Comps Practice

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Contents

Comps Lemma

Problem Comps Lemma. Let M, N be smooth, connected, n-manifolds, and $f : M \to N$ a (smooth) immersion. If M is compact and nonempty, then N is compact and f is a (smooth) covering map.

Let M, N be smooth, connected n-manifolds and $f : M \to N$ an immersion. Assume that M is compact and nonempty. Since $\dim N = n$ and f is an immersion, $\operatorname{rank} df_p = n$ at every $p \in M$. Hence, by the Inverse Function Theorem, f is a local diffeomorphism. Since local diffeomorphisms are open maps, $f(M)$ is open in N. On the other hand, since the continuous image of compact sets is compact, $f(M)$ is compact in N. Since N is Hausdorff, $f(M)$ is closed in N. Since N is connected, $f(M) = N$. Therefore, N is compact.
Now, let $q \in N$, and consider $f^{-1}(q) \subset M$. For each $x \in f^{-1}(q)$, let U_x be an open neighborhood of M containing x. Since M is Hausdorff, we can shrink each U_x so that these neighborhoods are pairwise disjoint. This means that each $x \in f^{-1}(q)$ is isolated, and hence $f^{-1}(q)$ is discrete. Since M is compact, we conclude that $f^{-1}(q)$ must be finite; let $f^{-1}(q) = \{x_1, \dots, x_s\}$. As noted above, for each $j = 1, \dots, s$, let U_j be a neighborhood of x_j such that $f|_{U_j} : U_j \to V_j \subset N$ is a diffeomorphism. Then by the Hausdorff condition on M, shrink each U_j so that $U_i \cap U_j = \varnothing$ for all $i \neq j$; f remains a diffeomorphism on these shrunken neighborhoods. Setting $V = \bigcap_1^s f(U_j)$ and taking $\widetilde{U}_j = f^{-1}(V) \cap U_j$ gives us an evenly covered neighborhood of q in N.

January 2025

Problem 2025-J-I-1 (Algebra). Let R be a UFD (unique factorization domain). Let F be its quotient field. Let $p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in F[x]$ be a monic polynomial with coefficients in R admitting a root $a \in F$. Prove that $a \in R$.

Let R be a UFD, and F its quotient field. Let $p(x) = x^n + b_{n-1}x^{n-1} + \cdots + b_0 \in F[x]$ be a monic polynomial with coefficients in R admitting a root $a \in F$. Let $a = c/d$, where $c, d \in R \setminus \{0\}$ so that $\gcd(c, d) = 1$. By definition of a root, we must have

$$0 = p(a) = \left(\frac{c}{d}\right)^n + b_{n-1}\left(\frac{c}{d}\right)^{n-1} + \cdots + b_0. \tag{1}$$

Multiplying both sides by d^n,

$$c^n + d(b_{n-1}c^{n-1} + b_{n-2}c^{n-2}d \cdots + b_0 d^{n-1}) = 0 \implies c^n = -d(b_{n-1}c^{n-1} + \cdots + b_0 d^{n-1}). \tag{2}$$

From this, we observe that $d \mid c^n$. If d is not a unit in R, then every nonidentity irreducible divisor of d is an irreducible divisor of c^n, and hence an irreducible divisor of c. But this contradicts our hypothesis that $\gcd(c, d) = 1$. Hence, d has to be a unit of R. If $v \in R \setminus \{0\}$ such that $dv = vd = 1$, then

$$a = \frac{c}{d} = \frac{c}{d} \cdot \frac{v}{v} = cv \in R. \tag{3}$$

Hence, this concludes the proof.

Problem 2025-J-I-2 (Real Analysis). Let $\{f_n\}_{n \geq 1}$ be a sequence of Lebesgue-measurable functions on $[0, 1]$. Suppose that

$$\int_0^1 f_n^2 dm \leq \frac{1}{n^2} \quad \text{for all } n \geq 1.$$

Prove that f_n converges to 0 a.e. on $[0, 1]$.

Let $\{f_n\}_{n \geq 1}$ be a sequence of Lebesgue-measurable functions on $[0, 1]$ so that

$$\int_0^1 f_n^2 dm \leq \frac{1}{n^2} \quad \text{for all } n \geq 1. \tag{4}$$

Consider the sequence $\{\sum_1^m f_n^2\}$, which is increasing and converges a.e. to $\sum_1^\infty f_n^2$. Hence, by the Monotone Convergence Theorem,

$$\sum_1^\infty \int_0^1 f_n^2 = \lim_{m\to\infty} \sum_1^m \int_0^1 f_n^2 = \lim_{m\to\infty} \int_0^1 \sum_1^m f_n^2 = \int_0^1 \sum_1^\infty f_n^2 \le \int_0^1 \sum_1^\infty \frac{1}{n^2} < \infty. \tag{5}$$

Therefore, $\sum_1^\infty f_n^2 \in L^1(\mathbb{R})$, which means that $\sum_1^\infty f_n^2 < \infty$ a.e. on $[0,1]$. Hence, $\sum_{n=1}^\infty f_n^2$ converges a.e. on $[0,1]$. This implies that $f_n^2 \to 0$ a.e. on $[0,1]$, and hence $f_n \to 0$ a.e. on $[0,1]$.

Problem 2025-J-I-3 (Geometry/Topology). Let M be an orientable, connected, and compact smooth n-manifold with boundary. Show that there is no (smooth) retraction to the boundary, that is, there does not exist a smooth map $f : M \to \partial M$ such that $f(x) = x$ when $x \in \partial M$.

Let M be an orientable, connected, and compact smooth n-manifold with boundary. Assume to the contrary that there exists a smooth map $f : M \to \partial M$ such that $f(x) = x$ when $x \in \partial M$. Let $\omega \in \Omega^{n-1}(\partial M)$ be a volume form for the boundary of M. Since volume forms are closed (hence, ω is closed), we have by Stokes's theorem

$$0 = \int_M f^* d\omega = \int_M \partial(f^*\omega) = \int_{\partial M} f^*\omega = \int_{\partial M} \omega > 0, \tag{6}$$

which is a contradiction. Hence, by contradiction, there cannot exist a smooth retraction to the boundary.

Problem 2025-J-II-3 (Algebra). Let V be a vector space of dimension n over $\mathbb{Q}$. Let $T : V \to V$ be a linear transformation with minimal polynomial $x^4 - x^2 - 2$ over $\mathbb{Q}$. Show that n must be even.

Consider V as a module over the ring $\mathbb{Q}[x]$ by letting a polynomial $f(x) \in \mathbb{Q}[x]$ act as the linear operator $f(T)$. Since $\dim V = n$, this module is finitely generated. By the structure theorem for finitely generated modules over principal ideal domains, V is isomorphic to a direct sum of modules of the form $\mathbb{Q}[x]/(p(x))^e$, where $p(x) \in \mathbb{Q}[x]$ is irreducible. Moreover, each $p(x)$ must divide the minimal polynomial of T. We note that over $\mathbb{Q}$,

$$x^4 - x^2 - 2 = (x^2 - 2)(x^2 + 1), \tag{7}$$

where both factors are irreducible over $\mathbb{Q}$. Therefore, the only choices for $p(x)$ are $x^2 - 2$ and $x^2 + 1$. Therefore, $\mathbb{Q}[x]/(p(x))^e$ has dimension $\deg p \cdot e = 2e$ for each choice of p. Since 2 divides these dimensions, we conclude that 2 must divide n. Hence, n is even.

Problem 2025-J-II-4 (Topology). Let Σ_2 be a compact oriented surface of genus 2. Is there a submersion $f : \Sigma_2 \to S^1 \times S^1$, where S^1 denotes the unit circle?

Assume to the contrary that there exists a submersion $f : \Sigma_2 \to S^1 \times S^1$, where S^1 denotes the unit circle. Since $\dim \Sigma_2 = \dim S^1 \times S^1 = 2$, df_p must have constant rank 2 at every $p \in \Sigma_2$. Hence, f is a local diffeomorphism. Since f is a local diffeomorphism, $f(\Sigma_2)$ is compact in $S^1 \times S^1$; since $S^1 \times S^1$ is Hausdorff, $f(\Sigma_2)$ must be closed in $S^1 \times S^1$. On the other hand, since local diffeomorphisms are open maps, $f(\Sigma_2)$ is open in $S^1 \times S^1$. Therefore, since $S^1 \times S^1$ is connected, $f(\Sigma_2) = S^1 \times S^1$; i.e., f is surjective. Therefore, f is a covering map. This means that the induced homomorphism, $f_* : \pi_1(\Sigma_2) \to \pi_1(S^1 \times S^1)$ is injective, and so $f_*(\pi_1(\Sigma_2)) \cong \operatorname{img} f_* \le \pi_1(S^1 \times S^1)$. However, $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$ is an abelian group and cannot have any nonabelian subgroups, whereas $\pi_1(\Sigma_2)$ is nonabelian. Hence, by contradiction, f cannot be a submersion.

Problem 2025-J-II-5 (Analysis). Let V be a topological vector space whose topology is Hausdorff. Let X_1 and X_2 be two Banach spaces, and assume there exist continuous linear bijections $F_1 : X_1 \to V$ and $F_2 : X_2 \to V$. Show that there is a continuous linear bijection $G : X_1 \to X_2$.

Assume the given hypotheses. Let $G = F_2^{-1} \circ F_2$. Since F_1, F_2 are bijections, we conclude that G is a bijection. Likewise, since F_1, F_2 are linear, G must also be linear. It suffices to prove that G is continuous. By the Closed Graph Theorem, continuity of G is equivalent to the graph of G being a closed subspace of $X_1 \times X_2$. Let $\{x_n\} \subset X_1$ be a sequence in X_1 such that $x_n \to x$ and $y_n = Gx_n \to y$. We need to show that $y = Gx$. By continuity of F_1, $F_1 x_n \to Fx$. By continuity of F_2,

$$F_2 y = \lim F_2 y_n = \lim F_2 G x_n = \lim F_1 x_n = F_1 x. \tag{8}$$

Since F_2 is bijective, $y = F_2^{-1} F_1 x = Gx$. Hence, the graph of G is closed, which implies that G is continuous.

August 2025

Problem 2025-A-I-1 (Geometry/Topology). Let S be a closed orientable surface of genus 4 and C be an embedded circle that partitions S into two subsurfaces of genus 2. Does S retract to C?

We claim that the answer is no; assume to the contrary that there exists a retraction $r : S \to C$. Let $i : C \hookrightarrow S$ be the inclusion map so that $r \circ i = \mathrm{id}_C$. Now since C is an embedded circle, $H_1(C)$ (i.e., the first homology) is isomorphic to $H_1(S^1) = \mathbb{Z}$. On the other hand, since C is separating in S, its homology class in $H_1(S)$ is the zero element. Hence, the induced map $i_* : H_1(C) \to H_1(S)$ is the zero map. But this is impossible since if i_* is the zero map,

$$0 = r_* \circ i_* = (r \circ i)_* = \mathrm{id}_{H_1}(C), \tag{9}$$

which is a contradiction. Hence, no such retraction can exist.

Problem 2025-A-I-6 (Algebra). Let $f(x)$ be an irreducible polynomial of degree n over a field F, and let $g(x)$ be any polynomial in $F[x]$. Prove that every irreducible factor of the composition $f(g(x))$ has degree divisible by n.

Let $h(x)$ be an irreducible factor of $f(g(x))$ in $F[x]$ and let α be the root of $h(x)$ in some algebraic closure of F. Since h is irreducible and α is a root, the minimum polynomial of α over F is h. Therefore,

$$\deg h = [F(\alpha) : F]. \tag{10}$$

Now since α is a root of $h(x) = f(g(x))$, $f(g(\alpha)) = 0$. In particular, $g(\alpha)$ is a root of f. Since f is irreducible of degree n over F, the minimal polynomial of $g(\alpha)$ over α is f. Hence,

$$[F(g(\alpha)) : F] = n. \tag{11}$$

Since $F \subset F(g(\alpha)) \subset F(\alpha)$, by the Tower Law,

$$\deg h = [F : (\alpha) : F] = [F(\alpha) : F(g(\alpha))] \cdot [F(g(\alpha)) : F] = n[F(\alpha) : F(g(\alpha))], \tag{12}$$

so that $n \mid \deg h$. Hence, this concludes the proof.

Problem 2025-A-II-2 (Geometry/Topology). Consider the plane distribution in $\mathbb{R}^3$ spanned by two vector fields

$$V = \partial_x + 2xy\partial_z, \qquad W = x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z. \tag{13}$$

(i) Show that this distribution is integrable.

(ii) Does the pair of vector fields V and W generate a coordinate system on integral surfaces? If not, find a pair that can play this role for the local integral surfaces passing through points $(0, 0, z_0)$.

(i) Let D be the plane distribution in $\mathbb{R}^3$ spanned by the two vector fields V and W given above. Then by the Frobenius Theorem, D is integrable if and only if D is involutive, which is true if and only if the Lie Bracket of V and W is a smooth section of D at each $p \in \mathbb{R}^3$. We observe that:

$$\begin{aligned} V(W) &= (\partial_x + 2xy\partial_z)\,(x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z) \\ &= \partial_x + (4xy + 2x)\partial_z. \\ W(V) &= \left(x\partial_x + \partial_y + (2x^2y + x^2 - 2y)\partial_z\right)(\partial_x + 2xy\partial_z) \\ &= 2xy\partial_z + 2x\partial_z. \end{aligned} \tag{14}$$

Therefore, for any $p \in \mathbb{R}^3$,

$$[V, W] = V(W) - W(V) = \partial_x + 2xy\partial_z = V. \tag{15}$$

Since V is a smooth section of D, we conclude that D is involutive, and hence integrable.

(ii) Let $\mathcal{S}$ be an integral surface, and assume there are coordinates (u, v) on $\mathcal{S}$ such that $V|_{\mathcal{S}} = \partial_u$ and $W|_{\mathcal{S}} = \partial_v$. Then we observe that $[V|_{\mathcal{S}}, W|_{\mathcal{S}}] = \partial_u(\partial_v) - \partial_v(\partial_u) = 0$. On the other hand,

$$[V|_{\mathcal{S}}, W|_{\mathcal{S}}] = ([V, W])|_{\mathcal{S}} = V|_{\mathcal{S}} \neq 0, \tag{16}$$

which is a contradiction. Therefore, V and W cannot generate a coordinate system on integral surfaces. However, consider the fields $\widetilde{V} = V$ and $\widetilde{W} = W - xV$ on $\mathbb{R}^3$. Then since

$$[\widetilde{V}, \widetilde{W}] = V(W - xV) - (W - xV)(V) = VW - xVV - W(V) + xVV = 0, \tag{17}$$

and so this pair generates a coordinate system on all integral surfaces.

January 2024

Problem 2024-J-I-1 (Algebra). For distinct odd primes p and q, prove that every finite group of order $2pq$ is a semidirect product of a normal subgroup of order pq and a subgroup of order 2.

Let G be a group of order $2pq$, where p, q are distinct odd primes. Without loss of generality, assume $q > p$. By Sylow's Theorem,

$$n_q \in \{1, 2, p, 2p\} \cap \{1, q+1, \dots\} = 1, \tag{18}$$

since $q > 2$ and $q > p$. Therefore, G has a unique, normal, Sylow q-subgroup, which we denote as Q. Let P be a Sylow p-subgroup of G. By the Second Isomorphism Theorem, we conclude that $N = PQ$ is a subgroup of G of order $|P||Q| = pq$. Since $|G : N| = 2pq/(pq) = 2$, where 2 is the smallest prime dividing $|G|$, we conclude that N is a normal subgroup of G. Next, by Cauchy's Theorem, G contains an element of order 2. Let M be the subgroup generated by this element, which also must have order 2. By Lagrange's Theorem, $N \cap M = \{e\}$. Next,

$$|NM| = \frac{|N||M|}{|N \cap M|} = |N||M| = 2pq = |G|, \tag{19}$$

so that $G = NM$. Therefore, we conclude that $G = N \rtimes M$.

Problem 2024-J-I-2 (Geometry/Topology). Let $p : E \to B$ be a covering space map, with B and E path connected. Choose a point $e_0 \in E$ and $b_0 \in B$ such that $p(e_0) = b_0$. This gives us a subgroup $H = p_*\pi_1(E, e_0)$ of the fundamental group $G = \pi_1(B, b_0)$.
Construct a bijection between the fiber $p^{-1}(b_0)$ and the set of right cosets of H and prove that this is indeed a bijection. Prove that the number of sheets of p equals the index $(G : H)$.

Assume all of the given hypotheses. Let $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ be the lifting correspondence induced by p defined by $\phi([f]) = \tilde{f}(1)$, where $\tilde{f}$ is the lift of f, and let $\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$ be the map induced by ϕ. It suffices to prove that Φ is a bijection.

(i) Since E is path connected and $p : E \to B$ is a covering map, the lifting correspondence ϕ must be surjective. Hence, since Φ is induced by ϕ, it follows that Φ is also surjective.

(ii) Now we will show that Φ is injective. Let f and g be two paths in B, and $\widetilde{f}, \widetilde{g}$ their liftings to paths in E that begin at e_0. We must show that $\widetilde{f}(1) = \widetilde{g}(1)$ iff $[f] \in H * [g]$.

($\Leftarrow$) Suppose $[f] = [h * g]$, where $h = p \circ \tilde{h}$ for some loop $\tilde{h}$ in E based at e_0. Since $\tilde{g}$ is a path in E that *begins* at e_0, the product $\tilde{h} * \tilde{g}$ is well-defined. Since $[f] = [h * g]$, it follows that $\tilde{f}$ and $\tilde{h} * \tilde{g}$ must end at the same point. Hence, $\tilde{f}$ and $\tilde{g}$ end at the same point. Therefore, $\phi([f]) = \phi([g])$.

($\Rightarrow$) Suppose $\phi([f]) = \phi([g])$, which means that $\tilde{f}(1) = \tilde{g}(1)$. Then the product of $\tilde{f}$ with the reverse of $\tilde{g}$ is well-defined and is a loop $\tilde{h}$ in E based at e_0. By direct computation, $[\tilde{h} * \tilde{g}] = [\tilde{f}]$. If $\tilde{F}$ is a path homotopy between $\tilde{h} * \tilde{g}$ and $\tilde{f}$, then $p \circ \tilde{F}$ is a path homotopy between $h * g$ and f, which means that $[f] \in H * [g]$. Hence, this concludes the proof that Φ is injective.

Hence, $|p^{-1}(b_0)| = |G/H| = (G : H)$.

Problem 2024-J-I-4 (Algebra). For each field K, prove that the polynomial ring $K[x, y]$ in two variables is not a principal ideal domain.

Let K be a field, and consider the polynomial ring $K[x, y]$. Let (x, y) be the proper ideal of $K[x, y]$ generated by the monomials x and y. Assume to the contrary that $(x, y) = (f(x, y))$ where $f(x, y) \in K[x, y]$ is not a unit of the polynomial ring. Since $x \in (f(x, y))$, $f(x, y) \mid x$. By our assumption that f is not a unit, it follows that $f(x, y)$ is an associate of x. Likewise, $f(x, y)$ must be an associate of y. But this is impossible since x and y are not associates of each other. This forces $f(x, y)$ to be a unit, which means that $(f(x, y)) = K[x, y]$. But this contradicts the fact that $(x, y) = (f(x, y))$ is a proper ideal. Hence, by contradiction, (x, y) is not a principal ideal, and so $K[x, y]$ is not a principal ideal domain.

Problem 2024-J-I-5 (Geometry/Topology). Let α be a closed 1-form on $\mathbb{RP}^n$, $n > 1$. Show that if $f : [0, 1] \to \mathbb{RP}^n$ is a smooth function such that $f(0) = f(1)$, then

$$\int_{[0,1]} f^* \alpha = 0.$$

Include all calculations that are relevant to your solution.

We recall that $H^k(\mathbb{RP}^n) = 0$ for all $0 < k < n$ so that $H^1(\mathbb{RP}^n) = 0$ if $n > 1$. In particular, this means that α is also an exact 1-form on $\mathbb{RP}^n$. Let g be a smooth function on $\mathbb{RP}^n$ so that $\alpha = dg$. Then

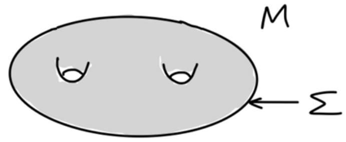
$$\int_0^1 f^* \alpha = \int_0^1 f^* dg = \int_0^1 d(f^* g) = g(f(1)) - g(f(0)) = 0, \tag{20}$$

where the last equality follows from the fact that $f(1) = f(0)$. Hence, the proof concludes.

Problem 2024-J-II-3 (Geometry/Topology). Let Σ be a genus 2 surface embedded in $\mathbb{R}^3$ as shown in the picture. Let M be the closure of the *unbounded* component of $\mathbb{R}^3 \setminus \Sigma$; in other words, M is the part of $\mathbb{R}^3$ which is *not* enclosed by Σ.

(a) Compute $\pi_1(M)$.

(b) Is Σ a retract of M?



(a)

Problem 2024-J-II-6 (Geometry/Topology). Let M be a smooth n-manifold, and let φ be a differential k-form on M which is closed, in the sense that $d\varphi = 0$. At each point $p \in M$, define

$$D_p = \left\{v \in T_pM : v \lrcorner \varphi = 0\right\}, \tag{21}$$

where $\lrcorner$ denotes the interior product. Assume $\ell := \dim D_p$, so that $D \subset TM$ is a rank-ℓ vector subbundle of the tangent bundle of M. Prove that D is an integrable distribution of ℓ-planes, in the sense of the Frobenius Theorem.

By the Frobenius Theorem, it suffices to prove that D is involutive, which is to say that if X, Y are smooth sections of D, then $[X, Y]$ is also a smooth section of D. Indeed, let X, Y be smooth sections of D, which means that $X \lrcorner \varphi, Y \lrcorner \varphi = 0$. Observe that,

$$[X, Y] \lrcorner \varphi = \mathcal{L}_X(Y \lrcorner \varphi) - Y \lrcorner (\mathcal{L}_X \varphi). \tag{22}$$

By hypothesis, $Y \lrcorner \varphi = 0$ so that $\mathcal{L}_X(Y \lrcorner \varphi) = 0$. On the other hand, by Cartan's Formula,

$$\mathcal{L}_X \varphi = d(X \lrcorner \varphi) + X \lrcorner d\varphi = 0, \tag{23}$$

by the hypotheses. Hence, this shows that $[X, Y] \lrcorner \varphi = 0$, and so $[X, Y]$ is a smooth section of D. Therefore, D is involutive, which means that it is Frobenius integrable.

August 2023

Problem 2023-A-II-1 (Algebra). A field extension K/L is called algebraic, if every element in K satisfies a polynomial equation with coefficients in L. Let F, K, L be fields such that $F \supset K \supset L$, and F/K and K/L are algebraic extensions. Prove that F/L is also an algebraic extension.

Since subfields of subfields is a subfield, L is a subfield of F. Hence, it suffices to show that every element in F satisfies a polynomial equation with coefficients in L. Let $a \in F$, and let

$$k(x) = k_n x^n + k_{n-1} x^{n-1} + \cdots + k_0 \in K[x] \tag{24}$$

such that $k(a) = 0$; this follows since F/K is an algebraic extension. Each $k_j \in K$, and hence is algebraic over L. Therefore, $L' = L(k_0, \ldots, k_n)$ is a finite extension of L. Since $k(a) = 0$ and $k(x)$ now has its coefficients in L', it follows that a is algebraic over L' so that $L'(a)$ is a finite extension of L. Then since

$$[L(a) : L] = [L(a) : L'][L' : L], \tag{25}$$

it follows that $L(a)$ is a finite extension of L. Therefore, a is algebraic over L. Since a was arbitrary, F/L is an algebraic extension.

January 2023

Problem 2023-J-II-4 (Geometry/Topology). Prove that $S^2 \times S^2$ is not diffeomorphic to $M_1 \times M_2 \times M_3$, where M_1, M_2, M_3 are smooth manifolds of nonzero dimension.

We begin with a technical lemma, that we will use to prove the desired result.

> (**Comps Lemma**) Let M, N be smooth, connected n-manifolds and $f : M \to N$ a (smooth) immersion. If M is compact and nonempty, then N is compact and f is a (smooth) covering map.
>
> *Proof.* Let M, N be smooth connected n-manifolds, $f : M \to N$ an immersion, and M compact and nonempty. Since $\dim N = n$ everywhere and f is an immersion, $df_p : T_pM \to T_{f(p)}N$ has constant rank n everywhere. Hence, by the Inverse Function Theorem, f is a local diffeomorphism. Since local diffeomorphisms are open maps, $f(M)$ is open in N. Next since the continuous image of compact sets is compact, $f(M)$ is compact in N. Since N is Hausdorff, $f(M)$ must be closed in N. Therefore, since N is connected, we conclude that $f(M) = N$. This means that N is compact and f is surjective. All that remains is to show that f is a covering map.
>
> Let $q \in N$, and consider $f^{-1}(q)$, which is closed in M. For each $x \in f^{-1}(q)$, there exists a neighborhood U_x of x such that $f|_{U_j}$ is a diffeomorphism. Since M is Hausdorff, we may shrink these neighborhoods so that they are pairwise disjoint. This means that each $x \in f^{-1}(q)$ is isolated. Hence, $f^{-1}(q)$ is discrete in M. Since discrete subspaces of compact spaces must be finite, it follows that $f^{-1}(q)$ is finite; let $f^{-1}(q) = \{x_1, \ldots, x_s\}$. As stated above, for each $j = 1, \ldots, s$, we can find a neighborhood U_j of x_j such that $f|_{U_j} : U_j \to V_j \subset N$ is a diffeomorphism. Since M is Hausdorff, we may shrink these neighborhoods so that $U_i \cap U_j = \varnothing$ for all $i \neq j$; f restricted to each of these new U_j's remains a diffeomorphism. Set $V = \bigcap_1^s f(U_j)$, and define $\widetilde{U}_j = f^{-1}(V) \cap U_j$. For each j, $f : \widetilde{U}_j \to V$ is a diffeomorphism and $V = \bigsqcup_1^s f(U_j)$. Hence, V is an evenly covered neighborhood of q, so that f is a covering map. □

Now, assume to the contrary that $f : S^2 \times S^2 \to M_1 \times M_2 \times M_3$ is a diffeomorphism; since diffeomorphisms preserve dimensions and M_1, M_2, M_3 have nonzero dimensions, it follows, without loss of generality, that M_1, M_2 are 1-dimensional and M_3 is 2-dimensional. Since diffeomorphisms of manifolds are immersions, by the Comps Lemma, $M_1 \times M_2 \times M_3$ must be compact and connected; by projecting onto each manifold, M_1, M_2, M_3 must be compact and connected. Moreover, the induced group homomorphism $f_* : \pi_1(S^2 \times S^2) \to \pi_1(M_1 \times M_2 \times M_3) = \pi_1(M_1) \times \pi_1(M_2) \times \pi_1(M_3)$ must be an isomorphism. Since S^2 is simply connected,

$$\pi_1(S^2 \times S^2) = \pi_1(S^2) \times \pi_1(S^2) = \{0\}. \tag{26}$$

On the other hand, since the only compact connected 1-manifold, up to diffeomorphism, is the unit circle S^1, and $\pi_1(S^1) \cong \mathbb{Z}$ is not trivial, $\pi_1(M_1 \times M_2 \times M_3)$ is not trivial. But this contradicts our claim that f_* is an isomorphism. Hence, by contradiction, f cannot be a diffeomorphism.

Problem 2023-J-II-3 (Geometry/Topology). Consider the form $\omega = (x^2 + x + y)dy \wedge dz$ on $\mathbb{R}^3$. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, and $i : S^2 \to \mathbb{R}^3$ be the inclusion map.

(a) Calculate $\int_{S^2} i^*\omega$.

(b) Construct a closed form α on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$, or show that such a form α does not exist.

Problem 2023-A-I-2 (Geometry/Topology). Let $f : T^2 \to S^2$ be a smooth map from the 2-torus to the 2-sphere. Can f be an immersion? If the answer is yes, given an explicit example. If the answer is no, then give a proof.

There cannot be an immersion $f : T^2 \to S^2$. To prove our answer, we will state and proof a technical lemma.

> **(Comps Lemma)** Let M, N be smooth, connected, n-manifolds and $f : M \to N$ a (smooth) immersion. If M is compact and nonempty, then f is a (smooth) covering map.

> *Proof.* Let M, N be smooth connected n-manifolds, M compact, and $f : M \to N$ an immersion. Since $\dim N = n$ everywhere and f is an immersion, $df_p : T_pM \to T_{f(p)}N$ has constant rank n everywhere. Hence, by the Inverse Function Theorem, f is a local diffeomorphism. Let $q \in N$ so that $f^{-1}(q) \subset M$ is closed. For each $x \in f^{-1}(q)$, there exists a neighborhood U_x such that $f|_{U_x} : U_x \to V_x \subset N$ is a diffeomorphism. Since M is Hausdorff, we can shrink these neighborhoods so that they are pairwise disjoint. Since every $x \in f^{-1}(q)$ is now isolated, it follows that $f^{-1}(q)$ is discrete. Since M is compact, we conclude that $f^{-1}(q)$ must be finite; let $f^{-1}(q) = \{x_1, \ldots, x_s\}$. As stated above, for each $j = 1, \ldots, s$, we can find a neighborhood U_j of x_j so that $f|_{U_j} : U_j \to V_j \subset N$ is a diffeomorphism. Again, since M is Hausdorff, we can shrink these neigborhoods so that $U_i \cap U_j = \varnothing$ for all $i \neq j$; f restricted to each of these shrunken neighborhoods remains a diffeomorphism. Now set $V = \bigcap_1^s f(U_j)$, and define $\widetilde{U}_j \subset M$ by $\widetilde{U}_j = f^{-1}(V) \cap U_j$ for each $j = 1, \ldots, s$. Hence, V is an evenly covered neighborhood of $q \in N$, which means f is a covering map. That f is surjective comes from recognizing that $f(M) = N$ due to connectedness of N. □

Now, assume $f : T^2 \to S^2$ is an immersion. Since T^2, S^2 are smooth, connected 2-manifolds, and T^2 is compact and nonempty, by the Comps Lemma, f is a covering map. Hence, the induced homomorphism $f_* : \pi_1(T^2) \to \pi_1(S^2)$ is injective. Since S^2 is simply connected, $\pi_1(S^2) \cong \{0\}$. On the other hand, $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$. Since the order of $\pi_1(T^2)$ is more than one, f_* cannot be injective. Hence, f cannot be an immersion.

Problem 2023-A-II-5 (Geometry/Topology). Let (t, x, y, z) be the standard coordinate system on $\mathbb{R}^4$, and let ϕ be the non-zero smooth 1-form on $\mathbb{R}^4$ defined by

$$\phi = dt + y dx + z dy.$$

Let D be the 3-plane field on $\mathbb{R}^4$ that consists of tangent vectors V such that $\phi(V) = 0$. Is D Frobenius integrable? Support your answer with a proof.

Let D be the 3-plane field on $\mathbb{R}^4$ defined as follows: for each $p \in \mathbb{R}^4$,

$$D_p = \left\{ v \in T_p\mathbb{R}^4 : \phi(v) = 0 \right\} =: \ker \phi_p. \tag{27}$$

Hence, by the Frobenius Theorem, D is Frobenius integrable if and only if $\phi \wedge d\phi = 0$. We compute:

$$d\phi = d(dt + y\,dx + z\,dy) = d^2t + dy \wedge dx + dz \wedge dy = dy \wedge dx + dz \wedge dy. \tag{28}$$

Therefore,

$$\phi \wedge d\phi = dt \wedge dy \wedge dx + dt \wedge dz \wedge dy + y dx \wedge dz \wedge dy. \tag{29}$$

Since $\phi \wedge d\phi$ is nowhere vanishing on $\mathbb{R}^4$, D is not Frobenius integrable.

Problem 2023-A-I-1 (Algebra). Let V be a n-dimensional vector space over a field F. An element $A \in \operatorname{End} V$ is called *nilpotent*, if $A^k = 0$ for some $k > 1$. Prove that A is nilpotent if and only if

$$\operatorname{Tr}(\Lambda^i A) = 0, \quad i = 1, \ldots, n, \tag{30}$$

where $\Lambda^i A$ denotes the induced action of A on the wedge product $\Lambda^i V$ for each i.

Let V be a n-dimensional vector space over a field F, and let $A \in \operatorname{End} V$. Recall that $\Lambda^i A$, the induced action of A on the wedge product $\Lambda^i V$, is defined to be

$$(\Lambda^i A)(v_1 \wedge \cdots \wedge v_i) = Av_1 \wedge \cdots \wedge Av_i, \qquad v_j \in V \text{ for all } j = 1, \dots, i. \tag{31}$$

Over an algebraic closure of F, A has eigenvalues $\lambda_1, \dots, \lambda_n$. Suppose A is diagonalizable, with the set of eigenvectors given by $\{v_1, \dots, v_n\}$. Then for each $i = 1, \dots, n$, since the collection

$$\left\{v_{j_1} \wedge \cdots \wedge v_{j_i} : 1 \le j_1 < \cdots < j_i \le n\right\}$$

is a basis of $\Lambda^i V$, and for each i-tuple, $\Lambda^i A(v_{j_1} \wedge \cdots \wedge v_{j_i}) = Av_{j_1} \wedge \cdots \wedge Av_{j_i} = (\lambda_{j_1} \cdots \lambda_{j_i})(v_{j_1} \wedge \cdots \wedge v_{j_i})$, it follows that the eigenvalues of $\Lambda^i A$ are the set of all products of the form $\lambda_{j_1} \cdots \lambda_{j_i}$ for $1 \le j_1 < \cdots < j_i \le n$, counting for multiplicity. Hence,

$$\operatorname{Tr}(\Lambda^i A) = \sum_{1 \le j_1 < \cdots < j_i \le n} \lambda_{j_1} \cdots \lambda_{j_i}. \tag{32}$$

If A is not diagonalizable, since the eigenvalues of $\Lambda^i A$ depend only on the eigenvalues of A, we may assume A is in Jordan normal form. Indeed, if $A = PJP^{-1}$, then

$$\Lambda^i(A) = \Lambda^i(PJP^{-1}) = \Lambda^i(P)\Lambda^i(J)\Lambda^i(P^{-1}), \tag{33}$$

so $\Lambda^i A$ and $\Lambda^i J$ are similar and therefore have the same eigenvalues. Thus it suffices to compute the eigenvalues of $\Lambda^i J$, which are exactly the products $\lambda_{j_1} \cdots \lambda_{j_i}$ of the eigenvalues of A.
If A is nilpotent so that $A^k = 0$ for some $k > 1$, then since $0 = A^k v = \lambda^k v$ for all eigenvectors v of A, it follows that every eigenvalue of A is zero. Therefore, the above expression implies that $\operatorname{Tr}(\Lambda^i A) = 0$ for all $i = 1, \dots, n$. On the other hand, expanding the characteristic polynomial for A is given by:

$$p_A(t) = \det(tI - A) = t^n - \operatorname{Tr}(\Lambda^1 A)t^{n-1} + \cdots + (-1)^n \operatorname{Tr}(\Lambda^n A). \tag{34}$$

If $\operatorname{Tr}(\Lambda^i A) = 0$ for all $i = 1, \dots, n$, then we conclude that the characteristic polynomial of A is precisely t^n. Therefore, A's eigenvalues are all zero. Hence, the minimal polynomial of A is of the form t^k for some $k \le n$. This implies that $A^k = 0$, and so A is nilpotent.

Problem 2023-A-II-6 (Complex Analysis). Find the number of solutions (counting multiplicity) to $z^8 - 5z^6 + 2z^3 - z - 1 = 0$ that lie inside the unit disk.

Recall Rouché's Formula, which states that

> For any two complex-valued functions f and g holomorphic inside some region K with closed and simple contour ∂K, if $|g(z)| < |f(z)|$ on ∂K, then f and $f+g$ have the same number of zeros inside K, where each zero is counted as many times as its multiplicity.

Pick $f(z) = 5z^6$ and set $h(z) = z^8 + 2z^3 - z - 1$ so that $p(z) = z^8 - 5z^6 + 2z^3 - z - 1 = h(z) - f(z)$. On the unit disk ∂S^1, we observe that

$$\begin{aligned} |f(z)| &= \left|5z^6\right| = 5 \\ &= 1 + 2 + 1 + 1 \\ &= |z^8| + 2|z^3| + |z| + |1| \\ &\ge |h(z)|. \end{aligned} \tag{35}$$

Hence, $p(z) = h(z) - f(z)$ has the same number of zeros, counting multiplicity, as $f(z)$. Since $f(z)$ has six zeros in the unit disk, we conclude that $p(z)$ must also have six zeros inside the unit disk.