 2003-J-I-6 (Algebra).**

- (a) Prove that a group of order  $p^2$ , where  $p$  is a prime number, is abelian.  
 (b) Classify groups of order  $p^2$  up to isomorphism.

- (a) Let  $G$  be a group of order  $p^2$ , and let  $Z(G)$  be its center. By Lagrange's Theorem,  $|Z(G)| \in \{1, p, p^2\}$ . If  $|Z(G)| = p^2$  and so  $G = Z(G)$ , which means  $G$  is abelian.  $|Z(G)| \neq p$  since otherwise  $|G/Z(G)| = p$  forcing  $G/Z$  to be cyclic and  $G$  to be abelian (which contradicts  $Z(G)$  being a proper subgroup of  $G$ ). Finally  $|Z(G)|$  cannot be one, since the center of any  $p$ -group must necessarily be nontrivial (by the class equation). Hence,  $Z(G) = G$ , which means  $G$  is abelian.  
 (b) Since every group of order  $p^2$  must necessarily be abelian, up to isomorphism, there must be exactly two groups, namely  $\mathbb{Z}_p \times \mathbb{Z}_p$  and  $\mathbb{Z}_{p^2}$ .

**Problem 2010-J-I-5 (Algebra).** Consider the following irreducible polynomial over  $\mathbb{Q}$ :  $p(x) = x^4 - 3x^2 - 1$ .

- (a) Describe the splitting field of  $p(x)$ .  
 (b) Consider the Galois group of  $p(x)$ . Compute its order and determine if it is abelian.

- (a) Let  $p(x) = x^4 - 3x^2 - 1$ . By the rational root test,  $p(x)$  has no roots over  $\mathbb{Q}$ . Moreover, it is straightforward to check that  $p(x)$  is not the product of irreducible quadratics with rational coefficients. Hence,  $p(x)$  is irreducible over  $\mathbb{Q}$ . We start by finding the roots of  $p(x)$ ; let  $u = x^2$ . Then

$$u^2 - 3u - 1 = 0 \implies u = \frac{3 \pm \sqrt{13}}{2} \implies x = \pm \sqrt{\frac{3 \pm \sqrt{13}}{2}}. \quad (162)$$

Let

$$\alpha = \sqrt{\frac{3 + \sqrt{13}}{2}}, \quad \beta = \sqrt{\frac{3 - \sqrt{13}}{2}}. \quad (163)$$

Observe that  $\alpha^2 \beta^2 = -1$  so that  $\beta = \pm \frac{i}{\alpha}$ . Therefore, the splitting field of  $p(x)$  is

$$\mathbb{Q}(\alpha, i). \quad (164)$$

Observe that the minimal polynomial of  $i$  is  $x^2 + 1$ , which is irreducible over  $\mathbb{Q}(\alpha)$  so that  $[\mathbb{Q}(\alpha, i) : \mathbb{Q}(\alpha)] = 2$ . On the other hand, the minimal polynomial of  $\alpha$  is a degree 4 polynomial so that  $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ . Hence, by the tower law,  $[\mathbb{Q}(\alpha, i) : \mathbb{Q}] = 8$ .

- (b) By the last work in (a), the order of the Galois group of  $p(x)$  is 8. Now, we will determine the Galois group of  $p(x)$ . Recall that elements of  $\text{Gal}(\mathbb{Q}(\alpha, i)/\mathbb{Q})$  are automorphisms  $\varphi$  of the field  $\mathbb{Q}(\alpha, i)$  with the constraints that: (1)  $\varphi$  fixes  $\mathbb{Q}$ , (2)  $\varphi(\alpha)$  must be another root of the minimal polynomial of  $\alpha$  over  $\mathbb{Q}$ , and (3)  $\varphi(i)$  must be another root of  $x^2 + 1$ . We will explicitly work through each of the elements.

- (i)  $\sigma : i \mapsto -i, \alpha \mapsto \alpha$ . This permutation has order 2 since  $\sigma^2(\alpha) = \sigma(\alpha) = \alpha$  and  $\sigma^2(i) = \sigma(-i) = i$ .  
 (ii)  $\tau : i \mapsto i, \alpha \mapsto -\alpha$ . Once again, this permutation has order 2.

(iii)  $\rho : i \mapsto -i, \alpha \mapsto \beta = \frac{i}{\alpha}$ . To compute the order of this permutation, observe that

$$\rho^2(\alpha) = \rho(i\alpha^{-1}) = (-i) \cdot \frac{1}{i/\alpha} = -\alpha \implies \rho^4(\alpha) = \rho^2(-\alpha) = \alpha. \quad (165)$$

Likewise,  $\rho^4(i) = \rho^2(i) = i$ . Hence,  $\rho$  has order 4.

Now, consider the three elements given above. We compute

$$\sigma\rho\sigma(i) = \sigma\rho(-i) = \sigma(i) = -i = \rho^{-1}(i). \quad (166)$$

Likewise,

$$\sigma\rho\sigma(\alpha) = \sigma\rho(\alpha) = \sigma(i)\sigma(\alpha)^{-1} = -\frac{i}{\alpha} = \rho^{-1}(\alpha). \quad (167)$$

Therefore,  $\sigma\rho\sigma = \rho^{-1}$ . Hence,

$$\text{Gal}(\mathbb{Q}(\alpha, i)/\mathbb{Q}) = \{1, \sigma, \rho, \rho^2, \rho^3, \sigma\rho, \sigma\rho^2, \sigma\rho^3\} \cong D_8. \quad (168)$$

Since the dihedral group is not abelian, we conclude that the Galois group for  $p(x)$  is non-abelian.

**Problem 2015-A-II-5 (Algebra).** Find the splitting field and the Galois group of the polynomial  $x^4 - 5x^2 + 5$  over  $\mathbb{Q}$ .

Let  $p(x) = x^4 - 5x^2 + 5$ . By the rational root test,  $p(x)$  has no rational roots. Moreover, it is straightforward to see that  $p(x)$  is not expressible as the product of irreducible quadratics. Hence,  $p(x)$  is irreducible over  $\mathbb{Q}$ . We find its four complex roots as follows. Let  $u = x^2$ . Then

$$u^2 - 5u + 5 = 0 \implies u = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{5 \pm \sqrt{5}}{2} \implies x = \pm \sqrt{\frac{5 \pm \sqrt{5}}{2}}. \quad (169)$$

Let

$$\alpha := \sqrt{\frac{5 + \sqrt{5}}{2}}, \quad \beta := \sqrt{\frac{5 - \sqrt{5}}{2}}. \quad (170)$$

We observe that

$$\alpha^2 = \frac{5}{2} + \frac{\sqrt{5}}{2} \quad \text{and} \quad \alpha^2\beta^2 = 5 \implies \beta = \pm \frac{5}{\alpha}. \quad (171)$$

Therefore, the splitting field is  $\mathbb{Q}(\sqrt{5}, \alpha)$ . Since the minimal polynomial of  $\sqrt{5}$  over  $\mathbb{Q}$  is  $x^2 - 5$ , which has degree 2,  $[\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2$ . On the other hand, the minimal polynomial of  $\alpha$  over  $\mathbb{Q}(\sqrt{5})$  is

$$x^2 - \frac{5 + \sqrt{5}}{2}, \quad (172)$$

so that  $[\mathbb{Q}(\sqrt{5}, \alpha) : \mathbb{Q}] = 2$ . Hence, by the Tower Law,  $[\mathbb{Q}(\sqrt{5}, \alpha) : \mathbb{Q}] = 4$ , which means that the corresponding Galois group has order 4; there are two groups, up to isomorphism, of order 4 (namely  $\mathbb{Z}_2 \times \mathbb{Z}_2$  and  $\mathbb{Z}_4$ ). The elements of  $\text{Gal}(\mathbb{Q}(\sqrt{5}, \alpha)/\mathbb{Q})$  are precisely the automorphisms on  $\mathbb{Q}(\sqrt{5}, \alpha)$  that fix  $\mathbb{Q}$  such that the automorphism group acts transitively on the roots. Consider the permutation  $\rho : \alpha \mapsto -\beta = -\frac{5}{\alpha}$  and  $\rho : \sqrt{5} \mapsto -\sqrt{5}$ . We observe that

$$\begin{aligned} \rho^2(\sqrt{5}) &= \rho(-\sqrt{5}) = \sqrt{5}. \\ \rho^2(\alpha) &= \rho\left(-\frac{5}{\alpha}\right) = -5\rho(\alpha)^{-1} = \alpha \\ \implies \rho^3(\alpha) &= \rho(\alpha) = -5\alpha^{-1} \\ \implies \rho^4(\alpha) &= -5\rho(\alpha)^{-1} = -5 \cdot \left(-\frac{\alpha}{5}\right) = \alpha. \end{aligned} \quad (173)$$

I.e.,  $\rho$  is an element of order 4. Therefore, since only  $\mathbb{Z}_4$  has an element of order 4, we conclude that  $\text{Gal}(\sqrt{5}, \alpha)/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$ .

**Problem 2003-A-II-4 (Algebra).** Let  $E$  be a splitting field of  $f(x) = x^3 + x^2 - 2x - 1$  over the field of rational numbers  $\mathbb{Q}$ . Find the Galois group of  $E/\mathbb{Q}$ . (Hint: first prove that  $f(x) : f(x^2 - 2)$ .) This was the exact notation used in the problem...

Let  $f(x) = x^3 + x^2 - 2x - 1$ . By the rational root test,  $f(x)$  has no rational roots and hence is irreducible over  $\mathbb{Q}$  (being a polynomial of degree 3). Consider the substitution  $x = y - 1/3$ :

$$f(y) = y^3 - \frac{7}{3}y - \frac{7}{27}. \quad (174)$$

The discriminant of this depressed cubic is

$$D = -4p^3 - 27q^2 = 4\left(\frac{7^3}{27}\right) - 27\left(\frac{7^2}{27^2}\right) = 7^2\left(\frac{28}{27} - \frac{1}{27}\right) = 7^2. \quad (175)$$

Since the discriminant is a perfect square, we conclude that the Galois group is  $A_3$ .

**Problem 2014-J-I-5 (Algebra).** Let  $K$  denote the splitting field for  $(x^5 - 1)(x^3 - 2)$  over the rational numbers  $\mathbb{Q}$ . Compute the cardinality of the Galois group  $G$  for the extension  $\mathbb{Q} \subset K$ , and show that  $G$  is not abelian.

Let  $K$  denote the splitting field for  $(x^5 - 1)(x^3 - 2)$ . We note that the splitting field for  $x^3 - 2$  is  $\mathbb{Q}(\sqrt[3]{2}, \zeta_3)$ , where  $\zeta_3$  is the primitive 3rd root of unity, and the splitting field for  $x^5 - 1$  is  $\mathbb{Q}(\zeta_5)$ , where  $\zeta_5$  is the primitive 5th root of unity. Now, since 3 and 5 are relatively prime, the 3rd primitive roots of unity cannot be expressed as a linear combination of 5th roots of unity. Likewise,  $\sqrt[3]{2} \notin \mathbb{Q}(\zeta_5)$ . Hence,  $\mathbb{Q}(\sqrt[3]{2}, \zeta_3) \cap \mathbb{Q}(\zeta_5) = \mathbb{Q}$ , which means that

$$\text{Gal}(K/\mathbb{Q}) \cong \text{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}) \times \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}). \quad (176)$$

From this, we see that the order of  $G$  is 24. Now consider  $\text{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q})$ . The corresponding minimal polynomial is  $x^3 - 2$ , which is a depressed cubic. Since its discriminant is  $-108$ , which is not a square, we conclude that  $\text{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}) \cong S_3$ , which is not abelian. Hence, we conclude that  $G$  is not abelian.

**Problem 2003-J-I-5 (Algebra).** Let  $f(x) = x^5 - 2$ . Find generators and relations for the Galois group  $G := \text{Gal}(F/\mathbb{Q})$  of the splitting field  $F$  of  $f(x)$  over the rational numbers  $\mathbb{Q}$ .

Let  $f(x) = x^5 - 2$ , which has no roots in  $\mathbb{Q}$  by the rational root test. It is also straightforward to check that  $x^5 - 2$  cannot be written as the product of an irreducible cubic and irreducible quadratic so that  $f(x)$  is indeed irreducible over  $\mathbb{Q}$ . The roots of this polynomial are  $\sqrt[5]{2}, \zeta_5 \sqrt[5]{2}, \dots, \zeta_5^4 \sqrt[5]{2}$ , where  $\zeta_5$  is the primitive 5th root of unity. Therefore, the splitting field  $F$  of  $f(x)$  must contain the field  $\mathbb{Q}(\sqrt[5]{2}, \zeta_5)$ . On the other hand, each of the roots mentioned above lie in this field so that  $F = \mathbb{Q}(\sqrt[5]{2}, \zeta_5)$ . Moreover, it follows that  $[F : \mathbb{Q}] = 5 \cdot 4 = 20$  so that  $G$  is a group of order 20. [!! Complete Later !!]

**Problem 2014-J-II-2 (Algebra).** Let  $H$  denote a normal subgroup of the finite group  $G$ . If  $P$  denotes a Sylow  $p$ -subgroup of  $H$ , then prove that  $G = N(P; G)H$  (where  $N(P; G)$  denotes the normalizer of  $P$  in  $G$ ).

Let  $H$  denote a normal subgroup of the finite group  $G$ , and let  $P$  be a Sylow  $p$ -subgroup of  $H$ . Let  $N_G(P) := N(P; G)$  denote the normalizer of  $P$  in  $G$ . Since  $H$  is normal,  $HK \leq G$  for any  $K \leq G$ . In particular,  $N_G(P)H \leq G$ . Hence, it suffices to show that  $G \leq N_G(P)H$ . Since  $P \leq H$  and  $H$  is a normal subgroup of  $G$ , for any  $g \in G$ ,  $gPg^{-1} \leq gHg^{-1} = H$  so that  $gPg^{-1}$  is another Sylow  $p$ -subgroup of  $H$ . On the other hand, we also know that all Sylow  $p$ -subgroups of  $H$  are conjugate by elements of  $H$ . Hence, for each  $g \in G$ , there exists a corresponding  $h \in H$  such that

$$hPh^{-1} = gPg^{-1} \implies (g^{-1}h)P(g^{-1}h)^{-1} = P. \quad (177)$$

I.e.,  $g^{-1}h \in N_G(P)$ , which means  $g \in HN_G(P) = N_G(P)H$ , where the equality stems from  $N_G(P)H$  being a subgroup of  $G$ . Hence, since  $g$  was arbitrary,  $G \subseteq N_G(P)H$ , which concludes the proof.

**Problem 2012-J-II-6 (Real Analysis).** Let  $(V, \|\cdot\|)$  be a normed vector space. Assume that for every sequence  $\{x_n\}_1^\infty$  in  $V$  with  $\sum_1^\infty \|x_n\| < \infty$ , the sequence of partial sums  $\{\sum_1^N x_n\}_1^\infty$  is convergent in  $V$ . Prove that  $V$  is complete.

Let  $(V, \|\cdot\|)$  be a normed vector space so that every absolutely convergent series is convergent. Let  $\{x_n\}_1^\infty$  be a Cauchy sequence in  $V$ . This means that we can find an increasing sequence of positive integers  $n_1 < n_2 < \dots$  such that for each  $j$  and  $n, m > n_j$ ,

$$\|x_n - x_m\| < 2^{-j}. \quad (178)$$

Let  $y_1 = x_{n_1}$ , and  $y_j = x_{n_j} - x_{n_{j-1}}$  for each  $j > 1$ . Then  $\sum_1^k y_{n_j} = x_{n_k}$ . Then we observe that

$$\sum_1^\infty \|y_j\| \leq \|y_1\| + \sum_2^\infty \|y_j\| \leq \|y_1\| + \frac{1}{2} < \infty. \quad (179)$$

Hence, by the hypothesis on  $V$ ,  $\sum_1^\infty y_n$  converges to  $\sum_1^\infty y_n$ . But this means that  $\{x_{n_j}\}$  converges in  $V$ . Since  $\{x_n\}$  is Cauchy, it follows that the sequence converges to the same limit. Hence,  $V$  is complete.

**Problem 2006-A-II-4 (Complex Analysis).** Show that  $z^7 - 4z^3 + z - 1$  has 3 zeros inside the unit circle (counted with multiplicity.)

Let  $f(z) = -4z^3$ , and  $g(z) = z^7 + z - 1$ , both of which are holomorphic functions. For all  $|z| = 1$ , we observe that

$$\begin{aligned} |g(z)| &\leq |z|^7 + |z| + 1 \\ &= 1 + 1 + 1 = 3 \\ &\leq 4 = 4|z|^3 = |f(z)|. \end{aligned} \quad (180)$$

Therefore, by Rouché's Theorem,  $f(z)$  and  $f(z) + g(z)$  have the same number of zeros inside the unit circle (counted with multiplicity).  $f(z)$  has 3 zeros (being a degree 3 polynomial), and  $f(z) + g(z) = z^7 - 4z^3 + z - 1$ . Hence, this concludes the claim.

**Problem 2012-A-I-4 (Complex Analysis).** Find, with proof, the precise number of zeros of the complex polynomial  $p(z) = z^9 - 2z^6 + z^2 - 8z + 2$  inside the annulus  $1 < |z| < 2$ .

Let  $p(z) = z^9 - 2z^6 + z^2 - 8z + 2$ . The number of solutions to  $p(z)$  inside the disk  $\mathbb{D}_2$  of radius 2 must be the sum of the number of the number of solutions inside the unit disk  $\mathbb{D}$  and the number of solutions in the annulus  $1 < |z| < 2$ . First, we compute the number of solutions inside  $\mathbb{D}_2$ .

Let  $f(z) = z^9$ , and  $g(z) = -2z^6 + z^2 - 8z + 2$ . Then on  $\partial\mathbb{D}_2$ ,

$$\begin{aligned} |g(z)| &\leq 2|z|^6 + |z|^2 + 8|z| + 2 \\ &= 2^7 + 2^2 + 16 + 2 = 2(3 + 8 + 2^6) \\ &= 2(11 + 64) = 2(75) = 150 < 2^9 = 8(64) = 512 = |f(z)|. \end{aligned} \quad (181)$$

Hence, by Rouché's Theorem, since  $p(z) = f(z) + g(z)$  and  $f(z)$  has nine roots,  $p(z)$  has nine roots inside  $\mathbb{D}_2$ . Now let  $f(z) = -8z$  and  $g(z) = z^9 - 2z^6 + z^2 + 2$ . On  $\partial\mathbb{D}$ ,

$$\begin{aligned} |g(z)| &\leq 1 + 2 + 1 + 2 = 6 \\ &< 8 = |f(z)|, \end{aligned} \quad (182)$$

Hence, by Rouché's Theorem,  $p(z)$  has the same number of roots as  $f(z)$  in  $\mathbb{D}$ , which is one. Therefore, we conclude that  $p(z)$  has a total of eight solutions inside the annulus  $1 < |z| < 2$ .

**Problem 2015-A-I-3 (Complex Analysis).** Let  $f$  be a holomorphic function defined on a neighborhood of the closed unit disk  $\bar{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ , and assume that  $|f(z)| < 1$  for all  $|z| = 1$ . Determine the number of fixed points of  $f$  in  $\bar{\mathbb{D}}$ .

Let  $f$  be a holomorphic function defined on a neighborhood of the closed unit disk  $\bar{\mathbb{D}}$  such that  $|f(z)| < 1$  for all  $|z| = 1$ . If  $z$  is a fixed point of  $f$  in  $\bar{\mathbb{D}}$ , then  $z$  is a solution to the equation  $f(z) = z$ . Hence, the number of fixed points of  $f$  in  $\bar{\mathbb{D}}$  is equal to the number of solutions to  $f(z) = z$  in  $\bar{\mathbb{D}}$ . Since  $|f(z)| < |z|$  for all  $z \in \partial\bar{\mathbb{D}}$  and  $z$  has exactly one root in  $\bar{\mathbb{D}}$ , we conclude that  $f$  has exactly one fixed point in  $\bar{\mathbb{D}}$ .

**Problem 2014-A-II-5 (Real Analysis).** Suppose  $f$  is a continuous function on  $[0, 1]$  and  $\int_0^1 f(t)t^n dt = 0$  for every  $n = 0, 1, 2, \dots$ . Prove that  $f$  is the zero function.

Let  $f$  be a continuous function on  $[0, 1]$ , and assume that  $\int_0^1 f(t)t^n dt = 0$  for all nonnegative integers  $n$ . We will use the Stone-Weierstraß Theorem in our argument. Let  $g(x)$  be an arbitrary continuous function on  $[0, 1]$ . Since  $[0, 1]$  is compact, and  $\mathbb{R}$  is Hausdorff (which implies that  $[0, 1]$  is Hausdorff), by the Stone-Weierstraß Theorem, there exists a sequence of polynomials  $\{p_j\}_1^\infty$  that converges to  $g$  uniformly. Writing

$$p_j(t) = \sum_{k=1}^{\deg p_j} a_k t^k, \quad (183)$$

we observe the following

$$\begin{aligned} \int_0^1 f(t)g(t) dt &= \int_0^1 \lim_{j \rightarrow \infty} f(t)p_j(t) dt \\ &= \lim_{j \rightarrow \infty} \int_0^1 f(t) \sum_{k=1}^{\deg p_j} a_k t^k dt \\ &= \lim_{j \rightarrow \infty} \left[ \sum_{k=1}^{\deg p_j} a_k \int_0^1 f(t)t^k dt \right] \\ &= \lim_{j \rightarrow \infty} (0) = 0, \end{aligned} \quad (184)$$

where (1) the second equality follows from uniform convergence of the  $p_j$  to  $f$ , and (2) the final line follows from the hypothesis. Hence, since  $g$  was arbitrary, taking  $g = f$  for example, forces  $f$  to be identically zero. Hence, the proof concludes.

**Problem 2016-J-II-3 (Real Analysis).** Suppose  $f$  is a continuous function on  $[0, 1]$  and  $\int_0^1 f(x)x^k dx = 0$  for  $k = 0, \dots, n$ . Prove that either  $f$  is identically zero or  $f$  must change signs at least  $n + 1$  times. (We say  $f$  changes sign  $n + 1$  times if there are points  $0 < x_1 < \dots < x_{n+1} < 1$  so that  $f(x_j)f(x_{j+1}) < 0$  for  $j = 1, \dots, n$ .)

Suppose  $f$  is a continuous function on  $[0, 1]$  so that  $\int_0^1 f(x)x^k dx = 0$  for  $k = 0, \dots, n$ . Suppose  $f$  changes sign  $m \leq n$  times. By definition of “change signs”, there exist  $m$  points  $0 \leq x_1 < x_2 < \dots < x_m \leq 1$  such that  $f(x_j) = 0$  for  $j = 1, \dots, m$ . Then the function  $g(x) = \pm f(x)(x - x_1)\cdots(x - x_m)$  is a continuous nonnegative function on  $[0, 1]$  for some choice of  $\pm$ , chosen so that the sign of the polynomial  $(x - x_1)\cdots(x - x_m)$  is the same as the sign of  $f(x)$  on each interval  $(0, x_1), \dots, (x_{m-1}, x_m)$ . Since  $(x - x_1)\cdots(x - x_m)$  is a polynomial of degree  $m \leq n$ , expanding out the polynomial, and interchanging the sum and the integral, the hypothesis gives us that

$$\int_0^1 g(x)dx = 0. \quad (185)$$

Since  $g(x) \geq 0$ , the above result forces  $g$  to be identically zero. Since  $(x - x_1)\cdots(x - x_m)$  is not identically zero, we conclude that  $f \equiv 0$ . Hence, the proof concludes.

**Problem 2016-J-II-6 (Real Analysis).** Let  $\mathcal{M}$  be a closed linear subspace of  $L^2([0,1];\mathbb{R})$  that is contained in  $C([0,1];\mathbb{R})$ .

- (a) Prove that there exists  $A > 0$  such that

$$\|f\|_u \leq A \|f\|_2$$

for all  $f \in \mathcal{M}$ . Here  $\|\cdot\|_u$  is the uniform norm.

- (b) Prove that  $\dim \mathcal{M} \leq A^2$ . (*Hint:* Show that if  $\{f_j\}$  is an  $L^2$  orthonormal basis of  $\mathcal{M}$ , then  $\sum |f_j|^2 \leq A^2$  for all  $x \in [0,1]$ .)

- (a) For this part of the problem, we required the Closed Graph Theorem, which states the following:

**(Closed Graph Theorem)** If  $\mathcal{X}$  and  $\mathcal{Y}$  are Banach spaces and  $T : \mathcal{X} \rightarrow \mathcal{Y}$  is a closed linear map, then  $T$  is bounded, where  $T$  is a closed linear map iff the following is true: if  $x_n \rightarrow x$  and  $Tx_n \rightarrow y$ , then  $y = Tx$ .

Let  $\iota : \mathcal{M} \hookrightarrow C([0,1];\mathbb{R})$  be the (linear) inclusion map. Since closed linear subspaces of Banach spaces is a Banach space,  $\mathcal{M}$  is a Banach space, itself. Therefore, it suffices to show that  $\iota$  is closed. Suppose  $f_n \rightarrow f$  in  $\mathcal{M}$  (hence,  $f_n \rightarrow f$  in  $L^2([0,1];\mathbb{R})$ ) and  $f_n \rightarrow g$  in  $C([0,1];\mathbb{R})$  (which means that  $f_n \rightarrow g$  in the supremum norm). By definition of the supremum norm, we observe that

$$0 \leq \|f_n - g\|_2^2 = \int_0^1 |f_n - g|^2 dx \leq \int_0^1 \sup_{x \in [0,1]} |f_n - g|^2 dx = \|f_n - g\|_u^2 \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (186)$$

Hence,  $f_n \rightarrow g$  in  $L^2$ . But since  $f_n \rightarrow f$  in  $L^2$  and limits in  $L^2$  are unique, we conclude that  $f = g$  a.e. Therefore,  $\iota$  is closed. By the Closed Graph Theorem, we conclude that  $\iota$  is bounded.

- (b) **[!! Complete Later !!]**

**Problem 2016-A-II-2 (Algebra).** Find the Galois group of the polynomial  $p(x) = x^3 - 2$  over the field  $\mathbb{Z}_{11} := \mathbb{Z}/11\mathbb{Z}$ .

We begin by looking for roots of  $p(x)$  over  $\mathbb{Z}_{11}$ . We find that  $p(x)$  has one root,  $x = 7$ . Therefore, using polynomial long division in  $\mathbb{Z}_{11}$ , we find that

$$p(x) = (x + 4)(x^2 + 7x + 5). \quad (187)$$

Then it is straightforward to verify that the quadratic  $x^2 + 7x + 5$  is irreducible over  $\mathbb{Z}_{11}$  since it has no roots contained in this field. Finding the roots of this quadratic and reducing it modulo 11,

$$x = -\frac{7}{2} \pm \frac{\sqrt{29}}{2} \equiv 2 \pm 6\sqrt{7} \pmod{11}. \quad (188)$$

Since  $\sqrt{7} \notin \mathbb{Z}_{11}$ , the splitting field of  $p(x)$  over  $\mathbb{Z}_{11}$  must contain the field  $\mathbb{Z}_{11}(\sqrt{7})$ . On the other hand, both of the aforementioned roots must lie in  $\mathbb{Z}_{11}(\sqrt{7})$ . Therefore, the splitting field of  $p(x)$  is  $\mathbb{Z}_{11}(\sqrt{7})$ . Since the minimal polynomial of  $\sqrt{7}$  over  $\mathbb{Z}_{11}$  is  $x^2 - 7$  (which is seen easily by verifying that this polynomial is irreducible over this field), it follows that  $[\mathbb{Z}(\sqrt{7}) : \mathbb{Z}] = 2$ . Hence, the Galois group of  $p(x)$  over the field  $\mathbb{Z}_{11}$  must be of order 2; since there is only one group of order 2, up to isomorphism, the Galois group must be  $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ .

**Problem 2016-J-I-2 (Geometry/Topology).** Let  $X = S^1 \times S^1$  be the 2-torus, and let  $Y = X \setminus \{p\}$  be the complement of one point in the 2-torus.

- (a) Prove that there is no covering map  $X \rightarrow Y$ .  
 (b) Prove that there is no covering map  $Y \rightarrow X$ .

- (a) Assume to the contrary that  $f : X \rightarrow Y$  is a covering map. Since  $X$  is compact,  $f(X) = Y$  must also be compact. But this is a contradiction since  $Y$  is not compact. Therefore,  $f$  cannot be surjective and hence not a covering map.
- (b) Suppose there exists a covering map  $f : Y \rightarrow X$ . We recall that  $\pi_1(Y) \cong \mathbb{Z} * \mathbb{Z}$ , while  $\pi_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ . The former is a nonabelian group, which implies that its image under the induced map  $f_*$  must be nonabelian in  $\pi_1(X)$ , which is a contradiction since  $\pi_1(X)$  is abelian. Hence, by contradiction,  $f$  cannot be a covering map.

**Problem 2016-A-I-1 (Algebra).** For  $p(x) = x^5 + 1$  over the field  $\mathbb{F}_{19}$ , find the splitting field  $K$ , find the Galois group  $\text{Aut}(K)$ , and find the action of  $\text{Aut}(K)$  on the 5 roots (up to labeling of the roots).

Let  $p(x) = x^5 + 1$ . We first note that  $p(x)$  has the root  $-1 \equiv 18 \pmod{19}$ . If  $\zeta$  is a fifth root of unity in some field extension of  $\mathbb{F}_{19}$ , then it follows that  $-\zeta_5^n$  is a root for all  $n$ . Since  $5 \mid 19^2 - 1$  but  $5 \nmid 19$ , it follows that  $\zeta_5 \in \mathbb{F}_{19^2}$ . Hence, the splitting field of  $p(x)$  is  $K = \mathbb{F}_{19^2}$ . This means that the Galois group  $\text{Aut}(K)$  has order  $[\mathbb{F}_{19^2} : \mathbb{F}_{19}] = 2$ . Since there is only one group of order 2 up to isomorphism, we conclude that  $\text{Aut}(K) \cong \mathbb{Z}_2$ . Hence, there is only one nontrivial automorphism, which maps each root to its inverse.

**Problem 2016-A-I-1 (Algebra I).** For  $q(x) = x^4 - 2$  over the field  $\mathbb{F}_{11}$ , find the splitting field  $K$ , the Galois group  $\text{Aut}(K)$ , and find the action of  $\text{Aut}(K)$  on the 4 roots (up to labeling of the roots).

Let  $q(x) = x^4 - 2$  over the field  $\mathbb{F}_{11}$ . It is straightforward to see that  $q(x)$  has no roots in this field. However,  $q(x)$  is reducible since

$$q(x) = (x^2 + 4x + 8)(x^2 + 7x + 8). \quad (189)$$

It is then straightforward to see that each of these quadratics are irreducible over  $\mathbb{F}_{11}$ . Let  $\alpha$  be a root of  $x^2 + 4x + 8$  so that  $\mathbb{F}_{11}(\alpha) \cong \mathbb{F}_{11^2}$  since the minimal polynomial of  $\alpha$  has degree 2. Since for each  $n \geq 1$ , there exists a unique extension of degree  $n$ , up to isomorphism, which is  $\mathbb{F}_{11^n}$ , we conclude that the other irreducible quadratic must also split over  $\mathbb{F}_{11^2}$ . Hence, the splitting field of  $q(x)$  is  $\mathbb{F}_{11^2}$ , which is a degree 2 extension. This means that the order of the Galois group of  $q(x)$  is 2. Hence,  $K \cong \mathbb{Z}_2$ . This implies that there exists only one non-trivial automorphism, which is the Frobenius automorphism  $\phi(x) = x^{11}$  maps a root of each irreducible quadratic to its conjugate root.

**Problem 2013-A-I-2 (Algebra).** Find the Galois group of  $x^3 - 2$  over the field  $\mathbb{Z}_5$ .

Let  $p(x) = x^3 - 2$ . We observe the following:

$$\begin{aligned} p(0) &= -2 \equiv 3 \pmod{5} & p(1) &= -1 \equiv 4 \pmod{5} & p(2) &= 6 \equiv 1 \pmod{5} \\ p(3) &= 25 \equiv 0 \pmod{5} & p(4) &= 62 \equiv 2 \pmod{5}. \end{aligned} \quad (190)$$

Hence,  $x = 3$  is a root of  $p(x)$ . In  $\mathbb{Z}_5$ , we can write

$$p(x) = (x + 2)(x^2 + 3x + 4). \quad (191)$$

It is straightforward to check that  $x^2 + 3x + 4$  is irreducible over  $\mathbb{Z}_5$  as it has no roots in this field. The roots of the irreducible quadratic are,

$$x = \frac{-3 \pm \sqrt{9 - 16}}{2} = \frac{-3}{2} \pm \frac{\sqrt{-7}}{2} \equiv 1 \pm 3\sqrt{3} \pmod{5}. \quad (192)$$

Since 3 is not a square in  $\mathbb{Z}_5$ , the splitting field of  $p(x)$  must contain the field  $\mathbb{Z}_5(\sqrt{3})$ . On the other hand, the field  $\mathbb{Z}_5(\sqrt{3})$  contains both of the aforementioned roots of the irreducible quadratic. Therefore, the splitting field of  $p(x)$  is  $\mathbb{Z}_5(\sqrt{3})$ . Since 3 is not a square in  $\mathbb{Z}_5$ , the polynomial  $x^2 - 3$  is the minimal polynomial of  $\sqrt{3}$ . Since this polynomial has degree 2,  $[\mathbb{Z}_5(\sqrt{3}) : \mathbb{Z}_5] = 2$ . Therefore, the Galois group of  $p(x)$  must be of order 2. Thus, we conclude that  $\text{Gal}(\mathbb{Z}_5(\sqrt{3})/\mathbb{Z}_5) \cong \mathbb{Z}_2$ .

**Problem 2014-A-II-1 (Algebra).** Determine the Galois group of the polynomial of the polynomial group  $f(x) = x^4 + 4$  over  $\mathbb{Q}$ .

Let  $f(x) = x^4 + 4$ . The roots of  $f(x)$  are  $\sqrt[4]{-4}, \zeta_4 \sqrt[4]{-4}, \zeta_4^2 \sqrt[4]{-4}$ , and  $\zeta_4^3 \sqrt[4]{-4}$ , where  $\zeta_4$  are the primitive fourth roots of unity. Observe that  $\zeta_4 = e^{2\pi i/4} = i$ . Moreover,

$$-4 = 4e^{i\pi} \implies \sqrt[4]{-4} = \sqrt{2}e^{i\pi/4} = \sqrt{2} \cdot \left( \frac{1+i}{\sqrt{2}} \right) = 1+i. \quad (193)$$

This means that the splitting field of  $f(x)$  must contain  $\mathbb{Q}(i)$ . On the other hand,  $\mathbb{Q}(i)$  contains all of the roots of  $f(x)$ . Hence, the splitting field of  $f(x)$  over  $\mathbb{Q}$  is  $\mathbb{Q}(i)$ . Since the minimal polynomial of  $i$  over  $\mathbb{Q}$  is  $x^2 + 1$ , which is a degree 2 polynomial,  $\text{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \cong \mathbb{Z}_2$ .

**Problem 2011-A-II-1 (Algebra).** Determine the Galois group of  $x^4 - 25$  over  $\mathbb{Q}$ .

Let  $f(x) = x^4 - 25$ . The roots of  $f(x)$  are  $\zeta_4^j \sqrt{5}$  for  $j = 0, \dots, 3$ , where  $\zeta_4 = e^{2\pi i/4} = i$  is the primitive fourth root of unity. Hence, the splitting field must contain the field  $\mathbb{Q}(i, \sqrt{5})$ . On the other hand, the field  $\mathbb{Q}(i, \sqrt{5})$  contains all of the aforementioned roots. Hence, the splitting field of  $f(x)$  over  $\mathbb{Q}$  is  $\mathbb{Q}(i, \sqrt{5})$ . Since the minimal polynomial of  $i$  over  $\mathbb{Q}$  is  $x^2 + 1$ ,  $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ . Likewise, since the minimal polynomial of  $\sqrt{5}$  over  $\mathbb{Q}(i)$  is  $x^2 - 5$ ,  $[\mathbb{Q}(i, \sqrt{5}) : \mathbb{Q}(i)] = 2$ . Therefore, by the tower law,  $\text{Gal}(\mathbb{Q}(i, \sqrt{5})/\mathbb{Q})$  is a group of order 4. Since the Galois group is the group of automorphisms of the field  $\mathbb{Q}(i, \sqrt{5})$  that fixes  $\mathbb{Q}$ , we have the following automorphisms:

- (1)  $e: i \mapsto i, \sqrt{5} \mapsto \sqrt{5}$ : This is the identity permutation.
- (2)  $\sigma: i \mapsto i, \sqrt{5} \mapsto -\sqrt{5}$ : This is a permutation of order 2.
- (3)  $\tau: i \mapsto -i, \sqrt{5} \mapsto \sqrt{5}$ : This is also a permutation of order 2.
- (4)  $\rho: i \mapsto -i, \sqrt{5} \mapsto -\sqrt{5}$ : This is equivalent to  $\sigma\tau = \tau\sigma$ .

Hence, it is straightforward to see that this group is isomorphic to the Klein-4 subgroup  $V$  of  $S_4$ . I.e.,  $\text{Gal}(\mathbb{Q}(i, \sqrt{5})/\mathbb{Q}) \cong V$ .

**Problem 2005-J-I-4 (Algebra).** Find the Galois group of the polynomial  $p(x) = x^3 - 3x + 1$  over  $\mathbb{Q}$ .

By the Rational Root Test,  $p(x)$  has no roots in  $\mathbb{Q}$ , and hence must be irreducible. Therefore, we shall examine its determinant:

$$D = -4p^3 - 27q^2 = -4(-3)^3 - 27(1)^2 = -27(-4+1) = 81. \quad (194)$$

Since 81 is a square in  $\mathbb{Q}$  (namely  $\sqrt{81} = 9$ ), we conclude that the Galois group of the polynomial over  $\mathbb{Q}$  is isomorphic to  $A_3$ .

**Problem 2003-J-I-5 (Algebra).** Let  $f(x) = x^5 - 2$ . Find generators and relations for the Galois group  $G := \text{Gal}(F/\mathbb{Q})$  of the splitting field  $F$  of  $f(x)$  over the rational numbers  $\mathbb{Q}$ .

Let  $f(x) = x^5 - 2$ , which has the roots  $\zeta_5^j \sqrt[5]{2}$  for  $j = 0, \dots, 4$ , where  $\zeta_5$  is the primitive fifth root of unity. The splitting field of  $f(x)$  over  $\mathbb{Q}$  must contain the field  $\mathbb{Q}(\sqrt[5]{2}, \zeta_5)$ . On the other hand, the field  $\mathbb{Q}(\sqrt[5]{2}, \zeta_5)$  contains all of the roots of  $f(x)$ . Hence, the splitting field of the polynomial over  $\mathbb{Q}$  is exactly  $\mathbb{Q}(\sqrt[5]{2}, \zeta_5)$ . Now, since  $\zeta_5^j$  are the roots to  $x^5 - 1$  over  $\mathbb{Q}$ , but  $x^5 - 1$  splits into the product  $(x-1)(x^4 + \dots + 1)$ , the minimal polynomial of  $\zeta_5$  is a degree 4 polynomial, which means that  $[\mathbb{Q}(\zeta_5) : \mathbb{Q}] = 4$ . On the other hand, since  $x^5 - 2$  is irreducible over  $\mathbb{Q}(\zeta_5)$ ,  $[\mathbb{Q}(\sqrt[5]{2}, \zeta_5) : \mathbb{Q}(\zeta_5)] = 5$ . Hence, from the Tower Law,

$$[\mathbb{Q}(\sqrt[5]{2}, \zeta_5) : \mathbb{Q}] = 20 \implies |\text{Gal}(\mathbb{Q}(\sqrt[5]{2}, \zeta_5))| = 20. \quad (195)$$

By classification of groups of order 20,  $G$  must be of the form  $\mathbb{Z}_5 \rtimes_{\phi} (\mathbb{Z}_2 \times \mathbb{Z}_2)$  or  $\mathbb{Z}_5 \rtimes_{\phi} \mathbb{Z}_4$  for some  $\phi \in \text{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$ . Since the automorphism group of  $\mathbb{Q}(\zeta_5)$ , which sends  $\zeta_5 \rightarrow \zeta_5^j$  for some  $j$  coprime

to 5, is cyclic of order 4, we deduce that  $G \cong \mathbb{Z}_5 \times_{\varphi} \mathbb{Z}_4$  for some  $\varphi \in \text{Aut}(\mathbb{Z}_5)$ . It is straightforward to see that the permutations  $\sigma : \sqrt[5]{2} \rightarrow \zeta_5 \sqrt[5]{2}$  (of order 5) and  $\tau : \zeta_5 \rightarrow \zeta_5^2$  are generators of  $G$ . Moreover,

$$\tau\sigma\tau^{-1}(\zeta_5) = \tau(\zeta_5^4) = \tau(\zeta_5^4) = \zeta_5, \quad \text{and} \quad \tau\sigma\tau^{-1}(\sqrt[5]{2}) = \tau(\sqrt[5]{(2)}) = \zeta_5^2 \sqrt[5]{2} = \sigma^2(\sqrt[5]{2}). \quad (196)$$

Hence, we obtain the relation  $\tau\sigma\tau^{-1} = \sigma^2$ . Therefore,

$$G = \langle \sigma, \tau : \sigma^5 = \tau^4 = 1, \tau\sigma\tau^{-1} = \sigma^2 \rangle. \quad (197)$$

**Problem 2010-A-II-2 (Algebra).** Find a rational number  $c$  such that the splitting field over  $\mathbb{Q}$  of the cubic polynomial  $x^3 + cx - 1$  has a nonabelian Galois group over  $\mathbb{Q}$ . For your value of  $c$ , compute the isomorphism type of the Galois group.

Let  $c = 1$  so that  $f(x) = x^3 + x - 1$ . Since  $f(1) = 1 + 1 - 1 = 1$  and  $f(-1) = -1 - 1 - 1 = -3$ , we conclude by the rational root test that  $f(x)$  has no rational roots, and hence is irreducible over  $\mathbb{Q}$ . Moreover, since the discriminant of  $f(x)$  is,

$$\Delta = -4p^3 - 27q^2 = -4(1)^3 - 27(-1)^2 = -4 - 27 = -31, \quad (198)$$

which is *not* a square in  $\mathbb{Q}$ , the Galois group of  $f(x)$  over  $\mathbb{Q}$  must be isomorphic to  $S_3$ , which is not abelian.

**Problem 2004-A-I-4 (Algebra).** Give an example of a Galois extension  $E$  of the rational numbers  $\mathbb{Q}$  such that  $\text{Gal}(E/\mathbb{Q})$  is  $\mathbb{Z}_7$  (the cyclic group of order 7).

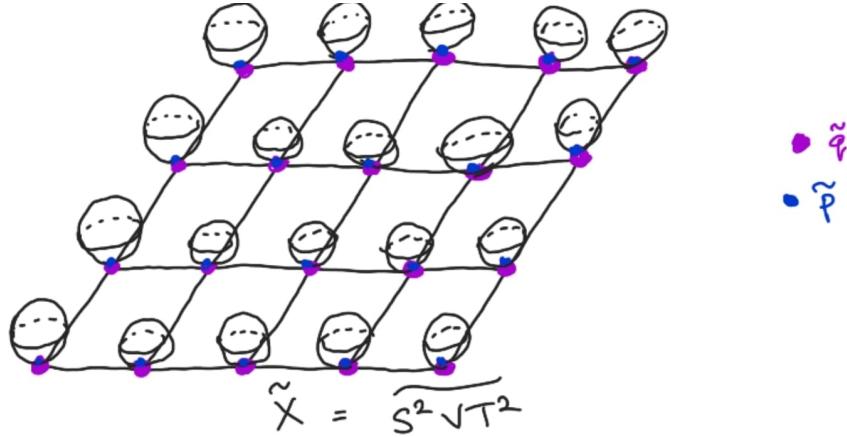
Consider the field extension  $\mathbb{Q}(\zeta_{29})$ , where  $\zeta_{29}$  is the primitive 29th root of unity. Since 29 is prime, we observe that

$$\text{Gal}(\mathbb{Q}(\zeta_{29})/\mathbb{Q}) \cong (\mathbb{Z}/29\mathbb{Z})^*, \quad (199)$$

which is abelian and cyclic of order 28. Since  $7 \mid 28$  and  $4 \mid 28$ ,  $\text{Gal}(\mathbb{Q}(\zeta_{29})/\mathbb{Q})$  has exactly one subgroup of order 7 and one subgroup of order 4. I.e., there exists a unique subfield  $E \subset \mathbb{Q}(\zeta_{29})$  such that  $[E : \mathbb{Q}] = 7$ . Hence, we conclude that  $\text{Gal}(E/\mathbb{Q})$  is a group of order 7, and hence is isomorphic to  $\mathbb{Z}_7$ .

**Problem 2016-A-II-1 (Geometry/Topology).** Let  $X$  be the topological space obtained by making a 2-sphere and a 2-torus touch at a single point. Give a concrete description of the universal cover  $\tilde{X}$  of  $X$ , accompanied by a drawing of  $\tilde{X}$ .

Let  $X = S^2 \vee T^2$ , obtained by identifying some  $p \in S^2$  with  $q \in T^2$ . We recall that the universal cover of  $S^2$  is  $S^2$  (which is a 1-fold covering of itself), and the universal cover of  $T^2$  is  $\mathbb{R}^2$ . The universal cover  $\tilde{X}$  of  $X$  is  $\mathbb{R}^2$ , whose integer lattice points are the lifts  $\tilde{q}$  of  $q$ , with a copy of  $S^2$  attached at each such point by identifying its basepoint (the lift of  $p$ ) with that lattice point. Visually:



**Problem 2011-J-I-3 (Geometry/Topology).** Let  $\Sigma$  be the double torus (i.e., a compact oriented surface of genus 2) and  $T = S^1 \times S^1$  the torus, where  $S^1$  is the circle. Suppose that  $f : \Sigma \rightarrow T$  is continuous. Show that  $f$  is not a local homeomorphism.

We begin with a modified version of the comps lemma.

**(Comps Lemma)** Let  $M$  and  $N$  be smooth connected  $n$ -manifolds, and  $f : M \rightarrow N$  a local homeomorphism. If  $M$  is compact and nonempty, then  $N$  is compact and  $f$  is a covering map.

*Proof.* Let  $M$  and  $N$  be smooth connected  $n$  manifolds,  $f : M \rightarrow N$  a local homeomorphism, and assume  $M$  is compact and nonempty. Since local homeomorphisms are open maps,  $f(M)$  is open in  $N$ . On the other hand, since the continuous image of a compact set is compact, and a compact subset of a Hausdorff space (like  $N$ ) is closed,  $f(M)$  is closed. Since  $f(M)$  is nonempty and  $N$  is connected, we conclude that  $f(M) = N$ . Hence,  $N$  is compact and  $f$  is surjective.

Now we show that  $f$  is a covering map. Let  $y \in N$ . Then for each  $x \in f^{-1}(y)$ , there exists a neighborhood  $U_x$  such that  $f|_{U_x}$  is a homeomorphism. Since  $M$  is Hausdorff, we may shrink these neighborhoods so that each  $x \in M$  is isolated. This means that  $f^{-1}(y)$  is discrete. Since discrete subspaces of compact spaces are finite,  $f^{-1}(y)$  is finite; let  $f^{-1}(y) = \{x_1, \dots, x_s\}$ . As identified earlier, for each  $j = 1, \dots, s$ , there exists a neighborhood  $U_j$  of  $x_j$  such that  $f|_{U_j}$  is a homeomorphism. Since  $M$  is Hausdorff, we may shrink these neighborhoods so that they are pairwise disjoint. Now set  $V = \bigcap_{j=1}^s f(U_j)$ , and let  $\tilde{U}_j = f^{-1}(V) \cap U_j$  for each  $j$  so that  $V$  is an evenly covered neighborhood of  $y$ . This shows that  $f$  is a covering map.  $\square$

Now assume to the contrary that  $f : \Sigma \rightarrow T$  is a local homeomorphism. By the Comps Lemma, it follows that  $f$  is a covering map. Hence, the induced group homomorphism  $f_* : \pi_1(\Sigma) \rightarrow \pi_1(S^1 \times S^1)$  is injective. We recall that  $\pi_1(\Sigma) \cong \mathbb{Z} * \mathbb{Z}$  is non-abelian, while  $\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$ . Since  $f_*$  is injective,  $f_*(\pi_1(\Sigma))$  is a nonabelian subgroup of  $\mathbb{Z} \times \mathbb{Z}$ , which is a contradiction. Hence, by contradiction,  $f$  cannot be a local homeomorphism.

**Problem 2004-A-I-5 (Geometry/Topology).** In this problem, the term “surface” means a compact orientable 2-manifold without boundary.

- (a) Does a surface of genus 1 have a covering space which is homeomorphic to a surface of genus 2? Explain.
- (b) Does a surface of genus 2 have a covering space which is homeomorphic to a surface of genus 3? Explain.
- (a) Let  $\Sigma_1$  denote a surface of genus 1 and  $\Sigma_2$  denote a surface of genus 2; let  $p : X \rightarrow \Sigma_1$  be a covering space. Assume to the contrary that  $X$  is homeomorphic to  $\Sigma_2$ ; this means that  $\pi_1(X) \cong \pi_1(\Sigma_2)$ . Since  $\pi_1(\Sigma_2)$  is nonabelian, the fundamental group of  $X$  must also be nonabelian. On the other hand, since  $p : X \rightarrow \Sigma_1$  is a covering space, the induced group homomorphism  $p_* : \pi_1(X) \rightarrow \pi_1(\Sigma_1)$  is injective. Since  $\pi_1(X)$  is non-abelian,  $p_*(\pi_1(X))$  must be a nonabelian subgroup of  $\pi_1(\Sigma_1)$ . But this is impossible since  $\pi_1(\Sigma_1) \cong \mathbb{Z} \times \mathbb{Z}$  is abelian. Hence,  $X$  cannot be homeomorphic to  $\Sigma_2$ .
- (b) The plot line of this solution is that every surface group is residually finite, and surface groups  $\pi_1(\Sigma_g)$  for  $g \geq 2$  contains subgroups of every finite index. Now suppose  $X$  is a covering space of  $\Sigma_2$  that is homeomorphic to  $\Sigma_3$ . If the covering is  $k$ -sheeted, then

$$\chi(X) = k \cdot \chi(\Sigma_2), \quad (200)$$

where  $\chi(\cdot)$  denotes the Euler characteristic. Since  $\chi(\Sigma_2) = 2 - 2(2) = -2$ , and  $\chi(X) = \chi(\Sigma_3) = 2 - 2(3) = -4$ , it follows that  $k = 2$ . Hence,  $X$  must be a connected 2-sheet covering of  $\Sigma_2$  if it exists. This is where the plot line becomes useful. Since  $\pi_1(\Sigma_2)$  contains a subgroup of index 2, and by classification of covering spaces, corresponds to a connected 2-sheeted covering. Hence,  $\Sigma_2$  does have a covering space which is homeomorphic to a surface of genus 3.

**Problem 2005-MAT530-II-4 (Geometry/Topology).** Determine the number of coverings of the space  $\mathbb{RP}^2 \times \mathbb{RP}^2$  (up to equivalence, with path-connected covering space.)

Let  $X = \mathbb{RP}^2 \times \mathbb{RP}^2$  so that  $\pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2) \cong \pi_1(\mathbb{RP}^2) \times \pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ . Since  $\mathbb{RP}^2 \times \mathbb{RP}^2$  has a universal cover (namely  $S^2 \times S^2$ ), the equivalence classes of coverings of  $\mathbb{RP}^2 \times \mathbb{RP}^2$  correspond to the conjugacy classes of subgroups of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . Since this group is abelian, the number of coverings of  $\mathbb{RP}^2 \times \mathbb{RP}^2$  corresponds to the number of subgroups of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . Since there are five such subgroups, it follows that there are five coverings spaces of  $\mathbb{RP}^2 \times \mathbb{RP}^2$  up to equivalence.

**Problem 2005-MAT530-II-4 (Geometry/Topology I).** Determine the number of coverings of the space  $\mathbb{RP}^1 \times \mathbb{RP}^1$  (up to equivalence, with path-connected covering space.)

Let  $X = \mathbb{RP}^1 \times \mathbb{RP}^1$  so that  $\pi_1(\mathbb{RP}^1 \times \mathbb{RP}^1) \cong \pi_1(\mathbb{RP}^1) \times \pi_1(\mathbb{RP}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ . Since  $\mathbb{RP}^1 \times \mathbb{RP}^1$  has a universal cover (namely  $\mathbb{R}^2$ ), the equivalence classes of coverings of  $\mathbb{RP}^1 \times \mathbb{RP}^1$  correspond to conjugacy classes of subgroups of  $\mathbb{Z} \oplus \mathbb{Z}$ . Since this group is abelian, the number of coverings of  $\mathbb{RP}^1 \times \mathbb{RP}^1$  correspond to the number of subgroups of  $\mathbb{Z} \oplus \mathbb{Z}$ . Since there are countably many such subgroups, we conclude that there are a countable infinitely many covering spaces of  $\mathbb{RP}^1 \times \mathbb{RP}^1$  up to equivalence.

**Problem 2005-J-II-5 (Geometry/Topology).** Give an example of a topological space admitting exactly 3 coverings up to equivalence (including the space itself).

Consider the Lens space  $L(4|1)$ , which has fundamental group  $\mathbb{Z}_4$ . Since  $L(4|1)$  has a universal cover (namely  $S^3$ ),  $\mathbb{Z}_4$  is abelian, and  $\mathbb{Z}_4$  has exactly 3 subgroups (namely  $\{0\}$ ,  $\mathbb{Z}_2$ , and  $\mathbb{Z}_4$ ), we conclude that  $L(4|1)$  has exactly 3 coverings up to equivalence.

## Other Qualifying Exams

**Problem RUT-2023-A-I-1 (Algebra).** Classify the groups of order  $2023 = 7 \cdot 17^2$  up to isomorphism. (You may use without proof the well-known result that if  $p$  is a prime, then every group of order  $p^2$  is abelian.)

Let  $G$  be a group of order  $2023 = 7 \cdot 17^2$ . By Sylow's Theorem,  $G$  contains a normal Sylow 7-subgroup and a normal Sylow 17-subgroup. Let  $H \cong \mathbb{Z}_7$  denote the Sylow 7-subgroup and  $K$  denote the Sylow 17-subgroup; note that either  $K \cong \mathbb{Z}_{17^2}$  or  $K \cong \mathbb{Z}_{17} \times \mathbb{Z}_{17}$ . Hence,  $G \cong H \rtimes_{\varphi} K$ , where  $\varphi \in \text{Aut}(H) \cong \mathbb{Z}_7^\times \cong \mathbb{Z}_6$ . We consider various cases.

- (I) Suppose  $K = \mathbb{Z}_{17^2}$ , which has a single generator, 1. Each homomorphism  $\varphi : K \rightarrow \mathbb{Z}_6$  is uniquely determined by where the generator 1 is mapped to, with the constraint that  $\varphi(1)$  is an element that divides the order of 1, namely  $17^2$ . Since the only such element is 0,  $\varphi$  is the trivial homomorphism, which means that the semidirect product is just the direct product, and so  $G \cong \mathbb{Z}_7 \times \mathbb{Z}_{17^2} \cong \mathbb{Z}_{2023}$ ; this is an abelian group.
- (II) Suppose  $K = \mathbb{Z}_{17} \times \mathbb{Z}_{17} = \langle a \rangle \times \langle b \rangle$ . Each homomorphism  $\psi : \mathbb{Z}_{17} \times \mathbb{Z}_{17} \rightarrow \mathbb{Z}_6$  is uniquely determined by  $\psi(a)$  and  $\psi(b)$  with the constraint that these elements divide the order of  $a$  and  $b$  in  $\mathbb{Z}_{17}$ , which is 17. Since there is only one such element, namely 0, we find that the semidirect is just the direct product, and  $G \cong \mathbb{Z}_7 \times \mathbb{Z}_{17} \times \mathbb{Z}_{17} \cong \mathbb{Z}_{289} \times \mathbb{Z}_7$ ; which is abelian.

Hence, up to isomorphism, there are exactly two groups of order 2023, both of which are abelian.

## Classification of Finite Groups

Some facts we will use to classify groups are:

- Every group of order  $p^2$ , where  $p$  is abelian, is abelian.
- Every group of order  $p$ , where  $p$  is prime, is isomorphic to  $\mathbb{Z}_p$ .

- (1)  $|G| = 1$ : This is the trivial group  $\{1\}$ .
- (2)  $|G| = 2$ : There is exactly one group, up to isomorphism, which is  $\mathbb{Z}/2\mathbb{Z}$ . This follows from Cauchy's Theorem which states that if  $p$  divides  $|G|$ , where  $p$  is prime, then  $G$  contains an element of order  $p$ .
- (3)  $|G| = 3$ : There is exactly one group, up to isomorphism, which is  $\mathbb{Z}/3\mathbb{Z}$ . This follows from Cauchy's Theorem, which states that if  $p$  divides  $|G|$ , where  $p$  is prime, then  $G$  contains an element of order  $p$ .
- (4)  $|G| = 4 = 2^2$ :  $\mathbb{Z}_4$  and  $\mathbb{Z}_2 \times \mathbb{Z}_2$  are both groups of order 4. Now let  $G$  be an arbitrary group of order 4; by Lagrange's Theorem, each element of  $G$  can have order 1, 2, or 4. Suppose  $G$  contains an element  $x$  of order 4. Then  $G = \langle g \rangle$ ; let  $\varphi : \mathbb{Z}_4 \rightarrow G$  be the map given by  $\varphi(n) \mapsto g^n$ ; this is easily seen to be a group isomorphism. Now suppose  $G$  has no element of order 4. Since the only element of  $G$  with order 1 is the identity (by uniqueness of group identities), the three nontrivial elements of  $G$  must have order 2. Consider the map  $\varphi : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow G$  defined as follows:

$$\begin{aligned} \varphi(0,0) &= 1_G, & \varphi(1,0) &= a, \\ \varphi(0,1) &= b, & \varphi(1,1) &= c, \end{aligned} \tag{201}$$

where  $a, b$ , and  $c$  are the three nonidentity elements of  $G$ ;  $\varphi$  is easily seen to be an isomorphism. Hence,  $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ .

- (5)  $|G| = 5$ : There is exactly one group, up to isomorphism, which is  $\mathbb{Z}/5\mathbb{Z}$ .
- (6)  $|G| = 6 = 2 \cdot 3$ . By Sylow's Theorem, there exists a normal Sylow 3-subgroup, which we denote by  $H$ . Let  $K$  be a Sylow 2-subgroup. By Lagrange's Theorem,  $H$  and  $K$  intersect trivially and  $|HK| = |H||K|/|H \cap K| = |H||K| = 6 = |G|$  so that  $G = HK$ . Hence, by the recognition theorem for semidirect products,  $G \cong H \rtimes_{\varphi} K$ , where  $\varphi \in \text{Aut}(H) \cong \mathbb{Z}_3^* \cong \mathbb{Z}_2$ . Hence, we look for homomorphisms  $\varphi : K \rightarrow \mathbb{Z}_2$ . Since  $K$  is a group of order 2,  $K \cong \mathbb{Z}_2$ . Homomorphisms  $\varphi : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$  are determined uniquely by where the generator 1 is sent to with the constraint that  $\varphi(1)$  divides the order of 1, which is 2. Hence, either  $\varphi_1(1) = 0$  (in which case, the homomorphism is trivial, the semidirect is just the direct product, and  $G$  is the abelian group  $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$ ), or  $\varphi_2(1) = 1$  (in which case, the homomorphism is nontrivial, and  $G$  is the nonabelian group  $\mathbb{Z}_3 \rtimes_{\varphi_2} \mathbb{Z}_2$ ). Hence, up to isomorphism, there are exactly two groups of order 6, one abelian and the other non-abelian.
- (7)  $|G| = 7$ : There is exactly one group, up to isomorphism, which is  $\mathbb{Z}/7\mathbb{Z}$ .
- (8)  $|G| = 8 = 2^3$ : **[!! Complete Later !!]**
- (9)  $|G| = 9 = 3^2$ : Every group of order  $p^2$  abelian. So by the Fundamental Theorem for Finitely Generated Abelian Groups,  $G \cong \mathbb{Z}_9$  or  $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ .
- (10)  $|G| = 10 = 2 \cdot 5$ : By Sylow's Theorem,  $G$  contains a normal Sylow 5-subgroup, which we denote by  $H$ . Let  $K$  be a Sylow 10-subgroup. Then by Lagrange's Theorem,  $H \cap K = \{e\}$  and  $G = HK$ . Therefore,  $G \cong H \rtimes_{\varphi} K$ , where  $\varphi \in \text{Aut } H \cong \mathbb{Z}_5^* \cong \mathbb{Z}_4$ . Hence, we look for homomorphisms  $\varphi : K \rightarrow \mathbb{Z}_4$ . Since  $K$  is a group of order 2,  $K \cong \mathbb{Z}_2$ . Homomorphisms  $\varphi : \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$  are determined uniquely by where the generator  $a$  is set to with the constraint that  $\varphi(a)$  divides the order of  $a$ , which is 2. The only such elements in  $\mathbb{Z}_4$  are 0 and 2. If  $\varphi_1 : 1 \mapsto 0$ , then  $\varphi_1$  is just the trivial homomorphism which means that the semidirect is just a direct product and  $G$  is isomorphic to the abelian group  $\mathbb{Z}_2 \times \mathbb{Z}_{10} \cong \mathbb{Z}_{10}$ . If  $\varphi_2 : 1 \mapsto 2$ , then  $\varphi$  is a nontrivial homomorphism, which means that  $G$  is the nonabelian group  $\mathbb{Z}_5 \rtimes_{\varphi_2} \mathbb{Z}_2$ . Hence, up to isomorphism, there are exactly 2 groups of order 10, only one of which is abelian.
- (11)  $|G| = 11$ : There is exactly one group of order 11, namely  $\mathbb{Z}/11\mathbb{Z}$ .

(12)  $|G| = 12 = 2^2 \cdot 3$ : By Sylow's Theorem,

$$\begin{aligned} n_3 &\in \{1, 2, 4\} \cap \{1, 4, \dots\} = \{1, 4\}. \\ n_2 &\in \{1, 3\} \cap \{1, 3, \dots\} = \{1, 3\}. \end{aligned} \tag{202}$$

Suppose  $n_3 =$ .

## Essential Review Notes

### Topological Vector Spaces

- **Def. (Topological Vector Space)** A vector space  $\mathcal{X}$  over a field  $K$  such that vector addition in  $\mathcal{X}$  and scalar multiplication are continuous maps from  $\mathcal{X} \times \mathcal{X}$  and  $K \times \mathcal{X}$ , respectively, to  $\mathcal{X}$ .
- **Def. (Weak Convergence)** A sequence  $\{x_n\}$  in a normed linear space  $\mathcal{X}$  converges weakly to  $x \in X$  if the sequence of scalars  $\{f(x_n)\}$  converges to  $f(x)$  for all  $f \in \mathcal{X}^*$ .
- **Def. (Weak\* Convergence)** Let  $\mathcal{X}$  be a normed linear space. A sequence  $\{f_n\} \subseteq \mathcal{X}^*$  is weak\* convergent to  $f \in \mathcal{X}^*$  if  $\{f_n(x)\}$  converges to  $f(x)$  for all  $x \in \mathcal{X}$ . Note, all this really says that the sequence of scalars  $\{\hat{x}(f_n)\} = \{f_n(x)\}$  converges to  $\hat{x}(f) = f(x)$  for all  $\hat{x} \in \mathcal{X}^{**}$  (read  $x \in \mathcal{X}$ ).