Comps Practice

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Analysis

Problem 2019-J-I-4. Let (X, μ) be a measure space and $f \in L^1(X, \mu) \cap L^\infty(X, \mu)$. Show that $f \in L^q(X, \mu)$ for all $q > 1$ and

$$\|f\|_\infty = \lim_{q\to\infty} \|f\|_q .$$

Let (X, μ) be a measure space, and assume $f \in L^1(X) \cap L^\infty(X)$. Let $q > 1$. Then since $|f|^q = |f| \cdot |f|^{q-1}$ and $|f| \leq \|f\|_\infty$ a.e., we observe that

$$\int |f|^q = \int |f| \cdot |f|^{q-1} \leq \|f\|_\infty^{q-1} \int |f| = \|f\|_1 \|f\|_\infty^{q-1} . \tag{1}$$

Taking the q^{th} roots on both sides:

$$\|f\|_q = \left[\int |f|^q\right]^{1/q} \leq \|f\|_1^{1/q} \|f\|_\infty^{1-1/q} . \tag{2}$$

Since $f \in L^1 \cap L^\infty$, $\|f\|_1 , \|f\|_\infty < \infty$ so that $\|f\|_q < \infty$. Therefore, $f \in L^q(X, \mu)$. From the above expression, it is clear to see that

$$\lim_{q\to\infty} \|f\|_q \leq \lim_{q\to\infty} \|f\|_1^{1/q} \|f\|_\infty^{1-1/q} = \|f\|_\infty . \tag{3}$$

Hence, it suffices to show that the reverse inequality is satisfied. Let $\varepsilon > 0$ and define the set

$$A_\varepsilon = \left\{x : |f(x)| \geq \|f\|_\infty - \varepsilon\right\}. \tag{4}$$

By definition of the essential supremum, A_ε is a set of positive measure. Therefore,

$$\|f\|_q^q = \int |f|^q \geq \int_{A_\varepsilon} |f|^q \geq \mu(A_\varepsilon) \cdot (\|f\|_\infty - \varepsilon)^q. \tag{5}$$

Taking the q^{th} roots on both sides,

$$\|f\|_q \geq \mu(A_\varepsilon)^{1/q} \cdot (\|f\|_\infty - \varepsilon). \tag{6}$$

Taking the limit $q \to \infty$ on both sides,

$$\lim_{q\to\infty} \|f\|_q \geq \|f\|_\infty - \varepsilon. \tag{7}$$

Since $\varepsilon > 0$ was arbitrary, taking $\varepsilon \to 0$ gives us the reverse inequality; namely, that $\lim_{q\to\infty} \|f\|_q \geq \|f\|_\infty$.

Problem 2024-J-I-6. Let f and g be Lebesgue-measurable functions on $\mathbb{R}$. Define the convolution

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)\, dy$$

for all x such that the integral exists. Prove that if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ with $p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, then $f * g$ is a bounded continuous function on $\mathbb{R}$.

Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, where $p, q \in (1, \infty)$ are conjugate exponents. Furthermore, define the convolution $(f * g)(x)$ as described above. Then by Hölder's inequality, for any fixed $x \in \mathbb{R}$,

$$|(f * g)(x)| = \left|\int_{\mathbb{R}} f(x-y)g(y)\, dy\right| \leq \int_{\mathbb{R}} |f(x-y)|\, |g(y)|\, dy \leq \|f(x-\cdot)\|_p \|g\|_q . \tag{8}$$

Since L^p norms are translation invariant, $\|f(x-\cdot)\|_p = \|f\|_p$. Therefore, for any given $x \in \mathbb{R}$, $|(f*g)(x)| \leq \|f\|_p \|g\|_q = M < \infty$, where M is finite since $f \in L^p$ and $g \in L^q$. Hence, this shows that $\|f * g\|_\infty \leq M$, which means that $f * g$ is bounded. To show continuity, we use the fact that translations are continuous in the L^p norm; i.e., if $\tau_y f(x) = f(x-y)$ denote the translation of $f(x)$, then $\lim_{y\to 0} \left\|\tau_y f - f\right\|_p \to 0$ for all $p \in (1,\infty)$. In our case,

$$\left\|\tau_y(f*g) - (f*g)\right\|_\infty = \left\|(\tau_y f - f) * g\right\|_\infty \le \left\|\tau_y f - f\right\|_p \|g\|_q \longrightarrow 0 \text{ as } y \to 0, \tag{9}$$

which shows that $f * g$ is uniformly continuous, and hence continuous on $\mathbb{R}$.

Problem 2023-A-I-6. Suppose $f : [-1,1] \to \mathbb{R}$ is odd and C^1. Show

$$\int_{-1}^{1} |f(x)|^2 \, dx \le \int_{-1}^{1} |f'(x)|^2 \, dx.$$

Let $f : [-1,1] \to \mathbb{R}$ be odd and C^1. Then since $|f(x)|^2$ is even,

$$\int_{-1}^{1} |f(x)|^2 \, dx = 2\int_{0}^{1} |f(x)|^2 \, dx. \tag{10}$$

Hence, it suffices to examine the integral $\int_0^1 |f(x)|^2 \, dx$. Using Cauchy-Schwarz, we observe that

$$\begin{aligned} |f(x)|^2 &= \left|\int^x f'(t)\, dt\right|^2 \le \left(\int^x 1^2 \, dt\right) \cdot \left(\int^x |f'(t)|^2 \, dt\right) \\ &= x \cdot \int^x |f'(t)|^2 \, dt. \end{aligned} \tag{11}$$

Integrating over $[0,1]$,

$$\begin{aligned} \int_0^1 |f(x)|^2 \, dx &= \int_0^1 x \int_0^x |f'(t)|^2 \, dt \, dx \\ &= \int_0^1 |f'(t)|^2 \int_t^1 x \, dx \, dt \\ &= \int_0^1 |f'(t)|^2 \cdot \frac{1-t^2}{2} \, dt \le \frac{1}{2}\int_0^1 |f'(t)|^2 \, dt \\ \implies \int_{-1}^1 |f'(x)|^2 \, dx = 2\int_0^1 \int |f(x)|^2 \, dx &\le \int_0^1 |f'(t)|^2 \, dt \le \int_{-1}^1 |f'(t)|^2 \, dt =: \int_{-1}^1 |f'(x)|^2 \, dx, \end{aligned} \tag{12}$$

where we used Fubini/Tonelli to swap the integrals in the second line.

Problem 2024-J-II-5. Let P be the vector space over $\mathbb{R}$ of (finite degree) polynomials in the variable $x \in (-\infty,\infty)$. Show that P cannot be a Banach space with respect to any norm, that is, if $\|\cdot\|$ is some norm on P, then P is not complete under this norm. Hint: You may use the Baire Category Theorem.

Let P be the vector space over $\mathbb{R}$ of finite-degree polynomials in the variable $x \in (-\infty,\infty)$. We may write

$$P = \bigcup_{n\in\mathbb{N}} P_n, \tag{13}$$

where P_n denotes the space of all polynomials in P with degree at most n. Assume to the contrary that P is a Banach space with respect to the norm $\|\cdot\|$. Then since any norm induces a metric, P is a complete metric space. By the Baire Category Theorem, there exists at least one positive integer m such that P_m is not nowhere dense in P; i.e., its closure $\overline{P}_m$ has nonempty interior. Since P_m is a finite-dimensional vector subspace of the normed space P, it follows that P_m is closed in P. Therefore,

$P_m = \overline{P}_m$ has nonempty interior. Let $p \in P_m$; since P_m has nonempty interior, it contains a ball $B(p,r) = \{q \in P : \|p-q\| < r\}$ for some $r > 0$. Let $u \in P \setminus \{0\}$ be an arbitrary finite-degree polynomial, and set

$$u' = p + \frac{r \cdot u}{2\,\|u\|}, \tag{14}$$

so that $u' \in B(p,r) \subset P_m$. But since P_m is closed, it follows that $u = (u'-p) \cdot \frac{2\|u\|}{r} \in P_m$. Since $u \in P$ was arbitrary and $P_m \subseteq P$, it follows that $P = P_m$, which is impossible. Therefore, every P_n has empty interior, which contradicts Baire's Category Theorem. Hence, P cannot be a Banach space with respect to the norm $\|\cdot\|$.

Problem (Steinhaus' Theorem). Let $E \subset \mathbb{R}^n$ be a set of positive Lebesgue measure. Then the set $E - E = \{a - b : a, b \in E\}$ contains a neighborhood around the origin.

Let $E \subset \mathbb{R}^n$ be a set of positive Lebesgue measure. Then by the Lebesgue Differentiation Theorem, almost every point of E is a Lebesgue point, which is to say that there exists at least one point $x \in E$ such that for all $\varepsilon > 0$, there exists a $r > 0$ so that

$$\frac{m(E \cap B(r,x))}{m(B(r,x))} \geq (1-\varepsilon) \implies m(E \cap B(r,x)) \geq (1-\varepsilon)m(B(r,x)). \tag{15}$$

Suppose for contradiction that $E-E$ does not contain a neighborhood of the origin. Then there exists a sequence $x_n \to 0$ such that $x_n \notin E-E$ for any n. Equivalently, $(E+x_n) \cap E = \varnothing$ for all n. Taking n large enough so that $|x_n| < \varepsilon r$,

$$m(B(r,x) \cap (B(r,x) - x_n)) \geq (1-\varepsilon)m(B(r,x)). \tag{16}$$

By translation invariance of the Lebesgue measure,

$$m((E+x_n) \cap B(r,x)) = m(E \cap (B(r,x) - x_n)). \tag{17}$$

Therefore,

$$\begin{aligned} m((E+x_n) \cap B(r,x)) &\geq m(B(r,x) \cap (B(r,x)-x_n)) - m(B(r,x) \setminus E) - m((B(r,x)-x_n) \setminus E) \\ &\geq (1-\varepsilon)m(B(r,x)) - 2\varepsilon m(B(r,x)) \\ &\geq 0 \end{aligned} \tag{18}$$

for ε small enough. Therefore, $(E+x_n) \cap E \cap B(r,x) \neq \varnothing$, contradicting $(E+x_n) \cap E_n = \varnothing$ for all n. Hence, $E - E$ contains a neighborhood of the origin.

Problem 2024-J-II-2. Suppose $E \subset \mathbb{R}^2$ is a set of positive Lebesgue measure. Show that there are points a, b, c in E such that their connecting segments form a right angle, i.e., $a-b$ is perpendicular to $c-b$ (as vectors in $\mathbb{R}^2$).

Differential Geometry

Problem 2019-J-II-6. Let X and Y be vector fields on $\mathbb{R}^3$, defined by

$$X = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \quad \text{and} \quad Y = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \tag{19}$$

Is there a coordinate chart $\varphi = (x_1, x_2, x_3) : U \to \mathbb{R}^3$ of the origin $0 \in \mathbb{R}^3$ such that

$$X|_U = \frac{\partial}{\partial x^1} \quad \text{and} \quad Y|_U = \frac{\partial}{\partial x^2}. \tag{20}$$

No, there exists no coordinate chart containing the origin $0 \in \mathbb{R}^3$ that satisfies the above conditions. To show this, we will compute the Lie Brackets of the given vector fields; let $\widetilde{X} = \partial/\partial x^1$ and $\widetilde{Y} = \partial/\partial x^2$. First, we observe that

$$\begin{aligned}
[X,Y] &= \frac{\partial}{\partial x}(y,z,1) + x\frac{\partial}{\partial y}(y,z,1) + y\frac{\partial}{\partial z}(y,z,1) - y\frac{\partial}{\partial x}(1,x,y) - z\frac{\partial}{\partial y}(1,x,y) - \frac{\partial}{\partial z}(1,x,y) \\
&= x(1,0,0) + y(0,1,0) - y(0,1,0) - z(0,0,1) \\
&= x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}.
\end{aligned} \tag{21}$$

This means that the Lie Bracket of X and Y is not identically zero on any neighborhood of the origin. On the other hand, it is straightforward to see that the Lie Bracket of $\widetilde{X}$ and $\widetilde{Y}$ is identically zero on *all* of U. This is a contradiction. Therefore, such a coordinate chart cannot exist.

Problem 2019-A-I-5. Let H^3 be the 3-dimensional Heisenberg group consisting of upper triangular 3×3 matrices, with 1's on the diagonal, and with the group operation being matrix multiplication. Let $\Gamma \subset H^3$ be the subgroup consisting of matrices all of whose entries are integers. Show that the quotient space $N = H^3/\Gamma$ is a closed 3-dimensional manifold. Show that there is a fiber bundle projection $P : N \to \mathbb{T}^2$ to a 2-dimensional torus $\mathbb{T}^2 = S^1 \times S^1$ with fiber S^1. Hint: Consider the center Z of H^3.

Let H^3 be the 3-dimensional Heisenberg group consisting of upper triangular 3×3 matrices, with 1's on the diagonal, and with the group operation being matrix multiplication, and let Γ be the subgroup consisting of matrices all of whose entries are integers. We proceed with the proof over several steps:

(1) (H^3 is a Lie Group): We claim that there is a one-to-one correspondence between the elements of H^3 and $\mathbb{R}^3$, given by the map

$$\Phi : (a,b,c) \in \mathbb{R}^3 \longmapsto \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \tag{22}$$

It is straightforward to verify that Φ is a bijection. In particular, since the coordinate functions of Φ are polynomials (as is for the inverse of Φ), it follows that Φ is actually a diffeomorphism. Therefore, since $\mathbb{R}^3$ is a smooth manifold, H^3 must also be a smooth manifold. Now consider the product of two matrices in H^3:

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+a' & b+b'+ac' \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

By means of the one-to-one correspondence Φ, the multiplication becomes

$$m : \mathbb{R}^3 \times \mathbb{R}^3, \qquad m((a,b,c)\cdot(a',b',c')) = (a+a', b+b'+ac', c+c'). \tag{24}$$

Since all of the coordinate functions are polynomials, m is a smooth map. Therefore, by way of Φ, the multiplication map on H^3 is also smooth. Likewise, the inverse map can be seen to be a smooth map on H^3. Therefore, we conclude that H^3 is three-dimensional Lie group.

(2) (Γ is a discrete Lie Group): Trivially, every discrete group is a Lie group so that Γ is a discrete Lie subgroup of H^3.

(3) (Action of Γ on H^3): Let Γ act on the Lie group H^3 by left multiplication. First, we show that this action is free:

$$\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & a' & b'\\ 0 & 1 & c'\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & a' & b'\\ 0 & 1 & c'\\ 0 & 0 & 1\end{pmatrix} \iff \begin{cases}a+a'=a',\\ b+b'+ac'=b',\\ c+c'=c'\end{cases} \iff a=0, b=0, c=0. \tag{25}$$

Hence, $\begin{pmatrix}1 & a & b\\ 0 & 1 & c\\ 0 & 0 & 1\end{pmatrix}$ is the identity. This proves that the action is free. That the group action is smooth follows trivially from the fact that matrix multiplication is smooth. Finally, we show that the group action is properly discontinuous. I.e., for a compact set $K \subset \mathbb{R}^3$, we want to show that the set $\{\gamma \in \Gamma : (\gamma \cdot K) \cap K \neq \varnothing\}$ is a finite set. By the one-to-one correspondence between H^3 and $\mathbb{R}^3$ we demonstrated in **(1)**, our goal is to show that there exist finitely many 3-tuples $(m, n, p) \in \mathbb{Z}^3$ such that if K is a compact set and $[-R, R]^3$ is a cube containing K, then $(m, n, p) \cdot K \cap K \neq \varnothing$. If $(m, n, p) \cdot K$ intersects K, then for some $(x, y, z) \in K$,

$$\begin{aligned} -R \le x+m \le R &\implies -2R \le m \le 2R.\\ -R \le z+p \le R &\implies -2R \le p \le 2R.\\ -R \le n+y+mz \le R &\implies -2R \le n+mz \le 2R.\end{aligned} \tag{26}$$

Since m, n, p are integers and $R < \infty$, there exist only finitely many 3-tuples (m, n, p) that satisfy the above conditions. Therefore, it follows that the action is properly continuous.

Therefore, by the Quotient Manifold Theorem (see Lee *Introduction to Smooth Manifolds*, Theorem 9.16), $N = H^3/\Gamma$ is a smooth manifold of dimension $\dim H^3 - \dim\Gamma = 3 - 0 = 3$. [!! Complete Later !!]

Problem 2023-A-II-5. Let (t, x, y, z) be the standard coordinate system on $\mathbb{R}^4$, and let ϕ be the non-zero smooth 1-form on $\mathbb{R}^4$ defined by

$$\Phi = dt + y\,dx + z\,dy.$$

Let D be the 3-plane field on $\mathbb{R}^4$ that consists of tangent vectors V such that $\Phi(V) = 0$. Is D Frobenius integrable? Support your answer with a proof.

Let $D = \ker\Phi \subset T\mathbb{R}^4$, which must be a smooth 3-plane field on $\mathbb{R}^4$ since ϕ is nowhere zero. Note that since $\dim T_p\mathbb{R}^4 = 4$ at any p and $\dim D_p = \dim\ker\Phi_p = 4 - 1 = 3$, it follows that $\operatorname{codim} D = 3$. By the Frobenius Theorem, a codimension-one distribution $D = \ker\Phi$ is integrable iff $\Phi \wedge d\Phi = 0$. Computing $d\Phi$ first, we observe that:

$$\begin{aligned} d\Phi &= d(dt) + dy \wedge dx + dz \wedge dy\\ &= -dx \wedge dy + dz \wedge dy.\end{aligned} \tag{27}$$

Therefore,

$$\begin{aligned} \Phi \wedge d\Phi &= (dt + y\,dx + z\,dy) \wedge (-dx \wedge dy + dz \wedge dy)\\ &= -dt \wedge dx \wedge dy + dt \wedge dz \wedge dy + y\,dx \wedge dz \wedge dy,\end{aligned} \tag{28}$$

which is not identically zero everywhere on $\mathbb{R}^4$. Therefore, since $\Phi \wedge d\Phi \neq 0$, the Frobenius integrability condition fails, and so D is not Frobenius integrable. **Remark:** Here, D had codimension one. *If* D was a smooth distribution of codimension k, we can write $D = \ker\phi^1, \ldots, \phi^k$, where $\phi^1, \ldots, \phi^k$ are k smooth 1-forms that are pointwise linearly independent, then D is Frobenius integrable if and only if $\phi^1 \wedge \cdots \wedge \phi^k \wedge d\phi^i = 0$ for all i.

Problem 2023-A-I-2. Let $f : T^2 \to S^2$ be a smooth map from the 2-torus to the 2-sphere. Can f be an immersion. If the answer is yes, give an explicit example. If the answer is no, then give a proof.

Let $f : T^2 \to S^2$ be a smooth map from the 2-torus to the 2-sphere, and assume for the sake of an argument, that f is an immersion. Since $\dim T^2 = \dim S^2 = 2$, f must have constant rank 2. That is, for each $p \in T^2$,

$$df_p : T_pT^2 \to T_pS^2 \tag{29}$$

is an isomorphism. Hence, by the Inverse Function Theorem, f is a local diffeomorphism near p. Since local diffeomorphisms are open maps, it follows that $f(T^2)$ is an open subset of S^2. On the other hand, since the impact of compact sets under continuous maps is compact, and T^2 is compact, $f(T^2)$ is a compact, hence closed, subset of S^2. Since S^2 is connected, this implies that $f(T^2) = S^2$. Therefore, f is a surjective local diffeomorphism, which means that f is a covering map. Because S^2 is simply connected, any covering map onto S^2 must be a diffeomorphism. This implies that $T^2 \cong S^2$. However, this is a contradiction since $\pi_1(T^2) \cong \mathbb{Z}^2$ and $\pi_1(S^2) = \{0\}$ and diffeomorphisms preserve fundamental groups. Hence, by contradiction, f cannot be an immersion.

Problem 2023-J-I-3. Show that if M is a closed manifold that has an even dimensional sphere S^{2n} as its universal cover, then its fundamental group $\pi_1(M)$ is either trivial or $\mathbb{Z}_2$.

Problem 2018-J-I-6. Consider the distribution in $\mathbb{R}^3$ spanned by the two vector fields

$$V = \partial_x + 2xy\partial_z, \qquad W = x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z.$$

Show that this distribution is integrable and find an explicit formula for the integral submanifold passing through the point $(0, 0, z_0)$.

Let D be the distribution in $\mathbb{R}^3$ spanned by the two vector fields V and W describe above. By the Frobenius Theorem, to show that this distribution is integrable, it suffices to show that the distribution is involutive. In other words, we merely have to show that the Lie Bracket of V and W is a smooth local section of D. For ease of notation, we denote V and W as $(1, 0, 2xy)$ and $(x, 1, (2x^2y + x^2 - 2y))$, respectively. Then their Lie Bracket is:

$$\begin{aligned}[V, W] &= \left\{(1, 0, 2xy) \cdot (x, 1, (2x^2y + x^2 - 2y))\right\} - \left\{(x, 1, (2x^2y + x^2 - 2y)) \cdot (1, 0, 2xy)\right\} \\ &= (1, 0, 4xy + 2x) - (0, 0, 2xy) - (0, 0, 2x) = (1, 0, 2xy) = V.\end{aligned} \tag{30}$$

Hence, this shows that the distribution D is involutive, and hence completely integrable. To find an explicit formula for the integral submanifold at some point $p \in \mathbb{R}^3$, since a 2-dimensional distribution in $\mathbb{R}^3$ is the kernel of a single 1-form ω (because the codimension of the distribution is 1), we start by finding an annihilator 1-form; i.e., a 1-form such that $\omega(V) = \omega(W) = 0$. Let $\omega = A\,dx + B\,dy + dz$. Then

$$\begin{aligned} 0 &= \omega(V) = A + 2xy \implies A = -2xy. \\ 0 &= \omega(W) = Ax + B + (2x^2y + x^2 - 2y) \implies B = 2x^2y - 2x^2y - x^2 + 2y = -x^2 + 2y.\end{aligned} \tag{31}$$

Hence, our annihilator 1-form is

$$\omega = -2xy\,dx - (x^2 - 2y)\,dy + dz. \tag{32}$$

On integral surfaces, $\omega = 0$. Hence, we observe that

$$dz = 2xy\,dx + (x^2 - 2y)\,dy \implies \frac{\partial z}{\partial x} = 2xy \text{ and } \frac{\partial z}{\partial y} = x^2 - 2y. \tag{33}$$

From the first differential equation, we find

$$z = x^2y + f(y). \tag{34}$$

From the second equation,

$$x^2 + \frac{df}{dy} = x^2 - 2y \implies \frac{df}{dy} = -2y \implies f(y) = -y^2 + c. \tag{35}$$

Therefore, we obtain

$$z = x^2 y - y^2 + c. \tag{36}$$

Plugging in the point $(0,0,z_0)$, we obtain $c = z_0$. Hence, the explicit formula for the integral submanifold passing through the point $(0,0,z_0)$ is given by

$$x^2 y - y^2 - z = z_0. \tag{37}$$

Problem 2019-J-11-6. Let X and Y be vector fields on $\mathbb{R}^3$, defined by

$$X = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \qquad \text{and} \qquad Y = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Is there a coordinate chart $\varphi = (x_1, x_2, x_3) : U \to \mathbb{R}^3$ of the neighborhood of the origin $0 \in \mathbb{R}^3$ such that

$$X|_U = \frac{\partial}{\partial x_1} \qquad \text{and} \qquad Y|_U = \frac{\partial}{\partial x_2}.$$

We claim that there does *not* exist such a coordinate chart. Suppose to the contrary that there do exist such coordinates on a neighborhood U of the origin $0 \in \mathbb{R}^3$. We observe that

$$[X|_U, Y|_U] = \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial x_1}\right) = 0. \tag{38}$$

I.e., the Lie Bracket of the vector fields in these coordinates vanish everywhere on U. On the other hand, computing the Lie Bracket of X and Y in the original coordinates,

$$\begin{aligned} [X, Y] &= (1, x, y) \cdot (y, z, 1) - (y, z, 1) \cdot (1, x, y) \\ &= (0,0,0) + (x,0,0) + (0,y,0) - (0,y,0) - (0,0,z) - (0,0,0) \\ &= (x, 0, -z) = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z}. \end{aligned} \tag{39}$$

Since the Lie Bracket of X and Y is not identically zero on U, we have reached a contradiction. Therefore, by contradiction, we see that such a coordinate chart cannot exist.

Problem 2019-A-I-4. Let $f : \mathbb{RP}^3 \to \mathbb{T}^3 = S^1 \times S^1 \times S^1$ be a smooth map. Show that f is not an immersion.

Let $f : \mathbb{RP}^3 \to \mathbb{T}^3 = S^1 \times S^1 \times S^1$ be a smooth map. Assume to the contrary that f is an immersion. First, we prove the *comps lemma*, which we will use to develop our argument.

> **(Comps Lemma)** Let M and N be smooth connected n-manifolds, and $f : M \to N$ a (smooth) immersion. If M is compact and nonempty, then N is compact and f is a (smooth) covering map.

Proof. Let M, N be smooth connected n-manifolds, $f : M \to N$ an immersion, and M compact and nonempty. Since f is an immersion, the map $df_p : T_pM \to T_{f(p)}N$ is injective at each $p \in M$ so that f is a local diffeomorphism. This implies that $f(M)$ is open in N (continuous image of an open set) and $f(M)$ is closed in N (continuous image of a compact set is compact, and compact subsets of Hausdorff spaces are closed). Since N is connected and M is nonempty, $f(M) = N$. Therefore, N is compact.

To show that f is a covering map, it remains to be shown that N is evenly covered. Let $q \in N$ be arbitrary but fixed. Since f is a local diffeomorphism, $f^{-1}(q) \subset M$ is closed and discrete, which means $f^{-1}(q) = \{x_1, \ldots, x_s\}$ for some finite s and $x_j \in M$. Since M is Hausdorff, we may pick a collection $\{U_j\}_{j=1}^{s}$ of open subsets of M such that $x_j \in U_j$ for each j, and $U_i \cap U_j = \varnothing$ for each $i \neq j$; shrink each U_j if needed so that $f|_{U_j} \to f(U_j)$ is a diffeomorphism. Set $V = \bigcap_{j=1}^{s} f(U_j)$ so that V is an evenly covered neighborhood of $q \in N$. $\square$

In our case, $\mathbb{RP}^3$ is a smooth connected compact nonempty 3-manifold, and $\mathbb{T}^3$ is a smooth connected 3-manifold. By the Comps Lemma, $\mathbb{T}^3$ is compact and f is a smooth covering map. Consider the induced *injective* homomorphism $f_* : \pi_1(\mathbb{RP}^3) \to \pi_1(\mathbb{T}^3)$. Since $\pi_1(\mathbb{RP}^3) = \mathbb{Z}/2\mathbb{Z}$, $\pi_1(\mathbb{T}^3) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, and the former has torsion while the latter does not since each subgroup of $\mathbb{Z}^3$ is free abelian, there cannot exist such an injective homomorphism. Hence, by contradiction, we must have that f cannot be a immersion.

Problem 2017-J-II-1. Let $f : M \to \mathbb{R}$ be a smooth function on a smooth manifold M. In an arbitrary smooth local coordinate chart $\pi : U \to \mathbb{R}^n$ of M, define

$$\mathcal{D}f := \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}. \tag{40}$$

Does $\mathcal{D}f$ give a well-defined vector field on M?

No, $\mathcal{D}f$ does *not* give a well-defined vector field on M. In fact, we claim that $\mathcal{D}f$ does not transform covariantly. Let $(U, (x^i))$ and $(V, (\widetilde{x}^i))$ be smooth local coordinate charts on M, and let $p \in U \cap V$. In the remainder of the proof, we shall use Einstein Summation Convention. We find that

$$\begin{aligned} \mathcal{D}F &= \frac{\partial f}{\partial x^i}(p) \frac{\partial}{\partial x^i}\bigg|_p \\ &= \left(\frac{\partial f}{\partial \widetilde{x}^j}(\hat{p}) \frac{\partial \widetilde{x}^j}{\partial x^i}\bigg|_p \frac{\partial \widetilde{x}^k}{\partial x^i} \right) \frac{\partial}{\partial \widetilde{x}^k}\bigg|_{\hat{p}} \\ &\neq \frac{\partial f}{\partial \widetilde{x}^k}(\hat{p}) \frac{\partial}{\partial \widetilde{x}^k}\bigg|_{\hat{p}} = \mathcal{D}f, \end{aligned} \tag{41}$$

which is a contradiction. Therefore, $\mathcal{D}f$ does not give a well-defined vector field on M.

Problem 2012-J-I-4. Let x, y, z be the usual coordinates on $\mathbb{R}^3$. Consider the 1-form on $\mathbb{R}^3$ given by

$$\varphi = dx + ydz.$$

Is it possible to find smooth functions u and v on $\mathbb{R}^3$ such that $\varphi = udv$? Why?

Let $\varphi_1 = dx + ydz$ and $\varphi_2 = udv$ for some smooth functions u and v on $\mathbb{R}^3$. Then, we observe that

$$\begin{aligned} d\varphi_1 &= d(dx + ydz) = d(dx) + d(ydz) \\ &= dy \wedge dz. \\ \varphi_1 \wedge d\varphi_1 &= (dx + ydz) \wedge (dy \wedge dz) = dx \wedge dy \wedge dz. \\ d\varphi_2 &= d(udv) = du \wedge dv. \\ \varphi_2 \wedge d\varphi_2 &= udv \wedge (du \wedge dv) = 0. \end{aligned} \tag{42}$$

We observe that $\varphi_1 \wedge d\varphi_1$ is the volume form on $\mathbb{R}^3$ and hence is identically nonzero everywhere on $\mathbb{R}^3$. On the other hand, $\varphi_2 \wedge d\varphi_2$ is zero everywhere on $\mathbb{R}^3$. This means that it is not possible to find smooth functions u and v on $\mathbb{R}^3$ such that $\varphi = udv$.

Problem 2011-A-II-5. On $\mathbb{R}^4$, equipped with coordinates (x, y, z, t), let X and Y be the vector fields given by
$$X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} \qquad \text{and} \qquad Y = x\frac{\partial}{\partial z} + \frac{\partial}{\partial t}.$$
If $f : \mathbb{R}^4 \to \mathbb{R}$ is a smooth function satisfying $Xf = Yf = 0$, show that f is constant.

Let $f : \mathbb{R}^4 \to \mathbb{R}$ be a smooth function satisfying $Xf = Yf = 0$, where X, Y are the vector fields described above. First, we observe that $[X, Y]f = 0$, where

$$\begin{aligned}[X, Y] &= \partial_x(x\partial_z) + \partial_x(\partial_t) + z\partial_y(x\partial_z) + z\partial_y(\partial_t) - \big(x\partial_z(\partial_x) + x\partial_z(z\partial_y) + \partial_t(\partial_x) + \partial_t(z\partial_y)\big) \\ &= \partial_z - x\partial_y.\end{aligned} \tag{43}$$

Since $[X, Y]f = 0$, $[X, [X, Y]]f = 0$, where

$$[X, [X, Y]] = -\partial_y - \partial_y = -2\partial_y. \tag{44}$$

This immediately implies that $-2\partial_y f = 0 \implies \partial_y f = 0$ everywhere on $\mathbb{R}^4$. Hence, since $[X, Y]f = 0$ everywhere, $0 = \partial_z f - x\partial_y f = \partial_z f$ everywhere on $\mathbb{R}^4$. Likewise, $0 = Xf = \partial_x f + z\partial_y f = \partial_x f$. Then since $0 = Yf = x\partial_z f + \partial_t f = \partial_t f$, we see that $\partial_t f = 0$ everywhere. Hence, since all of the first order partial derivatives of f are zero everywhere on $\mathbb{R}^4$, it follows that f is constant.

Problem 2010-J-I-2. Consider the differential 1-form $\varphi = dx^1 + x^2 dx^3$ on $\mathbb{R}^3$. Is it possible to find a smooth coordinate system (y^1, y^2, y^3) on a neighborhood of the origin such that $\varphi = f\, dy^1$ in these new coordinates, for some smooth function $f(y^1, y^2, y^3)$? Support your answer with a proof.

Let $f = f(y^1, y^2, y^3)$ be a smooth function on $\mathbb{R}^3$, where (y^1, y^2, y^3) is a smooth coordinate system on some neighborhood of the origin. Let $\varphi_1 = f\, dy^1$. Then we observe that

$$\begin{aligned} d\varphi_1 &= d(f\, dy^1) = df \wedge dy^1 \\ &= \left(\frac{\partial f}{\partial y^1} dy^1 + \frac{\partial f}{\partial y^2} dy^2 + \frac{\partial f}{\partial y^3} dy^3\right) \wedge dy^1 \\ &= -\left(\frac{\partial f}{\partial y^2} dy^1 \wedge dy^2 + \frac{\partial f}{\partial y^3}\, dy^1 \wedge dy^3\right). \\ \varphi_1 \wedge d\varphi_1 &= 0. \\ d\varphi &= d(dx^1 + x^2 dx^3) = dx^2 \wedge dx^3. \\ \varphi \wedge d\varphi &= dx^1 \wedge dx^2 \wedge dx^3. \end{aligned} \tag{45}$$

I.e., $\varphi \wedge d\varphi$ is the volume form on $\mathbb{R}^3$, and hence is nonzero everywhere. On the other hand, $\varphi_1 \wedge d\varphi_1$ is zero *everywhere*. Therefore, it is not possible to find a smooth coordinate system (y^1, y^2, y^3) on a neighborhood of the origin such that $\varphi = f\, dy^1$ for some smooth function $f = f(y^1, y^2, y^3)$.

Problem 2010-J-I-3. Let $f : S^1 \times S^1 \to \mathbb{RP}^2$ be a smooth map from the 2-torus to the projective plane. Prove that f cannot be an immersion.

Our strategy for this problem is to develop and prove the *Comps Lemma*, which we will then use to prove the problem.

> **(Comps Lemma)** Let M and N be smooth connected n-manifolds, and $f : M \to N$ a (smooth) immersion. If M is compact and nonempty, then N is compact and f is a (smooth) covering map.
>
> ***Proof*** Let M and N be smooth connected n-manifolds and let $f : M \to N$ be an immersion, and consider the map $df_p : T_pM \to T_{f(p)}N$, for $p \in M$. Since $\dim T_pM = \dim T_{f(p)}N = n$ for every $p \in M$ and f is an immersion, it follows that df_p has constant rank n. Therefore, the

Inverse Function Theorem implies that f is a local diffeomorphism. Since local diffeomorphisms are open maps, $f(M)$ is open in N. On the other hand, since the continuous image of compact sets is compact and compact subsets of Hausdorff spaces are closed, $f(M)$ is closed in N. Since $f(M)$ is nonempty and N is connected, $f(M) = N$, which proves that N is compact.

Now it remains to be shown that N is evenly covered. Let $q \in N$ and consider $f^{-1}(q)$. Since f is a local diffeomorphism, $f^{-1}(q)$ is closed in M. Moreover, for each $x \in f^{-1}(q)$, we can find a neighborhood U_x of x such that $f|_{U_x}$ is a diffeomorphism. Therefore, U_x contains no other point in the preimage of f. This means that every point in the preimage of q is isolated, which means that $f^{-1}(q)$ is discrete. Since M is compact and discrete subsets of compact spaces are finite, we conclude that $f^{-1}(q)$ is a finite set $\{x_1, \ldots, x_s\}$. By the local diffeomorphism property of f, we can find open subsets $U_1, \ldots, U_s$ such that each U_j contains x_j and $f|U_j$ is a diffeomorphism. Since M is Hausdorff, we may shrink these sets so that they are pairwise disjoint. Now set $V = \bigcap_{j=1}^{s} f(U_j)$, which is an evenly covered neighborhood of q in N. Therefore, N is evenly covered.

Now assume to the contrary that $f : S^1 \times S^1 \to \mathbb{RP}^2$ is a smooth immersion. Since $S^1 \times S^1$ is a smooth, connected, compact, and nonempty 2-manifold, while $\mathbb{RP}^2$ is a smooth connected 2-manifold, it follows from the Comps Lemma that $f : M \to N$ is a smooth covering map. This implies that the induced homomorphism, $f_* : \pi_1(S^1 \times S^1) \to \pi_1(\mathbb{RP}^2)$ is injective. We recall the following facts about the fundamental group:

$$\pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z} \quad \text{and} \quad \pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}. \tag{46}$$

Here, we observe that the fundamental group of $S^1 \times S^1$ is countably infinite, while the fundamental group of $\mathbb{RP}^2$ is finite. Therefore, the induced homomorphism cannot be injective, which is a contradiction. Hence, f cannot be an immersion.

Problem 2002-J-II-1 (Comps Lemma). Let M and N be smooth, connected n-dimensional manifolds, and let $f : M \to N$ be an immersion. (That is, assume that the derivative of f always sends nonzero tangent vectors to nonzero tangent vectors.) If M is compact and nonempty, show that N is compact, and that f is a covering map.

Let M, N be smooth connected n-manifolds, and $f : M \to N$ an immersion. Assume M is compact and nonempty, and consider the differential $df_p : T_pM \to T_{f(p)}N$. Since $\dim T_pM = \dim T_{f(p)}N = n$ at every $p \in M$ and f is an immersion, df_p has constant rank n. Hence, the Inverse Function Theorem implies that f is a local diffeomorphism. Since local diffeomorphisms are open maps, $f(M)$ is open in N. On the other hand, since the continuous image of compact sets is compact and compact subsets of Hausdorff spaces are closed, $f(M)$ is closed in N. Since $f(M)$ is nonempty and N is connected, $f(M) = N$, which proves that N is compact.
It remains to be shown that N is evenly covered. Let $q \in N$, and consider $f^{-1}(q)$. Since f is a local diffeomorphism, N is Hausdorff, and singletons are closed in Hausdorff spaces, $f^{-1}(q)$ is closed, and hence, compact in M. Moreover, since f is a local diffeomorphism, for each $x \in f^{-1}(q)$, there exists a neighborhood U_x such that $f|_{U_x}$ is a diffeomorphism. By the Hausdorff property on M, we may shrink each neighborhood U_x to some smaller neighborhood U'_x so that $f|_{U'_x}$ is still a diffeomorphism and the neighborhoods are pairwise disjoint. This means that every $x \in f^{-1}(q)$ is isolated, so that the preimage of q is discrete. Since discrete, compact subsets must necessarily be finite, $f^{-1}(q) = \{x_1, \ldots, x_s\}$ for finitely many $x_j \in M$. As suggested above, for each j, let U_j be a neighborhood of x_j in M such that, after possibly shrinking each set, $U_i \cap U_j = \varnothing$ for all $i \neq j$ and $f|_{U_j}$ is a diffeomorphism. Set $V = \bigcap_{j=1}^{s} f(U_j)$, which is then seen to be an evenly covered neighborhood of $q \in N$. Hence, N is evenly covered, concluding our proof that f is a covering map.

Problem 2025-A-II-2. Consider the plane distribution in $\mathbb{R}^3$ spanned by two vector fields

$$V = \partial_x + 2xy\partial_z \quad \text{and} \quad W = x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z.$$

(i) Show that this distribution is integrable.
(ii) Does the pair of vector fields V and W generate a coordinate system on integral surfaces? If not, find a pair that can play this role for the local integral surfaces passing through points $(0,0,z_0)$.

(i) Let D be the plane distribution in $\mathbb{R}^3$ spanned by the two vector fields V and W described above. By the Frobenius theorem, D is integrable if and only if D is involutive. Therefore, we will verify that D is involutive, for which it suffices to check that the Lie Bracket of V and W is a smooth section of D. For ease of notation, we shall write $V = (1,0,2xy)$ and $W = (x,1,2x^2y + x^2 - 2y)$. Then we observe the Lie Bracket of the vector fields to be

$$\begin{aligned}
[V,W] &= V(W) - W(V) \\
&= (1,0,2xy)\cdot(x,1,2x^2y + x^2 - 2y) - (x,1,2x^2y + x^2 - 2x)\cdot(1,0,2xy) \\
&= (1,0,4xy + 2x) + (0,0,0) + (0,0,0) - (0,0,2xy) - (0,0,2y) - (0,0,0) \\
&= (1,0,2xy) = V.
\end{aligned} \tag{47}$$

Hence, we conclude that D is involutive, which then proves that D is integrable.

(ii) We claim that V and W does *not* generate a coordinate system on integral surfaces. Assume to the contrary that V and W generates coordinates (u,v) on an integral surface $\mathcal{S}$ such that $V|_{\mathcal{S}} = \partial/\partial u$ and $W|_{\mathcal{S}} = \partial/\partial v$. Then it is straightforward to see that $V|_{\mathcal{S}}$ and $W|_{\mathcal{S}}$ commute with each other, which means that their Lie Bracket is identically zero. However, since $\mathcal{S}$ is an integral surface spanned by the distribution D, $[V|_{\mathcal{S}}, W|_{\mathcal{S}}] = ([V,W])|_{\mathcal{S}}$; but this is a contradiction since $[V,W]$ is nonvanishing everywhere (from (i)). Hence, by contradiction, it follows that V and W do *not* generate a coordinate system on integral surfaces. To see the second part, consider the vector fields $\widetilde{V} = V$ and $\widetilde{W} = W - xV$, which do commute with each other.

Problem 2003-J-11-5. Is the unit three-sphere S^3 diffeomorphic to the product of two other smooth manifolds of dimensions > 0?

We claim that S^3 *cannot* be diffeomorphic to the product of two other smooth manifolds of nonzero dimensions. Assume to the contrary: let M_1, M_2 be smooth manifolds of nonzero dimensions such that $S^3 \cong_f M_1 \times M_2$. Since diffeomorphisms preserve dimensions, $3 = \dim S^3 = \dim M_1 \times M_2 = \dim M_1 + \dim M_2$. Since neither of M_1, M_2 are 0-manifolds, without loss of generality, we must have $\dim M_1 = 1$ and $\dim M_2 = 2$. Next, since S^3 is compact and connected, it follows that $M_1 \times M_2$ must be compact and connected, so that M_1 and M_2 are compact and connected. Third, since $f : S^3 \to M_1 \times M_2$ is a diffeomorphism, the induced homomorphism $f_* : \pi_1(S^3) \longmapsto \pi_1(M_1 \times M_2) = \pi_1(M_1) \times \pi_1(M_2)$ is an isomorphism. Since S^n is simply connected for all $n \geq 2$, $\pi_1(S^3) = \{0\}$. On the other hand, since the only compact connected smooth 1-manifold, up to diffeomorphism, is the circle S^1, $\pi_1(M_1) \cong \pi_1(S^1) \cong \mathbb{Z}$. This implies that $\pi_1(M_1 \times M_2)$ is not a trivial group, which contradicts our claim that f_* is an isomorphism. Hence, by contradiction, S^3 cannot be diffeomorphic to the product of two smooth manifolds of nonzero dimensions.