

1 Basic Category Theory

Remark 1.1. The notion of “class” in ZFC requires a definition. This can be done (“parametrically definable classes”), but for now let’s blackbox the concept and just say “classes are a generalization of sets which are allowed to be ‘large’”, e.g. so a “class of all sets” makes sense even though a “set of all sets” does not. This shouldn’t matter ultimately anyways.

Definition 1.1. A **Category**, \mathcal{C} , is a triple of classes, $\text{obj}(\mathcal{C})$, $\text{Hom}(\mathcal{C})$, and \circ AKA $\text{Comp}(\mathcal{C})$. For every $a, b \in \text{obj}(\mathcal{C})$, we have some set $\text{Hom}_{\mathcal{C}}(a, b) \subset \text{Hom}(\mathcal{C})$ and ok blabla composition is a map $\text{Hom}_{\mathcal{C}}(a, b) \times \text{Hom}_{\mathcal{C}}(b, c) \rightarrow \text{Hom}_{\mathcal{C}}(a, c)$ such that it’s associative, and each $\text{Hom}_{\mathcal{C}}(a, a)$ has an identity id_a under composition.

Definition 1.2. Given a category \mathcal{C} , we define its opposite category \mathcal{C}^{op} to be the category

Definition 1.3. Given two categories, \mathcal{C} and \mathcal{D} , a **functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ is

Now, sometimes we wish to talk about things which don’t *respect* composition, but instead reverse it. These still get to be called functors, but we call those “contravariant functors” and (even if we *really* think of them as “composition reversing functors $\mathcal{C} \rightarrow \mathcal{D}$ ”) it’ll end up being nice to think of these quite formally as functors $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ —e.g. they “preserve reverse composition of arrows”.

Now we know what a functor is, we should talk about some examples and basic properties thereof.

Definition 1.4. A functor, $F : \mathcal{C} \rightarrow \mathcal{D}$, is said to be:

- **faithful** if the induced map $\text{Hom}_{\mathcal{C}}(a, b) \rightarrow \text{Hom}_{\mathcal{D}}(Fa, Fb)$ is injective for all $a, b \in \text{obj}(\mathcal{C})$
- **full** if the induced map $\text{Hom}_{\mathcal{C}}(a, b) \rightarrow \text{Hom}_{\mathcal{D}}(Fa, Fb)$ is surjective for all $a, b \in \text{obj}(\mathcal{C})$
- **bijective** if the induced map $\text{Hom}_{\mathcal{C}}(a, b) \rightarrow \text{Hom}_{\mathcal{D}}(Fa, Fb)$ is bijective for all $a, b \in \text{obj}(\mathcal{C})$

Now, in many contexts we want to compare two categories, talk about whether or not they’re the “same”. In such circumstances we *could* require that we have a pair of inverse functors— an “isomorphism of categories”— but the problem is that this is way too strong (to the point of coming down to “how strict are we about what sets we use to define our objects”). It is, for instance, perfectly valid linear algebra- wise to imagine all finite \mathbb{R} -vector spaces as just “vector spaces \mathbb{R}^n ”, but the two categories *aren’t* equivalent because one only has a \mathbb{Z}^+ worth of sets while the other has way more. The notion of “isomorphism” is thus the “wrong definition”. For a better definition, rather than being “bijective on objects”, we require a sort of weaker notion:

Definition 1.5. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be **essentially surjective** if, for every $d \in \text{obj}(\mathcal{D})$, there exists some $c \in \text{obj}(\mathcal{C})$ such that $Fc \cong d$.

That is to say, F “reaches all of \mathcal{D} (up to abstract isomorphism)” With this in mind, we can formulate the following:

Definition 1.6. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be a **(weak) equivalence of categories** if F is fully faithful and essentially surjective.

This is to say: morphisms in \mathcal{C} and \mathcal{D} can be thought of interchangeably—so long as you’re willing to transport along abstract isomorphisms in \mathcal{D} . Subscribing to the philosophy that “objects are but vessels for morphisms” (hey, I don’t wanna bother with the yoneda lemma here), this tells us the two categories are “morally” the same. The reason for the parenthetical “weak” is more or less a “you can choose to be careful about the axiom of choice”— in particular, there’s a natural notion of a “(homotopy) equivalence of categories”— cf. ??, which is nice and cool but requires a bit more terminology. With Axiom of choice these are the same, but without axiom of choice this notion is “weaker”. Jason has said he often means “weak equivalence” when he says equivalence though (even though he’s distinguished them in the past?), so mostly it’s a “have the homotopic notion in mind morally, and the weak notion in mind for like 80% of practical examples.”

Definition 1.7. Given two functors, $F, G : \mathcal{C} \rightarrow \mathcal{D}$, a **natural transformation** $\eta_{\bullet} : F \rightarrow G$ is

Remark 1.2. *If the notation was not itself suggestive enough (and perhaps it wasn't at this high a level of abstraction, alas), a “natural transformation” is a good concept of “map between functors”– this should already be a somewhat familiar notion to those who know some representation theory: A representation of G over k (take $k = \mathbb{R}, \mathbb{C}$ for comfort if you like) is a sort of map (indeed, literally a functor) $G \rightarrow \mathbf{Vect}_k$, and an equivariant map between two representations is analogous to (indeed, literally is) a natural transformation between them. People also sometimes like to illustrate these with a “globular diagram”*

$$\begin{array}{ccc} & F & \\ \swarrow & \Downarrow \eta \bullet & \searrow \\ \mathcal{C} & & \mathcal{D} \\ \nwarrow & G & \nearrow \end{array}$$

, which in particular should make you imagine a sort of “directed” homotopy from F to G (and indeed, one can kind of formulate homotopies of maps of topological spaces in this “maps between maps” language blabla)– in a bit, we'll see some concepts which are roughly “homotopy equivalences of categories”, so this intuition is by no means off the mark.

“why do we care about this?” well, to curry your question into a different question, we define another thing

Definition 1.8. *A pair of functors $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{D} \rightarrow \mathcal{C}$ are said to be an **adjoint pair** if any of the following equivalent definitions hold:*

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Remark 1.3. *Each of these definitions gives a slightly different insight into what a “adjunction” really is, namely:*

Remark 1.4. *Technically, a proof that these definitions are equivalent is required. Practically, it's pure abstract nonsense and not particularly insightful. For completeness, I provide the proofs here (actually idk maybe I don't, the proof's on nlab and wikipedia probably) , but you shouldn't need to remember them nearly so much as (the intuition behind) the definitions.*

Lemma 1.1. *(the different concepts of adjunction are equivalent)*

Now that we have the requisite language, we can finally formulate a “(homotopy) equivalence of categories”:

Definition 1.9. *Two categories, \mathcal{C}, \mathcal{D} are said to be **equivalent** if there exists pair of functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ and a pair of natural isomorphisms $\eta : \text{id}_{\mathcal{C}} \rightarrow F \circ G$ and $\varepsilon : G \circ F \rightarrow \text{id}_{\mathcal{D}}$.*

Remark 1.5. *Any equivalence is an adjunction by the counit-unit definition above, and indeed: an equivalence is precisely an adjunction for which the counit and unit transformations are iso.*

2 Tensor products (even over potentially non-commutative rings) and more miscellaneous module theory

Definition 2.1. ***R -balanced maps***

Definition 2.2. ***tensor product***

That's a universal property or what have you– really, it's the same philosophy as viewing the “tensor product of V and W ” as “the multilinear maps from $V \times W$ ”, we're just being cautious about commutativity and being a smidgeon more careful since not every module is free here. Anyways, “how/why does this universal object actually exist?”, well, we just take a super big free module and the mod out by what it means to be R -balanced, aka by what it means to be (non-commutatively) multilinear. We give the proof of existence as follows:

Lemma 2.1. *(tensor products exist)*

The tensor product has, itself, a deep relation to \mathbf{Hom} -functors– namely, we have the *tensor-hom adjunction*:

Theorem 2.1.

3 (co)homological algebra

Definition 3.1. If $F : {}_R\text{Mod} \rightarrow {}_S\text{Mod}$ is a functor of module categories (it can be different sided module categories too, it doesn't matter), F is said to be additive if, given any two $\varphi, \psi \in \text{Hom}_R(M, N)$, we have $F(\varphi + \psi) = F(\varphi) + F(\psi)$ —that is, on each Hom it's a homomorphism of additive groups.

I'm really only mentioning the word “additive” here so that I can say “additive” later on without needing to refer you to the stuff on abelian categories (at least, not yet).

Definition 3.2. A module is said to be **projective** if

4 Generalities on “Abelian Categories”

Remark 4.1. Jason might require us to technically be able to unravel the definition, but, in practice “abelian category” just means “category which we can think of like a category of left/right R -modules”. This is possible to formalize by the ??Freyd-Mitchell Embedding Theorem, which is technically useful. It's also complete abstract nonsense, and for every almost every single thing we talk about, you can use the words “abelian category” interchangeably with “category of R -modules” (or, if you really want, “category of chain complexes of R -modules”), so just think in terms of whatever's easiest.

Theorem 4.1. (Freyd-Mitchell Embedding) Let \mathcal{A} be a (small) abelian category. Then, there is a ring R (not necessarily commutative) and a full, faithful, exact (hence additive) functor $i : \mathcal{A} \hookrightarrow {}_R\text{Mod}$ the category of left R -modules.

You do *not* need to memorize this (nor will we see a proof), but I feel it necessary to mention it as a name to look up/be referred to.