

# Geometry Crash Course

Aniruddha Madhava

December 21, 2025

---

## Contents

<b>1</b>	<b>Smooth Manifolds</b>	<b>3</b>
1.1	Topological Manifolds	3
1.2	Smooth Manifolds	3
<b>2</b>	<b>Smooth Maps</b>	<b>5</b>
2.1	Smooth Functions and Smooth Maps	5
2.2	Smooth Covering Maps	6
2.3	Proper Maps	7
<b>3</b>	<b>Tangent Vectors</b>	<b>8</b>
3.1	Tangent Vectors on a Manifold	8
3.2	Pushforwards	8
3.3	Computation in Coordinates	9
<b>4</b>	<b>Vector Fields</b>	<b>11</b>
<b>5</b>	<b>Cotangent Bundle</b>	<b>13</b>
5.1	Covectors	13
5.2	Tangent Covectors on Manifolds	15
5.3	The Cotangent Bundle	15
5.4	The Differential of a Function	15
5.5	Pullbacks	18
5.6	Line Integrals	18
5.7	Conservative Vector Fields	20
<b>6</b>	<b>Submersions, Immersions, and Embeddings</b>	<b>21</b>
6.1	Maps of Constant Rank	21
<b>7</b>	<b>Tensors</b>	<b>22</b>
7.1	The Algebra of Tensors	22
7.2	Tensors and Tensor Fields on Manifolds	24
7.3	Pullbacks of Smooth Tensor Fields	25
<b>8</b>	<b>Homotopy and the Fundamental Group</b>	<b>26</b>
8.1	Homotopy	26
8.2	Homomorphisms Induced by Continuous Maps	26
8.3	Homotopy Equivalence	27

<b>9</b>	<b>Differential Forms</b>	<b>28</b>
9.1	The Geometry of Volume Measurement . . . . .	28
9.2	The Algebra of Alternating Tensors . . . . .	28
9.3	The Wedge Product . . . . .	30
9.4	Differential Forms on Manifolds . . . . .	31
9.5	Orientations . . . . .	31
<b>10</b>	<b>Problems</b>	<b>32</b>
10.1	Smooth Maps . . . . .	32
	Problem 2-5 . . . . .	32
	Problem 2-6 . . . . .	32
10.2	Tangent Vectors . . . . .	33
	Problem 3-1 . . . . .	33
	Problem 3-3 . . . . .	33
	Problem 3-4 . . . . .	33
10.3	The Cotangent Bundle . . . . .	33
	Problem 6-1 . . . . .	34
	Problem 6-2 . . . . .	34
10.4	Comps . . . . .	35
	Problem 2017-J-II-1 . . . . .	35
	Problem 2023-J-II-4 . . . . .	35
	Problem 2024-J-I-5 . . . . .	36

---

# 1 Smooth Manifolds

## 1.1 Topological Manifolds

- **Def. (Topological Manifold)** A topological space  $M$  with the following properties:
  1.  $M$  is Hausdorff;
  2.  $M$  is second countable (i.e., has a countable basis for its topology);
  3.  $M$  is locally Euclidean of dimension  $n$  (i.e., for each  $p \in M$ , there exists a neighborhood  $U \subset M$ , an open set  $\tilde{U} \subset \mathbb{R}^n$ , and a homeomorphism  $\varphi : U \rightarrow \tilde{U}$ ).

**Exercise 1.1.** Show that equivalent definitions of locally Euclidean spaces are obtained if instead of requiring  $U$  to be homeomorphic to an open subset of  $\mathbb{R}^n$ , we require it to be homeomorphic to an open ball in  $\mathbb{R}^n$ , or to  $\mathbb{R}^n$  itself.

Let  $M$  be a topological space that satisfies conditions (1) and (2). ( $\Leftarrow$ ) Suppose that for each  $p \in M$ , there exists a neighborhood  $U$  of  $p$  that is homeomorphic to an open ball in  $\mathbb{R}^n$ , or to  $\mathbb{R}^n$  itself. Since each of these are open subsets of  $\mathbb{R}^n$ , it follows that  $M$  satisfies condition (3). ( $\Rightarrow$ ) Suppose that  $M$  satisfies conditions (1) - (3). Suppose that for some  $p \in U \subset M$ ,  $U \cong_{\varphi} \tilde{U} \subseteq \mathbb{R}^n$ . Since every open subset of  $\mathbb{R}^n$  is the countable union of open balls in  $\mathbb{R}^n$ , suppose that  $\tilde{U} = \bigcup_1^{\infty} B_j$ . Pick some ball  $B_{j_0}$  containing  $\varphi(p)$ . Then  $V = \varphi^{-1}(B_{j_0})$  is an open neighborhood of  $p$  in  $M$  that is homeomorphic, under the map  $\varphi|_V : V \rightarrow B_{j_0}$ , to the open ball  $B_{j_0}$ .

- **Def. (Coordinate Chart)** A pair  $(U, \varphi)$ , where  $U$  is an open subset of  $M$  and  $\varphi : U \rightarrow \tilde{U}$  is a homeomorphism from  $U$  to an open subset  $\tilde{U} = \varphi(U) \subseteq \mathbb{R}^n$ .
- **Def. (Precompact Subset)** Let  $X$  be a topological space. A subset  $K \subset X$  is said to be *precompact* (or *relatively compact*) in  $X$  if its closure in  $X$  is compact. E.g., the subsets  $(-1, 1)$ ,  $(2, 3]$ ,  $(4, 5) \cup \{6\}$  are all precompact in  $\mathbb{R}$ , but the subset  $(-1, \infty)$  is not.
- **Lem 1.6. (Topological Manifolds have Precompact Basis)** Every topological manifold has a countable basis of precompact coordinate balls.

Let  $M$  be a topological  $n$ -manifold. Suppose  $\varphi : M \rightarrow \tilde{U} \subset \mathbb{R}^n$  is a global coordinate map. Let  $\mathcal{B}$  be the collection of all open balls  $B_r(x) \subset \mathbb{R}^n$  such that (1)  $r$  is rational, (2)  $x$  has rational coordinates, and (3)  $\overline{B_r(x)} \subset \tilde{U}$ . By definition, each such ball is precompact in  $\tilde{U}$  and  $\mathcal{B}$  is a countable basis for the topology of  $\tilde{U}$ . Since  $\varphi$  is a homeomorphism, the collection  $\mathcal{B}^{-1} = \{\varphi^{-1}(B) : B \in \mathcal{B}\}$  is a countable basis for the topology of  $M$ . Moreover, each of the sets in this collection is precompact in  $M$ : for each  $B \in \mathcal{B}$ ,  $\varphi^{-1}(\overline{B}) = \overline{\varphi^{-1}(B)} \subset M$ ; since  $\varphi^{-1}$  is continuous,  $\varphi^{-1}(\overline{B})$  is compact in  $M$ . The restrictions of  $\varphi$  are the coordinate maps. In this case, we assumed that  $M$  had a global coordinate map, which might not necessarily be true in general.

So, let  $M$  be an arbitrary topological  $n$ -manifold. By definition, every point of  $M$  is contained in the domain of a chart. Since every open cover of a second countable space has a countable subcover,  $M$  is covered by countably many charts  $\{(U_i, \varphi_i)\}$ . By the preceding argument, for each  $i$ ,  $U_i$  has a countable basis of precompact coordinate balls, and the union of all these balls is a countable basis for the topology on  $M$ . Suppose  $V \subset U_i$  is one of these precompact balls. Since the closure of  $V$  in  $U_i$  is compact, the closure must be closed in  $M$ . Hence, the closure of  $V$  in  $M$  is the same as the closure of  $V$  in  $U_i$ , so that  $V$  is precompact in  $M$ .

## 1.2 Smooth Manifolds

- **Def. (Transition Map between Charts)** Let  $M$  be a topological  $n$ -manifold. Let  $(U, \varphi), (V, \psi)$  be two charts such that  $U \cap V \neq \emptyset$ . The composite map  $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$  is called the *transition map* from  $\varphi$  to  $\psi$ .

- **Def. (Smoothly Compatible Charts)** Two charts  $(U, \varphi)$  and  $(V, \psi)$  are said to be *smoothly compatible* if either  $U \cap V = \emptyset$  or the transition map  $\psi \circ \varphi^{-1}$  is a diffeomorphism.
- **Def. ((Smooth) Atlases)** Let  $M$  be a topological  $n$ -manifold. (1) An *atlas*  $\mathcal{A}$  for  $M$  is a collection of charts whose domains cover  $M$ ; (2)  $\mathcal{A}$  is said to be a *smooth atlas* if any two charts in  $\mathcal{A}$  are smoothly compatible.
- **Def. (Maximal Atlas):** A smooth atlas  $\mathcal{A}$  on  $M$  is said to be *maximal* iff it is not contained in any strictly larger smooth atlas. I.e., any chart that is smoothly compatible with every chart in  $\mathcal{A}$  is already contained in  $\mathcal{A}$ . A *smooth structure* on  $M$  is a maximal atlas.
- **Lem 1.10. (Smooth Atlases)** Let  $M$  be a topological manifold.
  - (a) Every smooth atlas for  $M$  is contained in a unique maximal smooth atlas.
  - (b) Two smooth atlases for  $M$  determine the same maximal smooth atlas if and only if their union is a smooth atlas.

The proof of (b) was left as an exercise (see below). The proof of (a) is given. Let  $\mathcal{A}$  be a smooth atlas for  $M$ , and let  $\bar{\mathcal{A}}$  denote the set of all charts that are smoothly compatible with every chart in  $\mathcal{A}$ . To show that  $\bar{\mathcal{A}}$  is a smooth atlas, we need to show that any two charts of  $\bar{\mathcal{A}}$  are smoothly compatible with each other, which is to say that for any  $(U, \varphi), (V, \psi) \in \bar{\mathcal{A}}$ ,  $\psi \circ \varphi^{-1} : \varphi(U \cap V) \rightarrow \psi(U \cap V)$  is smooth.

Let  $x = \varphi(p) = \varphi(U \cap V)$  be arbitrary. Because the domains of the charts in  $\mathcal{A}$  cover  $M$ , there is some chart  $(W, \theta)$  in  $\mathcal{A}$  such that  $p \in W$ . Since every chart in  $\bar{\mathcal{A}}$  is smoothly compatible with  $(W, \theta)$ , both of the maps  $\theta \circ \varphi^{-1}$  and  $\psi \circ \theta^{-1}$  are smooth where they are defined. Since  $p \in U \cap V \cap W$ , it follows that

$$\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1}) \quad (1)$$

is smooth on a neighborhood of  $x$ . Hence,  $\psi \circ \varphi^{-1}$  is smooth in a neighborhood of each point in  $\varphi(U \cap V)$ . This concludes that  $\bar{\mathcal{A}}$  is a smooth atlas. Now, we need to show that  $\bar{\mathcal{A}}$  is maximal. But this is straightforward to see: any chart that is smoothly compatible with every chart contained in  $\bar{\mathcal{A}}$  must be smoothly compatible with every chart contained in  $\mathcal{A}$ , and hence, must be contained in  $\bar{\mathcal{A}}$ . Therefore,  $\bar{\mathcal{A}}$  is maximal. Uniqueness also follows in a straightforward way: suppose  $\mathcal{B}$  is another maximal atlas containing  $\mathcal{A}$ . Then since every chart in  $\mathcal{B}$  is smoothly compatible with every chart in  $\mathcal{A}$ , it follows that  $\mathcal{B} \subset \bar{\mathcal{A}}$ . Hence by maximality of  $\mathcal{B}$ ,  $\mathcal{B} = \bar{\mathcal{A}}$ .

**Exercise 1.4.** Prove Lemma 1.10(b).

Let  $M$  be a topological  $n$ -manifold,  $\mathcal{A}_1, \mathcal{A}_2$  be two smooth atlas on  $M$ , and  $\bar{\mathcal{A}}_1, \bar{\mathcal{A}}_2$  the maximal smooth atlases determined by the two smooth atlases, respectively. This means that among all the smooth atlases that contain  $\mathcal{A}_{1,2}$ ,  $\bar{\mathcal{A}}_{1,2}$  are maximal, respectively. ( $\Rightarrow$ ) Suppose that  $\bar{\mathcal{A}}_1 = \bar{\mathcal{A}}_2$ . This means that every chart contained in  $\bar{\mathcal{A}}_2$  is smoothly compatible with every chart in  $\mathcal{A}_1$ ; since  $\mathcal{A}_2 \subset \bar{\mathcal{A}}_2$ , this implies that every chart of  $\mathcal{A}_2$  is smoothly compatible with every chart of  $\mathcal{A}_1$ . Likewise, since every chart in  $\bar{\mathcal{A}}_1$  is smoothly compatible with every chart in  $\mathcal{A}_2$ , and  $\mathcal{A}_1 \subset \bar{\mathcal{A}}_1$ , it follows that every chart in  $\mathcal{A}_1$  is smoothly compatible with every chart in  $\mathcal{A}_2$ . Hence, it follows that every pair of charts in  $\mathcal{A}_1 \cup \mathcal{A}_2$  is smoothly compatible, showing that the union is a smooth atlas. ( $\Leftarrow$ ) Suppose that  $\mathcal{A}_1 \cup \mathcal{A}_2$  is a smooth atlas. This implies that every chart in  $\mathcal{A}_2$  is smoothly compatible with every chart in  $\mathcal{A}_1$ , and thus,  $\mathcal{A}_2 \subset \bar{\mathcal{A}}_1$ ; by maximality of  $\bar{\mathcal{A}}_2$ ,  $\bar{\mathcal{A}}_1 \subset \bar{\mathcal{A}}_2$ . Likewise, we can show that  $\bar{\mathcal{A}}_1 \subset \bar{\mathcal{A}}_2$ ; by maximality of  $\bar{\mathcal{A}}_2$ ,  $\bar{\mathcal{A}}_2 \subset \bar{\mathcal{A}}_1$ . Therefore,  $\bar{\mathcal{A}}_1 = \bar{\mathcal{A}}_2$ .

## 2 Smooth Maps

### 2.1 Smooth Functions and Smooth Maps

- **Def. (Smooth Function)** Let  $M$  be a smooth  $n$ -manifold. A function  $f : M \rightarrow \mathbb{R}^k$  is *smooth* if for every  $p \in M$ , there exists a smooth chart  $(U, \varphi)$  for  $M$  whose domain contains  $p$  and such that the composite function  $f \circ \varphi^{-1}$  is smooth on the open subset  $\tilde{U} = \varphi(U) \subset \mathbb{R}^n$ .

**Exercise 2.3.** Suppose  $M$  is a smooth manifold and  $f : M \rightarrow \mathbb{R}^k$  is a smooth function. Show that  $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{R}^k$  is smooth for every smooth chart  $(U, \varphi)$  for  $M$ .

Suppose  $M$  is a smooth manifold and  $f : M \rightarrow \mathbb{R}^k$  is a smooth function. Let  $(U, \varphi)$  be a smooth chart for  $M$ . By definition of a smooth function, for every  $p \in U$ , there exists a smooth chart  $(V_p, \psi_p)$  for  $M$  containing  $p$  in its domain such that  $f \circ \psi_p^{-1} : \psi_p(V_p) \rightarrow \mathbb{R}^k$  is smooth. Since

$$U = \bigcup_{p \in U} (U \cap V_p) \implies \varphi(U) = \varphi \left( \bigcup_{p \in U} (U \cap V_p) \right) = \bigcup_{p \in U} \varphi(U \cap V_p), \quad (2)$$

it suffices to show that  $f \circ \varphi^{-1}$  is smooth on  $\varphi(U \cap V_p)$  for each  $p$ . Indeed, since  $(V_p, \psi_p)$  and  $(U, \varphi)$  are smoothly compatible for all  $p$ ,  $\psi_p \circ \varphi^{-1} : \varphi(U \cap V_p) \rightarrow \psi_p(U \cap V_p)$  is smooth. Since  $f \circ \psi_p^{-1}$  is smooth on  $\psi_p(V_p)$ , it must be smooth on the subset  $\psi_p(U \cap V_p)$ . Therefore,

$$f \circ \varphi^{-1} = (f \circ \psi_p^{-1}) \circ (\psi_p \circ \varphi^{-1}) : \varphi(U \cap V_p) \rightarrow \mathbb{R}^k \quad (3)$$

is smooth for all  $p$ . Thus, we conclude that  $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{R}^k$  is smooth.

- **Def. (Coordinate Representation)** Given a function  $f : M \rightarrow \mathbb{R}^k$  and a chart  $(U, \varphi)$  for  $M$ , the function  $\hat{f} : \varphi(U) \rightarrow \mathbb{R}^k$  defined by  $\hat{f}(x) = f \circ \varphi^{-1}(x)$  is called the *coordinate representation* of  $f$ . By definition,  $f$  is smooth iff its coordinate representation is *smooth* in some smooth chart of  $M$ ; but by the preceding exercise, the coordinate representation of  $f$  is smooth in every smooth chart of  $M$ .
- **Def. (Smooth Map between Manifolds)** Let  $M, N$  be smooth manifolds, and let  $F : M \rightarrow N$  be any map.  $F$  is a *smooth map* if for every  $p \in M$ , there exist smooth charts  $(U, \varphi)$  containing  $p$  and  $(V, \psi)$  containing  $F(p)$  such that  $F(U) \subset V$  and the composite map  $\psi \circ F \circ \varphi^{-1}$  is smooth from  $\varphi(U)$  to  $\psi(V)$ .

**Exercise 2.4.** Let  $M$  and  $N$  be smooth manifolds, and let  $F : M \rightarrow N$  be a map. If every point  $p \in M$  has a neighborhood  $U$  such that the restriction  $F|_U$  is smooth, show that  $F$  is smooth. Conversely, if  $F$  is smooth, show that its restriction to any open subset is smooth.

Let  $M$  and  $N$  be smooth manifolds, and let  $F : M \rightarrow N$  be a map.

- Let  $p \in M$ , and let  $W$  be a neighborhood of  $p$  such that  $F|_W$  is smooth. This means that there exist smooth charts  $(U, \varphi)$ , where  $p \in U \subset W$ , and  $(V, \psi)$  containing  $F(p)$  such that  $F(U) \subset V$  and the composite function  $\psi \circ (F|_W) \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$  is smooth. Since  $U \subset W$ , it follows that  $(F|_W)|_U = F|_U$ . This means that  $\psi \circ F|_U \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$  is smooth. Hence, since  $p$  was arbitrary, we conclude that  $F$  is smooth.
- Now assume that  $F$  is smooth, and let  $W$  be an arbitrary open subset of  $M$ . By definition of smoothness, for each  $p \in W$ , there exist smooth charts  $(U, \varphi)$  for  $M$  containing  $p$  and  $(V, \psi)$  for  $N$  containing  $F(p)$  such that  $F(U) \subset V$  and the composite function  $\psi \circ F \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$  is smooth. Since  $\varphi(U \cap W)$  is an open subset of  $\varphi(U)$ , it follows that  $\psi \circ F \circ \varphi^{-1}$  is smooth on  $\varphi(U \cap W)$ ; that is,  $F|_{(U \cap W)}$  is smooth. Hence, we have shown that for every  $p \in W$ , there exists a neighborhood of  $p$  such that the restriction of  $F$  to this neighborhood is smooth. Therefore, we conclude that  $F|_W$  is smooth.

- **Lem 2.1. (Constructing Smooth Maps)** Let  $M$  and  $N$  be smooth manifolds, and let  $\{U_\alpha\}_{\alpha \in A}$  be an open cover of  $M$ . Suppose that for each  $\alpha \in A$ , we are given a smooth map  $F_\alpha : U_\alpha \rightarrow N$  such that the maps agree on overlaps  $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$  for all  $\alpha$  and  $\beta$ . Then there exists a unique smooth map  $F : M \rightarrow N$  such that  $F|_{U_\alpha} = F_\alpha$  for each  $\alpha \in A$ .

- **Lem. 2.2 (Smoothness Implies Continuity)** Every smooth map between smooth manifolds is continuous.

Suppose  $F : M \rightarrow N$  is smooth. By definition of smoothness, for each  $p \in M$ , we can choose smooth charts  $(U, \varphi)$  containing  $p$  and  $(V, \psi)$  containing  $F(p)$  such that  $F(U) \subset V$  and  $\psi \circ F \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$  is a smooth map, and hence continuous. Since  $\varphi : U \rightarrow \varphi(U)$  and  $\psi : V \rightarrow \psi(V)$  are homeomorphisms, this implies in turn that

$$F|_U = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi : U \rightarrow V, \quad (4)$$

which is a composition of continuous maps, is continuous. Hence, since  $F$  is continuous in a neighborhood of each point, it is continuous on  $M$ .

- **Def. (Coordinate Representation)** Let  $F : M \rightarrow N$  be a smooth map, and  $(U, \varphi)$ ,  $(V, \psi)$  be any smooth charts for  $M$  and  $N$ , respectively. Then we call  $\hat{F} = \psi \circ F \circ \varphi^{-1}$  the coordinate representation of  $F$  with respect to the given coordinates.

**Exercise 2.6.** Suppose  $F : M \rightarrow N$  is a smooth map between smooth manifolds. Show that the coordinate representation of  $F$  with respect to any pair of smooth charts for  $M$  and  $N$  is smooth.

Let  $F : M \rightarrow N$  be a smooth map between smooth manifolds, and let  $(U, \varphi)$ ,  $(V, \psi)$  be any pair of smooth charts for  $M$  and  $N$ . Without loss of generality, assume that  $F(U) \subset V$ . Our task is to show that  $\psi \circ F \circ \varphi^{-1}$  is smooth. Let  $p \in U$ . Since  $F$  is smooth, there exist smooth charts  $(W, \theta)$  and  $(R, \vartheta)$  containing  $p$  and  $F(p)$ , respectively, such that  $F(W) \subset V \cap R$  and the composite function  $\vartheta \circ F \circ \theta^{-1} : \theta(W) \rightarrow \vartheta(R)$  is smooth. Since  $U \cap W$  is nonempty and the corresponding charts are smoothly compatible, the transition map  $\theta \circ \varphi^{-1} : \varphi(U \cap W) \rightarrow \theta(U \cap W)$  is smooth. Likewise, the transition map  $\psi \circ \vartheta^{-1}$  is smooth. Hence, the composite function:

$$\psi \circ F \circ \varphi^{-1} = (\psi \circ \vartheta^{-1}) \circ (\vartheta \circ F \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1}) \quad (5)$$

is smooth on  $\varphi(U \cap W)$ . By locality of smoothness, since for each  $p \in U$ , there exists a neighborhood on which  $\psi \circ F \circ \varphi^{-1}$  is smooth, we conclude that the coordinate representation of  $F$  with respect to the given coordinates is smooth.

## 2.2 Smooth Covering Maps

- **Def. (Covering Map)** A surjective continuous map  $\pi : \widetilde{M} \rightarrow M$  between connected, locally path connected spaces with the property that for every  $p \in M$ , there exists a neighborhood  $U$  that is *evenly covered* (i.e.,  $U$  is connected, and each component of  $\pi^{-1}(U)$  is mapped homeomorphically onto  $U$  by  $\pi$ ).
- **Def. (Smooth Covering Map)** Let  $\widetilde{M}$  and  $M$  be connected smooth manifolds. A smooth covering map  $\pi : \widetilde{M} \rightarrow M$  is a smooth surjective map with the property that every  $p \in M$  has a connected neighborhood  $U$  such that each component of  $\pi^{-1}(U)$  is mapped *diffeomorphically* onto  $U$  by  $\pi$ . In this instance also, we say that  $U$  is evenly covered.
- **Prop 2.9. (Properties of Smooth Coverings)**
  - (a) Any smooth covering map is a local diffeomorphism and an open map.
  - (b) An injective smooth covering map is a diffeomorphism.
  - (c) A topological covering map is a smooth covering map if and only if it is a local diffeomorphism.

**Exercise 2.12.** If  $\pi_1 : \widetilde{M}_1 \rightarrow M_1$  and  $\pi_2 : \widetilde{M}_2 \rightarrow M_2$  are smooth covering maps, show that  $\pi_1 \times \pi_2 : \widetilde{M}_1 \times \widetilde{M}_2 \rightarrow M_1 \times M_2$  is a smooth covering map.

Since  $\widetilde{M}_{1,2}$  and  $M_{1,2}$  are all connected smooth manifolds,  $\widetilde{M}_1 \times \widetilde{M}_2$  and  $M_1 \times M_2$  are all connected smooth manifolds. Now let  $(p, q) \in M_1 \times M_2$ . Since  $\pi_1$  is surjective, there exists  $\tilde{p} \in \widetilde{M}_1$  such that  $\pi_1(\tilde{p}) = p$ ; likewise, there exists  $\tilde{q} \in \widetilde{M}_2$  such that  $\pi_2(\tilde{q}) = q$ . Hence,  $\pi_1 \times \pi_2 : (\tilde{p}, \tilde{q}) \mapsto (p, q)$ , which shows that  $\pi_1 \times \pi_2$  is surjective. Likewise, since  $\pi_1, \pi_2$  are smooth,  $\pi_1 \times \pi_2$  is smooth. Now we need to verify the evenly covered property for  $\pi_1 \times \pi_2$ .

Let  $(p, q) \in M_1 \times M_2$ . By the definition of smooth covering maps, there exist connected neighborhoods  $U \subset M_1$  and  $V \subset M_2$  such that each component of  $\pi_1^{-1}(U)$  and  $\pi_2^{-1}(V)$  is mapped diffeomorphically onto  $U$  and  $V$  by  $\pi_1$  and  $\pi_2$ , respectively. Since the product of connected open sets is connected,  $U \times V$  is a connected neighborhood of  $(p, q)$ . Then since  $(\pi_1 \times \pi_2)^{-1}(U \times V) = \pi_1^{-1}(U) \times \pi_2^{-1}(V)$ , the components of  $(\pi_1 \times \pi_2)^{-1}(U \times V)$  are just the products of the components of  $\pi_1^{-1}(U)$  with the components of  $\pi_2^{-1}(V)$ . Hence, since  $\pi_1$  ( $\pi_2$ ) maps each component of  $\pi_1^{-1}(U)$  ( $\pi_2^{-1}(V)$ ) diffeomorphically onto  $U$  ( $V$ ), it follows that  $\pi_1 \times \pi_2$  maps each component of  $\pi_1^{-1}(U) \times \pi_2^{-1}(V)$  diffeomorphically onto  $U \times V$ . Therefore,  $\pi_1 \times \pi_2$  is a smooth covering map.

- **Def. (Section of a Continuous Map)** If  $\pi : \widetilde{M} \rightarrow M$  is any continuous map, a *section* of  $\pi$  is a continuous map  $\sigma : M \rightarrow \widetilde{M}$  such that  $\pi \circ \sigma = \text{Id}_M$ :

$$\begin{array}{c} \widetilde{M} \\ \downarrow \pi \quad \nearrow \sigma \\ M \end{array}$$

Figure 1: Section of  $\pi$ .

- **Def. (Local Section of a Continuous Map)** A continuous map  $\sigma : U \subset M \rightarrow \widetilde{M}$  such that  $\pi \circ \sigma = \text{Id}_U$ .

## 2.3 Proper Maps

- **Def. (Proper Maps)** Let  $M, N$  be topological spaces.  $F : M \rightarrow N$  is *proper* if for every compact set  $K \subset N$ ,  $F^{-1}(K)$  is compact.

- **Lem. 2.14 (Sufficient Condition for Proper Map I)** Suppose  $M$  is a compact space and  $N$  is Hausdorff space. Then every continuous map  $F : M \rightarrow N$  is proper.

Let  $K \subset N$  be compact; since  $N$  is Hausdorff,  $K$  is closed. Then by continuity of  $F$ ,  $F^{-1}(K)$  is closed in  $M$ . Since  $M$  is compact,  $F^{-1}(K)$  must be compact in  $K$ .

- **Def. (Saturated Subset)** A subset  $A \subset M$  is said to be saturated with respect to a map  $F : M \rightarrow N$  if  $A = F^{-1}(F(A))$ .

- **Lem. 2.15. (Sufficient Condition for Proper Map II)** Suppose  $F : M \rightarrow N$  is a proper map between topological spaces, and  $A \subset M$  is any subset that is saturated with respect to  $F$ . Then  $F|_A : A \rightarrow F(A)$  is proper.

Let  $K \subset F(A)$  be compact. Since  $A$  is saturated,  $(F|_A)^{-1}(K) = F^{-1}(K)$ , which is compact since  $F$  is proper.

### 3 Tangent Vectors

- **Def. (Derivation at a Point)** Let  $a \in \mathbb{R}^n$ . A linear map  $X : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$  is called a *derivation at  $a$*  iff it satisfies the following product rule:

$$X(fg) = f(a)Xg + g(a)Xf. \quad (6)$$

- **Lem 3.1. (Properties of Derivations)** Suppose  $a \in \mathbb{R}^n$  and  $X \in T_a(\mathbb{R}^n)$ .
  - (a) If  $f$  is a constant function, then  $Xf = 0$ .
  - (b) If  $f(a) = g(a) = 0$ , then  $X(fg) = 0$ .

(a) It suffices to show that if  $f \equiv 1$ , then  $Xf = 0$ . Indeed,

$$Xf = X(ff) = f(a)Xf + f(a)Xf = 2f(a)Xf = 2Xf, \quad (7)$$

whence  $Xf = 0$ .

(b) From the product rule,  $X(fg) = f(a)Xg + g(a)Xf = 0 + 0 = 0$ .

#### 3.1 Tangent Vectors on a Manifold

- **Def. (Derivations on Manifolds)** Let  $M$  be a smooth manifold and  $p \in M$ . A linear map  $X : C^\infty(M) \rightarrow \mathbb{R}$  is called a *derivation at  $p$*  if it satisfies

$$X(fg) = f(p)Xg + g(p)Xf \quad (8)$$

for all  $f, g \in C^\infty(M)$ . The set of all derivations at  $p$  is called the *tangent space* to  $M$  at  $p$ , and is denoted by  $T_pM$ .

- **Lem 3.4. (Properties of Tangent Vectors on Manifolds)** Let  $M$  be a smooth manifold, and suppose  $p \in M$  and  $X \in T_pM$ .
  - (a) If  $f$  is a constant function, then  $Xf = 0$ .
  - (b) If  $f(p) = g(p) = 0$ , then  $X(fg) = 0$ .

#### 3.2 Pushforwards

- **Def. (Pushforward associated with a Map)** Let  $M, N$  be smooth manifolds and  $F : M \rightarrow N$  a smooth map. For each  $p \in M$ , we define a map  $F_* : T_pM \rightarrow T_{F(p)}(N)$ , called the *pushforward* associated with  $F$  as follows:

$$(F_*X)(f) = X(f \circ F). \quad (9)$$

It is straightforward to see that the pushforward is linear. It is also a derivation at  $p$ :

$$\begin{aligned} (F_*X)(fg) &= X(fg \circ F) = X((f \circ F)(g \circ F)) \\ &= (f \circ F)(p)X(g \circ F) + (g \circ F)(p)X(f \circ F) \\ &= (f \circ F)(p)(F_*X)(g) + (g \circ F)(p)(F_*X)(f). \end{aligned} \quad (10)$$

- **Lem 3.5. (Properties of Pushforwards)** Let  $F : M \rightarrow N$  and  $G : N \rightarrow P$  be smooth maps, and let  $p \in M$ .
  - (a)  $F_* : T_pM \rightarrow T_{F(p)}N$  is linear.
  - (b)  $(G \circ F)_* = G_* \circ F_* : T_pM \rightarrow T_{(G \circ F)(p)}P$ .
  - (c)  $(\text{Id}_M)_* = \text{Id}_{T_pM} : T_pM \rightarrow T_pM$ .
  - (d) If  $F$  is a diffeomorphism, then  $F_* : T_pM \rightarrow T_{F(p)}N$  is an isomorphism.

**Exercise 3.2.** Prove Lemma 3.5.

(a) Let  $f \in C^\infty(N)$ ,  $X, Y \in T_p(M)$ ,  $c_{1,2} \in \mathbb{R}^n$ . Then

$$\begin{aligned} (F_*(c_1X + c_2Y))(f) &= (c_1X + c_2Y)(f \circ F) \\ &= c_1X(f \circ F) + c_2Y(f \circ F) = c_1F_*(X)(f) + c_2F_*(Y)(f). \end{aligned} \quad (11)$$

(b) Let  $f \in C^\infty(N)$ , and  $X \in T_p(M)$ . Then

$$\begin{aligned} ((G \circ F)_*X)(f) &= X(f \circ (G \circ F)) = X((f \circ G) \circ F) \\ &= (F_*X)(f \circ G) \\ &= (G_*(F_*X))(f) = ((G_* \circ F_*)X)(f). \end{aligned} \quad (12)$$

(c) Let  $f \in C^\infty(N)$ , and  $X \in T_pM$ . Then

$$(\text{Id}_M_*X)(f) = X(f \circ \text{Id}_M) = X(f). \quad (13)$$

- **Prop. 3.6. (Tangent Space is Local)** Suppose  $M$  is a smooth manifold,  $p \in M$ , and  $X \in T_pM$ . If  $f$  and  $g$  are smooth functions in  $M$  that agree on some neighborhood of  $p$ , then  $Xf = Xg$ .

Let  $h = f - g$ . It suffices to show that  $Xh = 0$  by linearity of  $X$  whenever  $h$  vanishes in a neighborhood of  $p$ . Let  $\psi \in C^\infty(M)$  be a smooth function that is identically 1 on the support of  $h$  and supported in  $M \setminus \{p\}$ . Because  $\psi \equiv 1$  where  $h$  is nonzero, the product  $\psi h$  is identically equal to  $h$ . Since  $h(p) = \psi(p) = 0$ , Lemma 3.4(b) implies that  $Xh = X(\psi h) = 0$ .

### 3.3 Computation in Coordinates

- **Def. (Basis for  $T_pM$  in Coordinates)** Let  $(U, \varphi)$  be a smooth coordinate chart on  $M$ ; in particular,  $\varphi : U \rightarrow \tilde{U} \subset \mathbb{R}^n$  is a diffeomorphism. This implies that  $\varphi_* : T_pM \rightarrow T_{\varphi(p)}\mathbb{R}^n$  is an isomorphism. We've seen that  $T_{\varphi(p)}\mathbb{R}^n$  has as a basis consisting of all the derivations  $\partial_{x^i}|_{\varphi(p)}$ ,  $i = 1, \dots, n$ . Therefore, the pushforward of these vectors under  $(\varphi^{-1})_*$  form a basis for  $T_pM$ . We use the following notation:

$$\frac{\partial}{\partial x^i} \Big|_p = (\varphi^{-1})_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)}. \quad (14)$$

Indeed, if  $f : U \rightarrow \mathbb{R}$  is smooth, then

$$\frac{\partial}{\partial x^i} \Big|_p f = \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} (f \circ \varphi^{-1}) = \frac{\partial \hat{f}}{\partial x^i}(\hat{p}), \quad (15)$$

where  $\hat{f}$  is the coordinate representation of  $f$ , and  $\hat{p} = (p^1, \dots, p^n) = \varphi(p)$  is the coordinate representation of  $p$ .

- **Def. (Pushforward in Coordinates I)** Consider a smooth map  $F : U \rightarrow V$ , where  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^m$  are open subsets of Euclidean spaces. Let  $p \in U$ . We will use  $(x^1, \dots, x^n)$  to denote the coordinates in the domain and  $(y^1, \dots, y^m)$  to denote the coordinates in the range. Then using the chain rule,

$$\begin{aligned} \left( F_* \frac{\partial}{\partial x^i} \Big|_p \right) f &= \frac{\partial}{\partial x^i} \Big|_p (f \circ F) \\ &= \frac{\partial f}{\partial y^j}(F(p)) \frac{\partial F^j}{\partial x^i}(p) \\ &= \left( \frac{\partial F^j}{\partial x^i} \frac{\partial}{\partial y^j} \Big|_{F(p)} \right) f. \end{aligned} \quad (16)$$

Since  $f$  was arbitrary, we conclude that

$$F_* \frac{\partial}{\partial x^i} \Big|_p = \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j} \Big|_{F(p)}. \quad (17)$$

In other words, the matrix of  $F_*$  in terms of the standard coordinate basis is given by

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}. \quad (18)$$

This is precisely the Jacobian matrix of  $F$ .

- **Def. (Pushforward in Coordinates II)** Let  $F : M \rightarrow N$  be an arbitrary smooth map. Choosing smooth coordinate charts  $(U, \varphi)$  for  $M$  near  $p$  and  $(V, \psi)$  for  $N$  near  $F(p)$ , we obtain the coordinate representation  $\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \rightarrow \psi(V)$ . Now we apply the chain rule:

$$\begin{aligned} F_* \frac{\partial}{\partial x^i} \Big|_p &= F_* \left( (\varphi^{-1})_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) = (F \circ \varphi^{-1})_* \left( \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) \\ &= (\psi^{-1})_* \left( \hat{F}_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) = (\psi^{-1})_* \left( \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j} \Big|_{\hat{F}(\varphi(p))} \right) = \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p}) \frac{\partial}{\partial y^j} \Big|_{F(p)}. \end{aligned} \quad (19)$$

In other words, the pushforward of  $F$  is precisely the Jacobian matrix of its coordinate representation.

- **Obs. (Transformation of Vectors)** Suppose  $(U, \varphi)$  and  $(V, \psi)$  are two smooth charts on  $M$ , and let  $p \in U \cap V$ . We have two bases for the tangent space at  $p$ , namely  $\{\partial/\partial x^i|_p\}$ , where the coordinate functions of  $\varphi$  are  $(x^i)$ , and  $\{\partial/\partial \tilde{x}^i|_p\}$ , where the coordinate functions of  $\psi$  are  $(\tilde{x}^i)$ . By **Def. (Pushforward in Coordinates I)**, we have

$$(\psi \circ \varphi^{-1})_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} = \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j} \Big|_{\psi(p)}. \quad (20)$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial x^i} \Big|_p &= (\varphi^{-1})_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} = (\psi^{-1} \circ \psi \circ \varphi^{-1})_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \\ &= (\psi^{-1})_* \left( (\psi \circ \varphi^{-1})_* \frac{\partial}{\partial x^i} \Big|_{\varphi(p)} \right) \\ &= (\psi^{-1})_* \left( \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j} \Big|_{\psi(p)} \right) = \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \frac{\partial}{\partial \tilde{x}^j} \Big|_p. \end{aligned} \quad (21)$$

In particular, for any  $X \in T_p M$ , if

$$X = X^i \frac{\partial}{\partial x^i} \Big|_p = \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i} \Big|_p, \quad (22)$$

then by the above result,

$$\tilde{X}^i = X^i \frac{\partial \tilde{x}^i}{\partial x^i}(\varphi(p)) = X^i \frac{\partial \tilde{x}^i}{\partial x^i}(\hat{p}), \quad (23)$$

where  $\hat{p} = \varphi(p)$  is the representation of  $p$  in  $x^i$ -coordinates.

## 4 Vector Fields

- **Def. (Tangent Bundle)** Let  $M$  be a smooth manifold. Then the *tangent bundle* of  $M$  is the disjoint union of the tangent spaces at all points of  $M$ :

$$TM := \coprod_{p \in M} T_p M. \quad (24)$$

A typical element of the tangent bundle is of the form  $(p, X)$ , where  $p \in M$  and  $X \in T_p M$ .

- **Lem. 4.1: (Tangent Bundle is a Manifold)** For any smooth  $n$ -manifold  $M$ , the tangent bundle  $TM$  has a natural topology and smooth structure that make it into a  $2n$ -dimensional smooth manifold. With this structure, the canonical projection map  $\pi : TM \rightarrow M$ , defined as the map  $\pi : (p, X) \mapsto p$ , is a smooth map.

We start by defining the smooth charts that will give  $TM$  its smooth structure. For some given smooth chart  $(U, \varphi)$  for  $M$ , let  $(x^1, \dots, x^n)$  denote the coordinate functions of  $\varphi$ , and define the map  $\tilde{\varphi} : \pi^{-1}(U) \rightarrow \mathbb{R}^{2n}$  by

$$\tilde{\varphi} \left( v^i \frac{\partial}{\partial x^i} \Big|_p \right) = (x^1(p), \dots, x^n(p), v^1, \dots, v^n). \quad (25)$$

More precisely, the image set of  $\tilde{\varphi}$  is the set  $\varphi(U) \times \mathbb{R}^n$ , which is an open subset of  $\mathbb{R}^{2n}$ . [!! Complete Later !!]

**Exercise 4.2.** Suppose  $F : M \rightarrow N$  is a smooth map. By examining the local expression (3.6) for  $F_*$  in coordinates, show that  $F_* : TM \rightarrow TN$  is a smooth map.

Let  $F : M \rightarrow N$  be a smooth map, and consider its pushforward  $F_* : TM \rightarrow TN$ . Our goal is to show that  $F_*$  is a smooth map. Let  $p \in M$ ; by smoothness there exist smooth charts  $(U, \varphi)$  containing  $p$  in its domain, and  $(V, \psi)$  containing  $F(p)$  in its domain such that  $F(U) \subset V$  and the composite function  $\psi \circ F \circ \varphi^{-1}$  is smooth. Let  $(x^i)$  denote the coordinate functions of  $\varphi$  and  $(y^j)$  denote the coordinate functions of  $\psi$ . Let  $(\pi^{-1}(U), \tilde{\varphi})$  and  $(\pi^{-1}(V), \tilde{\psi})$  be the corresponding smooth charts for  $TM$  and  $TN$ , respectively, where  $\pi$  is the canonical projection map from the tangent bundle of a manifold onto the manifold. These charts are equipped with the standard coordinates  $(x^i, v^i)$  and  $(y^j, w^j)$ , respectively. Then in coordinates, the local expression for  $F_*$  is given by,

$$F_* \frac{\partial}{\partial x^i} \Big|_p = \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j} \Big|_{F(p)}. \quad (26)$$

This implies that

$$F_* : (p, v) \mapsto \left( F^j(p), v^i \frac{\partial F^j}{\partial x^i}(p) \right). \quad (27)$$

Since  $F$  is smooth, each coordinate function  $F^j$  must be smooth. Likewise, each  $\partial F^j / \partial x^i(p)$ . Since the map  $(x^i, v^i) \mapsto v^i \frac{\partial F^j}{\partial x^i}(x)$  is the finite sum of smooth functions, it must be smooth as well. Hence, we conclude that  $F_*$  is smooth.

- **Def. (Vector Field)** A section of the map  $\pi : TM \rightarrow M$ ; i.e., a vector field is a continuous map  $Y : M \rightarrow TM$ , usually written  $p \mapsto Y_p$ , with the property that

$$\pi \circ Y = \text{Id}_M. \quad (28)$$

- **Def. (Smooth Vector Field)** A smooth vector field.

- **Lem. 4.2 (Smoothness Criterion for Vector Fields)** Let  $M$  be a smooth manifold, and let  $Y : M \rightarrow TM$  be a rough vector field. If  $(U, (x^i))$  is *any* smooth coordinate chart on  $M$ , then  $Y$  is smooth on  $U$  if and only if its component functions with respect to this chart are smooth.

Let  $(x^i, v^i)$  be the standard coordinates on  $\pi^{-1}(U) \subset TM$  associated with the chart  $(U, (x^i))$ . By definition of the standard coordinate representation of  $Y$ ,

$$\hat{Y}(x) = (x^1, \dots, x^n, Y^1(x), \dots, Y^n(x)), \quad (29)$$

where  $Y^i$  is the  $i$ th component function of  $Y$  in  $x^i$ -coordinates. Hence, smoothness of  $Y$  is equivalent to smoothness of the component functions.

- **Lem 4.5. (Extending a Tangent Vector)** Let  $M$  be a smooth manifold. If  $p \in M$  and  $X \in T_p M$ , there is a smooth vector field  $\tilde{X}$  on  $M$  such that  $\tilde{X}_p = X$ .

Let  $(x^i)$  be smooth coordinates on a neighborhood  $U$  of  $p$ , and let  $X^i \partial / \partial x^i|_p$  be the coordinate expression for  $X$ . Let  $\psi$  be a smooth bump function supported in  $U$  and with  $\psi(p) = 1$ . Then the vector field  $\tilde{X}$  defined by

$$\tilde{X}_q = \begin{cases} \psi(q) X^i \frac{\partial}{\partial x^i} \Big|_q, & q \in U, \\ 0, & q \notin \text{supp } \psi \end{cases} \quad (30)$$

is a smooth vector field whose value at  $p$  is equal to  $X$ .

- **Def. (Set of all Smooth Vector Fields)** Let  $\mathcal{T}(M)$  denote the set of all smooth vector fields on  $M$ ;  $\mathcal{T}(M)$  is a vector space under pointwise addition and scalar multiplication:

$$(aY + bZ)_p = aY_p + bZ_p. \quad (31)$$

If  $f \in C^\infty(M)$  and  $Y \in \mathcal{T}(M)$ , we define  $fY : M \rightarrow TM$  by

$$(fY)_p = f(p)Y_p. \quad (32)$$

**Exercise 4.3.** If  $Y$  and  $Z$  are smooth vector fields on  $M$  and  $f, g \in C^\infty(M)$ , show that  $fY + gZ$  is a smooth vector field.

Let  $(U, (x^i))$  be a smooth coordinate chart on  $M$ . Then in these coordinates,

$$Y = Y^i \frac{\partial}{\partial x^i}, \quad Z = Z^i \frac{\partial}{\partial x^i}. \quad (33)$$

Then

$$fY + gZ = (fY^i + gZ^i) \frac{\partial}{\partial x^i}. \quad (34)$$

Since  $f, g, Y^i, Z^i$  are all smooth, and the product/sum of smooth functions is smooth,  $fY^i + gZ^i$  is smooth for all  $i$ . Hence, since the component functions of  $fY + gZ$  are smooth on any smooth coordinate chart on  $M$ , it follows that  $fY + gZ$  is a smooth vector field on  $M$ .

- **Def. (Action of Vector Field on Functions)** If  $Y \in \mathcal{T}(M)$  and  $f$  is a smooth real-valued function defined on an open set  $U \subset M$ , we obtain a new function  $Yf : U \rightarrow \mathbb{R}$  defined by

$$Yf(p) = Y_p f. \quad (35)$$

## 5 Cotangent Bundle

### 5.1 Covectors

- **Def. (Covector)** Let  $V$  be a finite-dimensional vector space. A *covector* on  $V$  is a real-valued linear functional on  $V$ ; i.e., a linear map  $\omega : V \rightarrow \mathbb{R}$ . The vector space of all covectors on  $V$  is denoted by  $V^*$  and called the *dual* space to  $V$ .
- **Prop. 6.1. (Dual Basis)** Let  $V$  be a finite-dimensional vector space. If  $(E_1, \dots, E_n)$  is any basis for  $V$ , then the covectors  $(\varepsilon^1, \dots, \varepsilon^n)$ , defined by

$$\varepsilon^i(E_j) = \delta_j^i = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \quad (36)$$

form a basis for  $V^*$ , called the dual basis to  $(E_i)$ . Therefore,  $\dim V^* = \dim V$ .

**Exercise 6.1.** Prove Proposition 6.1.

Assume the given hypotheses of Proposition 6.1. We must show that  $(\varepsilon^i)$  is a linearly independent collection of covectors that spans  $V^*$ ; we start by showing linear independence. Suppose  $a_1, \dots, a_n \in \mathbb{R}$  are scalars such that

$$a_1 \varepsilon^1 + \dots + a_n \varepsilon^n = 0. \quad (37)$$

Then allowing the left side to act on the  $V$ -basis vector  $E_j$ , for some  $j \in \{1, \dots, n\}$ ,

$$0 = (a_1 \varepsilon^1 + \dots + a_n \varepsilon^n)(E_j) = \sum_1^n a_i \varepsilon^i(E_j) = a_j. \quad (38)$$

Since this is true for all  $j \in \{1, \dots, n\}$ , we conclude that each  $a_j = 0$ . Therefore,  $(\varepsilon^i)$  is linearly independent. Now let  $\omega \in V^*$ . For each  $i = 1, \dots, n$ , let  $\omega(E_i) = a_i \in \mathbb{R}$ . Then we claim that  $\omega = a_i \varepsilon^i$  (where, we follow Einstein Summation Convention as usual). Indeed,

$$\begin{aligned} \omega(v) &= \omega(v^i E_i) = a_i v^i, \\ a_i \varepsilon^i(v) &= a_i \varepsilon^i(v^j E_j) = a_i v^j \varepsilon^i(E_j) = a_i v^j \delta_j^i = a_i v^i. \end{aligned} \quad (39)$$

Hence, it follows that  $(\varepsilon^i)$  spans  $V^*$ . Altogether, we have shown that this collection forms a basis for the dual space.

- **Def. (Dual Map)** Suppose  $V$  and  $W$  are vector spaces, and  $A : V \rightarrow W$  is a linear map. Define a linear map  $A^* : W^* \rightarrow V^*$ , called the *dual map* of  $A$  by,

$$(A^* \omega)(X) = \omega(AX) \quad \text{for } \omega \in W^*, X \in V. \quad (40)$$

**Exercise 6.2.** Show that  $A^* \omega$  is actually a linear functional on  $V$ , and that  $A^*$  is a linear map.

Let  $V, W$  be vector spaces,  $A : V \rightarrow W$  a linear map, and  $A^* : W^* \rightarrow V^*$  the dual map of  $A$ .

- (i) Let  $\omega \in W^*$  be a fixed covector, and let  $X, Y \in V$ ,  $a_1, a_2 \in \mathbb{R}$ . Then

$$\begin{aligned} (A^* \omega)(a_1 X + a_2 Y) &= \omega(A(a_1 X + a_2 Y)) \\ &= \omega(a_1 AX + a_2 AY) \\ &= \omega(a_1 AX) + \omega(a_2 AY) \\ &= a_1 \omega(AX) + a_2 \omega(AY) \\ &= a_1 (A^* \omega)(X) + a_2 (A^* \omega)(Y), \end{aligned} \quad (41)$$

where the second inequality follows from linearity of  $A$ , and the third and fourth inequalities follow from linearity of  $\omega$ . Hence,  $A^* \omega$  is a linear functional for each  $\omega \in W^*$ .

(ii) Now let  $X \in V$  be fixed, and let  $\omega_1, \omega_2 \in V^*$ ,  $a_1, a_2 \in \mathbb{R}$ . Then

$$\begin{aligned} (A^*(a_1\omega_1 + a_2\omega_2))(X) &= (a_1\omega_1 + a_2\omega_2)(AX) \\ &= a_1\omega_1(AX) + a_2\omega_2(AX) \\ &= (a_1(A^*\omega_1) + a_2(A^*\omega_2))(X). \end{aligned} \quad (42)$$

Since  $X$  was arbitrary, it follows that  $A^*$  is a linear map.

• **Prop. 6.2. (Properties of Dual Maps)** The dual map satisfies the following properties:

- (a)  $(A \circ B)^* = B^* \circ A^*$ .
- (b)  $(\text{Id}_V)^* : V^* \rightarrow V^*$  is the identity map of  $V^*$ .

**Exercise 6.3.** Prove the preceding proposition.

(a) Let  $B : V \rightarrow W$  and  $A : W \rightarrow Y$  be linear maps, and  $A^*, B^*$  their corresponding dual maps. Let  $\omega \in Y^*$  and  $X \in V$ . Then

$$\begin{aligned} ((B^* \circ A^*)\omega)(X) &= B^*(A^*\omega)(X) \\ &= A^*\omega(BX) = \omega(ABX) = \omega((A \circ B)X) \\ &= ((A \circ B)^*\omega)(X). \end{aligned} \quad (43)$$

Since  $X, \omega$  were arbitrary,  $(A \circ B)^* = B^* \circ A^*$ .

(b) Let  $\omega \in V^*$ , and  $X \in V$ . Then

$$((\text{Id}_V)^*\omega)(X) = \omega(\text{Id}_V X) = \omega(X). \quad (44)$$

Since  $X$  was arbitrary, we conclude that  $(\text{Id}_V)^*\omega = \omega$  for all  $\omega \in V^*$ .

• **Def. (Natural Basis-Independent Map)** For each vector space  $V$ , there is a natural, basis-independent map  $\xi : V \rightarrow V^{**}$ , defined as follows: for each vector  $X \in V$ , define a linear functional  $\xi(X) : V^* \rightarrow \mathbb{R}$  by

$$\xi(X)(\omega) = \omega(X), \quad \text{for } \omega \in V^*. \quad (45)$$

**Exercise 6.4.** Let  $V$  be a vector space.

- (a) For any  $X \in V$ , show that  $\xi(X)(\omega)$  depends linearly on  $\omega$ , so that  $\xi(X) \in V^{**}$ .
- (b) Show that the map  $\xi : V \rightarrow V^{**}$  is linear.

Let  $V$  be a vector space.

(a) Fix  $X \in V$ , and let  $\omega_1, \omega_2 \in V^*$ ,  $a_1, a_2 \in \mathbb{R}$ . Then

$$\begin{aligned} \xi(X)(a_1\omega_1 + a_2\omega_2) &= (a_1\omega_1 + a_2\omega_2)(X) = a_1\omega_1(X) + a_2\omega_2(X) \\ &= a_1\xi(X)(\omega_1) + a_2\xi(X)(\omega_2). \end{aligned} \quad (46)$$

Hence,  $\xi(X) \in V^{**}$ .

(b) Fix  $\omega \in V^*$ , and let  $X_1, X_2 \in V$ ,  $a_1, a_2 \in \mathbb{R}$ . Then

$$\begin{aligned} \xi(a_1X_1 + a_2X_2)(\omega) &= \omega(a_1X_1 + a_2X_2) = a_1\omega(X_1) + a_2\omega(X_2) \\ &= a_1\xi(X_1)(\omega) + a_2\xi(X_2)(\omega). \end{aligned} \quad (47)$$

Hence, since  $\omega \in V^*$  was arbitrary, we conclude that  $\xi : V \rightarrow V^{**}$  is linear.

• **Prop. 6.4 (Dual Dual Space is Isomorphic)** Let  $V$  be a finite-dimensional vector space. The map  $\xi : V \rightarrow V^{**}$  is an isomorphism.

Since  $V$  and  $V^{**}$  have the same dimension, it suffices to check that  $\xi$  is injective. Suppose  $X \in V \setminus \{0\}$ . Extend  $X$  to a basis  $(X = E_1, \dots, E_n)$ , and let  $(\varepsilon^1, \dots, \varepsilon^n)$  be the corresponding dual basis. Then

$$\xi(X)(\varepsilon^1) = \varepsilon^1(X) = \varepsilon^1(E_1) = 1 \neq 0, \quad (48)$$

so that  $\xi(X) \neq 0$ . Hence, the kernel is trivial, which proves injectivity.

## 5.2 Tangent Covectors on Manifolds

- **Def. (Cotangent Space)** Let  $M$  be a smooth manifold. For each  $p \in M$ , define the *cotangent space* at  $p$ , denoted by  $T_p^*M$ , to be the dual space to  $T_pM$ :  $T_p^*M = (T_pM)^*$ .
- **Obs. (Transformation Law for Covectors)** Suppose  $(U, \varphi)$  and  $(V, \psi)$  are two smooth charts on  $M$ , and let  $p \in U \cap V$ . As we saw before, we have two bases for the tangent space at  $p$ , namely  $\{\partial/\partial x^i|_p\}$ , where the coordinate functions of  $\varphi$  are  $(x^i)$ , and  $\{\partial/\partial \tilde{x}^i|_p\}$ , where the coordinate functions of  $\psi$  are  $(\tilde{x}^i)$ . Let  $(dx^i)$  and  $(d\tilde{x}^i)$  be the corresponding dual bases for the cotangent space at  $p$ . In particular, we have

$$\omega = \omega_i dx^i = \tilde{\omega}_j d\tilde{x}^j \iff \omega \left( \frac{\partial}{\partial x^i} \Big|_p \right) dx^i = \tilde{\omega}_j d\tilde{x}^j. \quad (49)$$

Using the transformation law for vectors,

$$\omega \left( \frac{\partial}{\partial x^i} \Big|_p \right) dx^i = \omega \left( \frac{\partial \tilde{x}^j}{\partial x^i}(p) \frac{\partial}{\partial \tilde{x}^j} \Big|_p \right) dx^i = \tilde{\omega}_j \frac{\partial \tilde{x}^j}{\partial x^i}(p) dx^i = \tilde{\omega}_j d\tilde{x}^j. \quad (50)$$

Therefore, we conclude that

$$d\tilde{x}^j = \frac{\partial \tilde{x}^j}{\partial x^i} dx^i. \quad (51)$$

Contrast this with the transformation law for vectors.

## 5.3 The Cotangent Bundle

- **Def. (Cotangent Bundle)** The disjoint union

$$T^*M = \coprod_{p \in M} T_p^*M. \quad (52)$$

- **Lem. 6.6. (Smoothness Criteria for Covector Fields)** Let  $M$  be a smooth manifold, and let  $\omega : M \rightarrow T^*M$  be a rough section.
  - If  $\omega = \omega_i \lambda^i$  is the coordinate representation for  $\omega$  in any smooth chart  $(U, (x^i))$  for  $M$ , then  $\omega$  is smooth on  $U$  if and only if its component functions  $\omega_i$  are smooth
  - $\omega$  is smooth if and only if for every smooth vector field  $X$  on an open subset  $U \subset M$ , the function  $\langle \omega, X \rangle : U \rightarrow \mathbb{R}$  defined by

$$\langle \omega, X \rangle(p) = \langle \omega_p, X_p \rangle = \omega_p(X_p) \quad (53)$$

is smooth.

**Exercise 6.5.** Prove Lemma 6.6.

## 5.4 The Differential of a Function

**Exercise 6.6.** Let  $f(x, y) = x^2$  on  $\mathbb{R}^2$ , and let  $X$  be the vector field

$$X = \text{grad}(f) = 2x \frac{\partial}{\partial x}. \quad (54)$$

Compute the coordinate expression of  $X$  in polar coordinates (on some open set on which they

are defined) using (6.4) and show that it is *not* equal to

$$\frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta}. \quad (55)$$

Recall that (6.4) stated the following:

$$\left. \frac{\partial}{\partial x^i} \right|_p = \frac{\partial \tilde{x}^j}{\partial x^i}(p) \left. \frac{\partial}{\partial \tilde{x}^j} \right|_p. \quad (56)$$

Remember that polar coordinates are given by  $(x, y) = (r \cos(\theta), r \sin(\theta))$ . In particular, by (6.4),

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta}. \quad (57)$$

Using the polar coordinates,

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{\partial}{\partial x}(\sqrt{x^2 + y^2}) \\ &= \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{r} = \cos(\theta). \\ \frac{\partial \theta}{\partial x} &= \frac{\partial}{\partial x}(\arctan(x^{-1}y)) \\ &= -\frac{y}{x^2 + y^2} = -\frac{r \sin(\theta)}{r^2} = -\frac{\sin(\theta)}{r}. \end{aligned} \quad (58)$$

Hence, this implies that

$$X = 2r \cos^2(\theta) \frac{\partial}{\partial r} - 2 \sin(\theta) \cos(\theta) \frac{\partial}{\partial \theta}. \quad (59)$$

On the other hand,

$$\frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} = 2r \cos^2(\theta) \frac{\partial}{\partial r} - 2r^2 \sin(\theta) \cos(\theta) \frac{\partial}{\partial \theta} \neq X. \quad (60)$$

- **Def. (Differential of a Function)** Let  $f$  be a smooth real-valued function on a smooth manifold  $M$ . Define the covector field  $df$ , called the *differential* of  $f$ , by

$$df_p(X_p) = X_p f \quad \text{for } X_p \in T_p M. \quad (61)$$

- **Lem. 6.7. (Differential is Smooth Covector Field)** The differential of a smooth function is a smooth covector field.
- **Obs. (Differential in Coordinates)** Let  $(x^i)$  be smooth coordinates on an open subset  $U \subset M$ , and let  $(\lambda_i)$  be the corresponding coframe on  $U$ . Suppose that in coordinates,  $df_p = A_i(p) \lambda^i|_p$  for some functions  $A_i : U \rightarrow \mathbb{R}$ . This implies the following:

$$\begin{aligned} A_i(p) &= A_i(p) \lambda^i|_p \left( \left. \frac{\partial}{\partial x^i} \right|_p \right) = df_p \left( \left. \frac{\partial}{\partial x^i} \right|_p \right) \\ &= \frac{\partial f}{\partial x^i}(p). \end{aligned} \quad (62)$$

This implies that

$$df_p = \frac{\partial f}{\partial x^i}(p) \lambda^i|_p. \quad (63)$$

Taking  $f$  to be  $x^j : U \rightarrow \mathbb{R}$ , we obtain

$$dx^j|_p = \frac{\partial x^j}{\partial x^i}(p) \lambda^i|_p = \lambda^j|_p. \quad (64)$$

Therefore, this proves that

$$df|_p = \frac{\partial f}{\partial x^i}(p) dx^i|_p \iff df = \frac{\partial f}{\partial x^i} dx^i. \quad (65)$$

- **Prop. 6.9 (Properties of Differentials)** Let  $M$  be a smooth manifold, and let  $f, g \in C^\infty(M)$ .
  - (a) For any constants  $a, b$ ,  $d(af + bg) = a df + b dg$ .
  - (b)  $d(fg) = f dg + g df$ .
  - (c)  $d(f/g) = (g df - f dg)/g^2$  on the set where  $g \neq 0$ .
  - (d) If  $J \subset \mathbb{R}$  is an interval containing the image of  $f$ , and  $h : J \rightarrow \mathbb{R}$  is a smooth function, then  $d(h \circ f) = (h' \circ f) df$ .
  - (e) If  $f$  is constant, then  $df = 0$ .

**Exercise 6.7.** Prove Proposition 6.9.

Let  $a, b \in \mathbb{R}$ , and  $f, g \in C^\infty(M)$ .

- (a) Let  $U \subseteq M$  be open,  $(x^i)$  smooth coordinates on  $U$ , and  $(dx^i)$  the corresponding coordinate coframe. Then

$$\begin{aligned} d(af + bg) &= \frac{\partial(af + bg)}{\partial x^i} dx^i = \left[ \frac{\partial(af)}{\partial x^i} + \frac{\partial(bg)}{\partial x^i} \right] dx^i \\ &= \left[ a \frac{\partial f}{\partial x^i} + b \frac{\partial g}{\partial x^i} \right] dx^i = a df + b dg. \end{aligned} \quad (66)$$

- (b) We will work in coordinates as in (a). Then we observe that

$$\begin{aligned} d(fg) &= \frac{\partial(fg)}{\partial x^i} dx^i = \left[ g \frac{\partial f}{\partial x^i} + f \frac{\partial g}{\partial x^i} \right] dx^i \\ &= g df + f dg. \end{aligned} \quad (67)$$

- (c) Let  $E = \{x \in M : g(x) \neq 0\}$ . Then let  $U \subseteq E$  be an open subset,  $(x^i)$  be smooth coordinates, and  $(dx^i)$  the corresponding coframe. Then

$$\begin{aligned} d(f/g) &= \frac{\partial(f/g)}{\partial x^i} dx^i = \frac{g \frac{\partial f}{\partial x^i} - f \frac{\partial g}{\partial x^i}}{g^2} dx^i \\ &= \frac{g df - f dg}{g^2}. \end{aligned} \quad (68)$$

- (d) Let  $J \subset \mathbb{R}$  be an interval containing the image of  $f$ , and  $h : J \rightarrow \mathbb{R}$  be smooth. Let  $U \subseteq M$  be open,  $(x^i)$  smooth coordinates on  $U$ , and  $(dx^i)$  the corresponding coordinate coframe. Then

$$d(h \circ f) = \frac{\partial(h \circ f)}{\partial x^i} dx^i = (h' \circ f) \cdot \frac{\partial f}{\partial x^i} dx^i = (h' \circ f) df, \quad (69)$$

where the second equality follows from the chain rule.

- (e) It suffices to show that if  $f = 1$ , then  $df = 0$ . Indeed,

$$df = d(ff) = f df + f df = 2f df = 2 df. \quad (70)$$

Hence, this proves that  $df = 0$ .

- **Prop. 6.10. (Functions with Vanishing Differentials)** If  $f$  is a smooth real-valued function on a smooth manifold  $M$ , then  $df = 0$  if and only if  $f$  is constant on each component of  $M$ .

It suffices to assume that  $M$  is connected and to show that  $df = 0$  if and only if  $f$  is constant. Indeed, assume  $f$  is constant. Then by Prop. 6.9(e),  $df = 0$ . Now suppose  $df = 0$ ,  $p \in M$ , and let  $\mathcal{C} = \{q \in M : f(p) = f(q)\}$ . If  $q$  is any point in  $\mathcal{C}$ , then let  $U$  be a smooth coordinate ball centered at  $q$ . By virtue of the differential being zero, we must have  $\partial f / \partial x^i = 0$  in  $U$  for each  $i$ . This implies that  $f$  is constant on  $U$ . Hence,  $\mathcal{C}$  is open. On the other hand, by continuity of  $f$ ,  $\mathcal{C}$  is closed. Since the only open and closed sets in a connected set are the empty set and  $M$ , it follows that  $\mathcal{C} = M$ ; i.e.,  $f$  is constant on  $M$ .

## 5.5 Pullbacks

- **Def. (Pullback of a Smooth Map)** Let  $F : M \rightarrow N$  be a smooth map, and  $F_* : T_p M \rightarrow T_{F(p)} N$  its pushforward. Then the pushforward induces a dual linear map  $F^* : T_{F(p)}^* N \rightarrow T_p^* M$  defined by

$$(F^* \omega)(X_p) = \omega(F_* X), \quad \text{for } \omega \in T_{F(p)}^* N, X \in T_p M. \quad (71)$$

- **Obs. (Pullback in Coordinates)** Let  $p \in M$  be arbitrary, and choose smooth coordinates  $(x^i)$  for  $M$  near  $p$  and  $(y^j)$  for  $N$  near  $G(p)$ . Then

$$G^* \omega = G^*(\omega_j dy^j) = (\omega_j \circ G) dG^j. \quad (72)$$

- **Ex. 6.14. (Example of Pullback)** Let  $G : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  be the map defined by

$$(u, v) = G(x, y, z) = (x^2 y, y \sin(z)), \quad (73)$$

and let  $\omega \in \mathcal{T}^*(\mathbb{R}^2)$  be the covector field

$$\omega = u \, dv + v \, du. \quad (74)$$

First, we shall compute the differentials.

$$\begin{aligned} du &= \frac{\partial u}{\partial x^i} dx^i = 2xy \, dx + x^2 \, dy. \\ dv &= \frac{\partial v}{\partial x^i} dx^i = \sin(z) \, dy + y \cos(z) \, dz. \end{aligned} \quad (75)$$

Therefore,

$$\begin{aligned} G^* \omega &= x^2 y [\sin(z) \, dy + y \cos(z) \, dz] + y \sin(z) [2xy \, dx + x^2 \, dy] \\ &= 2xy^2 \sin(z) \, dx + 2xy^2 \sin(z) \, dy + x^2 y^2 \cos(z) \, dz. \end{aligned} \quad (76)$$

## 5.6 Line Integrals

- **Prop. 6.16 (Diffeomorphism Invariance of the Integral)** Let  $\omega$  be a smooth covector field on the compact interval  $[a, b] \subset \mathbb{R}$ . If  $\varphi : [c, d] \rightarrow [a, b]$  is an increasing diffeomorphism (meaning that  $t_1 < t_2$  implies  $\varphi(t_1) < \varphi(t_2)$ ), then

$$\int_{[c,d]} \varphi^* \omega = \int_{[a,b]} \omega. \quad (77)$$

Let  $s$  be the standard coordinates on  $[c, d]$  and  $t$  be the standard coordinates on  $[a, b]$ . We may write  $\omega_t = f(t) \, dt$  for some smooth function  $f : [a, b] \rightarrow \mathbb{R}$ . Then using the pullback expression in local coordinates,

$$(\varphi^* \omega)_s = f(\varphi(s)) \varphi'(s) \, ds. \quad (78)$$

Therefore,

$$\int_{[c,d]} \varphi^* \omega = \int_c^d f(\varphi(s)) \varphi'(s) \, ds \stackrel{t:=\varphi(s)}{=} \int_a^b f(t) \, dt = \int_{[a,b]} \omega. \quad (79)$$

**Exercise 6.8.** If  $\varphi : [c, d] \rightarrow [a, b]$  is a decreasing diffeomorphism, show that  $\int_{[c,d]} \varphi^* \omega = - \int_{[a,b]} \omega$ .

The proof follows almost nearly identically to the proof from above. Suppose that  $s$  is the standard coordinate on  $[c, d]$ , and let  $t$  be the standard coordinate on  $[a, b]$ . We may assume that  $\omega_t = f(t) dt$  for some smooth function  $f : [a, b] \rightarrow \mathbb{R}$ . Note that because of the decreasing property,  $\varphi(c) = b$  and  $\varphi(d) = a$ . Hence,

$$\int_{[c,d]} \varphi^* \omega = \int_c^d f(\varphi(s)) \varphi'(s) ds = \int_b^a f(t) dt = - \int_{[a,b]} f(t) dt = - \int_{[a,b]} \omega. \quad (80)$$

- **Def. (Curve Segment)** Let  $M$  be a smooth manifold. A *curve segment* is a continuous curve  $\gamma : [a, b] \rightarrow M$  whose domain is a compact interval. It is a *smooth curve segment* if it has a smooth extension to an open interval containing  $[a, b]$ . A *piecewise smooth curve segment* is a piecewise smooth curve segment.
- **Def. (Line Integral)** Let  $\gamma : [a, b] \rightarrow M$  be a smooth curve segment and  $\omega$  a smooth covector field on  $M$ . The *line integral* of  $\omega$  over  $\gamma$  is defined to be the real number

$$\int_{\gamma} \omega = \int_{[a,b]} \gamma^* \omega. \quad (81)$$

If  $\gamma$  is *piecewise smooth*, then

$$\int_{\gamma} \omega = \sum_{i=1}^n \int_{[a_{i-1}, a_i]} \gamma^* \omega, \quad (82)$$

where  $\{a_i\}_0^n$  is a partition of  $[a, b]$ .

- **Prop. 6.18. (Properties of Line Integrals)** Let  $M$  be a smooth manifold. Suppose  $\gamma : [a, b] \rightarrow M$  is a piecewise smooth curve segment and  $\omega, \omega_1, \omega_2 \in \mathcal{T}^*(M)$ .

(a) For any  $c_1, c_2 \in \mathbb{R}$ ,

$$\int_{\gamma} (c_1 \omega_1 + c_2 \omega_2) = c_1 \int_{\gamma} \omega_1 + c_2 \int_{\gamma} \omega_2. \quad (83)$$

(b) If  $\gamma$  is a constant map, then  $\int_{\gamma} \omega = 0$ .

(c) If  $a < c < b$ , then

$$\int_{\gamma} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega, \quad (84)$$

where  $\gamma_1 = \gamma|_{[a,c]}$  and  $\gamma_2 = \gamma|_{[c,b]}$ .

**Exercise 6.9.** Prove Proposition 6.18

Assume all of the hypotheses given in the statement of the proposition. Let  $a = a_0 < a_1 < \dots < a_n = b$  be a partition of  $[a, b]$  such that  $\gamma$  is smooth on each subinterval.

(a) By linearity of pullbacks,

$$\begin{aligned} \int_{\gamma} (c_1 \omega_1 + c_2 \omega_2) &= \sum_{i=1}^n \int_{[a_{i-1}, a_i]} \gamma^* (c_1 \omega_1 + c_2 \omega_2) = \sum_{i=1}^n \int_{[a,b]} [\gamma^* (c_1 \omega_1) + \gamma^* (c_2 \omega_2)] \\ &= \sum_{i=1}^n \left[ \int_{[a_{i-1}, a_i]} \gamma^* (c_1 \omega_1) + \int_{[a_{i-1}, a_i]} \gamma^* (c_2 \omega_2) \right] \\ &= c_1 \int_{\gamma} \omega_1 + c_2 \int_{\gamma} \omega_2. \end{aligned} \quad (85)$$

(c) Let  $a < c < b$ ; let  $a = a'_0 < a'_1 < \dots < a'_n = c$  and  $c = a'_{n+1} < \dots < a'_m = b$  be partitions of  $[a, c]$  and  $[c, b]$ , respectively. Clearly  $\{a'_i\}_{i=0}^m$  is also a partition of  $[a, b]$ . Then

$$\int_{\gamma} \omega = \sum_{i=1}^m \int_{[a'_{i-1}, a'_i]} \gamma^* \omega = \sum_{i=1}^n \int_{[a'_{i-1}, a'_i]} \gamma^* \omega + \sum_{i=n+1}^m \int_{[a'_{i-1}, a'_i]} \gamma^* \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega. \quad (86)$$

- (b) It suffices to assume that  $\gamma$  is a smooth curve segment. Let  $s$  be the standard coordinates on  $[a, b]$ . Then in local coordinates,  $\gamma^*\omega = \omega(\gamma(s))\gamma'(s) ds = 0$  since  $\gamma'(s) = 0$  for all  $s$ . Hence,

$$\int_{\gamma} \omega = \int_{[a,b]} \gamma^*\omega = \int_{[a,b]} 0 ds = 0. \quad (87)$$

## 5.7 Conservative Vector Fields

- **Def. (Exact Smooth Covector Field)** Let  $\omega$  be a smooth covector field on a smooth manifold  $M$ .  $\omega$  is *exact* if it is the differential of some  $f \in C^\infty(M)$ . The function  $f$  is called a *potential* for  $\omega$ .
- **Def. (Conservative Covector Field)** A smooth covector field  $\omega$  is *conservative* if the line integral of  $\omega$  over *any* closed piecewise smooth curve segment is zero.
- **Lem. 6.23. (Conservative Covector Field Criterion I)** A smooth covector field  $\omega$  is conservative if and only if the line integral of  $\omega$  depends only on the endpoints of the curve, i.e.,  $\int_{\gamma} \omega = \int_{\tilde{\gamma}} \omega$  whenever  $\gamma$  and  $\tilde{\gamma}$  are piecewise smooth curves are piecewise smooth curve segments with the same starting and ending points.

**Exercise 6.10.** Prove Lemma 6.23. [Observe that this would be much harder to prove if we defined conservative fields in terms of smooth curves instead of piecewise smooth ones.]

## 6 Submersions, Immersions, and Embeddings

### 6.1 Maps of Constant Rank

- **Def. (Rank of a Smooth Map)** Let  $M$  and  $N$  be smooth manifolds, and  $F : M \rightarrow N$  a smooth map. The *rank* of  $F$  at  $p \in M$  is the rank of the linear map  $F_* : T_p M \rightarrow T_{F(p)} N$ ; this is equivalent to the rank of the matrix of partial derivatives of  $F$  in any smooth chart, or to the dimension of  $\text{Im } F_* \subset T_{F(p)} N$ . I.e., the rank is equivalent to the maximum number of linearly independent rows/columns of the corresponding matrix.
- **Def. (Submersion)** A smooth map  $F : M \rightarrow N$  such that  $F_*$  is surjective at each point, which is to say that  $\text{rank } F = \dim N$ .
- **Def. (Immersion)** A smooth map  $F : M \rightarrow N$  such that  $F_*$  is injective at each point; equivalently  $\text{rank } F = \dim M$ .
- **Def. (Smooth Embedding)** An immersion  $F : M \rightarrow N$  such that  $F : M \rightarrow F(M) \subset N$  is a homeomorphism.

**Exercise 7.2.** Show that a composition of submersions is a submersion, a composition of immersions is an immersion, and a composition of smooth embeddings is a smooth embedding.

- (i) Let  $F : M \rightarrow N$  and  $G : N \rightarrow P$  be submersions, where  $M$  is a smooth  $m$ -manifold,  $N$  is a smooth  $n$ -manifold, and  $P$  is a smooth  $p$ -manifold. Let  $U, V, T$  be smooth coordinate charts for  $M, N$ , and  $P$ , respectively, such that (wlog)  $F(U) \subset V$  and  $G(V) \subset T$ . Then since  $(G \circ F)_* = G_* \circ F_*$ , in local coordinates,  $(G \circ F)_*$  corresponds to the matrix product of an  $p \times n$  matrix with an  $n \times m$  matrix (the  $p \times n$  matrix representing  $G_*$ , and the  $n \times m$  matrix representing  $F_*$ ); the rank of the  $p \times n$  matrix is  $p$ , while the rank of the  $n \times m$  matrix is  $n$ . Hence, by the properties of the rank of a matrix product, the matrix representation of  $G_* \circ F_*$  has rank  $p$ , which proves that  $(G \circ F)$  is a submersion.
- (ii) Let  $F : M \rightarrow N$  and  $G : N \rightarrow P$  be immersions. Since  $F_*$  is injective and  $G_*$  is injective, and the composition of injective functions is injective,

$$(G \circ F)_* = G_* \circ F_* \quad (88)$$

is injective. Hence,  $G \circ F$  is an immersion.

- (iii) Let  $F : M \rightarrow N$  and  $G : N \rightarrow P$  be smooth embeddings. From (ii),  $G \circ F$  is an immersion. Finally, since the composition of homeomorphisms is always another homeomorphism, we conclude that  $G \circ F$  is a smooth embedding.

- **Prop. 7.4. (Smooth Embedding Criteria)** Suppose  $F : M \rightarrow N$  is an injective immersion. If either of the following condition holds, then  $F$  is a smooth embedding with closed image:
  - (a)  $M$  is compact.
  - (b)  $F$  is a proper map.

## 7 Tensors

### 7.1 The Algebra of Tensors

- **Def. (Multilinear Function)** Suppose  $V_1, \dots, V_n$  and  $W$  are vector spaces. A map  $F : V_1 \times \dots \times V_n \rightarrow W$  is said to be *multilinear* if it is linear as a function of each variable separately:

$$F(v_1, \dots, av_i + a'v'_i, \dots, v_k) = aF(v_1, \dots, v_i, \dots, v_k) + a'F(v_1, \dots, v'_i, \dots, v_k). \quad (89)$$

- **Def. (Covariant  $k$ -Tensor)** Let  $V$  be a finite-dimensional real vector space, and let  $k$  be a natural number. A *covariant  $k$ -tensor* on  $V$  is a real-valued multilinear function of  $k$  elements of  $V$ :

$$T : \underbrace{V \times \dots \times V}_{k \text{ copies}} \rightarrow \mathbb{R}. \quad (90)$$

The number  $k$  is called the *rank* of  $T$ .

- **Def. (Tensor Product)** We can build up covariant tensors of larger ranks as follows: let  $V$  be a finite-dimensional real vector space and let  $S \in T^k(V)$ ,  $T \in T^l(V)$ . Define a map  $S \otimes T : \underbrace{V \times \dots \times V}_{k+l \text{ copies}} \rightarrow \mathbb{R}$  by

$$S \otimes T(X_1, \dots, X_{k+l}) = S(X_1, \dots, X_k)T(X_{k+1}, \dots, X_{k+l}). \quad (91)$$

We will check multilinearity. WLOG, assume  $i \leq k$ . Then

$$\begin{aligned} S \otimes T(X_1, \dots, aX_i + a'X'_i, \dots, X_k, \dots, X_{k+l}) &= S(X_1, \dots, aX_i + a'X'_i, \dots, X_k)T(X_{k+1}, \dots, X_{k+l}) \\ &= [aS(X_1, \dots, X_i, \dots, X_k) + a'S(X_1, \dots, X'_i, \dots, X_k)]T(X_{k+1}, \dots, X_{k+l}) \\ &= a(S \otimes T)(X_1, \dots, X_i, \dots, X_{k+l}) + a'(S \otimes T)(X_1, \dots, X'_i, \dots, X_{k+l}). \end{aligned} \quad (92)$$

Hence,  $S \otimes T$  is a covariant  $(k+l)$ -tensor.

**Exercise 11.1.** Show that the tensor product operation is bilinear and associative. More precisely, show that  $S \otimes T$  depends linearly on each of the tensors  $S$  and  $T$ , and that  $(R \otimes S) \otimes T = R \otimes (S \otimes T)$ .

Let  $P, R, S, T$  be  $k, k, l, l$ -tensors, respectively. Then

$$\begin{aligned} (a_1P + a_2R) \otimes (a_3S + a_4T)(X_1, \dots, X_{k+l}) &= (a_1P + a_2R)(X_1, \dots, X_k) \cdot (a_3S + a_4T)(X_{k+1}, \dots, X_{k+l}) \\ &= a_1P(X_1, \dots, X_k) \cdot (a_3S + a_4T)(X_{k+1}, \dots, X_{k+l}) \\ &\quad + a_2R(X_1, \dots, X_k)(a_3S + a_4T)(X_{k+1}, \dots, X_{k+l}) \\ &= a_1a_3P(X_1, \dots, X_k)S(X_{k+1}, \dots, X_{k+l}) \\ &\quad + a_1a_4P(X_1, \dots, X_k)T(X_{k+1}, \dots, X_{k+l}) \\ &\quad + a_2a_3P(X_1, \dots, X_k)S(X_{k+1}, \dots, X_{k+l}) \\ &\quad + a_2a_4P(X_1, \dots, X_k)T(X_{k+1}, \dots, X_{k+l}). \end{aligned} \quad (93)$$

Using the definition of the tensor products, we can simplify the final expressions to see that the tensor product is, indeed, linear in each of the tensor terms.

- **Prop. 11.2. (Basis for  $T^k V$ )** Let  $V$  be a real vector space of dimension  $n$ , let  $(E_i)$  be any basis for  $V$ , and let  $(\varepsilon^i)$  be the dual basis. The set of all  $k$ -tensors of the form  $\varepsilon^{i_1} \otimes \dots \otimes \varepsilon^{i_k}$  for  $1 \leq i_1 \leq \dots \leq i_k \leq n$  is a basis for  $T^k V$ , which therefore has dimension  $n^k$ .

Let  $\mathcal{B}$  denote the set  $\{\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} : 1 \leq i_1 \leq \cdots \leq i_k \leq n\}$ . It suffices to show that  $\mathcal{B}$  is linearly independent and spans  $T^k V$ . Let  $T \in T^k(V)$ . For any  $k$ -tuple of integers  $(i_1, \dots, i_k)$ , where  $1 \leq i_j \leq n$  for all  $j = 1, \dots, k$ , define the number  $T_{i_1 \dots i_k}$  as follows:

$$T_{i_1 \dots i_k} = T(E_{i_1}, \dots, E_{i_k}). \quad (94)$$

We will show that  $T = T_{i_1 \dots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}$ . Indeed,

$$\begin{aligned} T_{i_1 \dots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} (E_{j_1}, \dots, E_{j_k}) &= T_{i_1 \dots i_k} \varepsilon^{i_1}(E_{j_1}) \cdots \varepsilon^{i_k}(E_{j_k}) \\ &= T_{j_1 \dots j_k} \\ &= T(E_{j_1}, \dots, E_{j_k}). \end{aligned} \quad (95)$$

By multilinearity, since a tensor is completely determined by its action on sequences of basis vectors, this proves the claim that  $\mathcal{B}$  spans  $T^k V$ . Now, we must prove linear independence. But this is straightforward to show by letting  $T_{i_1 \dots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} = 0$  act on a sequence of basis vectors.

- **Def. (Free Vector Space)** Let  $S$  be a set. The *free vector space on  $S$* , denoted by  $\mathbb{R}\langle S \rangle$ , is the set of all finite formal linear combinations of  $S$  with real coefficients. More precisely, a finite formal linear combination is a function  $\mathcal{F} : S \rightarrow \mathbb{R}$  such that  $\mathcal{F}(s) = 0$  for all but finitely many  $s \in S$ .

**Exercise 11.2 (Characteristic Property of Free Vector Spaces).** Let  $S$  be a set and  $W$  a vector space. Show that any map  $F : S \rightarrow W$  has a unique extension to a linear map  $\bar{F} : \mathbb{R}\langle S \rangle \rightarrow W$ .

Let  $S$  be a set,  $W$  a vector space, and  $F : S \rightarrow W$  an arbitrary map. Define the map  $\bar{F} : \mathbb{R}\langle S \rangle \rightarrow W$  as follows: given a formal sum  $\sum_{s \in S} \alpha_s s$ , where  $\alpha_s = 0$  for all but finitely many elements  $s \in S$ , let

$$\bar{F} \left( \sum_{s \in S} \alpha_s s \right) = \sum_{s \in S} \alpha_s F(s). \quad (96)$$

Since each  $\alpha_s \in \mathbb{R}$  and  $F(s) \in W$ , it follows that  $\sum_{s \in S} \alpha_s F(s) \in W$ . First, we must show that  $\bar{F}$  is a linear map. Let  $\sum_{s \in S} \alpha_s s, \sum_{s \in S} \beta_s s \in \mathbb{R}\langle S \rangle$ , where  $\alpha_s, \beta_s$  are zero for all but finitely elements (not necessarily the same) of  $S$ . Then

$$\begin{aligned} \bar{F} \left( \sum_{s \in S} \alpha_s s + \sum_{s \in S} \beta_s s \right) &= \bar{F} \left( \sum_{s \in S} (\alpha_s + \beta_s) s \right) \\ &= \sum_{s \in S} (\alpha_s + \beta_s) F(s) \\ &= \sum_{s \in S} \alpha_s F(s) + \sum_{s \in S} \beta_s F(s) \\ &= \bar{F} \left( \sum_{s \in S} \alpha_s s \right) + \bar{F} \left( \sum_{s \in S} \beta_s s \right). \end{aligned} \quad (97)$$

Hence,  $\bar{F}$  is linear. The proof of uniqueness follows as proceeds: if  $F$  extends to two linear maps  $\bar{F}_1$  and  $\bar{F}_2$ , let these linear maps act on each element of  $S$ . By construction of these maps, it follows that  $\bar{F}_1(s) = \bar{F}_2(s)$  for all  $s \in S$ . Since  $S$  is a basis for  $\mathbb{R}\langle S \rangle$  and  $\bar{F}_{1,2}$  are completely determined by their actions on the basis elements, we conclude that  $\bar{F}_1 = \bar{F}_2$ , and so this extension is unique.

- **Def. (Tensor Product of Vector Spaces)** Let  $V$  and  $W$  be finite-dimensional real vector spaces, and let  $\mathcal{R}$  be the subspace of the free vector space  $\mathbb{R}\langle V \times W \rangle$  spanned by all elements of the

following forms:

$$\begin{aligned}
 & \alpha(v, w) - (\alpha v, w), \\
 & \alpha(v, w) - (v, \alpha w), \\
 & (v, w) + (v', w) - (v + v', w) \\
 & (v, w) + (v, w') - (v, w + w'),
 \end{aligned} \tag{98}$$

for  $\alpha \in \mathbb{R}$ ,  $v, v' \in V$ , and  $w, w' \in W$ . Define the *tensor product of  $V$  and  $W$* , denoted  $V \otimes W$  to be the quotient space  $\mathbb{R}\langle V \times W \rangle / \mathcal{R}$ . The equivalence class of an element  $(v, w) \in V \otimes W$  is denoted by  $v \otimes w$ , and is called the *tensor product of  $v$  and  $w$* .

- **Prop. 11.3. (Characteristic Property of Tensor Products)** Let  $V$  and  $W$  be finite dimensional real vector spaces. IF  $A : V \times W \rightarrow X$  is a bilinear map into any vector space  $X$ , there is a unique linear map  $\tilde{A} : V \otimes W \rightarrow X$  such that the following diagram commutes:

$$\begin{array}{ccc}
 V \times W & \xrightarrow{A} & X \\
 \downarrow \pi & \nearrow \tilde{A} & \\
 V \otimes W & & 
 \end{array}$$

where  $\pi(v, w) = v \otimes w$ .

- **Prop. 11.4. (Other Properties of Tensor Products)** Let  $V, W$ , and  $X$  be finite-dimensional real vector spaces.
  - (a) The tensor product  $V^* \otimes W^*$  is canonically isomorphic to the space  $B(V, W)$  of bilinear maps from  $V \times W$  into  $\mathbb{R}$ .
  - (b) If  $(E_i)$  is a basis for  $V$  and  $(F_j)$  is a basis for  $W$ , then the set of all elements of the form  $E_i \otimes F_j$  is a basis for  $V \otimes W$ , which therefore has dimension  $(\dim V)(\dim W)$ .
  - (c) There is a unique isomorphism  $V \otimes (W \otimes X) \rightarrow (V \otimes W) \otimes X$  sending  $v \otimes (w \otimes x)$  to  $(v \otimes w) \otimes x$ .
- **Cor. 11.5. (Space of Covariant  $k$ -Tensors and Tensor Products)** If  $V$  is a finite-dimensional real vector space, the space  $T^k(V)$  of covariant  $k$ -tensors on  $V$  is canonically isomorphic to the  $k$ -fold tensor product  $V^* \otimes \cdots \otimes V^*$ .
- **Def. (Space of Contravariant  $k$ -Tensors)** Let  $V$  be a finite-dimensional real vector space, and define the space of all *contravariant  $k$ -tensors* on  $V$  to be the space

$$T_k(V) = \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}}. \tag{99}$$

Because of the canonical identification  $V = V^{**}$ , one may think of an element of  $T_k V$  as a multilinear function from  $V^* \times \cdots \times V^*$  into  $\mathbb{R}$ .

## 7.2 Tensors and Tensor Fields on Manifolds

- **Def. (Various Tensor Bundles)** Let  $M$  be a smooth manifold. Define the following:
  - (a) *Bundle of covariant  $k$ -tensors on  $M$ :*

$$T^k M = \coprod_{p \in M} T^k(T_p M). \tag{100}$$

- (b) *Bundle of contravariant  $l$ -tensors on  $M$ :*

$$T_l M = \coprod_{p \in M} T_l(T_p M). \tag{101}$$

(c) *Bundle of mixed tensors of type  $\binom{k}{l}$  on  $M$ :*

$$T_l^k M = \coprod_{p \in M} T_l^k(T_p M). \quad (102)$$

- **Def. (Smooth Tensor Fields)** A *smooth tensor field* is a smooth section of the above tensor bundles.
- **Obs. (Smooth Tensor Fields in Coordinates)** Given any smooth local coordinates  $(x^i)$  on  $M$ , sections of the above bundles can be written as:

$$\sigma = \begin{cases} \sigma_{i_1 \dots i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}, & \sigma \in \mathcal{T}^k(M); \\ \sigma^{j_1 \dots j_l} \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}}, & \sigma \in \mathcal{T}_l(M). \\ \sigma_{i_1 \dots i_k}^{j_1 \dots j_l} dx^{i_1} \otimes \dots \otimes dx^{i_k} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}}, & \sigma \in \mathcal{T}_l^k(M). \end{cases} \quad (103)$$

- **Lem. 11.6. (Equivalent Conditions for Smooth Tensor Fields)** Let  $M$  be a smooth manifold, and let  $\sigma : M \rightarrow T^k M$  be a rough section. The following are equivalent:
  - (a)  $\sigma$  is smooth.
  - (b) In any smooth coordinate chart, the composition functions of  $\sigma$  are smooth.
  - (c) If  $X_1, \dots, X_k$  are smooth vector fields defined on an open subset  $U \subset M$ , then the function  $\sigma(X_1, \dots, X_k) : U \rightarrow \mathbb{R}$ , defined by

$$\sigma(X_1, \dots, X_k)(p) = \sigma_p(X_1|_p, \dots, X_k|_p) \quad (104)$$

is smooth.

### 7.3 Pullbacks of Smooth Tensor Fields

- **Def. (Pullback of a Smooth Map in Relation to Tensor Fields)** If  $F : M \rightarrow N$  is a smooth map, for each integer  $k \geq 0$  and each  $p \in M$ , we obtain a map  $F_* : T^k(T_{F(p)}N) \rightarrow T^k(T_p M)$  called the pullback by

$$(F^*S)(X_1, \dots, X_k) = S(F_*X_1, \dots, F_*X_k). \quad (105)$$

- **Prop. 11.8. (Properties of Tensor Pullbacks)** Suppose  $F : M \rightarrow N$  and  $G : N \rightarrow P$  are smooth maps,  $p \in M$ ,  $S \in T^k(T_{F(p)}N)$ , and  $T \in T^l(T_{G(F(p))}P)$ .
  - (a)

## 8 Homotopy and the Fundamental Group

### 8.1 Homotopy

- **Def. (Homotopy of Maps)** Let  $X$  and  $Y$  be *topological spaces*, and  $f, g \in C(X, Y)$ . Then a *homotopy* from  $f$  to  $g$  is a continuous map  $H : X \times I \rightarrow Y$  such that  $H(x, 0) = f(x)$  and  $H(x, 1) = g(x)$  for all  $x \in X$ .
- **Def. (Homotopy Relative to a Subset)** Let  $X$  and  $Y$  be topological spaces, and  $A \subset X$  an arbitrary subspace. A homotopy  $H$  between maps  $f, g : X \rightarrow Y$  is called a *homotopy relative to  $A$*  if
 
$$H(x, t) = f(x), \quad \text{for all } x \in A, t \in I. \quad (106)$$
- **Def. (Path Homotopy)** Given two paths  $f, g$  on  $X$ , a *path homotopy* from  $f$  to  $g$  is a homotopy between the paths relative to the subset  $\{0, 1\} \subset I$ .
- **Def. (Fundamental Group)** The *fundamental group* of  $X$  based at  $q \in X$ , denoted by  $\pi_1(X, q)$  is the set of all path classes of loops based at  $q$ , with operation defined by concatenation.
- **Def. (Simply Connected Topological Space)** Let  $X$  be a topological space. If  $X$  is path connected and  $\pi_1(X)$  is trivial, then  $X$  is said to be *simply connected*.

**Exercise 7.2.** Let  $X$  be a topological space.

- Let  $f, g : I \rightarrow X$  be two paths from  $p$  to  $q$ . Show that  $f \sim g$  if and only if  $f \cdot g^{-1} \sim c_p$ .
- Show that  $X$  is simply connected if and only if any two paths in  $X$  with the same initial and terminal points are path homotopic.

Let  $X$  be a topological space.

- Let  $f, g : I \rightarrow X$  be two paths from  $p$  to  $q$ . Suppose  $f \sim g$ . Then since  $[f] = [g]$ ,

$$[g] \cdot [g^{-1}] = [c_p] \implies [f] \cdot [g^{-1}] = [c_p] \implies f \cdot g^{-1} \sim c_p. \quad (107)$$

On the other hand, if  $f \cdot g^{-1} \sim c_p$ , then

$$[f] \cdot [g^{-1}] = [c_p] = [g] \cdot [g^{-1}] \implies [f] = [g] \implies f \sim g. \quad (108)$$

- Suppose that  $X$  is a simply connected space, and let  $f, g : I \rightarrow X$  be two paths from  $p$  to  $q$ . Then the product  $f \cdot g^{-1}$  is well-defined and is a loop based at  $p$ . By simple connectedness,  $f \cdot g^{-1} \sim c_p$ . Hence, by part (a), we conclude that  $f \sim g$ . Now suppose that  $X$  is path connected and that any two paths in  $X$  that have the same initial and terminal points are path homotopic. Let  $\gamma$  be an arbitrary loop based at  $p \in X$ . Then by hypothesis,  $\gamma \sim c_p$ . Hence,  $\pi_1(X, p)$  is trivial. By path connectedness,  $\pi_1(X)$  is trivial, and so  $X$  is simply connected.

### 8.2 Homomorphisms Induced by Continuous Maps

- **Lem. 7.14. (Path Homotopy is Preserved by Composition with Continuous Maps)** The path homotopy relation is preserved by composition with continuous maps. That is, if  $f_0, f_1 : I \rightarrow X$  are path homotopic and  $\varphi : X \rightarrow Y$  is continuous, then  $\varphi \circ f_0$  and  $\varphi \circ f_1$  are path homotopic.

**Exercise 7.5.** Prove Lemma 7.14.

Suppose that  $f_0, f_1 : I \rightarrow X$  are path homotopic, and that  $\varphi : X \rightarrow Y$  is continuous. Let  $H : I \times I \rightarrow X$  be the path homotopy from  $f_0$  to  $f_1$ , and consider the map  $\varphi \circ H : I \times I \rightarrow Y$ . Since  $H$  and  $\varphi$  are continuous on their respective domains, it follows that  $\varphi \circ H$  is continuous. Moreover, for any  $s \in I$ ,

$$(\varphi \circ H)(s, 0) = (\varphi \circ f_0)(s), \quad \text{and} \quad (\varphi \circ H)(s, 1) = (\varphi \circ f_1)(s). \quad (109)$$

Hence,  $\varphi \circ H$  is a path homotopy from  $\varphi \circ f_0$  to  $\varphi \circ f_1$ .

- **Def. (Homomorphism Induced by a Continuous Map)** Let  $X$  and  $Y$  be topological spaces, and  $\varphi : X \rightarrow Y$  a continuous map. The map  $\varphi_* : \pi_1(X, q) \rightarrow \pi_1(X, \varphi(q))$  defined by  $\varphi_*([f]) = [\varphi \circ f]$  is a group homomorphism, and is called the *homomorphism induced by  $\varphi$* .

- **Prop. 7.16 (Properties of the Induced Homomorphism)**

- Let  $\varphi : X \rightarrow Y$  and  $\psi : Y \rightarrow Z$  be continuous maps. Then  $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ .
- If  $\text{Id}_X : X \rightarrow X$  denotes the identity map of  $X$ , then for any  $q \in X$ ,  $(\text{Id}_X)_*$  is the identity map of  $\pi_1(X, q)$ .

- Let  $\varphi : X \rightarrow Y$  and  $\psi : Y \rightarrow Z$  be continuous maps,  $p \in X$ , and  $[f] \in \pi_1(X, p)$ . Then

$$(\psi_* \circ \varphi_*)([f]) = \psi_*([\varphi \circ f]) = [(\psi \circ \varphi) \circ f] = (\psi \circ \varphi)_*([f]). \quad (110)$$

Since this is true for all  $[f] \in \pi_1(X, p)$ ,  $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ .

- Let  $[f] \in \pi_1(X, q)$ . Then

$$(\text{Id}_X)_*([f]) = [\text{Id}_X \circ f] = [f]. \quad (111)$$

Since this is true for all  $[f] \in \pi_1(X, q)$ , we conclude that  $(\text{Id}_X)_*$  is the identity map of  $\pi_1(X, q)$ .

- **Cor. 7.17 (Induced Isomorphism)** Homeomorphic spaces have isomorphic fundamental groups; namely, if  $\varphi : X \rightarrow Y$  is a homeomorphism, then  $\varphi_* : \pi_1(X, q) \rightarrow \pi_1(X, \varphi(q))$  is an isomorphism.
- **Def. (Retraction of a Space)** Let  $X$  be a topological space, and  $A$  a subspace of  $X$ . A continuous map  $r : X \rightarrow A$  is called a *retraction* if  $r|_A = \text{Id}_A$ . Equivalently,  $r$  is a retraction if  $r \circ \iota_A = \text{Id}_A$ , where  $\iota_A : A \hookrightarrow X$  is the inclusion map. If there exists a retraction from  $X$  to  $A$ , then we say that  $A$  is a *retract* of  $X$ .
- **Prop. 7.18. (Injective Induced Homomorphism)** Suppose  $A$  is a retract of  $X$ . If  $r : X \rightarrow A$  is any retraction, then for any  $q \in A$ ,  $(\iota_A)_* : \pi_1(A, q) \rightarrow \pi_1(X, q)$  is injective and  $r_* : \pi_1(X, q) \rightarrow \pi_1(A, q)$  is surjective.

Since  $r \circ \iota_A = \text{Id}_A$ ,  $r_* \circ (\iota_A)_*$  is the identity on  $\pi_1(A, q)$ , from which it follows that  $(\iota_A)_*$  is injective and  $r_*$  is surjective.

### 8.3 Homotopy Equivalence

- **Def. (Homotopy Equivalences)** Let  $\varphi : X \rightarrow Y$  and  $\psi : Y \rightarrow X$ .
  - $\psi$  is a *homotopy inverse* for  $\varphi$  if  $\psi \circ \varphi \simeq \text{Id}_X$  and  $\varphi \circ \psi \simeq \text{Id}_Y$ .
  - If  $\varphi$  has a homotopy inverse  $\psi$ , then  $\varphi$  is called a *homotopy equivalence*, and we say that  $X$  is *homotopically equivalent* to  $Y$ , or that  $X$  has the same *homotopy type* as  $Y$ . We denote  $X \simeq Y$ .
- **Def. (Deformation Retract)** A subspace  $A \subset X$  is said to be a *deformation retract* if there exists a retraction  $r : X \rightarrow A$  such that the identity of  $X$  is homotopic to  $\iota_A \circ r$ ; the homotopy  $H : \text{Id}_X \simeq \iota_A \circ r$  is called a *deformation retraction*. Intuitively, this means that points in  $A$  end up at the same position they started at. A deformation retraction is *strong* iff  $\text{Id}_X \simeq_A (r \circ \iota_A)$ , which is to say that the points of  $A$  remain *fixed* throughout the retraction.<sup>1</sup>
- **Def. (Contractible Space)** Let  $X$  be any topological space.  $X$  is said to be *contractible* iff the identity map of  $X$  is homotopic to a constant map (i.e., if  $\text{Id}_X$  is nullhomotopic).

<sup>1</sup>See [https://encycla.com/Deformation\\_retraction](https://encycla.com/Deformation_retraction) for a gif of a (strong) deformation retraction.

## 9 Differential Forms

### 9.1 The Geometry of Volume Measurement

- **Lem. 12.1. (Intuition Behind Using Alternating Tensors for Integration)** Suppose  $\Omega$  is a  $k$ -tensor on a vector space with the property that  $\Omega(X_1, \dots, X_k) = 0$  whenever  $X_1, \dots, X_k$  are linearly dependent. Then  $\Omega$  is alternating.

Let  $\Omega$  be a  $k$ -tensor on a vector space with the above property. Remember that an alternating  $k$ -tensor is a multilinear function  $\Omega : V \times \dots \times V \rightarrow \mathbb{R}$  such that

$$\Omega(X_1, \dots, X_i, \dots, X_j, \dots, X_k) + \Omega(X_1, \dots, X_j, \dots, X_i, \dots, X_k) = 0. \quad (112)$$

By our hypothesis, whenever two arguments of  $\Omega$  are the same, we ought to get zero. Therefore,

$$\begin{aligned} 0 &= \Omega(X_1, \dots, X_i + X_j, \dots, X_i + X_j, \dots, X_n) \\ &= \Omega(X_1, \dots, X_i, \dots, X_i, \dots, X_n) + \Omega(X_1, \dots, X_i, \dots, X_j, \dots, X_n) \\ &\quad + \Omega(X_1, \dots, X_j, \dots, X_i, \dots, X_n) + \Omega(X_1, \dots, X_j, \dots, X_j, \dots, X_n) \\ &= \Omega(X_1, \dots, X_i, \dots, X_j, \dots, X_n) + \Omega(X_1, \dots, X_j, \dots, X_i, \dots, X_n). \end{aligned} \quad (113)$$

Hence,  $\Omega$  is alternating.

### 9.2 The Algebra of Alternating Tensors

- **Obs. (Alternating 2-Tensors)** Any 2-tensor  $T$  can be expressed as the sum of an alternating tensor and a symmetric one.

Let  $T$  be an alternating tensor. Then we observe that

$$\begin{aligned} T(X, Y) &= \frac{1}{2} (T(X, Y) - T(Y, X)) + \frac{1}{2} (T(X, Y) + T(Y, X)) \\ &= A(X, Y) + S(X, Y). \end{aligned} \quad (114)$$

We claim that  $A$  is an alternating tensor, and  $S$  is a symmetric tensor. To see this, note that

$$\begin{aligned} A(X, Y) + A(Y, X) &= \frac{1}{2} (T(X, Y) - T(Y, X)) + \frac{1}{2} (T(Y, X) - T(X, Y)) = 0. \\ S(X, Y) - S(Y, X) &= \frac{1}{2} (T(X, Y) + T(Y, X)) - \frac{1}{2} (T(Y, X) + T(X, Y)) = 0. \end{aligned} \quad (115)$$

- **Def. (Alternating Projection)** Define the *alternating projection*,  $\text{Alt} : T^k(V) \rightarrow \Lambda^k(V)$  as follows:

$$\text{Alt}(T)(X_1, \dots, X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sgn } \sigma) T(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \iff \text{Alt } T = \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sgn } \sigma) (\sigma T) \quad (116)$$

- **Ex. (Examples of Alternating Projections)** Let  $T$  be any 1-tensor. Then  $\text{Alt } T = T$ . If  $T$  is a 2-tensor, then  $\text{Alt } T(X, Y) = (1/2)(T(X, Y) - T(Y, X))$ . If  $T$  is a 3-tensor, then

$$\begin{aligned} \text{Alt } T(X, Y, Z) &= \frac{1}{6} (T(X, Y, Z) - T(X, Z, Y) - T(Z, Y, X) \\ &\quad - T(Z, X, Y) + T(Y, Z, X) - T(Y, X, Z)). \end{aligned} \quad (117)$$

- **Lem. 12.3 (Properties of the Alternating Projection)**

- For any tensor  $T$ ,  $\text{Alt } T$  is alternating.
- $T$  is alternating if and only if  $\text{Alt } T = T$ .

**Exercise 12.2.** Prove Lemma 12.3.

- (a) Let  $T$  be an arbitrary  $k$ -tensor, and  $\text{Alt } T$  its alternating projection. Let  $\tau$  be the transposition  $(i j)$ . Then

$$\begin{aligned}
 \text{Alt } T(X_1, \dots, X_j, \dots, X_i, \dots, X_k) &= \text{Alt } T(X_{\tau(1)}, \dots, X_{\tau(i)}, \dots, X_{\tau(j)}, \dots, X_k) \\
 &= \frac{1}{k!} \sum_{\sigma \in S_k} (\text{sgn } \sigma \tau) T(X_{\sigma\tau(1)}, \dots, X_{\sigma\tau(k)}) \\
 &= -\frac{1}{k!} \sum_{\sigma \in S_k} (\text{sgn } \sigma) T(X_{\sigma\tau(1)}, \dots, X_{\sigma\tau(k)}) \\
 &= -\frac{1}{k!} \sum_{\sigma' \in S_k} (\text{sgn } \sigma') T(X_{\sigma'(1)}, \dots, X_{\sigma'(k)}) \\
 &= -\text{Alt } T(X_1, \dots, X_i, \dots, X_j, \dots, X_k).
 \end{aligned} \tag{118}$$

Hence, the alternating projection is indeed alternating.

- (b) If  $\text{Alt } T = T$ , then by (a),  $T$  is alternating.

- **Def. (Multi-Index)** Let  $k$  be a positive integer. An ordered  $k$ -tuple  $I = (i_1, \dots, i_k)$  of positive integers is called a *multi-index* of length  $k$ . If  $\sigma \in S_k$  is a permutation, then we write

$$I_\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(k)}). \tag{119}$$

- **Def. (Generalized Kronecker delta)** Let  $I$  and  $J$  be multi-indices of length  $k$ . Then we may define

$$\delta_I^J = \begin{cases} \text{sgn } \sigma, & \text{if neither } I \text{ nor } J \text{ has a repeated index} \\ & \text{and } J = I_\sigma \text{ for some } \sigma \in S_k \\ 0 & \text{if } I \text{ or } J \text{ has a repeated index} \\ & \text{or } J \text{ is not a permutation of } I. \end{cases} \tag{120}$$

- **Def. (Elementary Alternating Tensor/ $k$ -Covector)** Let  $V$  be an  $n$ -dimensional vector space, and suppose  $(\varepsilon^1, \dots, \varepsilon^n)$  be a basis for  $V^*$ . We will define a collection of alternating tensors on  $V$  that generalize the determinant function on  $\mathbb{R}^n$ . For each multi-index  $I = (i_1, \dots, i_k)$  of length  $k$  such that  $1 \leq i_1, \dots, i_k \leq n$ , define a covariant  $k$ -tensor  $\varepsilon^I$  by

$$\begin{aligned}
 \varepsilon^I(X_1, \dots, X_k) &= \det \begin{pmatrix} \varepsilon^{i_1}(X_1) & \dots & \varepsilon^{i_1}(X_k) \\ \vdots & & \vdots \\ \varepsilon^{i_k}(X_1) & \dots & \varepsilon^{i_k}(X_k) \end{pmatrix} \\
 &= \det \begin{pmatrix} X_1^{i_1} & \dots & X_k^{i_1} \\ \vdots & & \vdots \\ X_1^{i_k} & \dots & X_k^{i_k} \end{pmatrix}.
 \end{aligned} \tag{121}$$

I.e., if  $\mathbb{X}$  denotes the matrix whose columns are the components of the vectors  $X_1, \dots, X_k$  with respect to the basis  $(E_i)$  dual to the basis  $(\varepsilon^i)$ , then  $\varepsilon^I(X_1, \dots, X_k)$  is the determinant of the  $k \times k$  minor consisting of rows  $i_1, \dots, i_k$  of  $\mathbb{X}$ . Since the determinant is an alternating tensor,  $\varepsilon^I$  must also be an alternating tensor. We call  $\varepsilon^I$  an *elementary alternating tensor* or *elementary  $k$ -covector*.

- **Def. (Example of an Elementary  $k$ -Covector)** Let  $(e^1, e^2, e^3)$  be the standard dual basis for  $(\mathbb{R}^3)^*$ . Then

$$\mathbb{X} = \begin{pmatrix} X^1 & Y^1 \\ X^2 & Y^2 \\ X^3 & Y^3 \end{pmatrix} \implies \varepsilon^{13}(X, Y) = \det \begin{pmatrix} X^1 & Y^1 \\ X^3 & Y^3 \end{pmatrix} = X^1 Y^3 - Y^1 X^3. \tag{122}$$

- **Lem. 12.4. (Properties of Elementary Alternating Tensor)** Let  $(E_i)$  be a basis for  $V$ , let  $(\varepsilon^i)$  be the dual basis for  $V^*$ , and let  $\varepsilon^I$  be as defined above.
  - (a) If  $I$  has a repeated index, then  $\varepsilon^I = 0$ .
  - (b) If  $J = I_\sigma$  for some  $\sigma \in S_k$ , then  $\varepsilon^I = (\text{sgn } \sigma)\varepsilon^J$ .
  - (c)  $\varepsilon^I(E_{j_1}, \dots, E_{j_k}) = \delta_J^I$ .

- **Prop. 12.5. (Basis for  $\Lambda^k V$ )** Let  $V$  be an  $n$ -dimensional vector space. If  $(\varepsilon^i)$  is any basis for  $V^*$ , then for each positive integer  $k \leq n$ , the collection of  $k$ -covectors

$$\mathcal{E} = \{\varepsilon^I : I \text{ is an increasing multi-index of length } k\} \quad (123)$$

is a basis for  $\Lambda^k V$ . Therefore,

$$\dim \Lambda^k V = \begin{cases} \binom{n}{k} = \frac{n!}{k!(n-k)!}, & k \leq n \\ 0, & n < k. \end{cases} \quad (124)$$

- **Lem. 12. 6. (The Space  $\Lambda^n(V)$ )** Suppose  $V$  is an  $n$ -dimensional vector space and  $\omega \in \Lambda^n(V)$ . If  $T : V \rightarrow V$  is any linear map and  $X_1, \dots, X_n$  are arbitrary vectors in  $V$ , then

$$\omega(TX_1, \dots, TX_n) = (\det T)\omega(X_1, \dots, X_n). \quad (125)$$

Let  $(E_i)$  be any basis for  $V$ , and let  $(\varepsilon^i)$  be the corresponding dual basis for  $V^*$ . Let  $(T_i^j)$  denote the matrix of  $T$  with respect to this basis, and let  $T_i = TE_i = T_i^j E_j$ . It suffices to prove this relationship holds when  $X_i = E_i$  for each  $i$ . By Proposition 12.5,  $\dim \Lambda^n V = 1$ . This implies that  $\omega = c\varepsilon^{1 \cdots n}$  for some real number  $n$ . Then we observe that

$$\begin{aligned} (\det T)c\varepsilon^{1 \cdots n}(E_1, \dots, E_n) &= c \det T. \\ c\varepsilon^{1 \cdots n}(TE_1, \dots, TE_n) &= c\varepsilon^{1 \cdots n}(T_1, \dots, T_n) = c \det(\varepsilon^j(T_i)) = c \det T_i^j. \end{aligned} \quad (126)$$

Hence, this concludes the proof.

### 9.3 The Wedge Product

- **Def. (Wedge/Exterior Product)** Let  $\omega \in \Lambda^k V$  and  $\eta \in \Lambda^l V$ . Define the *wedge product*, or *exterior product*, of  $\omega$  and  $\eta$  to be the alternating  $(k+l)$ -tensor:

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} \text{Alt}(\omega \otimes \eta). \quad (127)$$

- **Lem. 12.7. (Wedge Product of Multi-Indices)** Let  $(\varepsilon^1, \dots, \varepsilon^n)$  be a basis for  $V^*$ . For any multi-indices  $I = (i_1, \dots, i_k)$  and  $J = (j_1, \dots, j_l)$ ,

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ}, \quad (128)$$

where  $IJ = (i_1, \dots, i_k, j_1, \dots, j_l)$  is the concatenated multi-index.

- **Prop. 12.8. (Properties of the Wedge Product)**

(a) BILINEARITY:

$$\begin{aligned} (a\omega' + a'\omega) \wedge \eta &= a(\omega \wedge \eta) + a'(\omega' \wedge \eta). \\ \eta \wedge (a\omega + a'\omega') &= a(\eta \wedge \omega) + a'(\eta \wedge \omega'). \end{aligned} \quad (129)$$

(b) ASSOCIATIVITY:

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi. \quad (130)$$

(c) ANTICOMMUTATIVITY: For any  $\omega \in \Lambda^k(V)$  and  $\eta \in \Lambda^l(V)$ ,

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega. \quad (131)$$

(d)

$$\varepsilon^{i_1} \wedge \dots \wedge \varepsilon^{i_k} = \varepsilon^I. \quad (132)$$

(e) For any covectors  $\omega^1, \dots, \omega^k$  and vectors  $X_1, \dots, X_k$ ,

$$\omega^1 \wedge \dots \wedge \omega^k(X_1, \dots, X_k) = \det(\omega^j(X_i)). \quad (133)$$

## 9.4 Differential Forms on Manifolds

- **Def. (Space of all Alternating  $k$ -Tensors)** Let  $M$  be an  $n$ -dimensional smooth manifold. The subset of  $T^k M$  consisting of alternating tensors is denoted by  $\Lambda^k M$ :

$$\Lambda^k M = \coprod_{p \in M} \Lambda^k(T_p M). \quad (134)$$

- **Note:** Look over Cartan's Lemma (Exercise 12-17 on pg. 323).

## 10 Orientations

## 11 Problems

### 11.1 Smooth Maps

**Problem 2-5.** Let  $M$  be a nonempty smooth manifold of dimension  $n \geq 1$ . Show that  $C^\infty(M)$  is infinite dimensional.

Let  $M$  be a nonempty smooth manifold of dimension  $n \geq 1$ . Let  $(U, \varphi)$  be a smooth chart for  $M$ , and let  $x_1, \dots, x_k$  be  $k$  distinct points contained in  $U$ . For each  $j = 1, \dots, k$ , define the smooth function real-valued function  $f_j$  with compact support inside  $\varphi(U)$  as follows:  $f_j(x_m) = \delta_{mj}$ . Then for each  $j$ , define the function  $g_j : M \rightarrow \mathbb{R}$  as follows:

$$g_j(x) = \begin{cases} f_j(\varphi(x)), & x \in U. \\ 0, & x \in M \setminus U. \end{cases} \quad (135)$$

Then since  $U$  is open, it follows that  $g_j$  is smooth for each  $j$ . Hence, we have obtained a linearly independent subset of  $C^\infty(M)$  consisting of  $k$  vectors. Since  $k$  was arbitrary, we conclude that  $C^\infty(M)$  is infinite dimensional.

**Problem 2-6.** For any topological space  $M$ , let  $C(M)$  denote the algebra of continuous functions  $f : M \rightarrow \mathbb{R}$ . If  $F : M \rightarrow N$  is a continuous map, define  $F^* : C(N) \rightarrow C(M)$  by  $F^* = f \circ F$ ,

- (a) Show that  $F^*$  is a linear map.
- (b) If  $M$  and  $N$  are smooth manifolds, show that  $F$  is smooth if and only if  $F^*(C^\infty(N)) \subset C^\infty(M)$ .
- (c) If  $F : M \rightarrow N$  is a homeomorphism between smooth manifolds, show that it is a diffeomorphism if and only if  $F^*$  restricts to an isomorphism from  $C^\infty(N)$  to  $C^\infty(M)$ .

- (a) Let  $a, b \in \mathbb{R}$  and  $f, g \in C(N)$ . Then

$$F^*(af + bg) = (af + bg) \circ F = a(f \circ F) + b(g \circ F) = aF^*(f) + bF^*(g). \quad (136)$$

- (b) Let  $M$  and  $N$  be smooth manifolds. Assume that  $F$  is smooth, and let  $f \in C^\infty(N)$ . Then

$$F^*(f) = (f \circ F) : M \rightarrow \mathbb{R} \quad (137)$$

is smooth since it is the composition of smooth functions. Hence,  $F^*(C^\infty(N)) \subset C^\infty(M)$ . Now we need to show the converse. Suppose  $F^*(C^\infty(N)) \subset C^\infty(M)$ . Let  $(U, \varphi)$  and  $(V, \psi)$  be smooth charts for  $M$  and  $N$ , respectively such that  $F(U) \subset V$ . Let  $\psi = (\psi^i)$ , where each coordinate function  $\psi^i : V \rightarrow \mathbb{R}$  is smooth. Note we can extend  $\psi^i$  to a smooth function on  $N$  by means of a smooth bump function. By our hypothesis,  $F^*(\psi^i) = \psi^i \circ F$  is smooth. Then since  $\varphi^{-1} : \varphi(U) \rightarrow M$  is smooth,

$$\psi^i \circ F \circ \varphi^{-1} \quad (138)$$

is smooth for each  $i$ , which means that  $\psi \circ F \circ \varphi^{-1}$  is smooth. Hence, we conclude that  $F$  is smooth on  $M$ .

- (c) Let  $F : M \rightarrow N$  be a homeomorphism between smooth manifolds. Suppose  $F^*$  restricts to an isomorphism from  $C^\infty(N)$  to  $C^\infty(M)$ . Then since  $F^*(C^\infty(N)) \subset C^\infty(M)$ ,  $F$  is a smooth map. Let  $G$  be the inverse function of  $F$ . Then  $G^*$  is the inverse function of  $F^*$ , and so  $G^*$  restricts to an isomorphism from  $C^\infty(M)$  to  $C^\infty(N)$ . In particular, this implies that  $G$  is a smooth map. Hence,  $F$  is a diffeomorphism. Now assume that  $F$  is a diffeomorphism. Let  $G$  be its inverse map. Since  $G : N \rightarrow M$  is smooth,  $G^*(C^\infty(M)) \subset C^\infty(N)$ . Let  $g \in C^\infty(M)$  and let  $C^\infty(N) \ni f = G^*(g)$ . Then

$$F^*(f) = f \circ F = g \circ G \circ F = g, \quad (139)$$

so that  $F^*$  is surjective. Now we show injectivity of  $F^*$ . Suppose  $F^*(f) = F^*(g) \iff f \circ F = g \circ F$ . Then  $(f \circ F) \circ G = (g \circ F) \circ G \iff f = g$ . Hence,  $F^*$  is injective. Using (a), we conclude that  $F^*$  restricts to an isomorphism from  $C^\infty(N)$  to  $C^\infty(M)$ .

## 11.2 Tangent Vectors

**Problem 3-1.** Suppose  $M$  and  $N$  are smooth manifolds with  $M$  connected, and  $F : M \rightarrow N$  is a smooth map such that  $F_* : T_p M \rightarrow T_{F(p)} N$  is the zero map for each  $p \in M$ . Show that  $F$  is a constant map.

Let  $M$  and  $N$  be smooth manifolds with  $M$  connected, and  $F : M \rightarrow N$  be a smooth map such that  $F_*$  is the zero map for each  $p \in M$ . Let  $p \in M$ , and define the subset

$$\mathcal{C} = \{q \in M : F(q) = F(p)\}. \quad (140)$$

Clearly this subset is nonempty since it at least contains  $p \in M$ . If  $q \in \mathcal{C}$ , let  $U$  be a smooth coordinate chart containing  $q$ . By hypothesis, for all  $r \in U$ ,  $F_*$  is the zero map; in local coordinates, this is possible iff all of the partial derivatives of the coordinate representation of  $F$  is zero at each  $r \in U$ . But this means that  $F$  is constant on  $U$ . Hence,  $U \subset \mathcal{C}$ , which means that  $\mathcal{C}$  is an open subset of  $M$ . By continuity of  $F$ ,  $\mathcal{C}$  is also a closed subset of  $M$ . Since  $M$  is connected and  $\mathcal{C}$  is nonempty, it then follows that  $\mathcal{C} = M$ . Hence,  $F$  is a constant map.

**Problem 3-3.** If a nonempty smooth  $n$ -manifold is diffeomorphic to an  $m$ -manifold, show that  $n = m$ .

Let  $M$  be a nonempty  $m$ -manifold and  $N$  a nonempty  $n$ -manifold; let  $F : M \rightarrow N$  be a diffeomorphism. Then since  $F$  is a local diffeomorphism, for each  $p \in M$ ,  $F_* : T_p M \rightarrow T_{F(p)} N$  is an isomorphism. Since  $\dim T_p M = m$  and  $\dim T_{F(p)} N = n$  for every  $p \in M$ , it then follows that  $m = n$ .

**Problem 3-4.** Let  $C \subset \mathbb{R}^2$  be the unit circle, and let  $S \subset \mathbb{R}^2$  be the boundary of the square of side 2 centered at the origin:

$$S = \{(x, y) : \max(|x|, |y|) = 1\}.$$

Show that there is a homeomorphism  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $F(C) = S$ , but there is no *diffeomorphism* with the same property. [Hint: Consider what  $F$  does to the tangent vector to a suitable curve in  $C$ .]

Let  $C \subset \mathbb{R}^2$  be the unit circle and  $S \subset \mathbb{R}^2$  the boundary of the square of side 2 centered at the origin. Consider the map  $G : S \rightarrow C$  defined as follows

$$G(x, y) = \frac{(x, y)}{\sqrt{x^2 + y^2}} \in S. \quad (141)$$

First, we show that  $G$  is injective. Suppose  $G(x_1, y_1) = G(x_2, y_2)$ . Since  $\sqrt{x_1^2 + y_1^2}, \sqrt{x_2^2 + y_2^2}$  are nonzero, multiplying both sides by  $\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$ , we get  $(x_1, y_1) = (x_2, y_2)$ . Hence,  $G$  is injective. Now we show that  $G$  is surjective. Let  $(\tilde{x}, \tilde{y}) \in C$ . We give a rough sketch for surjectivity, but the idea is clear. Consider the ray connecting the origin  $(0, 0)$  to the point  $(\tilde{x}, \tilde{y})$ . Extend this ray indefinitely. Then this ray must intersect  $S$  at some point  $(x_0, y_0)$ . Since  $G$  radially projects all of the points in  $S$  inwards onto  $C$ , it follows that  $G(x_0, y_0) = (\tilde{x}, \tilde{y})$ . Hence,  $G$  is bijective. By calculus,  $G$  is continuous. Since continuous bijections from compact spaces onto Hausdorff spaces is a homeomorphism,  $G$  is a homeomorphism. Notably, its inverse  $F$  must also be a homeomorphism, proving the claim. However, there can be no diffeomorphism between  $C$  and  $S$ . Suppose  $F$  was such a diffeomorphism, and let  $a$  be one of the corners of the square, and  $p = F^{-1}(a)$ . Since  $F$  is a diffeomorphism,  $T_p C \cong T_a S$  under the isomorphism  $F_*$ . As we showed before  $T_p C$  is 1-dimensional. On the other hand,  $T_a S$  is not well-defined. But this is a contradiction. Therefore,  $C$  and  $S$  are not diffeomorphic.

## 11.3 The Cotangent Bundle

**Problem 6-1.**

- (a) If  $V$  and  $W$  are finite-dimensional vector spaces and  $A : V \rightarrow W$  is any linear map, show that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{A} & W \\ \downarrow \xi_V & & \downarrow \xi_W \\ V^{**} & \xrightarrow{(A^*)^*} & W^{**}, \end{array}$$

where  $\xi_V$  and  $\xi_W$  denote the isomorphisms defined by (6.3) for  $V$  and  $W$ , respectively.

- (a) Assume all of the given hypotheses. Let  $X \in V$  and let  $\omega \in W^*$ . Then

$$\xi_W(AX)(\omega) = \omega(AX). \quad (142)$$

On the other hand, since  $A^*\omega \in V^*$ ,  $\xi_V(X)(A^*\omega) = A^*\omega(X)$ .

**Problem 6-2.**

- (a) If  $F : M \rightarrow N$  is a smooth map, show that  $F^* : T^*N \rightarrow T^*M$  is a smooth bundle map.  
 (b) Show that the assignment  $M \mapsto T^*M$ ,  $F \mapsto F^*$  defines a contravariant functor from the category of smooth manifolds to the category of smooth vector bundles.

## 11.4 Comps

**Problem 2017-J-II-1.** Let  $f : M \rightarrow \mathbb{R}$  be a smooth function on a smooth manifold  $M$ . In an arbitrary smooth local coordinate chart  $x : U \rightarrow \mathbb{R}^n$  of  $M$ , define

$$\mathcal{D}f := \sum_{i=1}^n \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}. \quad (143)$$

Does  $\mathcal{D}f$  give a well-defined vector field on  $M$ ?

Let  $f : M \rightarrow \mathbb{R}$  be a smooth function on a smooth manifold  $M$ , and define  $\mathcal{D}f$  as prescribed above. For any given smooth local coordinate chart  $(U, (x^i))$ , by smoothness of  $f$ , all of the partial derivatives  $\partial f / \partial x^i$  are smooth so that the component functions of  $\mathcal{D}f$  are all smooth; hence  $\mathcal{D}f$  is smooth in each smooth local coordinate chart. However, we need to check if  $\mathcal{D}f$  transforms like a vector field. Suppose  $p \in (U, (x^i)) \cap (V, (\tilde{x}^i))$ . Then

$$\begin{aligned} \mathcal{D}f &= \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i} \Big|_p \\ &= \frac{\partial f}{\partial \tilde{x}^j}(p) \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) \cdot \frac{\partial \tilde{x}^k}{\partial x^i}(\hat{p}) \frac{\partial}{\partial \tilde{x}^k} \Big|_p \\ &\neq \frac{\partial f}{\partial \tilde{x}^k}(p) \frac{\partial}{\partial \tilde{x}^k} \Big|_p, \end{aligned} \quad (144)$$

where  $\hat{p}$  is the coordinate representation of  $p$  in the  $(x^i)$  coordinates, and we used the contravariant vector transformation law in the second line. Therefore, since  $\mathcal{D}f$  does not transform as a vector field on  $M$ , it cannot be a well-defined vector field on  $M$ .

**Problem 2023-J-II-4.** Prove that  $S^2 \times S^2$  is not diffeomorphic to  $M_1 \times M_2 \times M_3$ , where  $M_1, M_2, M_3$  are smooth manifolds of nonzero dimension.

Assume to the contrary that  $S^2 \times S^2$  is diffeomorphic to  $M_1 \times M_2 \times M_3$ ; since diffeomorphisms preserve dimensions,  $\dim(S^2 \times S^2) = \dim S^2 + \dim S^2 = 4$ , and  $\dim M_{1,2,3} \neq 0$ , without loss of generality, we must have  $\dim M_1 = \dim M_2 = 1$  and  $\dim M_3 = 2$ . Additionally, since  $S^2$  is compact and connected,  $\times_{j=1}^3 M_j$  must be compact and connected, which then implies that each  $M_j$  must be compact and connected. Moreover, since diffeomorphisms induce isomorphisms between fundamental groups, we must have

$$\pi_1(S^2 \times S^2) \cong \pi_1\left(\bigtimes_{j=1}^3 M_j\right). \quad (145)$$

On the left side, since  $S^2$  is simply connected,  $\pi_1(S^2 \times S^2)$  is trivial. On the right side, since the only compact, connected, smooth 1-manifold, up to diffeomorphism, is  $S^1$ , and  $\pi_1(S^1) \cong \mathbb{Z}$ ,

$$\pi_1\left(\bigcup_{j=1}^3 M_j\right) = \mathbb{Z} \times \mathbb{Z} \times \pi_1(M_3), \quad (146)$$

which is clearly not isomorphic to the trivial group. But this is a contradiction. Hence, by contradiction,  $S^2 \times S^2$  cannot be diffeomorphic to  $M_1 \times M_2 \times M_3$ .

**Problem 2024-J-I-5.** Let  $\alpha$  be a closed 1-form on  $\mathbb{R}P^n$ ,  $n > 1$ . Show that if  $f : [0, 1] \rightarrow \mathbb{R}P^n$  is a smooth function such that  $f(0) = f(1)$ , then

$$\int_{[0,1]} f^* \alpha = 0.$$

Include all calculations that are relevant to your solution.

Let  $\alpha$  be a closed 1-form on  $\mathbb{R}P^n$ ,  $n > 1$ , and  $f : [0, 1] \rightarrow \mathbb{R}P^n$  a smooth function such that  $f(0) = f(1)$ . We should show this computation in light of the above problem, but we know that the  $k^{\text{th}}$  de Rham Cohomology group of  $\mathbb{R}P^n$  vanishes for all  $0 < k < n$ . Since

$$H^1(\mathbb{R}P^n) = \frac{\{\text{closed 1-forms on } \mathbb{R}P^n\}}{\{\text{exact 1-forms on } \mathbb{R}P^n\}} = 0, \quad (147)$$

it follows that a 1-form on  $\mathbb{R}P^n$  is closed iff it is exact. So, since  $\alpha$  is a closed 1-form, there exists a smooth function  $g$  on  $\mathbb{R}P^n$  such that  $\alpha = dg$ . Then  $f^* \alpha = f^* dg = d(g \circ f)$ . Therefore,

$$(*) := \int_{[0,1]} f^* \alpha = \int_0^1 d(g \circ f) = g(f(1)) - g(f(0)). \quad (148)$$

By hypothesis, since  $f(1) = f(0)$ ,  $g(f(1)) = g(f(0))$ . Therefore,  $(*) = 0$ , which concludes the proof.