Comps Practice

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Contents

Some Helpful Resources

(1) https://en.wikipedia.org/wiki/Main_Page
(2) https://encyclopediaofmath.org/wiki/Main_Page
(3) https://www.math3ma.com/
(4) https://mathworld.wolfram.com/
(5) https://math.stackexchange.com/
(6) Folland, *Real Analysis*
(7) Dummit & Foote, *Abstract Algebra*
(8)

Comps Lemma

Problem Comps Lemma. Let M, N be smooth, connected, n-manifolds, and $f : M \to N$ a (smooth) immersion. If M is compact and nonempty, then N is compact and f is a (smooth) covering map.

Let M, N be smooth, connected n-manifolds and $f : M \to N$ an immersion. Assume that M is compact and nonempty. Since $\dim N = n$ and f is an immersion, $\operatorname{rank} df_p = n$ at every $p \in M$. Hence, by the Inverse Function Theorem, f is a local diffeomorphism. Since local diffeomorphisms are open maps, $f(M)$ is open in N. On the other hand, since the continuous image of compact sets is compact, $f(M)$ is compact in N. Since N is Hausdorff, $f(M)$ is closed in N. Since N is connected, $f(M) = N$. Therefore, N is compact.
Now, let $q \in N$, and consider $f^{-1}(q) \subset M$. For each $x \in f^{-1}(q)$, let U_x be an open neighborhood of M containing x. Since M is Hausdorff, we can shrink each U_x so that these neighborhoods are pairwise disjoint. This means that each $x \in f^{-1}(q)$ is isolated, and hence $f^{-1}(q)$ is discrete. Since M is compact, we conclude that $f^{-1}(q)$ must be finite; let $f^{-1}(q) = \{x_1, \ldots, x_s\}$. As noted above, for each $j = 1, \ldots, s$, let U_j be a neighborhood of x_j such that $f|_{U_j} : U_j \to V_j \subset N$ is a diffeomorphism. Then by the Hausdorff condition on M, shrink each U_j so that $U_i \cap U_j = \varnothing$ for all $i \neq j$; f remains a diffeomorphism on these shrunken neighborhoods. Setting $V = \bigcap_1^s f(U_j)$ and taking $\widetilde{U}_j = f^{-1}(V) \cap U_j$ gives us an evenly covered neighborhood of q in N.

Problem (Comps Lemma - Local Homeomorphisms). Let M, N be smooth, connected n-manifolds and $f : M \to N$ a local homeomorphism. If M is compact and nonempty, then N is compact and f is a covering map.

Problem (Comps Lemma - Submersions). Let M, N be smooth, connected n-manifolds and $F : M \to N$ a submersion. If M is compact and nonempty, then N is compact and F is a covering map.

Let M, N be smooth, connected n-manifolds and $F : M \to N$ a submersion. Also assume M is compact and nonempty. Since submersions are open maps, $f(M)$ is open in N. On the other hand, since F is continuous, continuous images of compacts sets are compact, and compact subsets of Hausdorff spaces are closed, $F(M)$ is closed in N. Hence, since N is connected and $F(M)$ is nonempty, $F(M) = N$. This proves that N is compact. We also claim that F is a local diffeomorphism. Since F is a submersion, at every $p \in M$, $dF_p : T_pM \to T_{f(p)}N$ is surjective. Since $\dim M = \dim N = n$, it follows that dF_p is bijective. Hence, by the Inverse Function Theorem, F is a local diffeomorphism.
All that remains to be seen is that F is a covering map. Let $q \in N$ and consider the closed subset $F^{-1}(q) \subset M$. Since F is a local diffeomorphism, for each $x \in F^{-1}(q)$, there exists a neighborhood U_x such that $F|_{U_x}$ is a local diffeomorphism. Since M is Hausdorff, we may shrink these neighborhoods so that they are pairwise disjoint. This means that each $x \in F^{-1}(q)$ is isolated, and hence, $f^{-1}(q)$ is discrete. Since M is compact, $f^{-1}(q)$ is finite; let $f^{-1}(q) = \{x_1, \ldots, x_s\}$. For each $j = 1, \ldots, s$, let U_j be a neighborhood of x_j such that $F|_{U_j}$ is a diffeomorphism. Since M is Hausdorff, we shrink these neighborhoods such that they are pairwise disjoint; F remains a diffeomorphism on each shrunken U_j. Set $V = \bigcap_1^s f(U_j)$, and let $\widetilde{U}_j = f^{-1}(V) \cap U_j$. Hence, V is an evenly covered neighborhood of $q \in N$, which concludes the proof that F is a covering map.

Steinhaus Theorem

Problem (Steinhaus Theorem). Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ such that $m^n(E) > 0$, and let $v_1, \ldots, v_N$ be a finite collection of vectors in $\mathbb{R}^n$. Then there exists $R > 0$, depending on E, and $M = \max\{|v_1|, \ldots, |v_N|\}$ such that for all $0 < r < R$, there exists $p \in S$ so that the $(N+1)$-points, $p, p + rv_1, \ldots, p + rv_1 + \cdots + rv_n \in S$.

Let E be a measurable subset of $\mathbb{R}^n$ with positive Lebesgue measure. We recall that the Lebesgue measure is *regular* (which means it is both *inner* and *outer* regular). By inner regularity, there exists a compact set $K_1 \subset E$ such that $m^n(K_1) > 0$. Let $\beta < (2^N - 1)^{-1}$; by outer regularity, there exists an open set U containing K_1 such that

$$m^n(U) \le (1+\beta)m^n(K_1). \tag{1}$$

Since K_1 is compact, $d_1 = d(K_1, U^c) > 0$. Let $R = d_1/M$, and choose an arbitrary r such that $0 < r < R$. First, observe that the set $K_1 + rv_1$ is contained in U, since otherwise,

$$d(K_1, U^c) \le |rv_1| \le rM < d_1. \tag{2}$$

Therefore, $K_1 \cup (K_1 + rv_1) \subset U$, and so

$$m^n(U) \ge m^n(K_1 \cup (K_1 + rv_1)) = m^n(K_1) + m^n(K_1 + rv_1) - m^n(K_1 \cap (K_1 + rv_1)). \tag{3}$$

Since the Lebesgue measure is translation invariant,

$$m^n(K_1 \cap (K_1 + rv_1)) \ge 2m^n(K_1) - m^n(U) \ge 2m^n(K_1) - m^n(K_1) - \beta m^n(K_1) = (1-\beta)m^n(K_1). \tag{4}$$

Since $\beta < 1$, it follows that $m^n(K_1 \cap (K_1 + rv_1)) > 0$, and so $K_1 \cap (K_1 + rv_1) \ne \varnothing$. Now we proceed by induction. For each $i = 1, \dots, N$, let $K_{i+1} = K_i \cap (K_i + rv_i)$. Each $K_i + rv_i$ must be contained in U (by a generalization of the argument made above) and each $K_{i+1} \subset K_i \subset U$. We claim that for each i, $m^n(K_{i+1}) \ge (1 - (2^i - 1)\beta)m^n(K_1)$. We have already proven the base case $i = 1$. So assume the result holds for some $1 \le m < N$. Then

$$m^n(U) \ge m^n(K_i \cup (K_i + rv_i)) = m^n(K_i) + m^n(K_i + rv_i) - m^n(K_i \cap (K_i + rv_i)). \tag{5}$$

By translation invariance of the Lebesgue measure,

$$\begin{aligned} m^n(K_{i+1}) = m^n(K_i + rv_i) &\ge 2m^n(K_i) - m^n(U) \ge 2(1 - (2^i - 1)\beta)m^n(K_1) - (1+\beta)m^n(K_1) \\ &= m^n(K_1) - 2^{i+1}\beta m^n(K_1) + 2\beta m^n(K_1) - \beta m^n(K_1) \\ &= (1 - (2^{i+1} - 1)\beta)m^n(K_1). \end{aligned} \tag{6}$$

Hence, since $\beta < (2^N - 1)^{-1}$, we obtain a nested sequence of compact subsets $\varnothing \ne K_{N+1} \subset K_N \subset \cdots \subset K_1 \subset U$. Let $q \in K_{N+1}$ be arbitrary. Since $K_{N+1} = K_N \cap (K_N + rv_N)$, the point $q - rv_N$ is contained in K_N. Then since $K_N = K_{N-1} \cap (K_{N-1} \cap rv_{N-1})$, $q - rv_N - rv_{N-1} \in K_{N-1}$. Proceeding inductively, we obtain the sequence $\{q, q - rv_N, q - rv_N - rv_{N-1}, \dots, q - rv_N - \cdots - rv_1\} \subset K_1 \subset E$. Hence, the proof concludes.

Fat Cantor Set

Problem (Fat Cantor Set). There exists a closed nowhere dense subset $E \subset \mathbb{R}$ of positive Lebesgue measure.

Consider the interval $[0,1] \subset \mathbb{R}$. Delete the open set fourth $(\frac{3}{8}, \frac{5}{8})$, which leaves the two line segments

$$\left[0, \frac{3}{8}\right] \cup \left[\frac{5}{8}, 1\right]. \tag{7}$$

From each of these segments, remove the corresponding open middle fourths again, yielding the set

$$\left[0, \frac{5}{32}\right] \cup \left[\frac{7}{32}, \frac{3}{8}\right] \cup \left[\frac{5}{8}, \frac{25}{32}\right] \cup \left[\frac{27}{32}, 1\right]. \tag{8}$$

We repeat this procedure inductively, removing an interval of width 4^n from each of the remaining 2^{n-1} intervalsOverall, we remove intervals of total length

$$\sum_{n=0}^{\infty} \frac{2^n}{2^{2n+2}} = \frac{1}{2}, \tag{9}$$

which means that the Lebesgue measure of the overall set is $1/2 > 0$. Moreover, the set is the intersection of a sequence of closed sets so that it is closed. Finally, the set does not contain any intervals so that it has empty interior. Therefore, the fat Cantor set is a closed nowhere dense subset of $\mathbb{R}$ of positive Lebesgue measure.

Fundamental Group of the Projective Planes

Problem (Fundamental Group of the Projective Planes). Compute the fundamental group of $\mathbb{RP}^n$ for all $n \geq 1$.

We recall the definition of $\mathbb{RP}^n$ for $n \geq 1$.

> **(Real Projective Space)** $\mathbb{RP}^n$ is the quotient space obtained from S^n by identifying each point x of S^2 with its antipodal point $-x$.

Now we prove the following:

> **(Theorem)** For all $n \geq 1$, $\mathbb{RP}^n$ is compact, and the quotient map $p : S^n \to \mathbb{RP}^n$ is a covering map.
>
> *Proof.* First, we will show that $\mathbb{RP}^n$ is compact. Let $U \subset S^2$ be open. The antipodal map $a : S^n \to S^n$ given by $a(x) = -x$ is a homeomorphism of S^n, which means that $a(U)$ is open in S^n. Then since p is a quotient map,
>
> $$p^{-1}(p(U)) = U \cup a(U), \tag{10}$$
>
> the set is also open in S^n. Therefore, $p(U)$ is open in $\mathbb{RP}^n$. Hence, p is an open map. Likewise, we can show that p is a closed map. Since $\mathbb{RP}^n$ is connected, S^n is compact, and p is surjective, we conclude that $p(S^n) = \mathbb{RP}^n$, and so $\mathbb{RP}^n$ is compact.
>
> Now let $y \in \mathbb{RP}^n$, and choose $x \in p^{-1}(y)$. Then choose an ε-neighborhood U of x in S^n for some $\varepsilon < 1$, using the euclidean metric of $\mathbb{R}^n$. Then U contains no pair $\{z, a(z)\}$ of antipodal points of S^n, since $d(z, a(z)) = 2$. As a result, the map
>
> $$p : U \to p(U)$$
>
> is bijective. Since p is continuous and open, p is a homeomorphism. Likewise, $p : a(U) \to p(a(U)) = p(U)$ is a homeomorphism. The set $p^{-1}(p(U))$ is thus the union of the two disjoint open sets U and $a(U)$, each of which is homeomorphically mapped by p onto $p(U)$. Hence, $p(U)$ is a neighborhood of $p(x) = y$ that is evenly covered by p. □

Now we have the following theorem

> **(Theorem)** Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. If E is path connected, then the lifting correspondence
>
> $$\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$$
>
> is surjective. If E is simply connected, it is bijective.
>
> *Proof.* Let E be path connected. Then given $e_1 \in p^{-1}(b_0)$, there is a path $\widetilde{f}$ in E from e_0 to e_1. Then $f = p \circ \widetilde{f}$ is a loop in B at b_0, and $\phi([f]) = e_1$ by definition. Therefore, ϕ is surjective.
>
> Now suppose E is simply connected. Let $[f], [g]$ be two elements of $\pi_1(B, b_0)$ such that $\phi([f]) = \phi([g])$. Let $\widetilde{f}$ and $\widetilde{g}$ be the liftings of f and g, respectively, to paths in E that begin at e_0; then $\widetilde{f}(1) = \widetilde{g}(1)$. Since E is simply connected, there is a path homotopy $\widetilde{F}$ in E between $\widetilde{f}$ and $\widetilde{g}$. Hence, $F = p \circ \widetilde{F}$ is a path homotopy between f and g, which proves injectivity. □

Now, suppose $n \geq 2$. Then since S^n is simply connected, $\phi : \pi_1(\mathbb{RP}^n, b_0) \to p^{-1}(b_0)$ is bijective. But since $|p^{-1}(b_0)| = 2$, $|\pi_1(\mathbb{RP}^n, b_0)| = 2$ for all $b_0 \in \mathbb{RP}^n$. Therefore, $\pi_1(\mathbb{RP}^n, b_0) \cong \mathbb{Z}/2\mathbb{Z}$. Now suppose $n = 1$. Since $\mathbb{RP}^1$ is homeomorphic to S^1, their fundamental groups must be isomorphic. Therefore, $\pi_1(\mathbb{RP}^1, b_0) \cong \pi_1(S^1, p_0) \cong \mathbb{Z}$. Hence, we conclude that

$$\pi_1(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}, & n = 1, \\ \mathbb{Z}/2\mathbb{Z}, & n \geq 2. \end{cases} \tag{11}$$

Problem (Lifting Criterion). Let $f : (Y, y_0) \to (X, x_0)$ be a map with Y path-connected and locally path-connected. Then a lift of f exists if and only if

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)).$$

January 2025

Problem 2025-J-I-1 (Algebra). Let R be a UFD (unique factorization domain). Let F be its quotient field. Let $p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in F[x]$ be a monic polynomial with coefficients in R admitting a root $a \in F$. Prove that $a \in R$.

Let R be a UFD, and F its quotient field. Let $p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in F[x]$ be a monic polynomial with coefficients in R admitting a root $a \in F$. Let $a = c/d$, where $c, d \in R \setminus \{0\}$ so that $\gcd(c,d) = 1$. By definition of a root, we must have

$$0 = p(a) = \left(\frac{c}{d}\right)^n + b_{n-1}\left(\frac{c}{d}\right)^{n-1} + \cdots + b_0. \tag{12}$$

Multiplying both sides by d^n,

$$c^n + d(b_{n-1}c^{n-1} + b_{n-2}c^{n-2}d \cdots + b_0 d^{n-1}) = 0 \implies c^n = -d(b_{n-1}c^{n-1} + \cdots + b_0 d^{n-1}). \tag{13}$$

From this, we observe that $d \mid c^n$. If d is not a unit in R, then every nonidentity irreducible divisor of d is an irreducible divisor of c^n, and hence an irreducible divisor of c. But this contradicts our hypothesis that $\gcd(c,d) = 1$. Hence, d has to be a unit of R. If $v \in R \setminus \{0\}$ such that $dv = vd = 1$, then

$$a = \frac{c}{d} = \frac{c}{d} \cdot \frac{v}{v} = cv \in R. \tag{14}$$

Hence, this concludes the proof.

Problem 2025-J-I-2 (Real Analysis). Let $\{f_n\}_{n\geq 1}$ be a sequence of Lebesgue-measurable functions on $[0,1]$. Suppose that

$$\int_0^1 f_n^2 dm \leq \frac{1}{n^2} \quad \text{for all } n \geq 1.$$

Prove that f_n converges to 0 a.e. on $[0,1]$.

Let $\{f_n\}_{n\geq 1}$ be a sequence of Lebesgue-measurable functions on $[0,1]$ so that

$$\int_0^1 f_n^2 dm \leq \frac{1}{n^2} \quad \text{for all } n \geq 1. \tag{15}$$

Consider the sequence $\{\sum_1^m f_n^2\}$, which is increasing and converges a.e. to $\sum_1^\infty f_n^2$. Hence, by the Monotone Convergence Theorem,

$$\sum_1^\infty \int_0^1 f_n^2 = \lim_{m\to\infty} \sum_1^m \int_0^1 f_n^2 = \lim_{m\to\infty} \int_0^1 \sum_1^m f_n^2 = \int_0^1 \sum_1^\infty f_n^2 \leq \int_0^1 \sum_1^\infty \frac{1}{n^2} < \infty. \tag{16}$$

Therefore, $\sum_1^\infty f_n^2 \in L^1(\mathbb{R})$, which means that $\sum_1^\infty f_n^2 < \infty$ a.e. on $[0,1]$. Hence, $\sum_{n=1}^\infty f_n^2$ converges a.e. on $[0,1]$. This implies that $f_n^2 \to 0$ a.e. on $[0,1]$, and hence $f_n \to 0$ a.e. on $[0,1]$.

Problem 2025-J-I-3 (Geometry/Topology). Let M be an orientable, connected, and compact smooth n-manifold with boundary. Show that there is no (smooth) retraction to the boundary, that is, there does not exist a smooth map $f : M \to \partial M$ such that $f(x) = x$ when $x \in \partial M$.

Let M be an orientable, connected, and compact smooth n-manifold with boundary. Assume to the contrary that there exists a smooth map $f : M \to \partial M$ such that $f(x) = x$ when $x \in \partial M$. Let $\omega \in \Omega^{n-1}(\partial M)$ be a volume form for the boundary of M. Since volume forms are closed (hence, ω is closed), we have by Stokes's theorem

$$0 = \int_M f^* d\omega = \int_M d(f^*\omega) = \int_{\partial M} f^*\omega = \int_{\partial M} \omega > 0, \tag{17}$$

which is a contradiction. Hence, by contradiction, there cannot exist a smooth retraction to the boundary.

Problem 2025-J-II-3 (Algebra). Let V be a vector space of dimension n over $\mathbb{Q}$. Let $T : V \to V$ be a linear transformation with minimal polynomial $x^4 - x^2 - 2$ over $\mathbb{Q}$. Show that n must be even.

Consider V as a module over the ring $\mathbb{Q}[x]$ by letting a polynomial $f(x) \in \mathbb{Q}[x]$ act as the linear operator $f(T)$. Since $\dim V = n$, this module is finitely generated. By the structure theorem for finitely generated modules over principal ideal domains, V is isomorphic to a direct sum of modules of the form $\mathbb{Q}[x]/(p(x))^e$, where $p(x) \in \mathbb{Q}[x]$ is irreducible. Moreover, each $p(x)$ must divide the minimal polynomial of T. We note that over $\mathbb{Q}$,

$$x^4 - x^2 - 2 = (x^2 - 2)(x^2 + 1), \tag{18}$$

where both factors are irreducible over $\mathbb{Q}$. Therefore, the only choices for $p(x)$ are $x^2 - 2$ and $x^2 + 1$. Therefore, $\mathbb{Q}[x]/(p(x))^e$ has dimension $\deg p \cdot e = 2e$ for each choice of p. Since 2 divides these dimensions, we conclude that 2 must divide n. Hence, n is even.

Problem 2025-J-II-4 (Topology). Let Σ_2 be a compact oriented surface of genus 2. Is there a submersion $f : \Sigma_2 \to S^1 \times S^1$, where S^1 denotes the unit circle?

Assume to the contrary that there exists a submersion $f : \Sigma_2 \to S^1 \times S^1$, where S^1 denotes the unit circle. Since $\dim \Sigma_2 = \dim S^1 \times S^1 = 2$, df_p must have constant rank 2 at every $p \in \Sigma_2$. Hence, f is a local diffeomorphism. Since f is a local diffeomorphism, $f(\Sigma_2)$ is compact in $S^1 \times S^1$; since $S^1 \times S^1$ is Hausdorff, $f(\Sigma_2)$ must be closed in $S^1 \times S^1$. On the other hand, since local diffeomorphisms are open maps, $f(\Sigma_2)$ is open in $S^1 \times S^1$. Therefore, since $S^1 \times S^1$ is connected, $f(\Sigma_2) = S^1 \times S^1$; i.e., f is surjective. Therefore, f is a covering map. This means that the induced homomorphism, $f_* : \pi_1(\Sigma_2) \to \pi_1(S^1 \times S^1)$ is injective, and so $f_*(\pi_1(\Sigma_2)) \cong \operatorname{img} f_* \leq \pi_1(S^1 \times S^1)$. However, $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$ is an abelian group and cannot have any nonabelian subgroups, whereas $\pi_1(\Sigma_2)$ is nonabelian. Hence, by contradiction, f cannot be a submersion.

Problem 2025-J-II-5 (Analysis). Let V be a topological vector space whose topology is Hausdorff. Let X_1 and X_2 be two Banach spaces, and assume there exist continuous linear bijections $F_1 : X_1 \to V$ and $F_2 : X_2 \to V$. Show that there is a continuous linear bijection $G : X_1 \to X_2$.

Assume the given hypotheses. Let $G = F_2^{-1} \circ F_2$. Since F_1, F_2 are bijections, we conclude that G is a bijection. Likewise, since F_1, F_2 are linear, G must also be linear. It suffices to prove that G is continuous. By the Closed Graph Theorem, continuity of G is equivalent to the graph of G being a closed subspace of $X_1 \times X_2$. Let $\{x_n\} \subset X_1$ be a sequence in X_1 such that $x_n \to x$ and $y_n = Gx_n \to y$. We need to show that $y = Gx$. By continuity of F_1, $F_1 x_n \to Fx$. By continuity of F_2,

$$F_2 y = \lim F_2 y_n = \lim F_2 G x_n = \lim F_1 x_n = F_1 x. \tag{19}$$

Since F_2 is bijective, $y = F_2^{-1} F_1 x = Gx$. Hence, the graph of G is closed, which implies that G is continuous.

August 2025

Problem 2025-A-I-1 (Geometry/Topology). Let S be a closed orientable surface of genus 4 and C be an embedded circle that partitions S into two subsurfaces of genus 2. Does S retract to C?

We claim that the answer is no; assume to the contrary that there exists a retraction $r : S \to C$. Let $i : C \hookrightarrow S$ be the inclusion map so that $r \circ i = \mathrm{id}_C$. Now since C is an embedded circle, $H_1(C)$ (i.e., the first homology) is isomorphic to $H_1(S^1) = \mathbb{Z}$. On the other hand, since C is separating in S, its homology class in $H_1(S)$ is the zero element. Hence, the induced map $i_* : H_1(C) \to H_1(S)$ is the zero map. But this is impossible since if i_* is the zero map,

$$0 = r_* \circ i_* = (r \circ i)_* = \mathrm{id}_{H_1}(C), \tag{20}$$

which is a contradiction. Hence, no such retraction can exist.

Problem 2025-A-II-1 (Geometry/Topology). For $n \geq 2$,

(i) Calculate the fundamental group of the real n-dimensional projective space $\mathbb{P}^n$;

(ii) Show that any continuous map $f : \mathbb{P}^n \to \mathbb{T}^n$ from $\mathbb{P}^n$ to the n-dimensional torus is homotopic to a constant map.

Let $n \geq 2$.

(i) Let $\sim$ be an equivalence relation on S^n that identifies each $x \in S^n$ with its antipodal point $-x$, and let $\mathbb{RP}^n = S^n / \sim$ with quotient map $p : S^n \to \mathbb{RP}^n$. We begin by proving that $p : S^n \to \mathbb{RP}^n$ is a covering map.

Let $y \in \mathbb{RP}^n$. For each $x \in p^{-1}(y)$, choose an ε-neighborhood U in S^n for $\varepsilon < 1$ using the euclidean metric on $\mathbb{R}^n$. Let $a : S^n \to S^n$ be the antipodal map given by $a(x) = -x$, which is clearly seen to be a homeomorphism of S^n; this means that $a(U)$ is open in S^n. By construction of U, U does not contain an pair $\{z, a(z)\}$ of antipodal points of S^n since $d(z, a(z)) = 2$. Therefore, the map $p : U \to p(U)$ is bijective. Since p is continuous and open, p is a homeomorphism. Likewise, $p : a(U) \to p(a(U)) = p(U)$ is a homeomorphism. The set $p^{-1}(p(U)) = U \sqcup a(U)$, each of which is homeomorphically mapped by p onto $p(U)$. Therefore, $p(U)$ is a neighborhood of $p(x) = y$ that is evenly covered by p. Hence, p is a covering map.

Now we recall that if $p : E \to B$ is a covering map, $b_0 \in B$, $e_0 \in p^{-1}(b_0)$, and E is simply connected, then the map $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ defined by $\phi([f]) = \widetilde{f}(1)$, where $\widetilde{f}$ is a lift of f to a path in E beginning at e_0, is bijective. Since $p^{-1}(b_0)$ contains only two elements for every $b_0 \in \mathbb{RP}^n$ and S^n is simply connected, $\pi_1(\mathbb{RP}^n, b_0)$ is a group of two elements. Therefore, the fundamental group of $\mathbb{RP}^n$ is simply $\mathbb{Z}_2$.

(ii) Now let $f : \mathbb{RP}^n \to \mathbb{T}^n$ be a continuous map; f induces a group homomorphism $f_* : \pi_1(\mathbb{RP}^n) \longrightarrow \pi_1(\mathbb{T}^n)$. We observe that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$, while $\pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$. Since $\mathbb{Z}^n$ has no subgroup of order 2, we conclude that f_* be the trivial homomorphism. By the lifting criterion, this then implies that f lifts to a path $\widetilde{f} : \mathbb{RP}^n \to \mathbb{R}^n$, where $\mathbb{R}^n$ is the universal cover of S^n. But since $\mathbb{R}^n$ is contractible, $\widetilde{f}$ is homotopic to a constant map. Hence, projecting down, we conclude that f is homotopic to a constant map.

Problem 2025-A-I-6 (Algebra). Let $f(x)$ be an irreducible polynomial of degree n over a field F, and let $g(x)$ be any polynomial in $F[x]$. Prove that every irreducible factor of the composition $f(g(x))$ has degree divisible by n.

Let $h(x)$ be an irreducible factor of $f(g(x))$ in $F[x]$ and let α be the root of $h(x)$ in some algebraic closure of F. Since h is irreducible and α is a root, the minimum polynomial of α over F is h. Therefore,

$$\deg h = [F(\alpha) : F]. \tag{21}$$

Now since α is a root of $h(x) = f(g(x))$, $f(g(\alpha)) = 0$. In particular, $g(\alpha)$ is a root of f. Since f is irreducible of degree n over F, the minimal polynomial of $g(\alpha)$ over α is f. Hence,

$$[F(g(\alpha)) : F] = n. \tag{22}$$

Since $F \subset F(g(\alpha)) \subset F(\alpha)$, by the Tower Law,

$$\deg h = [F : (\alpha) : F] = [F(\alpha) : F(g(\alpha))] \cdot [F(g(\alpha)) : F] = n[F(\alpha) : F(g(\alpha))], \tag{23}$$

so that $n \mid \deg h$. Hence, this concludes the proof.

Problem 2025-A-II-2 (Geometry/Topology). Consider the plane distribution in $\mathbb{R}^3$ spanned by two vector fields

$$V = \partial_x + 2xy\partial_z, \qquad W = x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z. \tag{24}$$

(i) Show that this distribution is integrable.

(ii) Does the pair of vector fields V and W generate a coordinate system on integral surfaces? If not, find a pair that can play this role for the local integral surfaces passing through points

$(0, 0, z_0)$.

(i) Let D be the plane distribution in $\mathbb{R}^3$ spanned by the two vector fields V and W given above. Then by the Frobenius Theorem, D is integrable if and only if D is involutive, which is true if and only if the Lie Bracket of V and W is a smooth section of D at each $p \in \mathbb{R}^3$. We observe that:

$$\begin{aligned} V(W) &= (\partial_x + 2xy\partial_z)\left(x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z\right) \\ &= \partial_x + (4xy + 2x)\partial_z. \\ W(V) &= \left(x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z\right)(\partial_x + 2xy\partial_z) \\ &= 2xy\partial_z + 2x\partial_z. \end{aligned} \tag{25}$$

Therefore, for any $p \in \mathbb{R}^3$,

$$[V, W] = V(W) - W(V) = \partial_x + 2xy\partial_z = V. \tag{26}$$

Since V is a smooth section of D, we conclude that D is involutive, and hence integrable.

(ii) Let $\mathcal{S}$ be an integral surface, and assume there are coordinates (u, v) on $\mathcal{S}$ such that $V|_{\mathcal{S}} = \partial_u$ and $W|_{\mathcal{S}} = \partial_v$. Then we observe that $[V|_{\mathcal{S}}, W|_{\mathcal{S}}] = \partial_u(\partial_v) - \partial_v(\partial_u) = 0$. On the other hand,

$$[V|_{\mathcal{S}}, W|_{\mathcal{S}}] = ([V, W])|_{\mathcal{S}} = V|_{\mathcal{S}} \neq 0, \tag{27}$$

which is a contradiction. Therefore, V and W cannot generate a coordinate system on integral surfaces. However, consider the fields $\widetilde{V} = V$ and $\widetilde{W} = W - xV$ on $\mathbb{R}^3$. Then since

$$[\widetilde{V}, \widetilde{W}] = V(W - xV) - (W - xV)(V) = VW - xVV - W(V) + xVV = 0, \tag{28}$$

and so this pair generates a coordinate system on all integral surfaces.

January 2024

Problem 2024-J-I-1 (Algebra). For distinct odd primes p and q, prove that every finite group of order $2pq$ is a semidirect product of a normal subgroup of order pq and a subgroup of order 2.

Let G be a group of order $2pq$, where p, q are distinct odd primes. Without loss of generality, assume $q > p$. By Sylow's Theorem,

$$n_q \in \{1, 2, p, 2p\} \cap \{1, q+1, \dots\} = 1, \tag{29}$$

since $q > 2$ and $q > p$. Therefore, G has a unique, normal, Sylow q-subgroup, which we denote as Q. Let P be a Sylow p-subgroup of G. By the Second Isomorphism Theorem, we conclude that $N = PQ$ is a subgroup of G of order $|P||Q| = pq$. Since $|G : N| = 2pq/(pq) = 2$, where 2 is the smallest prime dividing $|G|$, we conclude that N is a normal subgroup of G. Next, by Cauchy's Theorem, G contains an element of order 2. Let M be the subgroup generated by this element, which also must have order 2. By Lagrange's Theorem, $N \cap M = \{e\}$. Next,

$$|NM| = \frac{|N||M|}{|N \cap M|} = |N||M| = 2pq = |G|, \tag{30}$$

so that $G = NM$. Therefore, we conclude that $G = N \rtimes M$.

Problem 2024-J-I-2 (Geometry/Topology). Let $p : E \to B$ be a covering space map, with B and E path connected. Choose a point $e_0 \in E$ and $b_0 \in B$ such that $p(e_0) = b_0$. This gives us a subgroup $H = p_*\pi_1(E, e_0)$ of the fundamental group $G = \pi_1(B, b_0)$.
Construct a bijection between the fiber $p^{-1}(b_0)$ and the set of right cosets of H and prove that this is indeed a bijection. Prove that the number of sheets of p equals the index $(G : H)$.

Assume all of the given hypotheses. Let $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ be the lifting correspondence induced by p defined by $\phi([f]) = \tilde{f}(1)$, where $\tilde{f}$ is the lift of f, and let $\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$ be the map induced by ϕ. It suffices to prove that Φ is a bijection.

(i) Since E is path connected and $p : E \to B$ is a covering map, the lifting correspondence ϕ must be surjective. Hence, since Φ is induced by ϕ, it follows that Φ is also surjective.

(ii) Now we will show that Φ is injective. Let f and g be two paths in B, and $\widetilde{f}, \widetilde{g}$ their liftings to paths in E that begin at e_0. We must show that $\widetilde{f}(1) = \widetilde{g}(1)$ iff $[f] \in H * [g]$.

($\Leftarrow$) Suppose $[f] = [h * g]$, where $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ in E based at e_0. Since $\tilde{g}$ is a path in E that *begins* at e_0, the product $\tilde{h} * \tilde{g}$ is well-defined. Since $[f] = [h * g]$, it follows that $\tilde{f}$ and $\tilde{h} * \tilde{g}$ must end at the same point. Hence, $\tilde{f}$ and $\tilde{g}$ end at the same point. Therefore, $\phi([f]) = \phi([g])$.

($\Rightarrow$) Suppose $\phi([f]) = \phi([g])$, which means that $\tilde{f}(1) = \tilde{g}(1)$. Then the product of $\tilde{f}$ with the reverse of $\tilde{g}$ is well-defined and is a loop $\tilde{h}$ in E based at e_0. By direct computation, $[\tilde{h} * \tilde{g}] = [\tilde{f}]$. If $\tilde{F}$ is a path homotopy between $\tilde{h} * \tilde{g}$ and $\tilde{f}$, then $p \circ \tilde{F}$ is a path homotopy between $h * g$ and f, which means that $[f] \in H * [g]$. Hence, this concludes the proof that Φ is injective.

Hence, $|p^{-1}(b_0)| = |G/H| = (G : H)$.

Problem 2024-J-I-3 (Complex Analysis). Suppose f is continuous on the plane and holomorphic on $\mathbb{C} \setminus \mathbb{R}$. Prove that f is holomorphic on the whole plane.

Let $f : \mathbb{C} \to \mathbb{C}$ be continuous on $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \mathbb{R}$. We show that f is holomorphic on all of $\mathbb{C}$.
By Morera's Theorem, it suffices to prove that

$$\oint_\gamma f(z)\, dz = 0$$

for every closed piecewise C^1 curve $\gamma \subset \mathbb{C}$.
If γ lies entirely in the upper or lower half-plane, then f is holomorphic on a neighborhood of γ, and by the Cauchy–Goursat theorem,

$$\oint_\gamma f(z)\, dz = 0.$$

Now suppose that γ intersects the real axis. For $\varepsilon > 0$, construct a closed piecewise C^1 curve γ_ε by modifying γ so that it avoids the real axis by small detours of height $\pm\varepsilon$. Then $\gamma_\varepsilon \subset \mathbb{C} \setminus \mathbb{R}$, so f is holomorphic on a neighborhood of γ_ε, and hence

$$\oint_{\gamma_\varepsilon} f(z)\, dz = 0.$$

Since f is continuous on $\mathbb{C}$, it is uniformly continuous on compact sets, and the total length of the detours tends to 0 as $\varepsilon \to 0$. Therefore,

$$\lim_{\varepsilon \to 0} \oint_{\gamma_\varepsilon} f(z)\, dz = \oint_\gamma f(z)\, dz.$$

Thus $\oint_\gamma f(z)\, dz = 0$.
Since this holds for every closed piecewise C^1 curve in $\mathbb{C}$, Morera's Theorem implies that f is holomorphic on all of $\mathbb{C}$.

Problem 2024-J-I-4 (Algebra). For each field K, prove that the polynomial ring $K[x, y]$ in two variables is not a principal ideal domain.

Let K be a field, and consider the polynomial ring $K[x,y]$. Let (x,y) be the proper ideal of $K[x,y]$ generated by the monomials x and y. Assume to the contrary that $(x,y) = (f(x,y))$ where $f(x,y) \in K[x,y]$ is not a unit of the polynomial ring. Since $x \in (f(x,y))$, $f(x,y) \mid x$. By our assumption that f is not a unit, it follows that $f(x,y)$ is an associate of x. Likewise, $f(x,y)$ must be an associate of y. But this is impossible since x and y are not associates of each other. This forces $f(x,y)$ to be a unit, which means that $(f(x,y)) = K[x,y]$. But this contradicts the fact that $(x,y) = (f(x,y))$ is a proper ideal. Hence, by contradiction, (x,y) is not a principal ideal, and so $K[x,y]$ is not a principal ideal domain.

Problem 2024-J-I-5 (Geometry/Topology). Let α be a closed 1-form on $\mathbb{RP}^n$, $n > 1$. Show that if $f : [0,1] \to \mathbb{RP}^n$ is a smooth function such that $f(0) = f(1)$, then

$$\int_{[0,1]} f^*\alpha = 0.$$

Include all calculations that are relevant to your solution.

We recall that $H^k(\mathbb{RP}^n) = 0$ for all $0 < k < n$ so that $H^1(\mathbb{RP}^n) = 0$ if $n > 1$. In particular, this means that α is also an exact 1-form on $\mathbb{RP}^n$. Let g be a smooth function on $\mathbb{RP}^n$ so that $\alpha = dg$. Then

$$\int_0^1 f^*\alpha = \int_0^1 f^* dg = \int_0^1 d(f^*g) = g(f(1)) - g(f(0)) = 0, \tag{31}$$

where the last equality follows from the fact that $f(1) = f(0)$. Hence, the proof concludes.

Problem 2024-J-I-6 (Real Analysis). Let f and g be Lebesgue-measurable functions on $\mathbb{R}$. Define the convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\, dy$$

for all x such that the integral exists. Prove that if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ with $p, q \in (1,\infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, then $f * g$ is a bounded continuous function on $\mathbb{R}$.

Assume the given hypotheses. Then by Hölder's inequality, for any $x \in \mathbb{R}$,

$$|(f * g)(x)| \le \int_{\mathbb{R}} |f(x-y)g(y)|\, dy \le \|f(x-\cdot)\|_p \|g\|_q. \tag{32}$$

Since L^p norms are translation invariant, $\|f(x-\cdot)\|_p = \|f\|_p$. Hence, $|(f*g)(x)| \le \|f\|_p \|g\|_q = M < \infty$ for all $x \in \mathbb{R}$. Hence, we conclude that $f * g$ is a bounded function on $\mathbb{R}$. Next, let τ_z be the translation operator defined by $\tau_z f = f(x-z)$. Since translation operators are continuous in the L^p norms, $\|\tau_z f - f\| \to 0$ as $z \to 0$, which implies that

$$\begin{aligned} \|\tau_z(f*g) - (f*g)\|_\infty &= \|(\tau_z f - f) * g\|_\infty & (33)\\ &\le \|\tau_z f - f\|_p \|g\|_q \longrightarrow 0 \text{ as } z \to 0. & (34) \end{aligned}$$

Hence, $f * g$ is uniformly continuous, and therefore continuous on $\mathbb{R}$. Note that the inequality used in the second line of the above equation comes from *Young's convolution inequality*, which states the following:

> **(Young's Convolution Inequality)** Let $f \in L^p$, $g \in L^q$, and $p^{-1} + q^{-1} = r^{-1} + 1$. Then $\|f * g\|_r \le \|f\|_p \|g\|_q$.

In our case, we had $r = \infty$ so that $r^{-1} = 0$.

Problem 2024-J-II-2. Suppose $E \subset \mathbb{R}^2$ is a set of positive Lebesgue measure. Show that there are points a, b, c in E such that their connecting segments form a right angle, i.e., $a - b$ is perpendicular to $c - b$ (as vectors in $\mathbb{R}^2$).

Let $E \subset \mathbb{R}^2$ be a set of positive Lebesgue measure; let m^2 denote the Lebesgue measure on $\mathbb{R}^2$. Let $\{v_1, v_2, v_3\}$ be a collection of vectors in $\mathbb{R}^2$ such that $v_1 \perp v_2$, and $v_3 = -v_1$. Without loss of generality, assume that $\|v_j\| = 1$ for all $j = 1, \ldots, 3$. By inner regularity of the Lebesgue measure, there exists a compact subset $K_1 \subset E$ such that $m^2(K_1) > 0$. Taking $\beta < 1/7$, by outer regularity of the Lebesgue measure, there exists an open set U containing K_1 such that $m^2(U) \le (1+\beta)m^2(K_1)$.
Since K_1 is compact, $d_1 = d(K_1, U^c) > 0$. Hence, let $R = d_1$. Fix some $r \in (0, R)$ and consider the set $K_1 + rv_1$. We claim that $K_1 + rv_1 \subset U$ since if otherwise,

$$d(K_1, U^c) \le |rv_1| = r < d_1, \text{ which is a contradiction.} \tag{35}$$

Hence, $K_1 \cup (K_1 + rv_1) \subset U$, which means that

$$m^2(U) \ge m^2(K_1 \cup (K_1 + rv_1)) = m^2(K_1) + m^2(K_1 + rv_1) - m^2(K_1 \cap (K_1 + rv_1)). \tag{36}$$

By translation invariance of the Lebesgue measure, $m^2(K_1) + m^2(K_1 + rv_1) = 2m^2(K_1)$ so that

$$m^2(K_1 \cap (K_1 + rv_1)) = 2m^2(K_1) - m^2(U) \ge (1-\beta)m^2(K_1). \tag{37}$$

Since $\beta < 1$, $m^2(K_1 \cap (K_1 + rv_1)) > 0$ so that the set is nonempty. For $i = 1, \ldots, 3$, define $K_{i+1} = K_i \cap (K_i + rv_i)$. Generalizing the argument from above shows that each $K_{i+1} \subset U$. We claim that $m^2(K_{i+1}) \ge (1 - (2^i - 1)\beta)m^2(K_1)$ for each i; the above work establishes the result for $i = 1$. Now assume the result holds for some $1 \le j < 3$. Then

$$m^2(U) \ge m^2(K_j \cup (K_j + rv_j)) = m^2(K_j) + m^2(K_j + rv_j) - m^2(K_j \cap (K_j + rv_j)) = 2m^2(K_j) - m^2(K_j \cap (K_j + rv_j)). \tag{38}$$

Therefore,

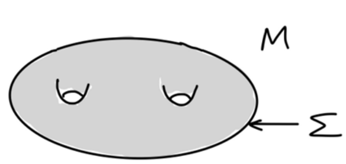
$$\begin{aligned} m^2(K_j \cap (K_j + rv_j)) &= 2m^2(K_j) - m^2(U) \\ &\ge 2m^2(K_1) - 2^{j+1}\beta m^2(K_1) + 2\beta m^2(K_1) - m^2(K_1) - \beta m^2(K_1) \\ &= (1 - (2^{j+1} - 1)\beta)m^2(K_1). \end{aligned} \tag{39}$$

Since $\beta < (2^3 - 1)^{-1} = 7^{-1}$, we conclude that each K_i is nonempty. Hence, we obtain a nested sequence $\varnothing \ne K_4 \subset \cdots \subset K_1 \subset E$. Let $q \in K_4$; since $K_4 = K_3 \cap (K_3 + rv_3)$, $q - rv_3 \in K_3$. Following inductively, we obtain a sequence of points $\{p, p + rv_1, p + rv_1 + rv_2, p + rv_1 + rv_2 + rv_3\} \subset E$, with $p \in K_1$, and $p + rv_j \in K_j$ for $j = 1, 2, 3$ (note we have renamed $q - rv_1 - \cdots - rv_3 = p$, and so on). Let $a = p$, $b = p + rv_1$, and $c = p + rv_1 + rv_2$. Then $a - b = -rv_1$ and $c - b = rv_2$. By hypothesis on v_1 and v_2, $a - b$ is orthogonal to $c - b$.

Problem 2024-J-II-3 (Geometry/Topology). Let Σ be a genus 2 surface embedded in $\mathbb{R}^3$ as shown in the picture. Let M be the closure of the *unbounded* component of $\mathbb{R}^3 \smallsetminus \Sigma$; in other words, M is the part of $\mathbb{R}^3$ which is *not* enclosed by Σ.

(a) Compute $\pi_1(M)$.

(b) Is Σ a retract of M?



(a)

Problem 2024-J-II-5 (Real Analysis). Let P be the vector space over $\mathbb{R}$ of (finite degree) polynomials in the variable $x \in (-\infty, \infty)$. Show that P cannot be a Banach space with respect to any norm, that is, if $\|\cdot\|$ is some norm on P, then P is not complete under this norm. Hint: You may use the Baire Category Theorem.

We recall the Baire Category Theorem:

> **(Baire Category Theorem)** Let X be a complete metric space.
> (a) If $\{U_n\}_1^\infty$ is a sequence of open dense subsets of X, then $\bigcap_1^\infty U_n$ is dense in X.
> (b) X is not a countable union of nowhere dense sets.

For each positive integer n, let P_n be the vector space of all polynomials of degree $\leq n$ so that $P = \bigcup_{n\in\mathbb{N}} P_n$. Let $\|\cdot\|$ be a norm on P and assume to the contrary that P is complete under this norm; this means that P is a complete metric space. Since X cannot be the countable union of nowhere dense sets, it follows that there exists some positive integer n_0 so that P_{n_0} is not nowhere dense; i.e., the closure of P_{n_0} has nonempty interior. Since any finite dimensional vector subspace of a normed vector space is closed, it follows that P_{n_0} is closed in P; i.e., $\overline{P}_{n_0} = P_{n_0}$. Hence, by our hypothesis, P_{n_0} has nonempty interior. Let $p \in P_{n_0}$ and $B(r,p)$ a ball of radius $r > 0$ centered at p that is contained *entirely within* P_{n_0}. Let $u \in P \smallsetminus \{0\}$ be arbitrary, and set

$$u' = p + \frac{r \cdot u}{2\,\|u\|} \implies u' \in B(r,p) \subset P_{n_0}. \tag{40}$$

But since P_{n_0} is a vector space, this implies that $u \in P_{n_0}$. Since u was arbitrary in P, this means that $P_{n_0} = P$, which is a contradiction. Hence, every P_n must have empty interior, which then contradicts the Baire Category Theorem. Hence, P cannot be a Banach space with respect to any norm.

Problem 2024-J-II-6 (Geometry/Topology). Let M be a smooth n-manifold, and let φ be a differential k-form on M which is closed, in the sense that $d\varphi = 0$. At each point $p \in M$, define

$$D_p = \left\{v \in T_pM : v \lrcorner\, \varphi = 0\right\}, \tag{41}$$

where $\lrcorner$ denotes the interior product. Assume $\ell \coloneqq \dim D_p$, so that $D \subset TM$ is a rank-ℓ vector subbundle of the tangent bundle of M. Prove that D is an integrable distribution of ℓ-planes, in the sense of the Frobenius Theorem.

By the Frobenius Theorem, it suffices to prove that D is involutive, which is to say that if X, Y are smooth sections of D, then $[X,Y]$ is also a smooth section of D. Indeed, let X, Y be smooth sections of D, which means that $X \lrcorner\, \varphi, Y \lrcorner\, \varphi = 0$. Observe that,

$$[X,Y] \lrcorner\, \varphi = \mathcal{L}_X(Y \lrcorner\, \varphi) - Y \lrcorner\, (\mathcal{L}_X \varphi). \tag{42}$$

By hypothesis, $Y \lrcorner\, \varphi = 0$ so that $\mathcal{L}_X(Y \lrcorner\, \varphi) = 0$. On the other hand, by Cartan's Formula,

$$\mathcal{L}_X \varphi = d(X \lrcorner\, \varphi) + X \lrcorner\, d\varphi = 0, \tag{43}$$

by the hypotheses. Hence, this shows that $[X,Y] \lrcorner\, \varphi = 0$, and so $[X,Y]$ is a smooth section of D. Therefore, D is involutive, which means that it is Frobenius integrable.

Problem 2024-J-II-4 (Algebra). Let $\alpha = \sqrt{2 + \sqrt{3}} \in \mathbb{C}$. Let K be the smallest Galois extension of $\mathbb{Q}$ which contains α. Describe the Galois group $\mathrm{Gal}(K/\mathbb{Q})$.

Let $\alpha = \sqrt{2+\sqrt{3}} \in \mathbb{C}$, and K the smallest Galois extension of $\mathbb{Q}$ that contains α. We start by finding the minimal polynomial of α. We observe that

$$\alpha^2 = 2 + \sqrt{3} \implies (\alpha^2 - 2)^2 - 3 = 0. \tag{44}$$

Simplifying,

$$\alpha^4 - 4\alpha^2 + 1 = 0. \tag{45}$$

I.e., the polynomial $x^4 - 4x^2 + 1$ is the minimal polynomial of α. Solving this polynomial over an algebraic closure of $\mathbb{Q}$, we obtain the four roots, $\pm\sqrt{2+\sqrt{3}}, \pm\sqrt{2-\sqrt{3}}$. Hence, the elements of the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ are the identity permutation, the permutation σ that fixes $\pm\sqrt{2-\sqrt{3}}$ and permutes $\pm\sqrt{2+\sqrt{3}}$, the permutation τ that fixes $\pm\sqrt{2+\sqrt{3}}$ and permutes $\pm\sqrt{2-\sqrt{3}}$, and the permutation $\sigma\tau$. Labeling these roots as $\alpha_1, \dots, \alpha_4$, we see that $\mathrm{Gal}(K/\mathbb{Q}) \cong \{1, (1\,2), (3\,4), (1\,2)(3\,4)\} \cong V \subset S_4$, where V is the Klein-4 subgroup.

August 2024

Problem 2024-A-I-1 (Geometry/Topology). Let M be a smooth compact manifold without boundary, and let φ be a smooth closed 1-form on M that has the property that $\varphi \neq 0$ at every point of M. Prove that the first de Rham cohomology $H^1_{\mathrm{dr}}(M)$ of the given manifold is non-zero.

Let M be a smooth compact manifold without boundary and let φ be a smooth closed 1-form on M that has the property that $\varphi \neq 0$ at every point of M. Suppose that φ is exact; i.e., assume there exists a smooth function f on M such that $\varphi = df$. By the Extreme Value Theorem, since M is compact, f must have either a maximum or minimum value at some point $p \in M$. Since all of the first-order partial derivatives of f must vanish at the point p where f attains its maximum/minimum value, $df|_p = 0$. This means that φ must also vanish at p, which contradicts our hypothesis that φ is nowhere vanishing. Hence, by contradiction, φ cannot be an exact form. Since $H^1_{\mathrm{dr}}(M) := \{\text{closed 1-forms on } M\}/\{\text{exact 1-forms on } M\}$ and we have shown the existence of a closed 1-form that is *not* an exact 1-form, we conclude that $H^1_{\mathrm{dr}}(M)$ is non-zero.

Problem 2024-A-I-2 (Geometry/Topology). Suppose that $f : \Sigma_2 \to \Sigma_1$ is a continuous map between a genus 2 closed orientable surface Σ_2 and a torus Σ_1. Prove that f is not a local homeomorphism. In other words, show that there exists a point $x \in \Sigma_2$ which does not have an open neighborhood $U \subset \Sigma_2$ on which the restriction $f|_U$ is a homeomorphism between U and $f(U)$.

Before presenting our argument, we will state and prove a quick technical lemma.

> (**Modified Comps Lemma**) Let M and N be smooth connected manifolds, and $f : M \to N$ a local homeomorphism. If M is compact and nonempty, then N is compact and f is a covering map.
>
> *Proof.* Let M and N be smooth connected manifolds, and $f : M \to N$ a local homeomorphism. Since f is an open map, $f(M)$ is open in M. Next since the continuous image of a compact set is compact and a compact subset of a Hausdorff space is closed, $f(M)$ is closed in N. Hence, since N is connected, $f(M) = N$, which means N is connected and f is surjective.
>
> Now let $q \in N$, and consider the closed subset $f^{-1}(q) \subset M$. For each $x \in f^{-1}(q)$, there exists a neighborhood U_x such that $f|_{U_x}$ is a homeomorphism. Since M is Hausdorff, we may shrink these neighborhoods so that they are pairwise disjoint. Hence, each $x \in f^{-1}(q)$ is isolated, which means $f^{-1}(q)$ is discrete. Since discrete subspaces of compact spaces is necessarily finite, $f^{-1}(q)$ is finite; let $\{x_1, \dots, x_s\} = f^{-1}(q)$. As stated above, for each $j = 1, \dots, s$, we may find a neighborhood U_j' such that $f|_{U_j'}$ is a homeomorphism. Using Hausdorff-ness of M, we may shrink these neighborhoods to obtain the collection $\{\widetilde{U}_j\}_1^s$ of pairwise disjoint open neighborhoods. Set $V = \bigcap_1^s U_j$, which is then an evenly covered neighborhood of q. Therefore, f is a covering map. $\square$

Now assume to the contrary that $f : \Sigma_2 \to \Sigma_1$ is a local homeomorphism; by the modified Comps Lemma, f is a covering map. Moreover, Σ_2 must be a k-sheeted covering space for some finite positive integer k, which means that $\chi(\Sigma_2) = k \cdot \chi(\Sigma_1)$. However, this is impossible since $\chi(\Sigma_1) = 0$, while $\chi(\Sigma_2) = 2 - 2(2) = 2 - 4 = -2$. Therefore, f cannot be a local homeomorphism.

Problem 2024-A-I-5 (Algebra). Determine whether or not the complex number $i = \sqrt{-1}$ is in the field $\mathbb{Q}(\alpha)$, where α is any complex number subject to the relation $\alpha^3 + \alpha + 1 = 0$. Justify your answer.

The polynomial $x^3 + x + 1$ has no roots in $\mathbb{Q}$ (by the rational root test), and so is irreducible (since it is a cubic). This means that $\mathbb{Q}(\alpha)$ is an extension of degree 3 over $\mathbb{Q}$. Therefore, it cannot contain the field $\mathbb{Q}(i)$, which has degree 2 over $\mathbb{Q}$ (since the minimal polynomial of i is $x^2 + 1$) since $2 \nmid 3$.

Problem 2024-A-II-1 (Geometry/Topology). Recall that S^n denotes the unit sphere in $\mathbb{R}^{n+1}$. Also recall that a smooth map is called a smooth submersion if its differential is everywhere surjective. Prove or disprove each of the following statements:

(a) There is a smooth submersion $F : S^3 \to S^1$.
(b) There is a smooth submersion $F : S^3 \to S^2$.

(a) [!! Complete Later !!]

Problem 2024-A-II-2 (Geometry/Topology). On $\mathbb{R}^5$, equipped with standard coordinates (v, w, x, y, z), consider the 1-form

$$\theta = dz + v\,dx + w\,dy.$$

Are there two smooth functions $f, g : \mathbb{R}^5 \to \mathbb{R}$ such that $\theta = f\,dg$? Justify your answer by means of concrete solutions.

We claim that there do *not* exist smooth functions $f, g : \mathbb{R}^5 \to \mathbb{R}$ such that $\theta = f\,dg$. Assume to the contrary. First, we observe that if $\theta = f dg$, then

$$d\theta = d(f dg) = df \wedge dg \implies \theta \wedge d\theta = f dg \wedge df \wedge dg = 0. \tag{46}$$

I.e., if $\theta = f dg$, then $\theta \wedge d\theta$ must be identically zero. However, since $\theta = dz + vdx + wdy$, we note that

$$d\theta = d^2 z + d(vdx) + d(wdy) = dv \wedge dx + dw \wedge dy \implies \theta \wedge d\theta = dz \wedge dv \wedge dx + dz \wedge dw \wedge dy + vdx \wedge dw \wedge dy + wdy \wedge dv \wedge dx, \tag{47}$$

which is nowhere vanishing on $\mathbb{R}^5$. Hence, by contradiction, there cannot exist two smooth functions $f, g : \mathbb{R}^5 \to \mathbb{R}$ such that $\theta = f dg$.

January 2023

Problem 2023-J-I-3 (Geometry/Topology). Show that if M is a closed manifold that has an even dimensional sphere S^{2n} as its universal cover, then its fundamental group is either trivial or $\mathbb{Z}_2$.

Let M be a closed manifold that has an even dimensional sphere S^{2n} as its universal cover. Then $M = S^{2n}/G$, where G is the group of deck transformations. Since S^{2n} is path connected, each nonidentity deck transformation g is a homeomorphism $S^{2n} \to S^{2n}$ with no fixed points. Then

$$\deg g = (-1)^{2n+1} = -1. \tag{48}$$

If G contains two nontrivial elements f and g, this implies $\deg(f \circ g) = \deg f \deg g = 1$. Hence, by our observation $f \circ g = \mathrm{id}_{S^{2n}}$, and so $f = g^{-1}$. Likewise, $g^2 = \mathrm{id}_{S^{2n}}$ so that $f = g$. Since S^{2n} is simply connected, we have $\pi_1(M) \cong G$, and therefore $\pi_1(M)$ is either trivial or $\mathbb{Z}_2$.

Problem 2023-J-II-4 (Geometry/Topology). Prove that $S^2 \times S^2$ is not diffeomorphic to $M_1 \times M_2 \times M_3$, where M_1, M_2, M_3 are smooth manifolds of nonzero dimension.

We begin with a technical lemma, that we will use to prove the desired result.

> (**Comps Lemma**) Let M, N be smooth, connected n-manifolds and $f : M \to N$ a (smooth) immersion. If M is compact and nonempty, then N is compact and f is a (smooth) covering map.
>
> *Proof.* Let M, N be smooth connected n-manifolds, $f : M \to N$ an immersion, and M compact and nonempty. Since $\dim N = n$ everywhere and f is an immersion, $df_p : T_pM \to T_{f(p)}N$ has constant rank n everywhere. Hence, by the Inverse Function Theorem, f is a local diffeomorphism. Since local diffeomorphisms are open maps, $f(M)$ is open in N. Next since the continuous image of compact sets is compact, $f(M)$ is compact in N. Since N is Hausdorff, $f(M)$ must be closed in N. Therefore, since N is connected, we conclude that $f(M) = N$. This means that N is compact and f is surjective. All that remains is to show that f is a covering map.
>
> Let $q \in N$, and consider $f^{-1}(q)$, which is closed in M. For each $x \in f^{-1}(q)$, there exists a neighborhood U_x of x such that $f|_{U_j}$ is a diffeomorphism. Since M is Hausdorff, we may shrink these neighborhoods so that they are pairwise disjoint. This means that each $x \in f^{-1}(q)$ is isolated. Hence, $f^{-1}(q)$ is discrete in M. Since discrete subspaces of compact spaces must be finite, it follows that $f^{-1}(q)$ is finite; let $f^{-1}(q) = \{x_1, \dots, x_s\}$. As stated above, for each $j = 1, \dots, s$, we can find a neighborhood U_j of x_j such that $f|_{U_j} : U_j \to V_j \subset N$ is a diffeomorphism. Since M is Hausdorff, we may shrink these neighborhoods so that $U_i \cap U_j = \varnothing$ for all $i \neq j$; f restricted to each of these new U_j's remains a diffeomorphism. Set $V = \bigcap_1^s f(U_j)$, and define $\widetilde{U}_j = f^{-1}(V) \cap U_j$. For each j, $f : \widetilde{U}_j \to V$ is a diffeomorphism and $V = \bigsqcup_1^s f(U_j)$. Hence, V is an evenly covered neighborhood of q, so that f is a covering map. □

Now, assume to the contrary that $f : S^2 \times S^2 \to M_1 \times M_2 \times M_3$ is a diffeomorphism; since diffeomorphisms preserve dimensions and M_1, M_2, M_3 have nonzero dimensions, it follows, without loss of generality, that M_1, M_2 are 1-dimensional and M_3 is 2-dimensional. Since diffeomorphisms of manifolds are immersions, by the Comps Lemma, $M_1 \times M_2 \times M_3$ must be compact and connected; by projecting onto each manifold, M_1, M_2, M_3 must be compact and connected. Moreover, the induced group homomorphism $f_* : \pi_1(S^2 \times S^2) \to \pi_1(M_1 \times M_2 \times M_3) = \pi_1(M_1) \times \pi_1(M_2) \times \pi_1(M_3)$ must be an isomorphism. Since S^2 is simply connected,

$$\pi_1(S^2 \times S^2) = \pi_1(S^2) \times \pi_1(S^2) = \{0\}. \tag{49}$$

On the other hand, since the only compact connected 1-manifold, up to diffeomorphism, is the unit circle S^1, and $\pi_1(S^1) \cong \mathbb{Z}$ is not trivial, $\pi_1(M_1 \times M_2 \times M_3)$ is not trivial. But this contradicts our claim that f_* is an isomorphism. Hence, by contradiction, f cannot be a diffeomorphism.

Problem 2023-J-II-3 (Geometry/Topology). Consider the form $\omega = (x^2 + x + y)dy \wedge dz$ on $\mathbb{R}^3$. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, and $i : S^2 \to \mathbb{R}^3$ be the inclusion map.

(a) Calculate $\int_{S^2} i^*\omega$.

(b) Construct a closed form α on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$, or show that such a form α does not exist.

(a) (**Method 1**) Consider the form $\omega = (x^2 + x + y)dy \wedge dz$ on $\mathbb{R}^3$, and let $i : S^2 \hookrightarrow \mathbb{R}^3$ be the inclusion map. Let $D = [0, \pi] \times [0, 2\pi]$, and $F : D \to S^2$ be the coordinate map defined by

$$F(\varphi, \theta) = (\sin(\varphi)\cos(\theta), \sin(\varphi)\sin(\theta), \cos(\varphi)). \tag{50}$$

Taking $D_1 = [0,\pi] \times [0,\pi]$ and $D_2 = [0,\pi] \times [\pi, 2\pi]$, and letting $F_1 = F|_{D_1}$ and $F_2 = F|D_2$, we observe that

$$\int_{S^2} i^*\omega = \int_{D_1} F_1^* i^*\omega + \int_{D_1} F_2^*\omega = \int_{D_1} (i \circ F_1)^*\omega + \int_{D_2} (i \circ F_2^*)\omega = \int_D F^*\omega, \tag{51}$$

where the last equality follows from the fact that $i \circ F_{1,2} = F_{1,2}$. We observe that

$$F^* dy = \cos(\varphi)\sin(\theta)d\varphi + \sin(\varphi)\cos(\theta)d\theta \quad \text{and} \quad F^* dz = -\sin(\varphi)d\varphi. \tag{52}$$

Therefore,

$$F^*\omega = \left[\sin^2(\varphi)\cos^2(\theta) + \sin(\varphi)\cos(\theta) + \sin(\varphi)\sin(\theta)\right]\sin^2(\varphi)\cos(\theta)d\varphi \wedge d\theta. \tag{53}$$

From this, we conclude that

$$\int_{S^2} i^*\omega = \int_0^{2\pi}\int_0^{\pi} \left[\sin^2(\varphi)\cos^2(\theta) + \sin(\varphi)\cos(\theta) + \sin(\varphi)\sin(\theta)\right]\sin^2(\varphi)\cos(\theta)d\varphi d\theta = \frac{4\pi}{3}. \tag{54}$$

(**Method 2**) Using Stokes Theorem,

$$\int_{S^2} i^*\omega = \int_{B^3} d\omega, \tag{55}$$

where B^3 indicates the 3-ball (recall that $S^1 = \partial B^3$). We compute, $d\omega = (2x+1)dx \wedge dy \wedge dz$ so that

$$\int_{S^2} i^*\omega = \int_{B^3} d\omega = \int_{B^3} 2x dx dy dz + \int_{B^3} dx dy dz = \int_{B^3} dx dy dz = \frac{4\pi}{3}, \tag{56}$$

where the first integral after the second inequality is zero due to symmetry.

(b) Suppose there exists a closed form α on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$. Since α is closed, $d\alpha = 0$. Hence,

$$\int_{S^2} i^*\alpha = \int_{B^3} d(i^*\alpha) = \int_{B^3} i^* d\alpha = 0 \neq \frac{4\pi}{3} = \int_{S^2} i^*\omega, \tag{57}$$

which is a contradiction. Hence, such a closed form cannot exist.

Problem 2023-J-I-5 (Algebra). Consider the following irreducible polynomial over $\mathbb{Q}$: $p(x) = x^4 - 3x^2 - 1$.

(a) Describe the splitting field of $p(x)$.

(b) Consider the Galois group of $p(x)$. Compute its order and determine if it is abelian.

(a) Let $p(x) = x^4 - 3x^2 - 1$. By the rational root test, $p(x)$ has no roots over $\mathbb{Q}$. Moreover, it is straightforward to check that $p(x)$ is not the product of irreducible quadratics with rational coefficients. Hence, $p(x)$ is irreducible over $\mathbb{Q}$. We start by finding the roots of $p(x)$; let $u = x^2$. Then

$$u^2 - 3u - 1 = 0 \implies u = \frac{3 \pm \sqrt{13}}{2} \implies x = \pm\sqrt{\frac{3 \pm \sqrt{13}}{2}}. \tag{58}$$

Let

$$\alpha = \sqrt{\frac{3+\sqrt{13}}{2}}, \qquad \beta = \sqrt{\frac{3-\sqrt{13}}{2}}. \tag{59}$$

Observe that $\alpha^2\beta^2 = -1$ so that $\beta = \pm\frac{i}{\alpha}$. Therefore, the splitting field of $p(x)$ is

$$\mathbb{Q}(\alpha, i). \tag{60}$$

Observe that the minimal polynomial of i is x^2+1, which is irreducible over $\mathbb{Q}(\alpha)$ so that $[\mathbb{Q}(\alpha,i) : \mathbb{Q}(\alpha)] = 2$. On the other hand, the minimal polynomial of α is a degree 4 polynomial so that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$. Hence, by the tower law, $[\mathbb{Q}(\alpha, i) : \mathbb{Q}] = 8$.

(b) By the last work in (a), the order of the Galois group of $p(x)$ is 8. Now, we will determine the Galois group of $p(x)$. Recall that elements of $\mathrm{Gal}(\mathbb{Q}(\alpha,i)/\mathbb{Q})$ are automorphisms φ of the field $\mathbb{Q}(\alpha,i)$ with the constraints that: (1) φ fixes $\mathbb{Q}$, (2) $\varphi(\alpha)$ must be another root of the minimal polynomial of α over $\mathbb{Q}$, and (3) $\varphi(i)$ must be another root of x^2+1. We will explicitly work through each of the elements.

(i) $\sigma : i \mapsto -i, \alpha \mapsto \alpha$. This permutation has order 2 since $\sigma^2(\alpha) = \sigma(\alpha) = \alpha$ and $\sigma^2(i) = \sigma(-i) = i$.
(ii) $\tau : i \mapsto i, \alpha \mapsto -\alpha$. Once again, this permutation has order 2.
(iii) $\rho : i \mapsto -i, \alpha \mapsto \beta = \frac{i}{\alpha}$. To compute the order of this permutation, observe that

$$\rho^2(\alpha) = \rho(i\alpha^{-1}) = (-i)\cdot\frac{1}{i/\alpha} = -\alpha \implies \rho^4(\alpha) = \rho^2(-\alpha) = \alpha. \tag{61}$$

Likewise, $\rho^4(i) = \rho^2(i) = i$. Hence, ρ has order 4.

Now, consider the three elements given above. We compute

$$\sigma\rho\sigma(i) = \sigma\rho(-i) = \sigma(i) = -i = \rho^{-1}(i). \tag{62}$$

Likewise,

$$\sigma\rho\sigma(\alpha) = \sigma\rho(\alpha) = \sigma(i)\sigma(\alpha)^{-1} = -\frac{i}{\alpha} = \rho^{-1}(\alpha). \tag{63}$$

Therefore, $\sigma\rho\sigma = \rho^{-1}$. Hence,

$$\mathrm{Gal}(\mathbb{Q}(\alpha,i)/\mathbb{Q}) = \{1,\sigma,\rho,\rho^2,\rho^3,\sigma\rho,\sigma\rho^2,\sigma\rho^3\} \cong D_8. \tag{64}$$

Since the dihedral group is not abelian, we conclude that the Galois group for $p(x)$ is non-abelian.

Problem 2023-J-I-5 (Algebra I). Determine the Galois group of x^3-x^2-4.

Let $p(x) = x^3-x^2-4$. We start by finding the roots of $p(x)$ over some algebraic closure of $\mathbb{Q}$. Observe that 2 is a solution. Using polnomial long division,

$$p(x) = (x-2)(x^2+x+2) \implies x = 2, \frac{-1\pm\sqrt{-7}}{2}. \tag{65}$$

Hence, the splitting field of $p(x)$ is $\mathbb{Q}(\sqrt{7}i)$. Now since $\mathrm{Gal}(\mathbb{Q}(\sqrt{7}i)/\mathbb{Q})$ is the group of automorphisms of the splitting field $\mathbb{Q}(\sqrt{7}i)$ that preserve $\mathbb{Q}$. Since there are exactly two automorphisms (namely, the identity permutation fixing $\sqrt{7}i$ and the conjugation map $\sqrt{7}i \mapsto -\sqrt{7}i$), we conclude that $\mathrm{Gal}(\mathbb{Q}(\sqrt{7}i)/\mathbb{Q}) \cong \mathbb{Z}_2$.

Problem 2023-J-I-5 (Algebra II). Determine the Galois group of x^3-2x+4.

Let $p(x) = x^3-2x+4$. We start by finding the roots of $p(x)$ over some algebraic closure of $\mathbb{Q}$. Clearly -2 is a root of $p(x)$. Using polynomial long division,

$$p(x) = (x+2)(x^2-2x+2) \implies x = -2, 1\pm\sqrt{-1}. \tag{66}$$

Hence, the splitting field of $p(x)$ is $\mathbb{Q}(i)$, which is a quadratic extension of $\mathbb{Q}$. Now since $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ is the group of automorphisms of the splitting field $\mathbb{Q}(i)$ that preserve $\mathbb{Q}$, and there exactly two such automorphisms (namely, the identity fixing i, and the conjugation map $i \mapsto -i$), we conclude that $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$.

Problem 2023-J-I-5 (Algebra III). Determine the Galois group of $x^3 - x + 1$.

Let $p(x) = x^3 - x + 1$. We start by finding the roots of x over some algebraic closure of $\mathbb{Q}$. Since the only possible rational roots of p over $\mathbb{Q}$ are ± 1 by the Rational Root Test, and neither of these are actually roots of p, we conclude that p is irreducible. Hence, a root of $f(x)$ generates an extension of degree 3 so that the degree of the splitting field of F is divisible by 3. Since the Galois group is a subgroup of S_3, either $\mathrm{Gal}(\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3)/\mathbb{Q}) \cong A_3$ or $\mathrm{Gal}(\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3)/\mathbb{Q}) \cong S_3$. Since p is already a depressed cubic, we calculate its discriminant to be $-4(-1)^3 - 27(1)^2 = -23$. Since the discriminant is not a perfect square in $\mathbb{Q}$, we conclude that $\mathrm{Gal}(\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3)/\mathbb{Q}) \cong S_3$.

Problem 2023-J-I-4 (Geometry/Topology). Let ω be a smooth nowhere vanishing 1-form on a smooth 3-manifold M^3.

(a) Show that the distribution defined at each point $p \in M$ by

$$\ker \omega_p = \left\{ v \in T_pM^3 : \omega_p(v) = 0 \right\} \tag{67}$$

is integrable if and only if $\omega \wedge d\omega = 0$.

(b) Give an example of a codimension one distribution on $\mathbb{R}^3$ that is not integrable.

(a) We recall that a distribution D is Frobenius integrable if and only if given two smooth sections X, Y of D, the Lie Bracket $[X,Y]$ is also a smooth section of D. Therefore, let X, Y be smooth sections of D, which means that $\omega(X), \omega(Y) = 0$ by definition of D. We recall that

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) = -\omega([X,Y]), \tag{68}$$

where the first two terms are identically zero by our hypothesis. Therefore, D is integrable if and only if $[X,Y]$ is a smooth section of D if and only if $\omega([X,Y]) = 0$. Now, if D were integrable, then for any field Z on $\mathbb{R}^3$,

$$\omega \wedge d\omega(X,Y,Z) = \omega(Z)d\omega(X,Y) = 0, \tag{69}$$

where the other terms vanish by assumption on X and Y. Hence, since $X, Y \in \ker\omega$ were arbitrary and Z was arbitrary, $\omega \wedge d\omega = 0$. On the other hand, if $\omega \wedge d\omega = 0$, let $p \in M$, $Z_p \in T_pM$ with $\omega_p(Z_p) \neq 0$ and $X_p, Y_p \in \ker\omega_p$. Then

$$0 = (\omega \wedge d\omega)_p(X_p, Y_p, Z_p) = \omega_p(Z_p)d\omega_p(X_p, Y_p). \tag{70}$$

Hece, $d\omega_p(X_p, Y_p) = 0$. This means that for smooth sections X, Y of $\ker\omega$, $d\omega(X,Y) = 0$, and so D is integrable.

(b) Consider the smooth nowhere vanishing 1-form $\omega = ydx + dy + dz$ on $\mathbb{R}^3$, and let D be the distribution on $\mathbb{R}^3$ defined at each point $p \in M$ by $D_p = \ker\omega_p$. By the rank-nullity theorem, $\dim D = \dim T_p\mathbb{R}^3 - \operatorname{rank}\omega = 3 - 1 = 2$. Hence, $\operatorname{codim} D = 3 - 2 = 1$. Next, we observe that $d\omega = dy \wedge dx$, which is identically not zero. Then $\omega \wedge d\omega = dz \wedge dy \wedge dx$, which is also not identically zero. Hence, by the conclusion in (a), D is not integrable.

Problem 2023-J-I-1 (Real Analysis). Give (with proof) an example of a Banach space X and a norm closed set $E \subset X$ that is not weakly closed.

Let $X = \mathcal{H}$, where $\mathcal{H}$ is any infinite-dimensional Hilbert space, and let $E = \{v_n\}$ be an infinite orthonormal set in $\mathcal{H}$. Since $\|x_i - x_j\| \geq 1$ for any two distinct vectors x_i, x_j, it follows that if $\|x_n - x\| \to 0$, then the sequence must eventually be constant, which means $x \in E$. However, E is not weakly closed: fix an arbitrary element $y \in \mathcal{H}$. Then by Bessel's inequality,

$$\sum_1^\infty |\langle y, v_n\rangle|^2 \leq \|y\|^2, \tag{71}$$

so that the sequence of inner products $a_n = \langle y, v_n \rangle$ is square summable, so $a_n \to 0$. This means that $v_n \to 0$ weakly. However, $0 \notin E$ since $\|0\| = 0 \neq 1$. Hence, E is not weakly closed.

Problem 2023-J-I-2 (Complex Analysis). Set $\mathbb{D} = \{z \in \mathbb{C} : \|z\| < 1\}$, $f : \mathbb{D} \to \left\{w \in \mathbb{C} : e^{-\pi/2} < |\omega| < e^{\pi/2}\right\}$ be a holomorphic map satisfying $f(0) = 1$. Show that $|f'(0)| \le 2$.

[!! Complete Later]

Problem 2023-J-II-1 (Real Analysis). Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Show that $f^{-1}(y) = \{x \in \mathbb{R} : f(x) = y\}$ has Lebesgue measure zero for Lebesgue almost all y.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and define the following:

$$\begin{aligned} \Gamma &:= \left\{(x, y) \in \mathbb{R}^2 : f(x) = y\right\}. \\ \Gamma_x &:= \{y \in \mathbb{R} : f(x) = y\} = \{f(x)\}. \\ \Gamma^y &:= \{x \in \mathbb{R} : f(x) = y\} = f^{-1}(y). \\ g &: \mathbb{R} \to [0, \infty) \quad \text{defined by} \quad g(y) = m(f^{-1}(y)). \end{aligned} \tag{72}$$

Since Γ_x is a singleton for every $x \in \mathbb{R}$, we conclude that $m(\Gamma_x) = 0$ for all x. By Fubini-Tonelli, we recall that

$$m^2(\Gamma) = \int_{\mathbb{R}^2} \chi_\Gamma \, dx\, dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_\Gamma \, dx\right) dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_\Gamma \, dy\right) dx. \tag{73}$$

However,

$$\int_{\mathbb{R}} \chi_\Gamma \, dy = m(\Gamma_x) = 0. \tag{74}$$

This means that

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_\Gamma \, dx\right) dy = 0 \implies \int_{\mathbb{R}} \chi_\Gamma \, dx = 0 \text{ a.e.} \tag{75}$$

Now, we observe that

$$\int_{\mathbb{R}} \chi_\Gamma \, dx = g(y) = m(f^{-1}(y)). \tag{76}$$

Hence, this means that $m(f^{-1}(y)) = 0$ a.e., which implies that $f^{-1}(y)$ has Lebesgue measure zero for Lebesgue almost all y.

Problem 2023-J-II-2 (Real Analysis). Suppose that f is continuous on $[0, 1]$ and $\int_0^1 f(x)x^k dx = 0$ for $k = 0, \ldots, n$. Prove that either f is either identically zero, or f must change sign at least $n + 1$ times. We say that f changes sign n times if there are points $x_1 < \cdots < x_{n+1}$ so that $f(x_j)f(x_{j+1}) < 0$ for $j = 1, \ldots, n$.

Suppose that f is continuous on $[0, 1]$ and $\int_0^1 f(x)x^k dx = 0$ for $k = 0, \ldots, n$. If f is identically zero, then the claim is trivial and we are done. So assume that $f \not\equiv 0$. Suppose f changes sign only n times. By the definition provided above, we can find n points $x_1, \ldots, x_n$ such that $f(x_j) = 0$ for each $j = 1, \ldots, n$. Consider the function $g(x) = \pm f(x) \cdot \prod_1^n (x - x_j)$, which must be continuous on $[0, 1]$ since it is the product of finitely many continuous functions. For some choice of $\pm$, $g(x) \ge 0$ for all $x \in [0, 1]$. Since $\prod_1^n (x - x_j)$ is a polynomial of degree n, we conclude by the hypothesis that

$$\int_0^1 g(x) dx = 0. \tag{77}$$

Since $g(x) \ge 0$, this forces $g(x) = 0$ and so f has to be identically zero, which contradicts our hypothesis. Hence, by contradiction, f has to change at least $n + 1$ times.

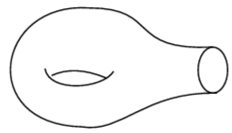
August 2023

Problem 2023-A-I-1 (Algebra). Let V be an n-dimensional vector space over a field F. An element $A \in \operatorname{End} V$ is called *nilpotent* if $A^k = 0$ for some $k > 1$. Prove that A is nilpotent if and only if

$$\operatorname{Tr}(\Lambda^i A) = 0, \quad i = 1, \dots, n$$

where $\Lambda^i A$ denotes the induced action of A on the wedge product $\Lambda^i V$ for each i.

Problem 2023-A-I-5 (Geometry/Topology). Let T be the 2-torus $S^1 \times S^1$ with an open 2-disk removed:



Show that there is no continuous retraction r onto its boundary (i.e., no continuous map $r : T \to \partial T$ satisfying $r^2 = r$).

Let T be the 2-torus $S^1 \times S^1$ with an open 2-disk removed, $\iota : \partial T \to T$ the inclusion map, and assume to the contrary that $r : T \to \partial T$ is a continuous retraction. Then the composition $r_* \circ \iota_* : \pi_1(\partial T) \to \pi_1(\partial T)$ must be the identity map. Since $\partial T \cong S^1$, $\pi_1(\partial T) = \mathbb{Z}$, and is generated by the element 1. By a direct computation, since $\partial_1(T) = \mathbb{Z} * \mathbb{Z}$ is the free product on two generators a and b ι_* maps 1 to the element $aba^{-1}b^{-1}$. But then r_* maps the commutator into the abelian group $\mathbb{Z}$, where the commutator must be zero. This contradicts our claim that $r_* \circ \iota_*$ is the identity map. Hence, by contradiction, there cannot be any continuous retraction of T onto its boundary.

Problem 2023-A-I-6 (Complex Analysis). Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk. Is there a holomorphic function f with $f(\mathbb{D}) = \mathbb{D}$, $f(0) = f'(0) = 2/3$? If so, give a formula. If not, prove that it cannot exist.

The problem lends itself nicely to an application of the Schwarz-Pick Theorem:

> (**Schwarz-Pick Theorem**) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. If $|f(z)| \le 1$ for all z, and $f(a) = b$ for some $a, b \in \mathbb{D}$, then
> $$|f'(a)| \le \frac{1 - |b|^2}{1 - |a|^2}.$$

Now assume that a holomorphic function f with $f(\mathbb{D}) = \mathbb{D}$, $f(0) = f'(0) = 2/3$ exists. Then by the Schwarz-Pick Lemma,

$$\frac{2}{3} \le \frac{1 - 4/9}{1 - 0} = \frac{5}{9} < \frac{2}{3}, \tag{78}$$

which is a contradiction. Hence, no such holomorphic function can exist.

Problem 2023-A-I-2 (Geometry/Topology). Let $f : T^2 \to S^2$ be a smooth map from the 2-torus to the 2-sphere. Can f be an immersion? If the answer is yes, give an explicit example. If the answer is no, then give a proof.

We begin by stating and proving a technical lemma, which we will then use in our argument.

> (**Comps Lemma**) Let M and N be smooth connected n-manifolds, and $f : M \to N$ a (smooth) immersion. If M is compact and nonempty, then N is compact and f is a (smooth) covering map.
>
> *Proof.* Let M and N be smooth connected n-manifolds, and $f : M \to N$ an immersion. Since $\dim M = \dim N = n$, and f is an immersion, the map $df_p : T_pM \to T_{f(p)}N$ has constant rank

n at every $p \in M$. Hence, by the Inverse Function Theorem, f is a local diffeomorphism. Since local diffeomorphisms are open maps, $f(M)$ is open in N. On the other hand, since continuous images of compact sets are compact, $f(M)$ is compact in N; since N is Hausdorff, $f(M)$ is closed in N. Since N is connected, it follows that $f(M) = N$. Therefore, N is compact. All that remains is to show is that f is a covering map.

Let $q \in N$; by continuity of f, $f^{-1}(q)$ is a closed subset of M. For each $x \in f^{-1}(q)$, there exists an open neighborhood U_x of x such that $f|_{U_x}$ is a diffeomorphism. Since M is Hausdorff, we can shrink these neighborhoods so that they are pairwise disjoint. This means that each $x \in f^{-1}(q)$ is isolated, implying that $f^{-1}(q)$ is discrete. Since M is compact, it follows that $f^{-1}(q)$ is finite; let $f^{-1}(q) = \{x_1, \dots, x_s\}$. As stated above, for each $j = 1, \dots, s$, we may find an open neighborhood U_j' so that $f|U_j'$ is a diffeomorphism. Moreover, we can shrink these neighborhoods to obtain a pairwise disjoint collection $\{\widetilde{U}_j\}_1^s$ of neighborhoods. Set $V = \bigcap_1^s f(\widetilde{U}_j)$. Then taking $U_j = f^{-1}(V) \cap \widetilde{U}_j$, V is an evenly covered neighborhood of p, so that f is a covering map. □

Now assume to the contrary that there exists an immersion $f : T^2 \to S^2$. By the Comps Lemma, f must be a covering map. Hence, the induced homomorphism of groups $f_* : \pi_1(T^2) \to \pi_1(S^2)$ must be injective. Since S^2 is simply connected, $\pi_1(S^2) \cong \{0\}$. However, $\pi_1(T^2)$ is not a trivial group (in fact, $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$). This means that f_* cannot be injective. Therefore, by contradiction, f cannot be an immersion. Hence, there exist no immersions from T^2 to S^2.

Problem 2023-A-II-1 (Algebra). A field extension K/L is called algebraic, if every element in K satisfies a polynomial equation with coefficients in L. Let F, K, L be fields such that $F \supset K \supset L$, and F/K and K/L are algebraic extensions. Prove that F/L is also an algebraic extension.

Since subfields of subfields is a subfield, L is a subfield of F. Hence, it suffices to show that every element in F satisfies a polynomial equation with coefficients in L. Let $a \in F$, and let

$$k(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_0 \in K[x] \tag{79}$$

such that $k(a) = 0$; this follows since F/K is an algebraic extension. Each $k_j \in K$, and hence is algebraic over L. Therefore, $L' = L(k_0, \dots, k_n)$ is a finite extension of L. Since $k(a) = 0$ and $k(x)$ now has its coefficients in L', it follows that a is algebraic over L' so that $L'(a)$ is a finite extension of L. Then since

$$[L(a) : L] = [L(a) : L'][L' : L], \tag{80}$$

it follows that $L(a)$ is a finite extension of L. Therefore, a is algebraic over L. Since a was arbitrary, F/L is an algebraic extension.

Problem 2023-A-I-2 (Geometry/Topology). Let $f : T^2 \to S^2$ be a smooth map from the 2-torus to the 2-sphere. Can f be an immersion? If the answer is yes, given an explicit example. If the answer is no, then give a proof.

There cannot be an immersion $f : T^2 \to S^2$. To prove our answer, we will state and proof a technical lemma.

(**Comps Lemma**) Let M, N be smooth, connected, n-manifolds and $f : M \to N$ a (smooth) immersion. If M is compact and nonempty, then f is a (smooth) covering map.

Proof. Let M, N be smooth connected n-manifolds, M compact, and $f : M \to N$ an immersion. Since $\dim N = n$ everywhere and f is an immersion, $df_p : T_pM \to T_{f(p)}N$ has constant rank n everywhere. Hence, by the Inverse Function Theorem, f is a local diffeomorphism. Let $q \in N$ so that $f^{-1}(q) \subset M$ is closed. For each $x \in f^{-1}(q)$, there exists a neighborhood U_x such that $f|_{U_x} : U_x \to V_x \subset N$ is a diffeomorphism. Since M is Hausdorff, we can shrink these neighborhoods so that they are pairwise disjoint. Since every $x \in f^{-1}(q)$ is now isolated, it follows that $f^{-1}(q)$ is discrete. Since M is compact, we conclude that $f^{-1}(q)$ must be finite; let $f^{-1}(q) = \{x_1, \ldots, x_s\}$. As stated above, for each $j = 1, \ldots, s$, we can find a neighborhood U_j of x_j so that $f|_{U_j} : U_j \to V_j \subset N$ is a diffeomorphism. Again, since M is Hausdorff, we can shrink these neigborhoods so that $U_i \cap U_j = \varnothing$ for all $i \neq j$; f restricted to each of these shrunken neighborhoods remains a diffeomorphism. Now set $V = \bigcap_1^s f(U_j)$, and define $\widetilde{U}_j \subset M$ by $\widetilde{U}_j = f^{-1}(V) \cap U_j$ for each $j = 1, \ldots, s$. Hence, V is an evenly covered neighborhood of $q \in N$, which means f is a covering map. That f is surjective comes from recognizing that $f(M) = N$ due to connectedness of N. □

Now, assume $f : T^2 \to S^2$ is an immersion. Since T^2, S^2 are smooth, connected 2-manifolds, and T^2 is compact and nonempty, by the Comps Lemma, f is a covering map. Hence, the induced homomorphism $f_* : \pi_1(T^2) \to \pi_1(S^2)$ is injective. Since S^2 is simply connected, $\pi_1(S^2) \cong \{0\}$. On the other hand, $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$. Since the order of $\pi_1(T^2)$ is more than one, f_* cannot be injective. Hence, f cannot be an immersion.

Problem 2023-A-II-5 (Geometry/Topology). Let (t, x, y, z) be the standard coordinate system on $\mathbb{R}^4$, and let ϕ be the non-zero smooth 1-form on $\mathbb{R}^4$ defined by

$$\phi = dt + ydx + zdy.$$

Let D be the 3-plane field on $\mathbb{R}^4$ that consists of tangent vectors V such that $\phi(V) = 0$. Is D Frobenius integrable? Support your answer with a proof.

Let D be the 3-plane field on $\mathbb{R}^4$ defined as follows: for each $p \in \mathbb{R}^4$,

$$D_p = \left\{ v \in T_p\mathbb{R}^4 : \phi(v) = 0 \right\} =: \ker \phi_p. \tag{81}$$

Hence, by the Frobenius Theorem, D is Frobenius integrable if and only if $\phi \wedge d\phi = 0$. We compute:

$$d\phi = d(dt + y\,dx + z\,dy) = d^2t + dy \wedge dx + dz \wedge dy = dy \wedge dx + dz \wedge dy. \tag{82}$$

Therefore,

$$\phi \wedge d\phi = dt \wedge dy \wedge dx + dt \wedge dz \wedge dy + ydx \wedge dz \wedge dy. \tag{83}$$

Since $\phi \wedge d\phi$ is nowhere vanishing on $\mathbb{R}^4$, D is not Frobenius integrable.

Problem 2023-A-I-1 (Algebra). Let V be a n-dimensional vector space over a field F. An element $A \in \operatorname{End} V$ is called *nilpotent*, if $A^k = 0$ for some $k > 1$. Prove that A is nilpotent if and only if

$$\operatorname{Tr}(\Lambda^i A) = 0, \quad i = 1, \ldots, n, \tag{84}$$

where $\Lambda^i A$ denotes the induced action of A on the wedge product $\Lambda^i V$ for each i.

Let V be a n-dimensional vector space over a field F, and let $A \in \operatorname{End} V$. Recall that $\Lambda^i A$, the induced action of A on the wedge product $\Lambda^i V$, is defined to be

$$(\Lambda^i A)(v_1 \wedge \cdots \wedge v_i) = Av_1 \wedge \cdots \wedge Av_i, \qquad v_j \in V \text{ for all } j = 1, \ldots, i. \tag{85}$$

Over an algebraic closure of F, A has eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose A is diagonalizable, with the set of eigenvectors given by $\{v_1, \ldots, v_n\}$. Then for each $i = 1, \ldots, n$, since the collection

$$\left\{ v_{j_1} \wedge \cdots \wedge v_{j_i} : 1 \leq j_1 < \cdots < j_i \leq n \right\}$$

is a basis of $\Lambda^i V$, and for each i-tuple, $\Lambda^i A(v_{j_1} \wedge \cdots \wedge v_{j_i}) = Av_{j_1} \wedge \cdots \wedge Av_{j_i} = (\lambda_{j_1} \cdots \lambda_{j_i})(v_{j_1} \wedge \cdots \wedge v_{j_i})$, it follows that the eigenvalues of $\Lambda^i A$ are the set of all products of the form $\lambda_{j_1} \cdots \lambda_{j_i}$ for $1 \leq j_1 < \cdots < j_i \leq n$, counting for multiplicity. Hence,

$$\operatorname{Tr}(\Lambda^i A) = \sum_{1 \leq j_1 < \cdots < j_i \leq n} \lambda_{j_1} \cdots \lambda_{j_i}. \tag{86}$$

If A is not diagonalizable, since the eigenvalues of $\Lambda^i A$ depend only on the eigenvalues of A, we may assume A is in Jordan normal form. Indeed, if $A = PJP^{-1}$, then

$$\Lambda^i(A) = \Lambda^i(PJP^{-1}) = \Lambda^i(P)\Lambda^i(J)\Lambda^i(P^{-1}), \tag{87}$$

so $\Lambda^i A$ and $\Lambda^i J$ are similar and therefore have the same eigenvalues. Thus it suffices to compute the eigenvalues of $\Lambda^i J$, which are exactly the products $\lambda_{j_1} \cdots \lambda_{j_i}$ of the eigenvalues of A.
If A is nilpotent so that $A^k = 0$ for some $k > 1$, then since $0 = A^k v = \lambda^k v$ for all eigenvectors v of A, it follows that every eigenvalue of A is zero. Therefore, the above expression implies that $\mathrm{Tr}(\Lambda^i A) = 0$ for all $i = 1, \ldots, n$. On the other hand, expanding the characteristic polynomial for A is given by:

$$p_A(t) = \det(tI - A) = t^n - \mathrm{Tr}(\Lambda^1 A)t^{n-1} + \cdots + (-1)^n \mathrm{Tr}(\Lambda^n A). \tag{88}$$

If $\mathrm{Tr}(\Lambda^i A) = 0$ for all $i = 1, \ldots, n$, then we conclude that the characteristic polynomial of A is precisely t^n. Therefore, A's eigenvalues are all zero. Hence, the minimal polynomial of A is of the form t^k for some $k \le n$. This implies that $A^k = 0$, and so A is nilpotent.

Problem 2023-A-II-6 (Complex Analysis). Find the number of solutions (counting multiplicity) to $z^8 - 5z^6 + 2z^3 - z - 1 = 0$ that lie inside the unit disk.

Recall Rouché's Formula, which states that

> For any two complex-valued functions f and g holomorphic inside some region K with closed and simple contour ∂K, if $|g(z)| < |f(z)|$ on ∂K, then f and $f+g$ have the same number of zeros inside K, where each zero is counted as many times as its multiplicity.

Pick $f(z) = 5z^6$ and set $h(z) = z^8 + 2z^3 - z - 1$ so that $p(z) = z^8 - 5z^6 + 2z^3 - z - 1 = h(z) - f(z)$. On the unit disk ∂S^1, we observe that

$$\begin{aligned} |f(z)| &= \left|5z^6\right| = 5 \\ &= 1 + 2 + 1 + 1 \\ &= |z^8| + 2|z^3| + |z| + |1| \\ &\ge |h(z)|. \end{aligned} \tag{89}$$

Hence, $p(z) = h(z) - f(z)$ has the same number of zeros, counting multiplicity, as $f(z)$. Since $f(z)$ has six zeros in the unit disk, we conclude that $p(z)$ must also have six zeros inside the unit disk.

Problem 2023-A-II-4 (Real Analysis). Let μ be a (positive) Borel probability measure on $[0,1]$, such that for all $t \in [0,1]$ we have $\mu(\{t\}) = 0$. Let μ_n be a (positive) Borel probability measure on $[0,1]$ for $n = 1, 2, \ldots$. Suppose $\mu_n \to \mu$ in the weak* topology. Let $F(t) = \mu([0,t])$ and $F_n(t) = \mu_n([0,t])$. Prove that $F_n \to F$ uniformly.

January 2022

Problem 2022-J-I-1 (Complex Analysis). Suppose $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function on the unit disk so that

$$f\left(\frac{1}{n}\right) = \frac{5}{n^2}, \qquad \forall n \in \mathbb{Z}, n \ge 2.$$

Find $f''(0)$.

This problem requires the notion of an analytic continuation:

> (**Analytic Continuation**) Let f be an analytic function defined on a non-empty open subset U of the complex plane $\mathbb{C}$. If V is a larger open subset of $\mathbb{C}$ containing U, and F is an analytic function defined on V such that $F(z) = f(z)$ for all $z \in U$, then F is called an *analytic continuation of f*. I.e., $F|_U = f$.

For our problem, since the function $F(z) = 5z^2$ agrees with $f(z)$ at a sequence of points converging in $\mathbb{D}$ and $\mathbb{D}$ is connected, $f(z) = 5z^2$. Hence, $f''(0) = 10$.

Problem 2022-J-I-3 (Algebra). Show that a group of order 1,000,000 contains a proper normal subgroup (i.e., is not simple).

Let G be a group of order $1{,}000{,}000 = 10^6 = 2^6 \cdot 5^6$. By Sylow's Theorem,

$$\begin{aligned} n_2 &\in \{1, 5, 5^2, 5^3, 5^4, 5^5, 5^6\} \cap \{2k+1 : k \in \mathbb{N}\}, \\ n_5 &\in \{1, 2, 4, 8, 16, 32, 64\} \cap \{5k+1 : k \in \mathbb{N}\} = \{1, 16\}. \end{aligned} \tag{90}$$

If $n_5 = 1$, then we are done since the unique Sylow 5-subgroup must necessarily be normal. So suppose $n_5 = 16$, and let G act on $\mathrm{Syl}_5(G)$ by conjugation. This induces a homomorphism $\varphi : G \to \varphi(G) \leq S_{16}$. However, $|G| = 10^6 \nmid 16! = |S_{16}|$. This means that φ cannot be an injective homomorphism since if otherwise, $|\varphi(G)| = |G|$, but this is impossible since $|G| \nmid |S_{16}|$. Therefore, $\ker\varphi$ is a nontrivial normal subgroup of G. If $\ker\varphi = G$, then every Sylow 5-subgroup of G is normal and is, in fact, unique, which contradicts our hypothesis that $n_5 = 16$. Hence, $\ker\varphi$ is a proper nontrivial normal subgroup of G, which means that G cannot be simple.

Problem 2022-J-II-3 (Real Analysis). Prove or give a counterexample: if $E \subset \mathbb{R}$ is a Lebesgue measurable subset of positive Lebesgue measure, then some countable union of translates of E covers $\mathbb{R}$.

The statement is not necessarily true. Consider the *fat Cantor* set, which is a subset of $\mathbb{R}$ that is nowhere dense and has positive Lebesgue measure $1/2$; call this set E. Assume to the contrary that $\mathbb{R}$ is the countable union of translates of E. Since E is nowhere dense, each translate of E must also be nowhere dense. Then $\mathbb{R}$ is the countable union of nowhere dense sets, which violates the Baire Category Theorem. Hence, the statement is not necessarily true.

Problem 2022-J-II-4 (Complex Analysis). Let $U \subset \mathbb{C}$ be an open subset. Suppose $f_i : U \to \mathbb{C}$ is a sequence of holomorphic functions converging uniformly on compact subsets to a function $f : U \to \mathbb{C}$. Show that f is also holomorphic. Justify each step clearly.

It is sufficient to show that f is holomorphic on any disk $\mathbb{D} \subset \mathbb{C}$ with $\overline{\mathbb{D}} \subset \mathbb{C}$. Since $f_i \to f$ uniformly on $\mathbb{D}$, f is continuous on $\mathbb{D}$. We will use Morera's Theorem, which states the following:

> **(Morera's Theorem)** A continuous, complex-valued function f defined on an open set D in the complex plane that satisfies
>
> $$\oint_\gamma f(z)\,dz = 0$$
>
> for every closed piecewise C^1 curve γ in D must be holomorphic on D.

Let γ be a closed piecewise C^1 curve in $\mathbb{D}$. Then

$$\oint_\gamma f(z)\,dz = \oint_\gamma \lim_{i\to\infty} f_i(z)\,dz = \lim_{i\to\infty} \oint_\gamma f_i(z)\,dz = \lim_{i\to\infty}(0) = 0. \tag{91}$$

The first equality follows from the hypothesis that $f_i \to f$; the second equality follows from the fact that this convergence is uniform; and the third equality follows from the fact that each f_i is holomorphic. Therefore, f is also holomorphic.

August 2022

Problem 2022-A-I-1 (Geometry/Topology). Let $Y \subset \mathbb{R}^3$ be a surface of genus two.

Let Z be the closure of the bounded component in $\mathbb{R}^3 \setminus Y$. (Thus, Z is the compact domain-with-boundary whose interior is the region inside of Y.) Prove that the inclusion homomorphism $\pi_1(Y, \mathrm{pt}) \to \pi_1(Z, \mathrm{pt})$ is surjective. Then use this to show that the fundamental group $\pi_1(Y, \mathrm{pt})$ of the surface Y is non-abelian.

Let $X \subset Y$ be a figure-eight along the top of the surface. Then Z is a deformation retract of Z and $X \hookrightarrow Z$ induces an isomorphism of fundamental groups. Hence, the image of $\pi_1(X)$ in $\pi_1(Y)$ maps isomorphically to $\pi_1(Z)$. In particular, $\pi_1(X) \to \pi_1(Z)$ is surjective. But since $\pi_1(Z) \cong \pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$ is non-abelian. Since any homomorphic image of an abelian group is abelian, it follows that $\pi_1(Y)$ cannot be abelian.

Problem 2022-A-I-3 (Real Analysis). Show that $L^4(X,\mu) \not\subseteq L^5(X,\mu)$ if and only if there are subsets of arbitrarily small measure.

($\Rightarrow$) Suppose X contains subsets of arbitrarily small positive measure. This means that for each positive integer n, there exists a measurable subset $\widetilde{E}_n$ of X such that $\mu(\widetilde{E}_n) = 2^{-n}$. From this collection, we can obtain a sequence of disjoint sets $\{E_n\}$ such that for each n, $0 < \mu(E_n) < 2^{-n}$. Define the function

$$f = \sum_1^\infty \mu(E_n)^{-1/5} \chi_{E_n}. \tag{92}$$

We claim that $f \in L^4(X,\mu)$:

$$\begin{aligned} \|f\|_4^4 &= \int_X |f|^4 = \int_X \sum_1^\infty |\mu(E_n)|^{-4/5} \chi_{E_n} \\ &= \sum_1^\infty \mu(E_n)^{-4/5} \int_X \chi_{E_n} \\ &= \sum_1^\infty \mu(E_n)^{1/5} = \sum_1^\infty 2^{-n/5} = \frac{2^{4/5}}{2^{4/5} - 2} < \infty. \end{aligned} \tag{93}$$

On the other hand, we claim that $f \notin L^5(X,\mu)$:

$$\begin{aligned} \|f\|_5^5 &= \int_X |f|^5 = \int_X \sum_1^\infty \mu(E_n)^{-1} \chi_{E_n} \\ &= \sum_1^\infty \mu(E_n)^{-1} \int_X \chi_{E_n} \\ &= \sum_1^\infty 1 = \infty. \end{aligned} \tag{94}$$

Hence, $L^4(X,\mu) \not\subseteq L^5(X,\mu)$.
($\Leftarrow$) Now suppose that $L^4(X,\mu) \not\subseteq L^5(X,\mu)$. Let $f \in L^4 \setminus L^5$, and for each positive integer n, define $E_n = \{x \in X : |f(x)| > n\}$. We observe that

$$\infty > \|f\|_4^4 = \int_X |f|^4 \ge \int_{E_n} |f|^4 \ge \int_{E_n} n^4 = n^4 \mu(E_n) \implies \mu(E_n) \le \frac{\|f\|_4^4}{n^4} \tag{95}$$

I.e., we observe that $\mu(E_n) \to 0$ as $n \to \infty$. It suffices to show that each E_j has positive measure (combined with the previous observation, this yields the result that X has arbitrarily small *positively* measured subsets). Assume to the contrary. Suppose there exists some j_0 for which $\mu(E_{j_0}) = 0$. Let $F_{j_0} = E_{j_0}^c$. Then we observe that since $|f| \leq n$ on F_{j_0},

$$\begin{aligned} \|f\|_q^q &= \int |f|^q = \int |f|^q \chi_{F_{j_0}} \\ &= \int |f|^{q-p} |f|^p \chi_{F_{j_0}} \leq n^{q-p} \int |f|^p \chi_{F_{j_0}} \\ &=: n^{q-p} \|f\|_p^p < \infty. \end{aligned} \tag{96}$$

This shows that $f \in L^p$, which contradicts our hypothesis that $f \in L^p \setminus L^q$. Therefore, by contradiction, $\mu(E_n) \to 0$ and $\mu(E_n) > 0$ for all n.

Problem 2022-A-I-4 (Geometry/Topology). Find the fundamental group of the space of unordered pairs of distinct points of S^n.

Let X be the space of unordered pairs of distinct points of S^n. We claim that X is homotopy equivalent to the real projective space $\mathbb{RP}^n$. Indeed the projective space of dimension n can be identified to the space of unordered pairs of antipodal points of S^n, and there is a deformation retraction of the whole space to this subspace: draw through the pair of distinct points a line and take the parallel line through the center of S^n. This means that $\pi_1(X) \cong \pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$, $\cong \mathbb{Z}$ for $n = 1$, and $\cong \{0\}$ for $n = 0$.

Problem 2022-A-I-5 (Complex Analysis). If $\Omega \subset \mathbb{C}$ is simply connected and $f : \Omega \to \mathbb{C} \setminus \{0\}$ is a holomorphic function, is there a holomorphic function $g : \Omega \to \mathbb{C}$ such that $f = \exp(g)$ on Ω? Prove or give a counterexample.

Yes, there exists a function $g : \Omega \to \mathbb{C}$ such that $f = \exp(g)$. Consider the function $f'/f : \Omega \to \mathbb{C}$, which is holomorphic since f is zero-free. Fix $z_0 \in \Omega$, and define

$$h(z) = \int_{\gamma|_z} \frac{f'(\zeta)}{f(\zeta)} d\zeta, \qquad z \in \Omega, \tag{97}$$

where $\gamma|_z$ is a smooth curve connecting z_0 to z; since Ω is simply connected, Cauchy's Theorem tells us that $h(z)$ is independent of the choice of curve $\gamma|_z$. Then, we note that

$$h'(z) = \frac{f'(z)}{f(z)}. \tag{98}$$

So consider the function $f \exp(-h)$. Then

$$\frac{d}{dz} f \exp(-h) = \exp(-h)(f' - h'f) = 0, \tag{99}$$

so that $f \exp(-h)$ is a non-zero constant c. So we must have $f = c\exp(h)$. Let $g = h + C$ where $e^C = c$. Then

$$\exp(g) = \exp(h + C) = e^C \exp(h) = c \exp(h) = f. \tag{100}$$

The proof concludes. The main ideas are simple connectedness and Cauchy's Theorem, and the fact that for every holomorphic function f, there exists a primitive holomorphic function g such that $g'(z) = f(z)$. But the existence of a primitive function holds only on simply connected regions.

Problem 2022-A-II-I (Real Analysis). Suppose $E \subset \mathbb{R}^2$ has positive Lebesgue area. Show that E contains 3 points that form the vertices of an equilateral triangle.

Let $E \subset \mathbb{R}^2$ be a set of positive Lebesgue measure (we will denote by m^2 the Lebesgue measure on $\mathbb{R}^2$). Let $\{v_1, v_2\}$ be a collection of unit vectors in $\mathbb{R}^2$ so that the angle between v_1 and v_2 is 120°, and let $\beta < 1/3$. By inner regularity of the Lebesgue measure, there exists a compact set $K_1 \subset E$ so that

$m^2(K_1) > 0$. Then by *outer* regularity of the Lebesgue measure, there exists an open set U containing K_1 such that $m^2(U) \leq (1+\beta)m^2(K_1)$.
Since K_1 is compact, $d_1 = d(K_1, U^c)$ is positive; so let $R = d_1$, pick an arbitrary $r \in (0, R)$, and consider the set $K_1 + rv_1$. $K_1 + rv_1$ has to be contained within U since otherwise,

$$d(K_1, U^c) < |rv_1| = r < d_1, \text{ which is a contradiction.} \tag{101}$$

Hence, $K_1 \cup (K_1 + rv_1) \subset U$, which means

$$m^2(U) \geq m^2(K_1 \cup (K_1+rv_1)) = m^2(K_1)+m^2(K_1+rv_1)-m^2(K_1 \cap (K_1+rv_1)) = 2m^2(K_1)-m^2(K_1 \cap (K_1+rv_1)), \tag{102}$$

where the last equality follows from translation invariance of the Lebesgue measure. Hence, $m^2(K_1 \cap (K_1+rv_1)) = 2m^2(K_1) - m^2(U) \geq (1-\beta)m^2(K_1) > 0$. Therefore, $K_2 \coloneqq K_1 \cap (K_1 + rv_1)$ is nonempty. Now define $K_3 = K_2 \cap (K_2 + rv_2)$. Using the same reasoning as above, we observe that $K_3 \neq \varnothing$ and $K_3 \subset K_2$. Hence, we obtain a nested sequence of sets $\varnothing \neq K_3 \subset K_2 \subset K_1 \subset E$. Let $M \in K_3$. Since $K_3 = K_2 \cap (K_2 + rv_1)$, $N = q - rv_2 \in K_2$. Likewise, $O = q - rv_2 - rv_1 \in K_1$. These three points form the vertices of a triangle. Then since

$$\|M - N\| = r, \qquad \|N - O\| = r, \qquad \|M - O\| = \|r(v_2 + v_1)\| = r\,\|v_2 + v_1\| = r. \tag{103}$$

Problem 2022-A-II-4 (Algebra). Let G be a finite group in which $(ab)^p = a^p b^p$ for every $a, b \in G$, where p is a prime dividing $|G|$. Prove that the Sylow p-subgroup of G is normal in G (and is in fact unique).

Let G be a finite group in which $(ab)^p = a^p b^p$ for every $a, b \in G$, where p is a prime dividing $|G|$. Consider the map $\varphi : G \to G$ defined by $\varphi(g) = g^p$. This map is a homomorphism since for any $g, h \in G$,

$$\varphi(gh) = (gh)^p = g^p h^p = \varphi(g)\varphi(h), \tag{104}$$

where the second equality follows from the hypothesis. Consider the map

$$\varphi^k \coloneqq \underbrace{\varphi \circ \cdots \circ \varphi}_{k \text{ copies}}, \tag{105}$$

which must also be a homomorphism since the composition of homomorphisms is a homomorphism. The kernel of φ^k consists exactly of those elements $x \in G$ whose order is a power of p (i.e., $x^{p^r} = 1$ for some positive integer r) since

$$\varphi^k(x) = x^{p^k} = x^{p^{r+(k-r)}} = \left(x^{p^r}\right)^{p^{k-r}} = 1^{p^{k-r}} = 1. \tag{106}$$

Hence, since every element with order equal to some order of p belongs in a Sylow p-subgroup of G,

$$\ker \varphi^k = \bigcup_{P \in \mathrm{Syl}_p(G)} P. \tag{107}$$

Moreover, $\ker \varphi^k$ must be a p-subgroup of G since if not, there exists a prime $p' \neq p$ dividing $|\ker \varphi^k|$, which means by Cauchy's Theorem that $\ker \varphi^k$ contains an element of order p' (which is impossible). Hence, since $\ker \varphi^k$ is a p-subgroup of G containing a Sylow p-subgroup, by maximality of Sylow p-subgroups, $\ker \varphi^k$ must be a Sylow p-subgroup of G. Hence, G has a unique Sylow p-subgroup. And since kernels of homomorphisms are normal subgroups, this Sylow p-subgroup must be normal.

Problem 2022-A-II-5. If $f : [-1, 2] \to \mathbb{R}$ is continuous and increasing, show that the set of $x \in [0, 1]$ where

$$\int_0^1 \frac{f(x+t) - f(x-t)}{t}\, dt = \infty,$$

has Lebesgue zero measure.

Since f is continuous and increasing on $[0,1]$, we must have

$$f(x) = a + \int_0^x d\mu(t), \tag{108}$$

where μ is a non-atomic measure. Therefore, $f'(x)$ exists and is finite for Lebesgue almost every x. Therefore, for almost every x, there exists a finite $M < \infty$ so that

$$|f(x+t) - f(x)| \le M\,|t|, \tag{109}$$

which means

$$\left|\int_0^1 \frac{f(x+t)-f(x-t)}{t}\,dt\right| \le \int_0^1 \frac{|f(x+t)-f(x-t)|}{t} dt \le \int_0^1 M\,dt < \infty \tag{110}$$

for Lebesgue almost x.

Problem 2021-A-II-6 (Real Analysis). Suppose μ is a finite positive measure of compact support and

$$\int_{\mathbb{R}} x^n d\mu(x) = 0$$

for every $n \in \{0,1,2,\ldots\}$. Show that μ is the zero measure.

Suppose μ is a finite positive measure of compact support and that

$$\int_{\mathbb{R}} x^n\,d\mu(x) = 0, \qquad \forall n \in \{0,1,2,\ldots\}. \tag{111}$$

Let E be the support of μ. Our strategy is to show that $\int_{\mathbb{R}} f\,d\mu(x) = 0$ for all continuous functions f on $\mathbb{R}$, for which we shall use the Stone-Weierstraß theorem:

> **(Stone-Weierstraß Theorem)** Let X be a compact Hausdorff space, $C(X,\mathbb{R})$ the space of all continuous functions on X. Suppose $\mathcal{B}$ is a subalgebra of $C(X,\mathbb{R})$ that separates points. If there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{B}$, then $\mathcal{B}$ is dense in $\{f \in C(X,\mathbb{R}) : f(x_0) = 0\}$. Otherwise, $\mathcal{B}$ is dense in $C(X,\mathbb{R})$.

Since $\mathbb{R}$ is Hausdorff, E is Hausdorff; by hypothesis, E is compact. Hence, the Stone-Weierstrass theorem is applicable for our case. Let $\mathcal{B}$ be the subalgebra of $C(E,\mathbb{R})$ that separates points. Then by the theorem, $\mathcal{B}$ is dense in $C(E,\mathbb{R})$, which means that any continuous function can be uniformly approximated by a sequence of polynomials in $\mathcal{B}$. Let $f \in C(E,\mathbb{R})$ be arbitrary, and consider a sequence $\{p_j(x)\}_1^\infty$ that uniformly converges to f. Then

$$\int_{\mathbb{R}} f(x)\,d\mu(x) = \int_E f(x)\,d\mu(x) = \int_E \lim_{j\to\infty} p_j(x)\,d\mu(x) = \lim_{j\to\infty}\int_E p_j(x)\,d\mu(x) = \lim_{j\to\infty}\left[\sum_{k=1}^{\deg p_j} a_k \int_E x^k\,d\mu(x)\right] = 0, \tag{112}$$

where the last equality follows from the hypothesis. Hence, since for *every* continuous function f, the integral over $\mathbb{R}$ with respect to μ is zero, we conclude that μ has to be the zero measure.

January 2021

Problem 2021-J-1-3 (Real Analysis). If $E \subset \mathbb{R}$ is Lebesgue measurable and $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz, then show that $f(E)$ is also Lebesgue measurable.

August 2021

Problem 2021-A-I-2 (Complex Analysis). Let $P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$. Prove that either there is $|z| = 1$ such that $|P(z)| > 1$ or $P(z) = z^n$.

This problem requires the Cauchy Integral Formula, which states the following:

> **(Cauchy's Integral Formula)** Let U be an open subset of the complex plane $\mathbb{C}$, and suppose the closed disk $\mathbb{D}$ defined as
> $$\mathbb{D} = \{z \in \mathbb{C} : |z - z_0| \leq r\} \tag{113}$$
> is completely contained in U. Let $f : U \to \mathbb{C}$ be a holomorphic function. Then for every $a \in \operatorname{int} \mathbb{D}$,
> $$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\partial \mathbb{D}} \frac{f(z)}{(z-a)^{n+1}}\, dz. \tag{114}$$

Returning to our original problem, assume that $|P(z)| \leq 1$ for all $|z| = 1$. Then by the Cauchy Integral Formula,

$$n! = P^{(n)}(0) = \frac{n!}{2\pi i} \oint_{|z|=1} \frac{P(z)}{z^{n+1}}\, dz = \tag{115}$$

[!! Complete Later !!]

Problem 2021-A-I-6 (Geometry/Topology). What connected spaces can be finitely-sheeted covering spaces of a sphere with three handles?

We claim that the finitely-sheeted covering spaces of a sphere with three handles are exactly the closed orientable connected surfaces of genus of the form $2k + 1$ for some positive integer k. Let M be a k-sheeted covering space of a sphere with three handles. If M were nonorientable, then since covering maps are local diffeomorphisms and local diffeomorphisms preserve orientability, the sphere with three handles must also be nonorientable, which is a contradiction. Hence, M has to be orientable. Next, since M is a k-sheeted covering space of the sphere with three handles, which has Euler characteristic $2 - 2(3) = -4$, we must have

$$2 - 2g_M = \chi(M) = -4k \implies g_M - 1 = 2k \implies g_M = 2k + 1. \tag{116}$$

Problem 2021-A-II-1 (Geometry/Topology). Let M be a compact manifold (without boundary) and $\pi : M \to S^1$ a submersion onto the circle. Show that the de Rham group $H^1_{\mathrm{dr}}(M) \neq 0$.

Let M be a compact manifold (without boundary) and $\pi : M \to S^1$ a submersion onto the circle. Assume to the contrary that $H^1_{\mathrm{dr}}(M) = 0$ which means that every closed form on M is an exact form. Since $H^1_{\mathrm{dr}}(S^1) \cong \mathbb{R}$, let $[\omega]$ be a generator of this cohomology group, where ω is a nowhere vanishing closed 1-form on S^1. Since π is a submersion, the 1-form $\pi^*\omega$ must also be a nowhere vanishing closed form on M. By our hypothesis on the de Rham cohomology group in degree one of M, $\pi^*\omega$ is exact, which means there exists a smooth function f such that $\pi^*\omega = df$. Since M is compact and f is smooth, f must attain either a maximum or minimum value at some $p_0 \in M$. This means that $df_{p_0} = 0$. But this contradicts our claim that $\pi^*\omega$ is nowhere vanishing. Hence, by contradiction, $H^1_{\mathrm{dr}}(M) \neq 0$.

January 2020

Problem 2020-J-I-1 (Algebra). Let G be a finite non-abelian group, and let $Z(G)$ denote its center. Prove that $|Z(G)| \leq \frac{1}{4}|G|$, and then give an example where equality holds.

Let G be a finite non-abelian group, and let $Z(G)$ denote its center. Assume to the contrary that $|Z(G)| > \frac{1}{4}|G| \implies |G|/|Z(G)| < 4$. Since $|Z(G)| \mid |G|$, $|G|/|Z(G)|$ is a positive integer. Therefore, one of the three must necessarily be true: (1) $|G|/|Z(G)| = 1$, (2) $|G|/|Z(G)| = 2$, (3) $|G|/|Z(G)| = 3$. If (1) were true, then since $|Z(G)| = |G|$, G has to be abelian, which contradicts our hypothesis. If (2) were true, then $G/Z \cong \mathbb{Z}/2\mathbb{Z}$ which is cyclic. Hence, G would have to be abelian, which is a contradiction. Finally, if (3) were true, then $G/Z \cong \mathbb{Z}/3\mathbb{Z}$ which is cyclic. Hence, G would have to be abelian, which ia a contradiction. Hence, $|Z(G)| \not> \frac{1}{4}|G|$, which means $|Z(G)| \leq \frac{1}{4}|G|$.

Problem 2020-J-I-4 (Geometry/Topology). Let θ be a closed smooth 1-form on a compact C^∞ manifold M with empty boundary, and let v be a smooth vector field on M. Prove that the Lie derivative $\mathcal{L}_v\theta$ vanishes at some point of M.

Let θ be a closed smooth 1-form on a compact C^∞ manifold M with empty boundary, and let v be a smooth vector field on M. By Cartan's Formula for the Lie derivative,

$$\mathcal{L}_v\theta = i_v(d\theta) + d(i_v\theta), \tag{117}$$

where $i_v(\cdot)$ denotes the interior product. Since θ is a closed 1-form, $d\theta = 0$. So $\mathcal{L}_v\theta = d(i_v\theta)$. Since θ is a 1-form, $i_v\theta$ is a 0-form on M, i.e., a smooth function on M. Since M is compact, $i_v\theta$ must attain a extrema at some point in M, which means that its differential $d(i_v\theta)$ must vanish where it achieves its maximum or minimum. This then implies that $\mathcal{L}_v\theta$ vanishes at this point.

August 2020

Problem 2020-A-II-1 (Complex Analysis). How many roots (counted with multiplicity) does the function

$$g(z) = 6z^3 + e^z + 1$$

have in the unit disk $|z| < 1$?

Let $g(z) = 6z^3 + e^z + 1$, which is holomorphic. Let $f(z) = 6z^3$ and $h(z) = e^z + 1$. Then on the unit circle $|z| = 1$,

$$\begin{aligned} |h(z)| &\leq |e^z| + 1 \leq e^{|z|} + 1 \\ &\leq e + 1 \\ &< 6 = 6|z|^3 = |f(z)|. \end{aligned} \tag{118}$$

Hence, by Rouché's Formula, $g(z)$ has the same number of zeros as $f(z)$. Counting multiplicity, $f(z)$ has three solutions in the unit disk, which means that $g(z)$ also has three solutions in the unit disk.

Problem 2020-A-II-4 (Geometry/Topology). Let M and N be compact connected orientable smooth manifolds and let $f : M \to N$ be a smooth mapping. Recall the degree of f is the integral

$$\deg(f) = \int_M f^*\omega$$

over M of the pullback $f^*\omega$ of any top-degree smooth form ω on N whose integral over N is one. Recall the degree is an integer, denote it by $\deg(f)$. Now consider the map

$$f_\# : \pi_1(M) \to \pi_1(N)$$

on fundamental groups induced by f. Suppose that the image of $f_\#$ has finite index, $\operatorname{ind}(f)$. Prove that $\operatorname{ind}(f)$ divides $\deg(f)$.

Let M, N be compact connected orientable smooth manifolds and let $f : M \to N$ be a smooth mapping. Suppose that $H := f_\#(\pi_1(M))$ is a subgroup of $\pi_1(N)$ of finite index k. This means there exists a k-sheeted covering $p : \widetilde{N} \to N$ so that $p_\#(\pi_1(\widetilde{N})) = H$. By the lifting criterion for coverings, f lifts to a smooth map

$$\widetilde{f} : M \to \widetilde{N} \tag{119}$$

such that $f = p \circ \widetilde{f}$. Let ω be a top-degree smooth form on N whose integral over N is one. Since $p : \widetilde{N} \to N$ is a k-sheeted covering of orientable manifolds, we must have $\deg(p) = k$. Therefore,

$$\deg(f) = \deg(p \circ \widetilde{f}) = \deg(p)\deg(\widetilde{f}) = \operatorname{ind}(f) \cdot \deg(\widetilde{f}). \tag{120}$$

Since $\deg(\widetilde{f})$ is an integer, we conclude that $\operatorname{ind}(f) \mid \deg(f)$.

Problem 2020-J-I-2 (Geometry/Topology). Let M and N be smooth compact connected oriented n-manifolds without boundary. Suppose that $\pi_1(M)$ is finite, but that $\pi_1(M)$ is infinite. Prove that every smooth map $\Psi : M \to N$ has degree zero.

Problem 2020-A-II-6 (Real Analysis). Let $g : \mathbb{R} \to \mathbb{R}$ be a strictly decreasing function.

(a) Prove there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)) = g(x), \qquad \text{for all } x \in \mathbb{R}.$$

(b) Show that there exists a function $f : [-1, 1] \to [-1, 1]$ with finitely many points of discontinuity such that

$$f(f(x)) = -x \qquad \text{for all } x \in [-1, 1].$$

(a) Let $g : \mathbb{R}$ be a strictly decreasing function, and assume to the contrary that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = g(x)$ for all $x \in \mathbb{R}$. Then $f(f(x))$ must be strictly decreasing as well. Suppose f is strictly decreasing. Then for any $x_1 < x_2$, $f(x_1) > f(x_2) \implies f(f(x_1)) < f(f(x_2))$ so that $f \circ f$ is strictly increasing. Suppose f is strictly increasing. Then for any $x_1 < x_2$, $f(x_1) < f(x_2)$ so that $f(f(x_1)) < f(f(x_2))$, so that f is again strictly increasing. Hence, we run into a contradiction, which means that such a continuous function f cannot exist.

(b) We explicitly construct such a function:

$$f(x) = \begin{cases} x - 0.5, & x \in [-0.5, 0] \cup [0.5, 1] \\ x + 0.5, & x \in [-1, -0.5] \cup [0, 0.5]. \end{cases} \tag{121}$$

It is easily checked that this function satisfies the desired properties.

January 2019

Problem 2019-J-I-1 (Algebra). Let A and B be $n \times n$ invertible matrices over complex numbers, satisfying
$$AB = \lambda BA \text{ for some } \lambda \in \mathbb{C}.$$
Prove that A^n and B commute.

Let A and B be $n \times n$ invertible matrices over complex numbers so that $AB = \lambda BA$ for some $\lambda \in \mathbb{C}$. Since A is invertible, left-multiplying both sides by A^{-1} yields,

$$B = \lambda A^{-1} BA. \tag{122}$$

So taking the determinant, we obtain:

$$\det B = \lambda^n \det A^{-1} \det B \det A = \lambda^n \det A^{-1} \det B \det A = \lambda^n \det B. \tag{123}$$

Since B is invertible, $\det B \neq 0$, which means that $\lambda^n = 1$ (i.e., λ is an n^{th} root of unity). Now, we claim that for any $m \in \mathbb{N}$, $A^m B = \lambda^m BA^m$. By hypothesis, this claim is true for the base case $m = 1$. Suppose the claim is true for some $m \geq 1$. Then

$$A^{m+1} B = A(A^m B) = \lambda^m (ABA^m) = \lambda^m (\lambda BA) A^m = \lambda^{m+1} BA^{m+1}. \tag{124}$$

Therefore, the claim is true by induction. This implies that

$$A^n B = \lambda^n BA^n = BA^n, \tag{125}$$

so that A^n and B commute.

Problem 2019-J-II-5. Let G be a finite group, and let H be a non-normal subgroup of G of index n. Show that if $|H|$ is divisible by a prime $p \geq n$, then G is not simple.

Let G be a finite group, H a non-normal subgroup of G of index n such that $|H|$ is divisible by a prime $p \geq n$. Let G act on the set of left cosets of H; this induces a group homomorphism $\varphi : G \to S_n$. Consider the kernel of this group action, $K = \ker \varphi$. *If* $K = G$, then for every $g \in G$, $gHg^{-1} = H$, which implies that H is a normal subgroup of G – a contradiction. Hence, $\ker \varphi$ is a proper normal subgroup of G. Likewise, $\ker \varphi \neq H$ since this equality also forces H to be normal. All that remains is to show that $\ker \varphi$ is not trivial. Since $p \mid |H|$, let P be a Sylow p-subgroup of H. [!! Complete Later !!]

August 2018

Problem 2018-A-II-3 (Analysis). Suppose E, F are two measurable subsets of the real numbers that both have positive measure. Prove that $E + F = \{x + y : x \in E, y \in F\}$ contains an interval.

Problem 2018-A-I-3 (Complex Analysis). Show that if $c > 1$, then the function
$$f(z) = ze^{c-z} - 1$$
has precisely one root in $\Delta = \{|z| < 1\}$, and this root is real and positive.

Let $c > 1$, and $f(z) = ze^{c-z} - 1$. Let $g(z) = ze^{c-z}$ and $h(z) = 1$. On $\partial\Delta$,

$$|h(z)| = 1 < e^{c-1} = |g(z)|, \tag{126}$$

so that by Rouché's Theorem, $f(z)$ has the same number of roots as $g(z)$. Since the exponential has no roots but the function z has one root inside the unit disk, we conclude that $f(z)$ has precisely one root in Δ. Now consider the real-valued function $\widetilde{f}(x) = xe^{c-z} - 1$ obtained by restricting $f(z)$ to the real line. We observe that

$$f(0) = -1 < 0 \qquad \text{and} \qquad f(1) = e^{c-1} - 1 > 0. \tag{127}$$

Since $f(z)$ is continuous, it follows from the intermediate value theorem that $f(x)$ must have a root inside the interval $(0,1)$. Such a root must necessarily be real and positive. Hence, the proof concludes.

January 2017

Problem 2017-J-I-1 (Geometry/Topology). Let Σ_1 be a torus and let Σ_2 be a genus-2 surface. Show that there is no submersion from Σ_2 to Σ_1.

Let Σ_1 be a torus and Σ_2 be a genus-2 surface. We begin with a second modification to the Comps Lemma. Assume to the contrary that F is a submersion from Σ_2 to Σ_1. By the second modification to the Comps Lemma, $F : \Sigma_2 \to \Sigma_1$ must be a k-sheeted covering map for some finite $k > 0$. This implies that $\chi(\Sigma_2) = k \cdot \chi(\Sigma_1)$, where $\chi(\cdot) = 2 - 2g$ denotes the Euler characteristic of a closed surface of genus g. But this is impossible since $\chi(\Sigma_2) = -2 < 0 = k \cdot 0 = k \cdot \chi(\Sigma_1)$. Hence, by contradiction, there cannot be any submersions from Σ_2 to Σ_1.

Problem 2017-J-I-6 (Geometry/Topology). Let M be a smooth 4-manifold, let ϕ be a 3-form on M, and let $U \subset M$ be the open set of points where $\varphi \neq 0$. Show that φ is closed if and only if, near any $p \in U$, one can find a smooth coordinate system (x^1, x^2, x^3, x^4) in which

$$\varphi = dx^1 \wedge dx^2 \wedge dx^3.$$

Assume the hypotheses of the problem. Recall that φ is closed if and only if $d\varphi$ is identically zero. Let $p \in U$ and suppose that we can find a smooth coordinate system (x^1, x^2, x^3, x^4) in some neighborhood of p in U so that $\varphi = dx^1 \wedge dx^2 \wedge dx^3$. Then $d\varphi_p = d^2x^1 \wedge dx^2 \wedge dx^3 + \cdots + dx^1 \wedge dx^2 \wedge d^2x^3 = 0$. Since this is true for all $p \in U$, we conclude that $d\varphi$ is identically zero on M, and hence φ is closed.
Now assume that φ is closed, which means that $\varphi \wedge d\varphi$ is identically zero. At each point $p \in U$, define

$$D_p = \ker \varphi_p,$$

which is Frobenius integrable by our previous observation. In particular, D_p is a 1-dimensional distribution. Since L is integrable, we can find smooth coordinates $(x^1, \ldots, x^4)$ near p such that $D_p = \operatorname{span}\{\partial_{x^4}\}$. Since φ annihilates ∂_{x^4}, it must be a linear combination of dx^1, dx^2, and dx^3. Suppose $\varphi = f dx^1 \wedge dx^2 \wedge dx^3$. Then

$$0 = d\varphi = f_{x^1} dx^1 \wedge dx^1 \wedge \cdots \wedge dx^3 + f_{x^2} dx^2 \wedge dx^1 \wedge \cdots \wedge dx^3 + \cdots + f_{x^4} \wedge dx^1 \wedge \cdots \wedge dx^4. \tag{128}$$

The first three terms are all zero. The last term is zero iff $f_{x^4} = 0$, which means $f = f(x^1, x^2, x^3)$. [!! Complete Later !!]

Problem 2017-J-II-1. Let $f : M \to \mathbb{R}$ be a smooth function on a smooth manifold M. In an arbitrary smooth local coordinate chart $x : U \to \mathbb{R}^n$ of M, define

$$\mathcal{D}f := \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

Does $\mathcal{D}f$ give a well-defined vector field on M?

We claim that $\mathcal{D}f$ does not give a well-defined vector field on M. Let $(U,(x^i))$ and $(V,(\widetilde{x}^i))$ denote two overlapping smooth local coordinate charts on M, and let $p \in U \cap V$. Then

$$\begin{aligned} \mathcal{D}f &= \frac{\partial f}{\partial x^i}(p)\frac{\partial}{\partial x^i}\bigg|_p \\ &= \frac{\partial f}{\partial \widetilde{x}^j}\frac{\partial \widetilde{x}^j}{\partial x^i}\bigg|_p \frac{\partial \widetilde{x}^k}{\partial x^i}\bigg|_p \frac{\partial}{\partial \widetilde{x}^k}\bigg|_{\hat{p}}, \end{aligned} \tag{129}$$

which is identically not equal to $(\partial_{\widetilde{x}^k} f)\partial_{\widetilde{x}^k}$, which is the expression for $\mathcal{D}f$ in the smooth coordinate chart $(V,(\widetilde{x}^j))$.

Problem 2017-J-II-2 (Real Analysis). Suppose $f : [0,1] \to \mathbb{R}$ is measurable. Suppose further that for all $g \in L^2([0,1])$, we have that $fg \in L^2([0,1])$. Show that f is in $L^\infty([0,1])$.

Let $f : [0,1] \to \mathbb{R}$ be measurable, and suppose that for all $g \in L^2([0,1])$, $fg \in L^2([0,1])$. Assume to the contrary that $f \notin L^\infty([0,1])$, which means that for every positive integer n, the set

$$E_n = \{x : |f_n(x)| \geq n\} \tag{130}$$

has positive measure. Consider the simple function

$$g = \sum_1^\infty \frac{1}{n\sqrt{m(E_n)}}\chi_{E_n} \tag{131}$$

so that

$$\|g\|_2^2 = \int_0^1 \sum_1^\infty \frac{1}{n^2 m(E_n)}\chi_{E_n} = \sum_1^\infty \frac{1}{n^2} < \infty. \tag{132}$$

On the other hand

$$\|fg\|_2^2 = \int_0^1 |f|^2 \sum_1^\infty \frac{1}{n^2 m(E_n)}\chi_{E_n} \geq \sum_1^\infty \int_{E_n} \frac{1}{m(E_n)} = \sum_1^\infty 1 > \infty, \tag{133}$$

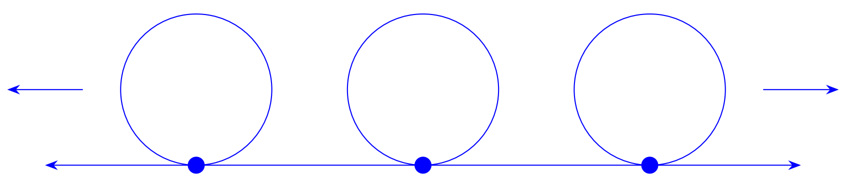
which means $fg \notin L^2$. This is a contradiction. Hence, by contradiction, $f \in L^\infty([0,1])$.

Problem 2017-J-II-4 (Geometry/Topology). Let $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$ be the "wedge" of two real projective planes, meaning the quotient space obtained from the disjoint union $\mathbb{RP}^2 \sqcup \mathbb{RP}^2$ by identifying a single point p in one copy of $\mathbb{RP}^2$ with a single point $\widetilde{p}$ in the other. (1) What is the fundamental group of X? (2) Give a concrete description of the universal cover $\widetilde{X}$ of X, accompanied by a drawing of X.

(1) Let $X = \mathbb{RP}^2 \vee \mathbb{RP}^2$. Since the fundamental group of the wedge product of two manifolds corresponds to the free product of the fundamental groups of each manifold, and the fundamental group of $\mathbb{RP}^2$ is $\mathbb{Z}/2\mathbb{Z}$, we conclude that

$$\pi_1(X) \cong (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}). \tag{134}$$

(2) Recall that S^2 is the universal covering of $\mathbb{RP}^2$; a base point in $\mathbb{RP}^2$ has two preimages in S^2. To obtain $\widetilde{X}$, wedge another sphere to each of the two preimages of the base point. Then wedge another sphere to each the new preimages of the base point, and keep repeating. $\widetilde{X}$ resembles the following diagram:



August 2017

Problem 2017-A-I-1 (Geometry/Topology). Let M be a smooth compact connected n-manifold (without boundary), and let $F : M \to \mathbb{R}^n$ be a smooth map. Does F necessarily have a critical point?

Let M be a smooth compact connected n-manifold (without boundary), and let $F : M \to \mathbb{R}^n$ be a smooth map. Suppose F has no critical points, which means that dF_p is surjective at every $p \in M$. I.e., $\operatorname{rank} dF_p = n$ for every $p \in M$. Let $F = (f_1, \dots, f_n)$, where each $f_j : M \to \mathbb{R}$ is a component function of F. Fix some f_j; since M is compact, f_j must attain a maximum or minimum at some point $p \in M$. This means that $df_j(p) = 0$. But since $dF_p = (df_1(p), \dots, df_j(p), \dots, df_n(p))$, $\operatorname{rank} dF_p \neq n$, which is a contradiction. Hence, F must have a critical point.

Problem 2017-A-II-1 (Real Analysis). Suppose that $f \in C([0,1])$ is a continuous real-valued function on $[0,1]$ for which

$$\int_0^1 x^n f(x) dx = 0$$

for all non-negative integers n. Does it follow that $f(x) = 0$ for all $x \in [0,1]$.

It is necessarily true that $f(x) = 0$ for all $x \in [0,1]$. By the Stone-Weierstraß Theorem, one can show that the hypothesis forces

$$\int_0^1 f(x)g(x) dx = 0, \tag{135}$$

for all continuous functions $g(x)$ on $[0,1]$. Taking $g(x) = f(x)$ so that $f(x)g(x) = f(x)^2$, we observe that

$$\int_0^1 f^2(x) dx = 0 \implies f(x)^2 = 0 \implies f(x) = 0. \tag{136}$$

Hence, the proof concludes.

Problem 2017-A-II-3 (Algebra). Let K denote the splitting field of $f(x) = x^4 + x^2 + 1$ over $\mathbb{Q}$. Compute the Galois group $\operatorname{Gal}(K/\mathbb{Q})$.

Let $f(x) = x^4 + x^2 + 1$; by the rational root test, $f(x)$ has no rational roots. However,

$$f(x) = x^4 + x + 1 = (x^2 + x + 1)(x^2 - x + 1), \tag{137}$$

where each quadratic factor is irreducible by the rational root test. The roots of these quadratic factors are

$$x = \pm\sqrt{\frac{-1 \pm \sqrt{-3}}{2}}. \tag{138}$$

Let $\alpha = \sqrt{\frac{-1+\sqrt{-3}}{2}}$ and $\beta = \sqrt{\frac{-1-\sqrt{-3}}{2}}$. We observe then that $\alpha^2\beta^2 = 1 \implies \beta = \pm\frac{1}{\alpha}$. On the other hand,

$$\alpha^2 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}, \tag{139}$$

so that $\alpha \in \mathbb{Q}(\sqrt{-3})$. Hence, we conclude that the splitting field of $f(x)$ over $\mathbb{Q}$ is $K = \mathbb{Q}(\sqrt{-3})$. Since the minimal polynomial of $\sqrt{-3}$ over $\mathbb{Q}$ has degree 2, $[\mathbb{Q}(\sqrt{3}) : \mathbb{Q}] = 2$. Hence, the Galois group $\operatorname{Gal}(K/\mathbb{Q})$ has order 2, which means $\operatorname{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$.

January 2013

Problem 2013-J-II-6 (Geometry/Topology). Let M be a smooth compact manifold, and suppose that there is a smooth map $F : M \to S^1$ whose derivative is non-zero at every point. Prove that the de Rham cohomology space $H^1_{\mathrm{dr}}(M)$ is non-zero.

Let M be a smooth compact manifold, and $F : M \to S^1$ a smooth map whose derivative is non-zero at every point. Assume to the contrary that the de Rham cohomology space $H^1_{\text{dr}}(M) = 0$, which means that every closed 1-form on M is exact. Since $H^1_{\text{dr}}(S^1) \cong \mathbb{R}$, there exists a nowhere vanishing closed 1-form ω on S^1 such that its equivalence class generates $H^1_{\text{dr}}(S^1)$. Then since F is a smooth map, $F^*\omega$ is a closed 1-form on M. Since $H^1_{\text{dr}}(M) = 0$, $F^*\omega$ is an exact form; i.e., there exists a smooth function: $f : M \to \mathbb{R}$ such that $F^*\omega = df$. Since f is smooth and M is compact, f must have a maximum or minimum at some point $p \in M$, which implies that $df_p = 0$ at $p \in M$. Therefore, $0 = (F^*\omega)_p = \omega_{F(p)} \circ dF_p$. Since ω is nowhere vanishing, we conclude that $dF_p = 0$. But this contradicts our assumption that dF is non-zero at every point. Hence, by contradiction, $H^1_{\text{dr}}(M) \neq 0$.

August 2013

Problem 2013-A-II-4 (Geometry/Topology). Let θ be a smooth 1-form on a manifold M such that $\theta \neq 0$ everywhere. Let $D \subset TM$ be the vector subbundle defined by

$$D = \ker\theta = \{v \in TM : \theta(v) = 0\}.$$

Prove that D is Frobenius integrable if and only if $\theta \wedge d\theta = 0$ everywhere.

Assume the hypotheses of the problem. We recall that D is Frobenius integrable if and only if for any pair of smooth sections X, Y of D, $[X, Y]$ is a smooth section of D. So let X, Y be smooth sections of D, which means that $\theta(X) = \theta(Y) = 0$ everywhere. Suppose that D is Frobenius integrable so that $\theta([X, Y]) = 0$. Since θ is not identically zero, for any $p \in M$, there exists a vector R_p with $\theta_p(R_p) = 1$. This means that locally one can choose a smooth vector field R with $\theta(R) = 1$. On this neighborhood, we have $T_pM = \mathbb{R}R_p \oplus D_p$. Now, we note that

$$\theta \wedge d\theta(X, Y, R) = \theta(X)d\theta(Y, R) + \theta(Y)d\theta(R, X) + \theta(R)d\theta(X, Y). \tag{140}$$

The first two terms are identically zero by our hypothesis. For the latter, we note that

$$d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]), \tag{141}$$

which is identically zero. Hence, this means that $\theta \wedge d\theta(R, X, Y)$ is zero. This means that $(\theta \wedge d\theta)_p = 0$ for all $p \in M$. Hence, $\theta \wedge d\theta$ is identically zero. Now suppose $\theta \wedge d\theta$ is identically zero. Let X, Y be smooth sections of D and pick a local vector field R such that $\theta(R) = 1$. We recover once again that

$$0 = \theta \wedge d\theta(X, Y, R) = -\theta(R)\theta([X, Y]) \implies \theta([X, Y]) = 0. \tag{142}$$

Hence, $[X, Y] \in \Gamma(D)$, which means that D is Frobenius integrable.

Problem 2013-J-II-5 (Real Analysis). Let $E \subset [0, 1]$ be a measurable set. Assume E has positive Lebesgue measure. Show that there are α and β such that all three numbers $\alpha, \alpha + \beta, \alpha + 2\beta \in E$.

Let $E \subset [0, 1]$ be measurable set with positive Lebesgue measure, and $\varepsilon < 1/3$. By inner regularity of the Lebesgue measure, there exists a compact set $K_1 \subset E$ so that $m(K_1) > 0$. By outer regularity of the Lebesgue measure, there exists an open set $U \supset K_1$ so that

$$m(U) \leq (1 + \varepsilon)m(K_1). \tag{143}$$

Since K_1 is compact, the quantity $D = d(K_1, U^c) > 0$. So let $R = D/2$, and pick an arbitrary $\beta \in (0, D/2)$. We first claim that $K_1 + \beta \subset U$, since if not, then

$$d(K_1, U^c) < \beta = \frac{D}{2} < D, \tag{144}$$

which is a contradiction. In particular, this means that $K_1 \cup (K_1 + \beta) \subset U$ so that

$$m(U) \geq m(K_1 \cup (K_1 + \beta)) = m(K_1) + m(K_1 + \beta) - m(K_1 \cap (K_1 + \beta)). \tag{145}$$

By translation invariance of the Lebesgue measure, $m(K_1) = m(K_1 + \beta)$ so that

$$m(K_1 \cap (K_1 + \beta)) \geq 2m(K_1) - m(U) \geq 2m(K_1) - (1 + \varepsilon)m(K_1) = (1 - \varepsilon)m(K_1). \tag{146}$$

Since $\varepsilon < 1$, we conclude that $m(K_1 \cap (K_1 + \beta)) > 0$ so that $K_1 \cap (K_1 + \beta) \neq \varnothing$. Now for $j = 1, 2$, define $K_{j+1} = K_j \cap (K_j + \beta)$. Generalizing the arguments from above, we see that $K_j + \beta \subset U$ for $j = 1, 2$ and $m(K_{j+1}) \geq (1 - \varepsilon(2^j - 1))m(K_1) > 0$ so that K_1, K_2, K_3 are nonempty. Hence, this produces a nested sequence of nonempty sets $\varnothing \neq K_3 \subset K_2 \subset K_1 \subset E$. Let $q \in K_3$ be arbitrary; since $K_3 = K_2 \cap (K_2 + \beta)$, $q - \beta \in K_2$. And since $K_2 = K_1 \cap (K_1 + \beta)$, $q - \beta - \beta = q - 2\beta \in K_1$, Let $\alpha = q - 2\beta$. This proves that $\{\alpha, \alpha + \beta, \alpha + 2\beta\} \subset E$, concluding the proof.

Textbook Problems

Problem Lee-7-5. Let M be a smooth compact manifold. Show that there is no submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.

Let M be a smooth compact manifold, and assume to the contrary that there exists a submersion $F : M \to \mathbb{R}^k$ for some $k > 0$. Since M is compact, F must attain either a maximum or minimum at some point $p \in M$, which means that $dF_p = 0$. But this is impossible since F is a submersion, which means that $\operatorname{rank} dF_p = \dim \mathbb{R}^k = k > 0$. Hence, by contradiction, F cannot be a submersion.

Problem D&F-14.6.2. Determine the Galois groups of the following polynomials:

(i) $x^3 - x^2 - 4$
(ii) $x^3 - 2x + 4$
(iii) $x^3 - x + 1$
(iv) $x^3 + x^2 - 2x - 1$.

(a) Let $f(x) = x^3 - x^2 - 4$. We note that f has a rational root $x = 2$ since $2^3 - 2^2 - 4 = 8 - 4 - 4 = 0$. Using polynomial long division, we find that $f(x)$ is reducible over $\mathbb{Q}$ as the product

$$f(x) = (x - 2)(x^2 + x + 2). \tag{147}$$

By the rational root test, the quadratic factor is irreducible and has complex roots

$$x_{1,2} = \frac{-1 \pm \sqrt{-7}}{2}. \tag{148}$$

Therefore, the splitting field of $f(x)$ is $\mathbb{Q}(\sqrt{-7})$, which has degree 2 since the minimal polynomial of $\sqrt{-7}$ is $x^2 + 7$. Therefore, the Galois group $\mathrm{Gal}(\mathbb{Q}(\sqrt{-7})/\mathbb{Q})$ has order 2; hence the Galois group is $\mathbb{Z}/2\mathbb{Z}$.

(b) Let $f(x) = x^3 - 2x + 4$. We note that $f(x)$ has a rational root $x = -2$ since $(-2)^3 - 2(-2) + 4 = -8 + 4 + 4 = 0$. Hence using polynomial long division,

$$f(x) = (x + 2)(x^2 - 2x + 2). \tag{149}$$

By the rational root test, $x^2 - 2x + 2$ is irreducible over $\mathbb{Q}$ with complex roots $1 \pm i$. Therefore, the splitting field of $f(x)$ is $\mathbb{Q}(i)$, which has degree 2 since the minimal polynomial of i is $x^2 + 1$. Therefore, the Galois group $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ has order 2; hence the Galois group is $\mathbb{Z}/2\mathbb{Z}$.

(c) Let $f(x) = x^3 - x + 1$; by the rational root test $f(x)$ is irreducible over $\mathbb{Q}$. However, since f is already a depressed cubic, we note that its discriminant is $-4p^3 - 27q^2 = 4 - 27 = -23$. Since -23 is not a perfect square, we conclude that the Galois group is S_3. In fact, the splitting field for this cubic is $\mathbb{Q}(\alpha, \sqrt{-23})$, where α is a root of $x^3 - x + 1$.

(d) Let $f(x) = x^3 + x^2 - 2x - 1$; by the rational root test $f(x)$ is irreducible over $\mathbb{Q}$. Therefore, we will now depress the cubic. Let $x = y - 1/3$. Then

$$x^3 + x^2 - 2x - 1 = y^3 - \frac{7}{3}y - \frac{7}{27}. \tag{150}$$

The discriminant of the depressed cubic is,

$$D = -4p^3 - 27q^2 = 4\left(\frac{7^3}{27}\right) - 27\left(\frac{7^2}{27^2}\right) = \frac{7^2}{27}(4 \cdot 7 - 1) = 7^2. \tag{151}$$

Since the discriminant is a square, we see that the Galois group of the polynomial is A_3.

Problem D&F-14.6.4. Determine the Galois group of $x^4 - 25$.

Let $f(x) = x^4 - 25$. The roots of $f(x)$ are $\zeta_4^0\sqrt[4]{25}$, $\zeta_4^1\sqrt[4]{25}$, $\zeta_4^2\sqrt[4]{25}$, and $\zeta_4^3\sqrt[4]{25}$, where ζ_4 is the primitive 4th root of unity. Here, we recall that the automorphisms in the Galois group of f act transitively on the roots of $f(x)$. Hence, the Galois group of $f(x)$ must contain the automorphism that maps $\sqrt[4]{25} \mapsto -\sqrt[4]{25}$ (i.e., a reflection) and $\sqrt[4]{25} \mapsto \zeta_4^j\sqrt[4]{25}$ (i.e., a rotation). Hence, the Galois group is D_8.

Problem D&F-14.6.5. Determine the Galois group of $x^4 + 4$.

Let $f(x) = x^4 + 4$, which is irreducible over $\mathbb{Q}$. However, the four roots of $f(x)$ are $\pm 1 \pm i$. This means that the splitting field of $f(x)$ is $\mathbb{Q}(i)$, which is a degree 2 extension over $\mathbb{Q}$. Hence, the Galois group is of order 2, which implies that the Galois group is the cyclic group $\mathbb{Z}/2\mathbb{Z}$.

Problem MAT532-F-4. Suppose $E \subset \mathbb{R}^2$ is Lebesgue measurable. For a square Q, let C_Q be the *white* squares of a (8×8) checkerboard fitted exactly in Q (so a white square has sidelength 1/8 the sidelength of Q). Suppose that for almost any $x \in E$, and any square Q_x with x in its lower left corner, we have that $E \cap C_{Q_x} = \varnothing$, i.e., E does not intersect the white squares of a checkerboard fitted to Q_x. Show $m(E) = 0$, where m is Lebesgue measure.

Let $E \subset \mathbb{R}^2$ be Lebesgue measurable, and set $A = \{x \in E : E \cap C_{Q_x} = \varnothing \text{ for any square } Q_x\}$; by hypothesis, A consists of almost every $x \in E$. Assume to the contrary that $m(E) \neq 0$ and pick $x \in A$. For this x, construct a family of sets $\{E_r\}_{r>0}$ as follows: for each r, let E_r be a square of sidelength $r/\sqrt{2}$ with x in its lower left corner. It is straightforward to see that for every $r > 0$, $E_r \subset B(x,r)$ and $m(E_r) = 2\pi^{-1}m(B(r,x))$. Hence, $\{E_r\}$ shrinks nicely to x. Now, by hypothesis, $m(E \cap E_r) \leq \frac{1}{2}m(E_r)$ for every r since E intersects at most half of E_r. This means that

$$\limsup_{r\to 0} \frac{m(E \cap E_r)}{m(E_r)} \leq \frac{1}{2}. \tag{152}$$

I.e., for almost every $x \in E$, the Lebesgue density is at most $1/2$, which contradicts the Lebesgue Density Theorem. Therefore, by contradiction, $m(E) = 0$.

Problem MAT532-7-4. Suppose a set $E \subset \mathbb{R}^3$ satisfies that for every $x \in \mathbb{R}^3$ and $r > 0$, there exists a point $z \in B(x,r)$ such that $E \cap B(z, r/2) \cap B(x, 2r) = \varnothing$. Show that $m(E) = 0$, where m is the Lebesgue measure on $\mathbb{R}^3$.

[!! Complete Later !!]

Problem (Algebra-Classification-I). Classify all groups of order 2026.

Let G be a group of order $2026 = 2 \cdot 1013$. By Sylow's Theorem, G must contain a normal Sylow 5-subgroup, which we denote by H. Let K be a Sylow 2-subgroup of G; note $K \cong \mathbb{Z}_2$. By Lagrange's Theorem, H and K must intersect trivially. Moreover, $|HK| = |H||K|/|H \cap K| = |H||K| = |G|$, so that $G = HK$. Hence, by the recognition theorem for semidirect products, $G \cong H \rtimes_\varphi \mathbb{Z}_2$, where $\varphi \in \operatorname{Aut} H = \mathbb{Z}_{1013}^* \cong \mathbb{Z}_{1012}$. So we look for homomorphisms $\varphi : \mathbb{Z}_2 \to \mathbb{Z}_{1012}$; each homomorphism is completely determined by where the generator 1 is mapped to.

(i) Consider the map $1 \mapsto 0$, which corresponds to the trivial homomorphism. Then the semidirect product is just the direct product, and so $G \cong \mathbb{Z}_{1013} \times \mathbb{Z}_2$.

(ii) Consider the map $\varphi : 1 \mapsto 506$, where 506 is the unique element of $\mathbb{Z}_{1012}$ with order 2. This is a non-trivial homomorphism with kernel $\{0\}$. Hence, this gives a non-abelian group $\mathbb{Z}_{1013} \rtimes_\varphi \mathbb{Z}_2$.

Hence, up to isomorphism, there are only two groups of order 2026.

Problem (Algebra-Classification-II). Classify all groups of order 1969.

Let G be a group of order $1969 = 11 \cdot 179$. By Sylow's Theorem, G must contain a normal Sylow 179-subgroup, which we denote by H. Let K be a Sylow 11-subgroup of G; note $K \cong \mathbb{Z}_{11}$. By Langrage's Theorem, H and K must intersect trivially and $G = HK$. Therefore, $G = H \rtimes_{\varphi} K$ for some $\varphi \in \operatorname{Aut} H = \mathbb{Z}_{179}^* \cong \mathbb{Z}_{178}$. So we look for homomorphisms $\varphi : \mathbb{Z}_{11} \to \mathbb{Z}_{178}$; each homomorphism is completely determine by where the generator 1 is mapped to.

(i) Consider the map $1 \mapsto 0$. This corresponds to the trivial homomorphism so that the semidirect product is just the direct product. Therefore, $G \cong \mathbb{Z}_{179} \times \mathbb{Z}_{11} \cong \mathbb{Z}_{1969}$ (by the Chinese Remainder Theorem).

(ii) Since 1 has order 11, 1 must map to some nonzero element of $\mathbb{Z}_{178}$ of order 11; but since 11 and 178 are relatively prime, there exists no such element.

Hence, we conclude that there is exactly one group of order 1969, which is precisely $\mathbb{Z}_{1969}$.

Problem 2008-J-I-3 (Algebra). Classify all groups of order 28.

Let G be a group of rder $28 = 2^2 \cdot 7$. By Sylow's Theorem, G contains a normal Sylow 7-subgroup, which we denote by H. Let K be a Sylow 2-subgroup, which has order 4. By Lagrange's Theorem, H and K must intersect trivially and $G = HK$. Hence, by the recognition theorem for semidirect products, $G = H \rtimes_{\varphi} K$ for some $\varphi \in \operatorname{Aut}(H) = \mathbb{Z}_7^* \cong \mathbb{Z}_6$. So we look for homomorphisms $\varphi : K \to \mathbb{Z}_6$, where K is a group of order 4. Up to isomorphism, there are precisely two groups of order 4: (1) $\mathbb{Z}_4$, and (2) $\mathbb{Z}_2 \times \mathbb{Z}_2$. We consider each case separately:

(I) Consider the case $K = \mathbb{Z}_4$, which has two generators: 1 and 3. Each homomorphism $\varphi : \mathbb{Z}_4 \to \mathbb{Z}_6$ is determined by where φ sends a generator with the constrain that 1 may be sent to only those elements of $\mathbb{Z}_6$ whose order divides 4 (namely $0, 3$).

  (i) Suppose $\varphi_1 : 1 \mapsto 0$. Then since $\varphi(3) = 3 \cdot \varphi(1) = 0$, φ is the trivial homomorphism. In this case, the semidirect product is the direct product and G is isomorphic to the abelian group $\mathbb{Z}_7 \times \mathbb{Z}_4$.

  (ii) Suppose $\varphi_2 : 1 \mapsto 3$. Then this is a nontrivial homomorphism with image consisting of $\{0, 3\}$ and kernel consisting of $\{0, 2\}$. Hence, this produces a non-abelian group $\mathbb{Z}_7 \rtimes_{\varphi_2} \mathbb{Z}_4$.

(II) Now consider the case $K = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$. $\psi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_6$ is determined uniquely by $\psi(a)$ and $\psi(b)$ provided that its order divides 2. This means $\psi(a), \psi(b) \in \{0, 3\}$.

  (i) Suppose $\psi_1(a) = \psi_1(b) = 0$. The semidirect is then a direct product and so $G \cong \mathbb{Z}_7 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_{14} \times \mathbb{Z}_2$.

  (ii) Suppose $\psi_2(a) = 0$ and $\psi_2(b) = 3$. This is a nontrivial homomorphism so that $G \cong \mathbb{Z}_7 \rtimes_{\psi_2} \mathbb{Z}_2^2$ is non-abelian.

  (iii) Suppose $\psi_3(a) = 3$ and $\psi_3(b) = 0$. Then $\psi_3 = \psi_2 \circ \theta$ where θ is the automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by $\theta(a) = b$ and $\theta(b) = a$. Hence, this semidirect product gives the same group as in case (ii).

  (iv) Suppose $\psi_4(a) = \psi_4(b) = 3$. Then $\psi_4 = \psi_3 \circ \theta$ where θ is the automorphism of $\mathbb{Z}_2 \times \mathbb{Z}_2$ given by $\theta(a) = a$ and $\theta(b) = ab$. Hence, this semidirect product gives the same group as in case (iii).

Altogether, we conclude that there are exactly four isomorphism classes of groups of order 28, namely $\mathbb{Z}_7 \times \mathbb{Z}_4$, $\mathbb{Z}_7 \rtimes_{\varphi_2} \mathbb{Z}_4$, $\mathbb{Z}_{14} \times \mathbb{Z}_2$, and $\mathbb{Z}_7 \rtimes_{\psi_2} \mathbb{Z}_2^2$, of which exactly two are abelian.

Problem 2010-J-II-5 (Algebra). Classify (up to isomorphism) all groups of order 45.

Let G be a group of order $45 = 3^2 \cdot 5$. By Sylow's Theorem, G has a normal Sylow 5-subgroup, which we denote by H. Let K denote a Sylow 3-subgroup of G, which has order 9. By Lagrange's Theorem, H, K intersect trivially and $|G| = |H||K|$ so that $G = HK$. Hence, $G \cong H \rtimes_{\varphi} K$ for some $\varphi \in \operatorname{Aut}(H) \cong \mathbb{Z}_5^* \cong \mathbb{Z}_4$. Hence, we look at homomorphisms $\varphi : K \to \mathbb{Z}_4$. There are exactly two groups of order 9, up to isomorphism; namely, these are $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$. Hence, we consider each separately.

(I) Let $K = \mathbb{Z}_9$, which has generators 1, 2, 4, 5, 7, and 8. Each homomorphism $\varphi : K \to \mathbb{Z}_4$ is determined uniquely by where φ sends a generator with the constraint that they may only be sent to those elements of $\mathbb{Z}_4$ whose order divides 9. There is only one such element, namely 0. Hence, the only group we get is the direct product $\mathbb{Z}_9 \times \mathbb{Z}_5 \cong \mathbb{Z}_{45}$, which is abelian.

(II) Let $K = \mathbb{Z}_3 \times \mathbb{Z}_3$. Each $\psi : \mathbb{Z}_3 \times \mathbb{Z}_3 = \langle a \rangle \times \langle b \rangle \to \mathbb{Z}_4$ is uniquely determined by $\psi(a)$ and $\psi(b)$ provided they divide 3. But there is only one such element in $\mathbb{Z}_4$, which is zero. Hence, we only get the direct product $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \cong \mathbb{Z}_3 \times \mathbb{Z}_{15}$, which is abelian.

Therefore, we find that (1) there are exactly two groups, up to isomorphism, of order 45; and (2) both groups are abelian.

Problem 2003-J-I-6 (Algebra).
(a) Prove that a group of order p^2, where p is a prime number, is abelian.
(b) Classify groups of order p^2 up to isomorphism.

(a) Let G be a group of order p^2, and let $Z(G)$ be its center. By Lagrange's Theorem, $|Z(G)| \in \{1, p, p^2\}$. If $|Z(G)| = p^2$ and so $G = Z(G)$, which means G is abelian. $|Z(G)| \neq p$ since otherwise $|G/Z(G)| = p$ forcing G/Z to be cyclic and G to be abelian (which contradicts $Z(G)$ being a proper subgroup of G). Finally $|Z(G)|$ cannot be one, since the center of any p-group must necessarily be nontrivial (by the class equation). Hence, $Z(G) = G$, which means G is abelian.

(b) Since every group of order p^2 must necessarily be abelian, up to isomorphism, there must be exactly two groups, namely $\mathbb{Z}_p \times \mathbb{Z}_p$ and $\mathbb{Z}_{p^2}$.

Problem 2010-J-I-5 (Algebra). Consider the following irreducible polynomial over $\mathbb{Q}$: $p(x) = x^4 - 3x^2 - 1$.
(a) Describe the splitting field of $p(x)$.
(b) Consider the Galois group of $p(x)$. Compute its order and determine if it is abelian.

(a) Let $p(x) = x^4 - 3x^2 - 1$. By the rational root test, $p(x)$ has no roots over $\mathbb{Q}$. Moreover, it is straightforward to check that $p(x)$ is not the product of irreducible quadratics with rational coefficients. Hence, $p(x)$ is irreducible over $\mathbb{Q}$. We start by finding the roots of $p(x)$; let $u = x^2$. Then

$$u^2 - 3u - 1 = 0 \implies u = \frac{3 \pm \sqrt{13}}{2} \implies x = \pm\sqrt{\frac{3 \pm \sqrt{13}}{2}}. \tag{153}$$

Let

$$\alpha = \sqrt{\frac{3 + \sqrt{13}}{2}}, \qquad \beta = \sqrt{\frac{3 - \sqrt{13}}{2}}. \tag{154}$$

Observe that $\alpha^2\beta^2 = -1$ so that $\beta = \pm\frac{i}{\alpha}$. Therefore, the splitting field of $p(x)$ is

$$\mathbb{Q}(\alpha, i). \tag{155}$$

Observe that the minimal polynomial of i is x^2+1, which is irreducible over $\mathbb{Q}(\alpha)$ so that $[\mathbb{Q}(\alpha, i) : \mathbb{Q}(\alpha)] = 2$. On the other hand, the minimal polynomial of α is a degree 4 polynomial so that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$. Hence, by the tower law, $[\mathbb{Q}(\alpha, i) : \mathbb{Q}] = 8$.

(b) By the last work in (a), the order of the Galois group of $p(x)$ is 8. Now, we will determine the Galois group of $p(x)$. Recall that elements of $\mathrm{Gal}(\mathbb{Q}(\alpha, i)/\mathbb{Q})$ are automorphisms φ of the field $\mathbb{Q}(\alpha, i)$ with the constraints that: (1) φ fixes $\mathbb{Q}$, (2) $\varphi(\alpha)$ must be another root of the minimal polynomial of α over $\mathbb{Q}$, and (3) $\varphi(i)$ must be another root of $x^2 + 1$. We will explicitly work through each of the elements.

(i) $\sigma : i \mapsto -i, \alpha \mapsto \alpha$. This permutation has order 2 since $\sigma^2(\alpha) = \sigma(\alpha) = \alpha$ and $\sigma^2(i) = \sigma(-i) = i$.
(ii) $\tau : i \mapsto i, \alpha \mapsto -\alpha$. Once again, this permutation has order 2.

(iii) $\rho : i \mapsto -i, \alpha \mapsto \beta = \frac{i}{\alpha}$. To compute the order of this permutation, observe that

$$\rho^2(\alpha) = \rho(i\alpha^{-1}) = (-i) \cdot \frac{1}{i/\alpha} = -\alpha \implies \rho^4(\alpha) = \rho^2(-\alpha) = \alpha. \tag{156}$$

Likewise, $\rho^4(i) = \rho^2(i) = i$. Hence, ρ has order 4.

Now, consider the three elements given above. We compute

$$\sigma\rho\sigma(i) = \sigma\rho(-i) = \sigma(i) = -i = \rho^{-1}(i). \tag{157}$$

Likewise,

$$\sigma\rho\sigma(\alpha) = \sigma\rho(\alpha) = \sigma(i)\sigma(\alpha)^{-1} = -\frac{i}{\alpha} = \rho^{-1}(\alpha). \tag{158}$$

Therefore, $\sigma\rho\sigma = \rho^{-1}$. Hence,

$$\mathrm{Gal}(\mathbb{Q}(\alpha, i)/\mathbb{Q}) = \{1, \sigma, \rho, \rho^2, \rho^3, \sigma\rho, \sigma\rho^2, \sigma\rho^3\} \cong D_8. \tag{159}$$

Since the dihedral group is not abelian, we conclude that the Galois group for $p(x)$ is non-abelian.

Problem 2015-A-II-5 (Algebra). Find the splitting field and the Galois group of the polynomial $x^4 - 5x^2 + 5$ over $\mathbb{Q}$.

Let $p(x) = x^4 - 5x^2 + 5$. By the rational root test, $p(x)$ has no rational roots. Moreover, it is straightforward to see that $p(x)$ is not expressible as the product of irreducible quadratics. Hence, $p(x)$ is irreducible over $\mathbb{Q}$. We find its four complex roots as follows. Let $u = x^2$. Then

$$u^2 - 5u + 5 = 0 \implies u = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{5 \pm \sqrt{5}}{2} \implies x = \pm\sqrt{\frac{5 \pm \sqrt{5}}{2}}. \tag{160}$$

Let

$$\alpha := \sqrt{\frac{5 + \sqrt{5}}{2}}, \qquad \beta := \sqrt{\frac{5 - \sqrt{5}}{2}}. \tag{161}$$

We observe that

$$\alpha^2 = \frac{5}{2} + \frac{\sqrt{5}}{2} \qquad \text{and} \qquad \alpha^2\beta^2 = 5 \implies \beta = \pm\frac{5}{\alpha}. \tag{162}$$

Therefore, the splitting field is $\mathbb{Q}(\sqrt{5}, \alpha)$. Since the minimal polynomial of $\sqrt{5}$ over $\mathbb{Q}$ is $x^2 - 5$, which has degree 2, $[\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2$. On the other hand, the minimal polynomial of α over $\mathbb{Q}(\sqrt{5})$ is

$$x^2 - \frac{5 + \sqrt{5}}{2}, \tag{163}$$

so that $[\mathbb{Q}(\sqrt{5}, \alpha) : \mathbb{Q}] = 2$. Hence, by the Tower Law, $[\mathbb{Q}(\sqrt{5}, \alpha) : \mathbb{Q}] = 4$, which means that the corresponding Galois group has order 4; there are two groups, up to isomorphism, of order 4 (namely $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$). The elements of $\mathrm{Gal}(\mathbb{Q}(\sqrt{5}, \alpha)/\mathbb{Q})$ are precisely the automorphisms on $\mathbb{Q}(\sqrt{5}, \alpha)$ that fix $\mathbb{Q}$ such that the automorphism group acts transitively on the roots. Consider the permutation $\rho : \alpha \mapsto -\beta = -\frac{5}{\alpha}$ and $\rho : \sqrt{5} \mapsto -\sqrt{5}$. We observe that

$$\begin{aligned} \rho^2(\sqrt{5}) &= \rho(-\sqrt{5}) = \sqrt{5}. \\ \rho^2(\alpha) &= \rho\left(-\frac{5}{\alpha}\right) = -5\rho(\alpha)^{-1} = \alpha \\ \implies \rho^3(\alpha) &= \rho(\alpha) = -5\alpha^{-1} \\ \implies \rho^4(\alpha) &= -5\rho(\alpha)^{-1} = -5 \cdot \left(-\frac{\alpha}{5}\right) = \alpha. \end{aligned} \tag{164}$$

I.e., ρ is an element of order 4. Therefore, since only $\mathbb{Z}_4$ has an element of order 4, we conclude that $\mathrm{Gal}(\sqrt{5}, \alpha)/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$.

Problem 2003-A-II-4 (Algebra). Let E be a splitting field of $f(x) = x^3 + x^2 - 2x - 1$ over the field of rational numbers $\mathbb{Q}$. Find the Galois group of $E/\mathbb{Q}$. (Hint: first prove that $f(x) : f(x^2 - 2)$.) This was the exact notation used in the problem...

Let $f(x) = x^3 + x^2 - 2x - 1$. By the rational root test, $f(x)$ has no rational roots and hence is irreducible over $\mathbb{Q}$ (being a polynomial of degree 3). Consider the substitution $x = y - 1/3$:

$$f(y) = y^3 - \frac{7}{3}y - \frac{7}{27}. \tag{165}$$

The discriminant of this depressed cubic is

$$D = -4p^3 - 27q^2 = 4\left(\frac{7^3}{27}\right) - 27\left(\frac{7^2}{27^2}\right) = 7^2\left(\frac{28}{27} - \frac{1}{27}\right) = 7^2. \tag{166}$$

Since the discriminant is a perfect square, we conclude that the Galois group is A_3.

Problem 2014-J-I-5 (Algebra). Let K denote the splitting field for $(x^5 - 1)(x^3 - 2)$ over the rational numbers $\mathbb{Q}$. Compute the cardinality of the Galois group G for the extension $\mathbb{Q} \subset K$, and show that G is not abelian.

Let K denote the splitting field for $(x^5-1)(x^3-2)$. We note that the splitting field for x^3-2 is $\mathbb{Q}(\sqrt[3]{(2)}, \zeta_3)$, where ζ_3 is the primitive 3rd root of unity, and the splitting field for $x^5 - 1$ is $\mathbb{Q}(\zeta_5)$, where ζ_5 is the primitive 5th root of unity. Now, since 3 and 5 are relatively prime, the 3rd primitive roots of unity cannot be expressed as a linear combination of 5th roots of unity. Likewise, $\sqrt[3]{2} \notin \mathbb{Q}(\zeta_5)$. Hence, $\mathbb{Q}(\sqrt[3]{2}, \zeta_3) \cap \mathbb{Q}(\zeta_5) = \mathbb{Q}$, which means that

$$\mathrm{Gal}(K/\mathbb{Q}) \cong \mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}). \tag{167}$$

From this, we see that the order of G is 24. Now consider $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q})$. The corresponding minimal polynomial is $x^3 - 2$, which is a depressed cubic. Since its discriminant is -108, which is not a square, we conclude that $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}) \cong S_3$, which is not abelian. Hence, we conclude that G is not abelian.

Problem 2003-J-I-5 (Algebra). Let $f(x) = x^5 - 2$. Find generators and relations for the Galois group $G \coloneqq \mathrm{Gal}(F/\mathbb{Q})$ of the splitting field F of $f(x)$ over the rational numbers $\mathbb{Q}$.

Let $f(x) = x^5 - 2$, which has no roots in $\mathbb{Q}$ by the rational root test. It is also straightforward to check that $x^5 - 2$ cannot be written as the product of an irreducible cubic and irreducible quadratic so that $f(x)$ is indeed irreducible over $\mathbb{Q}$. The roots of this polynomial are $\sqrt[5]{2}, \zeta_5\sqrt[5]{2}, \ldots, \zeta_5^4\sqrt[5]{2}$, where ζ_5 is the primitive 5th root of unity. Therefore, the splitting field F of $f(x)$ must contain the field $\mathbb{Q}(\sqrt[5]{2}, \zeta_5)$. On the other hand, each of the roots mentioned above lie in this field so that $F = \mathbb{Q}(\sqrt[5]{2}, \zeta_5)$. Moreover, it follows that $[F : \mathbb{Q}] = 5 \cdot 4 = 20$ so that G is a group of order 20. [!! Complete Later !!]

Problem 2014-J-II-2 (Algebra). Let H denote a normal subgroup of the finite group G. If P denotes a Sylow p-subgroup of H, then prove that $G = N(P; G)H$ (where $N(P; G)$ denotes the normalizer of P in G).

Let H denote a normal subgroup of the finite group G, and let P be a Sylow p-subgroup of H. Let $N_G(P) \coloneqq N(P; G)$ denote the normalizer of P in G. Since H is normal, $KH \leq G$ for any $K \leq G$. In particular, $N_G(P)H \leq G$. Hence, it suffices to show that $G \leq N_G(P)H$. Since $P \leq H$ and H is a normal subgroup of G, for any $g \in G$, $gPg^{-1} \leq gHg^{-1} = H$ so that gPg^{-1} is another Sylow p-subgroup of H. On the other hand, we also know that all Sylow p-subgroups of H are conjugate by elements of H. Hence, for each $g \in G$, there exists a corresponing $h \in H$ such that

$$hPh^{-1} = gPg^{-1} \implies (g^{-1}h)P(g^{-1}h)^{-1} = P. \tag{168}$$

I.e, $g^{-1}h \in N_G(P)$, which means $g \in HN_G(P) = N_G(P)H$, where the equality stems from $N_G(P)H$ being a subgroup of G. Hence, since g was arbitrary, $G \subseteq N_G(P)H$, which concludes the proof.

Problem 2012-J-II-6 (Real Analysis). Let $(V, \|\cdot\|)$ be a normed vector space. Assume that for every sequence $\{x_n\}_1^\infty$ in V with $\sum_1^\infty \|x_n\| < \infty$, the sequence of partial sums $\left\{\sum_1^N x_n\right\}_1^\infty$ is convergent in V. Prove that V is complete.

Let $(V, \|\cdot\|)$ be a normed vector space so that every absolutely convergent series is convergent. Let $\{x_n\}_1^\infty$ be a Cauchy sequence in V. This means that we can find an increasing sequence of positive integers $n_1 < n_2 < \cdots$ such that for each j and $n, m > n_j$,

$$\|x_n - x_m\| < 2^{-j}. \tag{169}$$

Let $y_1 = x_{n_1}$, and $y_j = x_{n_j} - x_{n_{j-1}}$ for each $j > 1$. Then $\sum_1^k y_{n_j} = x_{n_k}$. Then we observe that

$$\sum_1^\infty \|y_j\| \le \|y_1\| + \sum_2^\infty \|y_j\| \le \|y_1\| + \frac{1}{2} < \infty. \tag{170}$$

Hence, by the hypothesis on V, $\sum_1^N y_n$ converges to $\sum_1^\infty y_n$. But this means that $\{x_{n_j}\}$ converges in V. Since $\{x_n\}$ is Cauchy, it follows that the sequence converges to the same limit. Hence, V is complete.

Problem 2006-A-II-4 (Complex Analysis). Show that $z^7 - 4z^3 + z - 1$ has 3 zeros inside the unit circle (counted with multiplicity.)

Let $f(z) = -4z^3$, and $g(z) = z^7 + z - 1$, both of which are holomorphic functions. For all $|z| = 1$, we observe that

$$\begin{aligned} |g(z)| &\le |z|^7 + |z| + 1 \\ &= 1 + 1 + 1 = 3 \\ &\le 4 = 4|z|^3 = |f(z)|. \end{aligned} \tag{171}$$

Therefore, by Rouché's Theorem, $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside the unit circle (counted with multiplicity). $f(z)$ has 3 zeros (being a degree 3 polynomial), and $f(z) + g(z) = z^7 - 4z^3 + z - 1$. Hence, this concludes the claim.

Problem 2012-A-I-4 (Complex Analysis). Find, with proof, the precise number of zeros of the complex polynomial $p(z) = z^9 - 2z^6 + z^2 - 8z + 2$ inside the annulus $1 < |z| < 2$.

Let $p(z) = z^9 - 2z^6 + z^2 - 8z + 2$. The number of solutions to $p(z)$ inside the disk $\mathbb{D}_2$ of radius 2 must be the sum of the number of the number of solutions inside the unit disk $\mathbb{D}$ and the number of solutions in the annulus $1 < |z| < 2$. First, we compute the number of solutions inside $\mathbb{D}_2$.
Let $f(z) = z^9$, and $g(z) = -2z^6 + z^2 - 8z + 2$. Then on $\partial\mathbb{D}_2$,

$$\begin{aligned} |g(z)| &\le 2|z|^6 + |z|^2 + 8|z| + 2 \\ &= 2^7 + 2^2 + 16 + 2 = 2(3 + 8 + 2^6) \\ &= 2(11 + 64) = 2(75) = 150 < 2^9 = 8(64) = 512 = |f(z)|. \end{aligned} \tag{172}$$

Hence, by Rouché's Theorem, since $p(z) = f(z) + g(z)$ and $f(z)$ has nine roots, $p(z)$ has nine roots inside $\mathbb{D}_2$. Now let $f(z) = -8z$ and $g(z) = z^9 - 2z^6 + z^2 + 2$. On $\partial\mathbb{D}$,

$$\begin{aligned} |g(z)| &\le 1 + 2 + 1 + 2 = 6 \\ &< 8 = |f(z)|. \end{aligned} \tag{173}$$

Hence, by Rouché's Theorem, $p(z)$ has the same number of roots as $f(z)$ in $\mathbb{D}$, which is one. Therefore, we conclude that $p(z)$ has a total of eight solutions inside the annulus $1 < |z| < 2$.

Problem 2015-A-I-3 (Complex Analysis). Let f be a holomorphic function defined on a neighborhood of the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, and assume that $|f(z)| < 1$ for all $|z| = 1$. Determine the number of fixed points of f in $\overline{\mathbb{D}}$.

Let f be a holomorphic function defined on a neighborhood of the closed unit disk $\overline{D}$ such that $|f(z)| < 1$ for all $|z| = 1$. If z is a fixed point of f in $\overline{D}$, then z is a solution to the function $f(z) - z$. Hence, the number of fixed points of f in $\overline{\mathbb{D}}$ is equal to the number of solutions to $f(z) - z$ in $\overline{\mathbb{D}}$. Since $|f(z)| < |z|$ for all $z \in \partial\overline{\mathbb{D}}$ and z has exactly one root in $\overline{\mathbb{D}}$, we conclude that f has exactly one fixed point in $\overline{\mathbb{D}}$.

Problem 2014-A-II-5 (Real Analysis). Suppose f is a continuous function on $[0,1]$ and $\int_0^1 f(t)t^n\, dt = 0$ for every $n = 0, 1, 2, \ldots$. Prove that f is the zero function.

Let f be a continuous function on $[0,1]$, and assume that $\int_0^1 f(t)t^n\, dt = 0$ for all nonnegative integers n. We will use the Stone-Weierstraß Theorem in our argument. Let $g(x)$ be an arbitrary continuous function on $[0,1]$. Since $[0,1]$ is compact, and $\mathbb{R}$ is Hausdorff (which implies that $[0,1]$ is Hausdorff), by the Stone-Weierstraß Theorem, there exists a sequence of polynomials $\{p_j\}_1^\infty$ that converges to g uniformly. Writing

$$p_j(t) = \sum_{k=1}^{\deg p_j} a_k t^k, \tag{174}$$

we observe the following

$$\begin{aligned}
\int_0^1 f(t)g(t)\, dt &= \int_0^1 \lim_{j\to\infty} f(t)p_j(t)\, dt \\
&= \lim_{j\to\infty} \int_0^1 f(t) \sum_{k=1}^{\deg p_j} a_k t^k\, dt \\
&= \lim_{j\to\infty} \left[\sum_{k=1}^{\deg p_j} a_k \int_0^1 f(t)t^k\, dt \right] \\
&= \lim_{j\to\infty} (0) = 0,
\end{aligned} \tag{175}$$

where (1) the second equality follows from uniform convergence of the p_j to f, and (2) the final line follows from the hypothesis. Hence, since g was arbitrary, taking $g = f$ for example, forces f to be identically zero. Hence, the proof concludes.

Problem 2016-J-II-3 (Real Analysis). Suppose f is a continuous function on $[0,1]$ and $\int_0^1 f(x)x^k dx = 0$ for $k = 0, \ldots, n$. Prove that either f is identically zero of f must change signs at least $n+1$ times. (We say f changes sign $n+1$ times if there are points $0 < x_1 < \cdots < x_{n+1} < 1$ so that $f(x_j)f(x_{j+1}) < 0$ for $j = 1, \ldots, n$.)

Suppose f is a continuous function on $[0,1]$ so that $\int_0^1 f(x)x^k dx = 0$ for $k = 0, \ldots, n$. Suppose f changes sign $m \leq n$ times. By definition of "change signs", there exist m points $0 \leq x_1 < x_2 < \cdots < x_m \leq 1$ such that $f(x_j) = 0$ for $j = 1, \ldots, m$. Then the function $g(x) = \pm f(x)(x - x_1)\cdots(x - x_m)$ is a continuous nonnegative function on $[0,1]$ for some choice of $\pm$, chosen so that the sign of the polynomial $(x - x_1)\cdots(x - x_m)$ is the same as the sign of $f(x)$ on each interval $(0, x_1), \ldots, (x_{m-1}, x_m)$. Since $(x - x_1)\cdots(x - x_m)$ is a polynomial of degree $m \leq n$, expanding out the polynomial, and interchanging the sum and the integral, the hypothesis gives us that

$$\int_0^1 g(x)dx = 0. \tag{176}$$

Since $g(x) \geq 0$, the above result forces g to be identically zero. Since $(x - x_1)\cdots(x - x_m)$ is not identically zero, we conclude that $f \equiv 0$. Hence, the proof concludes.

Problem 2016-J-II-6 (Real Analysis). Let $\mathcal{M}$ be a closed linear subspace of $L^2([0,1];\mathbb{R})$ that is contained in $C([0,1];\mathbb{R})$.

(a) Prove that there exists $A > 0$ such that
$$\|f\|_u \leq A\,\|f\|_2$$
for all $f \in \mathcal{M}$. Here $\|\cdot\|_u$ is the uniform norm.

(b) Prove that $\dim \mathcal{M} \leq A^2$. (*Hint*: Show that if $\{f_j\}$ is an L^2 orthonormal basis of $\mathcal{M}$, then $\sum |f_j|^2 \leq A^2$ for all $x \in [0,1]$.)

(a) For this part of the problem, we required the Closed Graph Theorem, which states the following:

(**Closed Graph Theorem**) If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ is a closed linear map, then T is bounded, where T is a closed linear map iff the following is true: if $x_n \to x$ and $Tx_n \to y$, then $y = Tx$.

Let $\iota : \mathcal{M} \hookrightarrow C([0,1];\mathbb{R})$ be the (linear) inclusion map. Since closed linear subspaces of Banach spaces is a Banach space, $\mathcal{M}$ is a Banach space, itself. Therefore, it suffices to show that ι is closed. Suppose $f_n \to f$ in $\mathcal{M}$ (hence, $f_n \to f$ in $L^2([0,1];\mathbb{R})$) and $f_n \to g$ in $C([0,1];\mathbb{R})$ (which means that $f_n \to g$ in the supremum norm). By definition of the supremum norm, we observe that
$$0 \leq \|f_n - g\|_2^2 = \int_0^1 |f_n - g|^2\,dx \leq \int_0^1 \sup_{x\in[0,1]} |f_n - g|^2\,dx = \|f_n - g\|_u^2 \longrightarrow 0 \text{ as } n \to \infty. \tag{177}$$
Hence, $f_n \to g$ in L^2. But since $f_n \to f$ in L^2 and limits in L^2 are unique, we conclude that $f = g$ a.e. Therefore, ι is closed. By the Closed Graph Theorem, we conclude that ι is bounded.

(b) [!! Complete Later !!]

Problem 2016-A-II-2 (Algebra). Find the Galois group of the polynomial $p(x) = x^3 - 2$ over the field $\mathbb{Z}_{11} := \mathbb{Z}/11\mathbb{Z}$.

We begin by looking for roots of $p(x)$ over $\mathbb{Z}_{11}$. We find that $p(x)$ has one root, $x = 7$. Therefore, using polynomial long division in $\mathbb{Z}_{11}$, we find that
$$p(x) = (x+4)(x^2 + 7x + 5). \tag{178}$$
Then it is straightforward to verify that the quadratic $x^2 + 7x + 5$ is irreducible over $\mathbb{Z}_{11}$ since it has no roots contained in this field. Finding the roots of this quadratic and reducing it modulo 11,
$$x = -\frac{7}{2} \pm \frac{\sqrt{29}}{2} \equiv 2 \pm 6\sqrt{7} \pmod{11}. \tag{179}$$
Since $\sqrt{7} \notin \mathbb{Z}_{11}$, the splitting field of $p(x)$ over $\mathbb{Z}_{11}$ must contain the field $\mathbb{Z}_{11}(\sqrt{7})$. On the other hand, both of the aforementioned roots must lie in $\mathbb{Z}_{11}(\sqrt{7})$. Therefore, the splitting field of $p(x)$ is $\mathbb{Z}_{11}(\sqrt{7})$. Since the minimal polynomial of $\sqrt{7}$ over $\mathbb{Z}_{11}$ is $x^2 - 7$ (which is seen easily by verifying that this polynomial is irreducible over this field), it follows that $[\mathbb{Z}(\sqrt{7}) : \mathbb{Z}] = 2$. Hence, the Galois group of $p(x)$ over the field $\mathbb{Z}_{11}$ must be of order 2; since there is only one group of order 2, up to isomorphism, the Galois group must be $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$.

Problem 2016-J-I-2 (Geometry/Topology). Let $X = S^1 \times S^1$ be the 2-torus, and let $Y = X \setminus \{p\}$ be the complement of one point in the 2-torus.

(a) Prove that there is no covering map $X \to Y$.

(b) Prove that there is no covering map $Y \to X$.

(a) Assume to the contrary that $f : X \to Y$ is a covering map. Since X is compact, $f(X) = Y$ must also be compact. But this is a contradiction since Y is not compact. Therefore, f cannot be surjective and hence not a covering map.

(b) Suppose there exists a covering map $f : Y \to X$. We recall that $\pi_1(Y) \cong \mathbb{Z} * \mathbb{Z}$, while $\pi_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$. The former is a nonabelian group, which implies that its image under the induced map f_* must be nonabelian in $\pi_1(X)$, which is a contradiction since $\pi_1(X)$ is abelian. Hence, by contradiction, f cannot be a covering map.

Problem 2016-A-I-1 (Algebra). For $p(x) = x^5 + 1$ over the field $\mathbb{F}_{19}$, find the splitting field K, find the Galois group $\mathrm{Aut}(K)$, and find the action of $\mathrm{Aut}(K)$ on the 5 roots (up to labeling of the roots).

Let $p(x) = x^5 + 1$. We first note that $p(x)$ has the root $-1 \equiv 18 \pmod{19}$. If ζ is a fifth root of unity in some field extension of $\mathbb{F}_{19}$, then it follows that $-\zeta_5^n$ is a root for all n. Since $5 \mid 19^2 - 1$ but $5 \nmid 19$, it follows that $\zeta_5 \in \mathbb{F}_{19^2}$. Hence, the splitting field of $p(x)$ is $K = \mathbb{F}_{19^2}$. This means that the Galois group $\mathrm{Aut}(K)$ has order $[\mathbb{F}_{19^2} : \mathbb{F}_{19}] = 2$. Since there is only one group of order 2 up to isomorphism, we conclude that $\mathrm{Aut}(K) \cong \mathbb{Z}_2$. Hence, there is only one nontrivial automorphism, which maps each root to its inverse.

Problem 2016-A-I-1 (Algebra I). For $q(x) = x^4 - 2$ over the field $\mathbb{F}_{11}$, find the splitting field K, the Galois group $\mathrm{Aut}(K)$, and find the action of $\mathrm{Aut}(K)$ on the 5 roots (up to labeling of the roots).

Let $q(x) = x^4 - 2$ over the field $\mathbb{F}_{11}$. It is straightforward to check that $q(x)$ has no roots in $\mathbb{F}_{11}$. However, if ζ is a fourth root of unity in some field extension of $\mathbb{F}_{11}$ and $\sqrt[4]{2}$ is contained in this extension, then $\zeta_4^n \sqrt[4]{2}$ is a root for all n. Since $4 \nmid 11$ but $4 \mid 11^2 - 1$, it follows that $\zeta_4 \in \mathbb{F}_{11^2}$. Hence, the splitting field of $q(x)$ is $K = \mathbb{F}_{11^2}(\sqrt[4]{2})$. Hence,

$$|\mathrm{Aut}(K)| = [\mathbb{F}_{11^2}(\sqrt[4]{2}) : \mathbb{F}_{11}] = [\mathbb{F}_{11^2}(\sqrt[4]{2}) : \mathbb{F}_{11^2}] \cdot [\mathbb{F}_{11^2} : \mathbb{F}_{11}] = 4 \cdot 2 = 8. \tag{180}$$

I.e., the Galois group is of order 8.

Problem 2013-A-I-2 (Algebra). Find the Galois group of $x^3 - 2$ over the field $\mathbb{Z}_5$.

Let $p(x) = x^3 - 2$. We observe the following:

$$\begin{aligned} &p(0) = -2 \equiv 3 \pmod 5 \quad p(1) = -1 \equiv 4 \pmod 5 \quad p(2) = 6 \equiv 1 \pmod 5 \\ &p(3) = 25 \equiv 0 \pmod 5 \quad p(4) = 62 \equiv 2 \pmod 5. \end{aligned} \tag{181}$$

Hence, $x = 3$ is a root of $p(x)$. In $\mathbb{Z}_5$, we can write

$$p(x) = (x+2)(x^2 + 3x + 4). \tag{182}$$

It is straightforward to check that $x^2 + 3x + 4$ is irreducible over $\mathbb{Z}_5$ as it has no roots in this field. The roots of the irreducible quadratic are,

$$x = \frac{-3 \pm \sqrt{9 - 16}}{2} = \frac{-3}{2} \pm \frac{\sqrt{-7}}{2} \equiv 1 \pm 3\sqrt{3} \pmod 5. \tag{183}$$

Since 3 is not a square in $\mathbb{Z}_5$, the splitting field of $p(x)$ must contain the field $\mathbb{Z}_5(\sqrt{3})$. On the other hand, the field $\mathbb{Z}_5(\sqrt{3})$ contains both of the aforementioned roots of the irreducible quadratic. Therefore, the splitting field of $p(x)$ is $\mathbb{Z}_5(\sqrt{3})$. Since 3 is not a square in $\mathbb{Z}_5$, the polynomial $x^2 - 3$ is the minimal polynomial of $\sqrt{3}$. Since this polynomial has degree 2, $[\mathbb{Z}_5(\sqrt{3}) : \mathbb{Z}_5] = 2$. Therefore, the Galois group of $p(x)$ must be of order 2. Thus, we conclude that $\mathrm{Gal}(\mathbb{Z}_5(\sqrt{3})/\mathbb{Z}_5) \cong \mathbb{Z}_2$.

Problem 2014-A-II-1 (Algebra). Determine the Galois group of the polynomial of the polynomial group $f(x) = x^4 + 4$ over $\mathbb{Q}$.

Let $f(x) = x^4 + 4$. The roots of $f(x)$ are $\sqrt[4]{-4}, \zeta_4\sqrt[4]{-4}, \zeta_4^2\sqrt[4]{-4}$, and $\zeta_4^3\sqrt[4]{-4}$, where ζ_4 are the primitive fourth roots of unity. Observe that $\zeta_4 = e^{2\pi i/4} = i$. Moreover,

$$-4 = 4e^{i\pi} \implies \sqrt[4]{-4} = \sqrt{2}e^{i\pi/4} = \sqrt{2}\cdot\left(\frac{1+i}{\sqrt{2}}\right) = 1 + i. \tag{184}$$

This means that the splitting field of $f(x)$ must contain $\mathbb{Q}(i)$. On the other hand, $\mathbb{Q}(i)$ contains all of the roots of $f(x)$. Hence, the splitting field of $f(x)$ over $\mathbb{Q}$ is $\mathbb{Q}(i)$. Since the minimal polynomial of i over $\mathbb{Q}$ is x^2+1, which is a degree 2 polynomial, $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q}) \cong \mathbb{Z}_2$.

Problem 2011-A-II-1 (Algebra). Determine the Galois group of $x^4 - 25$ over $\mathbb{Q}$.

Let $f(x) = x^4 - 25$. The roots of $f(x)$ are $\zeta_4^j\sqrt{5}$ for $j = 0, \ldots, 3$, where $\zeta_4 = e^{2\pi i/4} = i$ is the primitive fourth root of unity. Hence, the splitting field must contain the field $\mathbb{Q}(i, \sqrt{5})$. On the other hand, the field $\mathbb{Q}(i, \sqrt{5})$ contains all of the aforementioned roots. Hence, the splitting field of $f(x)$ over $\mathbb{Q}$ is $\mathbb{Q}(i, \sqrt{5})$. Since the minimal polynomial of i over $\mathbb{Q}$ is x^2+1, $[\mathbb{Q}(i) : \mathbb{Q}] = 2$. Likewise, since the minimal polynomial of $\sqrt{5}$ over $\mathbb{Q}(i)$ is $x^2 - 5$, $[\mathbb{Q}(i, \sqrt{5}) : \mathbb{Q}(i)] = 2$. Therefore, by the tower law, $\mathrm{Gal}(\mathbb{Q}(i, \sqrt{5})/\mathbb{Q})$ is a group of order 4. Since the Galois group is the group of automorphisms of the field $\mathbb{Q}(i, \sqrt{5})$ that fixes $\mathbb{Q}$, we have the following automorphisms:

(1) e: $i \mapsto i$, $\sqrt{5} \mapsto \sqrt{5}$: This is the identity permutation.
(2) σ: $i \mapsto i$, $\sqrt{5} \mapsto -\sqrt{5}$: This is a permutation of order 2.
(3) τ: $i \mapsto -i$, $\sqrt{5} \mapsto \sqrt{5}$: This is also a permutation of order 2.
(4) ρ: $i \mapsto -i$, $\sqrt{5} \mapsto -\sqrt{5}$: This is equivalent to $\sigma\tau = \tau\sigma$.

Hence, it is straightforward to see that this group is isomorphic to the Klein-4 subgroup V of S_4. I.e., $\mathrm{Gal}(\mathbb{Q}(i, \sqrt{5})/\mathbb{Q}) \cong V$.

Problem 2005-J-I-4 (Algebra). Find the Galois group of the polynomial $p(x) = x^3 - 3x + 1$ over $\mathbb{Q}$.

By the Rational Root Test, $p(x)$ has no roots in $\mathbb{Q}$, and hence must be irreducible. Therefore, we shall examine its determinant:

$$D = -4p^3 - 27q^2 = -4(-3)^3 - 27(1)^2 = -27(-4+1) = 81. \tag{185}$$

Since 81 is a square in $\mathbb{Q}$ (namely $\sqrt{81} = 9$), we conclude that the Galois group of the polynomial over $\mathbb{Q}$ is isomorphic to A_3.

Problem 2003-J-I-5 (Algebra). Let $f(x) = x^5 - 2$. Find generators and relations for the Galois group $G := \mathrm{Gal}(F/\mathbb{Q})$ of the splitting field F of $f(x)$ over the rational numbers $\mathbb{Q}$.

Let $f(x) = x^5 - 2$, which has the roots $\zeta_5^j\sqrt[5]{2}$ for $j = 0, \ldots, 4$, where ζ_5 is the primitive fifth root of unity. The splitting field of $f(x)$ over $\mathbb{Q}$ must contain the field $\mathbb{Q}(\sqrt[5]{2}, \zeta_5)$. On the other hand, the field $\mathbb{Q}(\sqrt[5]{2}, \zeta_5)$ contains all of the roots of $f(x)$. Hence, the splitting field of the polynoomial over $\mathbb{Q}$ is exactly $\mathbb{Q}(\sqrt[5]{2}, \zeta_5)$. Now, since ζ_5^j are the roots to $x^5 - 1$ over $\mathbb{Q}$, but $x^5 - 1$ splits into the product $(x-1)(x^4 + \cdots + 1)$, the minimal polynomial of ζ_5 is a degree 4 polynomial, which means that $[\mathbb{Q}(\zeta_5) : \mathbb{Q}] = 4$. On the other hand, since $x^5 - 2$ is irreducible over $\mathbb{Q}(\zeta_5)$, $[\mathbb{Q}(\sqrt[5]{2}, \zeta_5) : \mathbb{Q}(\zeta_5)] = 5$. Hence, from the Tower Law,

$$[\mathbb{Q}(\sqrt[5]{2}, \zeta_5) : \mathbb{Q}] = 20 \implies \left|\mathrm{Gal}(\mathbb{Q}(\sqrt[5]{2}, \zeta_5))\right| = 20. \tag{186}$$

By classification of groups of order 20, G must be of the form $\mathbb{Z}_5 \rtimes_\varphi (\mathbb{Z}_2 \times \mathbb{Z}_2)$ or $\mathbb{Z}_5 \rtimes_\varphi \mathbb{Z}_4$ for some $\varphi \in \mathrm{Aut}(\mathbb{Z}_5) \cong \mathbb{Z}_4$. Since the automorphism group of $\mathbb{Q}(\zeta_5)$), which sends $\zeta_5 \to \zeta_5^j$ for some j coprime to 5, is cyclic of order 4, we deduce that $G \cong \mathbb{Z}_5 \rtimes_\varphi \mathbb{Z}_4$ for some $\varphi \in \mathrm{Aut}(\mathbb{Z}_5)$. It is straightforward to see that the permutations $\sigma : \sqrt[5]{2} \to \zeta_5 \sqrt[5]{2}$ (of order 5) and $\tau : \zeta_5 \to \zeta_5^2$ are generators of G. Moreover,

$$\tau\sigma\tau^{-1}(\zeta_5) = \tau\sigma(\zeta_5^4) = \tau(\zeta_5^4) = \zeta_5, \qquad \text{and} \qquad \tau\sigma\tau^{-1}(\sqrt[5]{2}) = \tau\sigma(\sqrt[5]{(2)}) = \zeta_5^2\sqrt[5]{2} = \sigma^2(\sqrt[5]{2}). \tag{187}$$

Hence, we obtain the relation $\tau\sigma\tau^{-1} = \sigma^2$. Therefore,

$$G = \langle \sigma, \tau : \sigma^5 = \tau^4 = 1, \tau\sigma\tau^{-1} = \sigma^2 \rangle. \tag{188}$$

Problem 2010-A-II-2 (Algebra). Find a rational number c such that the splitting field over $\mathbb{Q}$ of the cubic polynomial $x^3 + cx - 1$ has a nonabelian Galois group over $\mathbb{Q}$. For your value of c, compute the isomorphism type of the Galois group.

Let $c = 1$ so that $f(x) = x^3 + x - 1$. Since $f(1) = 1 + 1 - 1 = 1$ and $f(-1) = -1 - 1 - 1 = -3$, we conclude by the rational root test that $f(x)$ has no rational roots, and hence is irreducible over $\mathbb{Q}$. Moreover, since the discriminant of $f(x)$ is,

$$\Delta = -4p^3 - 27q^2 = -4(1)^3 - 27(-1)^2 = -4 - 27 = -31, \tag{189}$$

which is *not* a square in $\mathbb{Q}$, the Galois group of $f(x)$ over $\mathbb{Q}$ must be isomorphic to S_3, which is not abelian.

Problem 2004-A-I-4 (Algebra). Give an example of a Galois extension E of the rational numbers $\mathbb{Q}$ such that $\mathrm{Gal}(E/\mathbb{Q})$ is $\mathbb{Z}_7$ (the cyclic group of order 7).

Consider the field extension $\mathbb{Q}(\zeta_{29})$, where ζ_{29} is the primitive 29th root of unity. Since 29 is prime, we observe that

$$\mathrm{Gal}(\mathbb{Q}(\zeta_{29})/\mathbb{Q}) \cong (\mathbb{Z}/29\mathbb{Z})^*, \tag{190}$$

which is abelian and cyclic of order 28. Since $7 \mid 28$ and $4 \mid 28$, $\mathrm{Gal}(\mathbb{Q}(\zeta_{29})/\mathbb{Q})$ has exactly one subgroup of order 7 and one subgroup of order 4. I.e., there exists a unique subfield $E \subset \mathbb{Q}(\zeta_{29})$ such that $[E : \mathbb{Q}] = 7$. Hence, we conclude that $\mathrm{Gal}(E/\mathbb{Q})$ is a group of order 7, and hence is isomorphic to $\mathbb{Z}_7$.

Other Qualifying Exams

Problem RUT-2023-A-I-1 (Algebra). Classify the groups of order $2023 = 7 \cdot 17^2$ up to isomorphism. (You may use without proof the well-known result that if p is a prime, then every group of order p^2 is abelian.)

Let G be a group of order $2023 = 7 \cdot 17^2$. By Sylow's Theorem, G contains a normal Sylow 7-subgroup and a normal Sylow 17-subgroup. Let $H \cong \mathbb{Z}_7$ denote the Sylow 7-subgroup and K denote the Sylow 17-subgroup; note that either $K \cong \mathbb{Z}_{17^2}$ or $K \cong \mathbb{Z}_{17} \times \mathbb{Z}_{17}$. Hence, $G \cong H \rtimes_\varphi K$, where $\varphi \in \mathrm{Aut}(H) \cong \mathbb{Z}_7^\times \cong \mathbb{Z}_6$. We consider various cases.

(I) Suppose $K = \mathbb{Z}_{17^2}$, which has a single generator, 1. Each homomorphism $\varphi : K \to \mathbb{Z}_6$ is uniquely determined by where the generator 1 is mapped to, with the constraint that $\varphi(1)$ is an element that divides the order of 1, namely 17^2. Since the only such element is 0, φ is the trivial homomorphism, which means that the semidirect product is just the direct product, and so $G \cong \mathbb{Z}_7 \times \mathbb{Z}_{17^2} \cong \mathbb{Z}_{2023}$; this is an abelian group.

(II) Suppose $K = \mathbb{Z}_{17} \times \mathbb{Z}_{17} = \langle a \rangle \times \langle b \rangle$. Each homomorphism $\psi : \mathbb{Z}_{17} \times \mathbb{Z}_{17} \to \mathbb{Z}_6$ is uniquely determined by $\psi(a)$ and $\psi(b)$ with the constraint that these elements divide the order of a and b in $\mathbb{Z}_{17}$, which is 17. Since there is only one such element, namely 0, we find that the semidirect is just the direct product, and $G \cong \mathbb{Z}_7 \times \mathbb{Z}_{17} \times \mathbb{Z}_{17} \cong \mathbb{Z}_{289} \times \mathbb{Z}_7$; which is abelian.

Hence, up to isomorphism, there are exactly two groups of order 2023, both of which are abelian.

Classification of Finite Groups

Some facts we will use to classify groups are:

- Every group of order p^2, where p is abelian, is abelian.
- Every group of order p, where p is prime, is isomorphic to $\mathbb{Z}_p$.

(1) $|G| = 1$: This is the trivial group $\{1\}$.

(2) $|G| = 2$: There is exactly one group, up to ismorphism, which is $\mathbb{Z}/2\mathbb{Z}$. This follows from Cauchy's Theorem which states that if p divides $|G|$, where p is prime, then G contains an element of order p.

(3) $|G| = 3$: There is exactly one group, up to isomorphism, which is $\mathbb{Z}/3\mathbb{Z}$. This follows from Cauchy's Theorem, which states that if p divides $|G|$, where p is prime, then G contains an element of order p.

(4) $|G| = 4 = 2^2$: $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are both groups of order 4. Now let G be an arbitrary group of order 4; by Lagrange's Theorem, each element of G can have order 1, 2, or 4. Suppose G contains an element x of order 4. Then $G = \langle g \rangle$; let $\varphi : \mathbb{Z}_4 \to G$ be the map given by $\varphi(n) \mapsto g^n$; this is easily seen to be a group isomorphism. Now suppose G has no element of order 4. Since the only element of G with order 1 is the identity (by uniqueness of group identities), the three nontrivial elements of G must have order 2. Consider the map $\varphi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to G$ defined as follows:

$$\begin{aligned} \varphi(0,0) &= 1_G, \quad \varphi(1,0) = a, \\ \varphi(0,1) &= b, \quad \varphi(1,1) = c, \end{aligned} \tag{191}$$

where a, b, and c are the three nonidentity elements of G; φ is easily seen to be an isomorphism. Hence, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(5) $|G| = 5$: There is exactly one group, up to isomorphism, which is $\mathbb{Z}/5\mathbb{Z}$.

(6) $|G| = 6 = 2 \cdot 3$. By Sylow's Theorem, there exists a normal Sylow 3-subgroup, which we denote by H. Let K be a Sylow 2-subgroup. By Lagrange's Theorem, H and K intersect trivially and $|HK| = |H||K|/|H \cap K| = |H||K| = 6 = |G|$ so that $G = HK$. Hence, by the recognition theorem for semidirect products, $G \cong H \rtimes_\varphi K$, where $\varphi \in \mathrm{Aut}(H) \cong \mathbb{Z}_3^* \cong \mathbb{Z}_2$. Hence, we look for homomorphisms $\varphi : K \to \mathbb{Z}_2$. Since K is a group of order 2, $K \cong \mathbb{Z}_2$. Homomorphisms $\varphi : \mathbb{Z}_2 \to \mathbb{Z}_2$ are determined uniquely by where the generator 1 is sent to with the constraint that $\varphi(1)$ divides the order of 1, which is 2. Hence, either $\varphi_1(1) = 0$ (in which case, the homomorphism is trivial, the semidirect is just the direct product, and G is the abelian group $\mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_6$), or $\varphi_2(1) = 1$ (in which case, the homomorphism is *non*trivial, and G is the nonabelian group $\mathbb{Z}_3 \rtimes_{\varphi_2} \mathbb{Z}_2$). Hence, up to isomorphism, there are exactly two groups of order 6, one abelian and the other non-abelian.

(7) $|G| = 7$: There is exactly one group, up to isomorphism, which is $\mathbb{Z}/7\mathbb{Z}$.

(8) $|G| = 8 = 2^3$: [!! Complete Later !!]

(9) $|G| = 9 = 3^2$: Every group of order p^2 abelian. So by the Fundamental Theorem for Finitely Generated Abelian Groups, $G \cong \mathbb{Z}_9$ or $\mathbb{G} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

(10) $|G| = 10 = 2 \cdot 5$: By Sylow's Theorem, G contains a normal Sylow 5-subgroup, which we denote by H. Let K be a Sylow 10-subgroup. Then by Lagrange's Theorem, $H \cap K = \{e\}$ and $G = HK$. Therefore, $G \cong H \rtimes_\varphi K$, where $\varphi \in \mathrm{Aut}\, H \cong \mathbb{Z}_5^* \cong \mathbb{Z}_4$. Hence, we look for homomorphisms $\varphi : K \to \mathbb{Z}_4$. Since K is a group of order 2, $K \cong \mathbb{Z}_2$. Homomorphisms $\varphi : \mathbb{Z}_2 \to \mathbb{Z}_4$ are determined uniquely by where the generator a is set to with the constraint that $\varphi(a)$ divides the order of a, which is 2. The only such elements in $\mathbb{Z}_4$ are 0 and 2. If $\varphi_1 : 1 \mapsto 0$, then φ_1 is just the trivial homomorphism which means that the semidirect is just a direct product and G is isomorphic to the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_{10} \cong \mathbb{Z}_{10}$. If $\varphi_2 : 1 \mapsto 2$, then φ is a nontrivial homorphism, which means that G is the nonabelian group $\mathbb{Z}_5 \rtimes_{\varphi_2} \mathbb{Z}_2$. Hence, up to isomorphism, there are exactly 2 groups of order 10, only one of which is abelian.

(11) $|G| = 11$: There is exactly one group of order 11, namely $\mathbb{Z}/11\mathbb{Z}$.

(12) $|G| = 12 = 2^2 \cdot 3$: By Sylow's Theorem,

$$\begin{aligned} n_3 &\in \{1,2,4\} \cap \{1,4,\ldots\} = \{1,4\}. \\ n_2 &\in \{1,3\} \cap \{1,3,\ldots\} = \{1,3\}. \end{aligned} \tag{192}$$

Suppose $n_3 =$.

Essential Review Notes

Topological Vector Spaces

- **Def. (Topological Vector Space)** A vector space $\mathcal{X}$ over a field K such that vector addition in $\mathcal{X}$ and scalar multiplication are continuous maps from $\mathcal{X} \times \mathcal{X}$ and $K \times \mathcal{X}$, respectively, to $\mathcal{X}$.
- **Def. (Weak Convergence)** A sequence $\{x_n\}$ in a normed linear space $\mathcal{X}$ *converges weakly* to $x \in X$ if the sequence of scalars $\{f(x_n)\}$ converges to $f(x)$ for all $f \in \mathcal{X}^*$.
- **Def. (Weak* Convergence)** Let $\mathcal{X}$ be a normed linear space. A sequence $\{f_n\} \subseteq \mathcal{X}^*$ is *weak* convergent* to $f \in \mathcal{X}^*$ if $\{f_n(x)\}$ converges to $f(x)$ for all $x \in \mathcal{X}$. Note, all this really says that the sequence of scalars $\{\hat{x}(f_n)\} = \{f_n(x)\}$ converges to $\hat{x}(f) = f(x)$ for all $\hat{x} \in \mathcal{X}^{**}$ (read $x \in \mathcal{X}$).