



# Quantum States: Basic Quantum Mechanics Concepts for Quantum Computing

Ren Tristan A. de la Cruz Project Technical Specialist I Quantum Circuit Simulation Project DOST Advanced Science and Technology Institute 2023 November





## Computation is Information Processing



#### A <u>model of computation</u> defines the following:

- 1. representation of **information**
- 2. how information is **processed** (valid operations).

	Classical Computing Model	Quantum Computing Model
Unit of Information	Bit	Quantum Bit (Qubit)
Info. Representation	Bit Strings (Binary Strings)	Quantum State (Multi-Qubits)
Operations	Binary String Operations	Quantum Operations

The focus of this presentation is the introduction of quantum mechanics concepts related to *quantum states*.





## Classical Two-State System: Storing One Bit of Information



- A bit b can either be 0 or 1.
- Any system that have (at least) two states is needed. It is called classical two-state system.
- Let state A and state B be the two states of the system.
- Convention: b = 0 if the system is in state A
- Convention: b = 1 if the system is in state B

System	State A	State B
Toggle Switch	OFF	ON
Abacus Bead	Bead is down	Bead is up
Punched Tape	No Hole	Has Hole









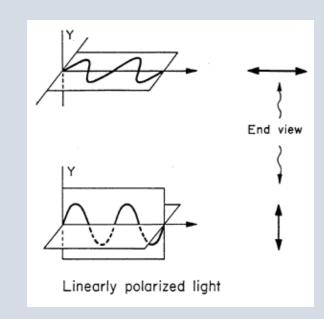


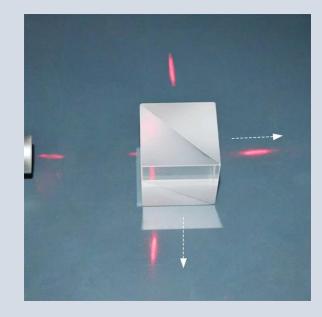
### **Quantum Two-State Systems**



- A qubit q can be | 0 or | 1 >.
- | 0 | pronounced "ket 0", | 1 | pronounced "ket 1"
- In a quantum two-state system, there are also two *observable* states.
- Let the observable states be state | A) and state | B)
- state | A > pronounced "ket A", state | B > pronounced "ket B"

System	State   A >	State B>
Light Polarity	horizontal	vertical
Photon Direction	horizontal	vertical







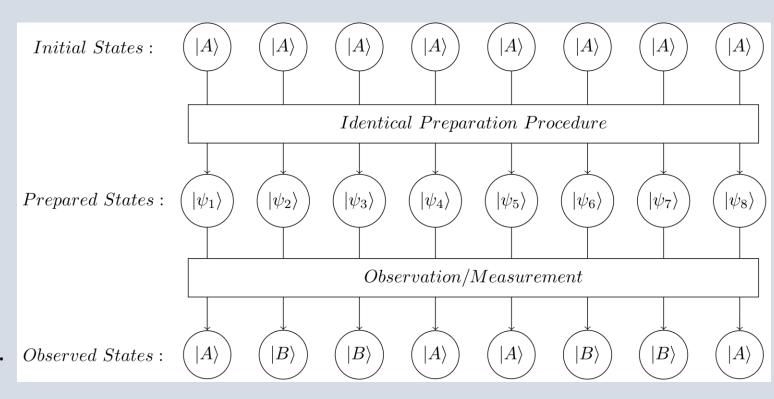


## Quantum Indeterminacy & Quantum Two-State Systems



## **Quantum Indeterminacy** Experiment Concept:

- Prepare multiple quantum twostate systems. The system should be identical and have the same initial state | A).
- 2. Apply an identical preparation procedure **U** to all systems.
- 3. Observe the state of each system. Observed States:



**Expectation**: The observed states should be identical.

**Results:** For quantum systems and certain procedures, the observed states are not guaranteed to be identical.





## Quantum Indeterminacy & Quantum Two-State Systems



The experiment is formulated in a such a way as to <u>eliminate external factors that can affect</u> the state of the quantum system.

**Conclusion:** After applying the preparation procedure but before observation or measurement, the <u>prepared states are not yet determined</u> (*indeterminate*). Only after observation or measurement will a state be in one of the two definite states.

**Question:** Why even use quantum two-state systems to store information and do computations if there are situations where the state of the system is indeterminate?

**Answer:** The indeterminacy of a quantum systems is to **not totally random**. By extending the experiment you extract information about the indeterminate state.



## Quantum Indeterminacy & Probabilities



#### **Quantum Indeterminacy** Experiment (extended):

Parameter: **n** – number of identical quantum systems

Parameter: initial state  $|\psi\rangle$  (select one:  $|\psi\rangle = |A\rangle$  or  $|\psi\rangle = |B\rangle$ )

Parameter: procedure **U** 

Output:  $n_A$  – number of systems observed to be in state  $|A\rangle$ 

Output:  $n_B$  – number of systems observed to be in state  $|B\rangle$ 

Compute:  $p_A = (n_A/n)$  – portion of the systems observed to be in state  $|A\rangle$ 

Compute:  $p_B = (n_B/n)$  – portion of the systems observed to be in state  $|B\rangle$ 

**Task**: Increase **n** and observe what values  $p_A$  and  $p_B$  approach.

**Result:** For the given parameters,  $|\psi\rangle$  and U,  $p_A$  and  $p_B$  approach some specific values.



### Probabilities, Superposition, and the Qubit



The prepared (aka process) quantum state can be **indeterminate** prior to observation. Knowing  $|\psi\rangle$  and U, we can still attach the information the following information to the quantum state:

- **p**<sub>A</sub> probability of observing | A)
- p<sub>B</sub> probability of observing | B)

 $|\psi'\rangle$  – be prepared/processed state, the state of the system after applying procedure **U** to initial quantum state  $|\psi\rangle$ 

 $|\psi'\rangle = \alpha |A\rangle + \beta |B\rangle$ , qubit  $|\psi'\rangle$  is a *superposition* of *basis states*  $|A\rangle$  and  $|B\rangle$ 

- $\alpha$  amplitude of state  $|A\rangle$ ,  $p_A = |\alpha|^2$
- $\beta$  amplitude of state  $|B\rangle$ ,  $p_B = |\beta|^2$

 $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ , we will now use the <u>logical basis states</u>  $|0\rangle$  and  $|1\rangle$ 





## Some Examples of Qubits



Qubit	Ampl. of $ 0 angle$	Ampl. of $ 1 angle$	Prob. of $ 0 angle$	Prob. of $ 1 angle$
$ \psi_1 angle=1 0 angle+0 1 angle= 0 angle$	1	0	$1^2 = 1$	$0^2 = 0$
$ \psi_2 angle=0 0 angle+1 1 angle= 1 angle$	0	1	$0^2 = 0$	$1^2 = 1$
$ \psi_3 angle=rac{1}{\sqrt{2}} 0 angle+rac{1}{\sqrt{2}} 1 angle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$	$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$
$ \psi_4 angle=rac{1}{\sqrt{4}} 0 angle+rac{\sqrt{3}}{\sqrt{4}} 1 angle$	$\frac{1}{\sqrt{4}}$	$\frac{\sqrt{3}}{\sqrt{4}}$	$\left(\frac{1}{\sqrt{4}}\right)^2 = \frac{1}{4}$	$\left(\frac{\sqrt{3}}{\sqrt{4}}\right)^2 = \frac{3}{4}$
$ \psi_5 angle=rac{\sqrt{5}}{\sqrt{8}} 0 angle+rac{\sqrt{3}}{\sqrt{8}} 1 angle$	$\frac{\sqrt{5}}{\sqrt{8}}$	$\frac{\sqrt{3}}{\sqrt{8}}$	$\left(\frac{\sqrt{5}}{\sqrt{8}}\right)^2 = \frac{5}{8}$	$\left(\frac{\sqrt{3}}{\sqrt{8}}\right)^2 = \frac{3}{8}$





#### Rule on Probabilities & the Born Rule



```
Qubit: |\psi\rangle = \alpha |0\rangle + \beta |1\rangle
```

- $\alpha$ , amplitude of state  $|0\rangle$
- $\beta$ , amplitude of state  $|1\rangle$

$$p_0 = |\alpha|^2$$
, probability of observing state  $|0\rangle$   
 $p_1 = |\beta|^2$ , probability of observing state  $|1\rangle$ 

```
Probability Rule: p_0 + p_1 = 1, alternately: p_0 = (1-p_1) or p_1 = (1-p_0)
Born Rule: |\alpha|^2 + |\beta|^2 = 1, or simply \alpha^2 + \beta^2 = 1 if \alpha and \beta are real numbers.
```

The Born Rule is the probability rule expressed in terms of amplitudes.





#### The Born Rule & Qubits as Points on a Circle



Qubit: 
$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

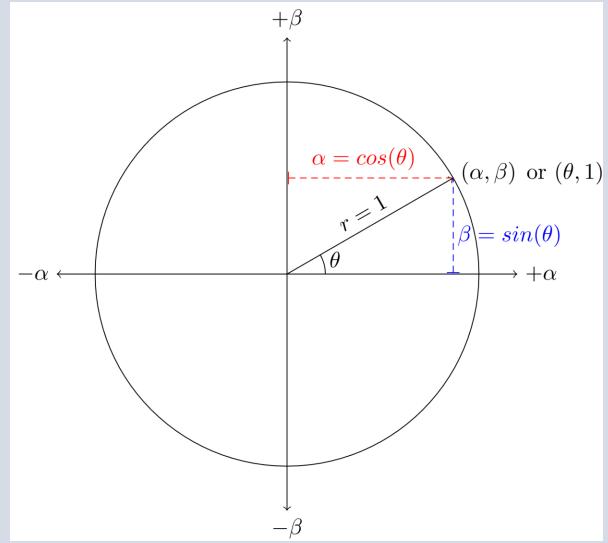
Born Rule: (for real numbers)

$$\alpha^2 + \beta^2 = 1$$

- circle on the  $\alpha\beta$ -plane
- center on (0,0)
- radius = 1

$$x^2 + y^2 = 1$$

- circle on the xy-plane
- center on (0,0)
- radius = **1**





#### Qubits as Points on a Circle



Qubit: 
$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

 $(\alpha,\beta)$ 

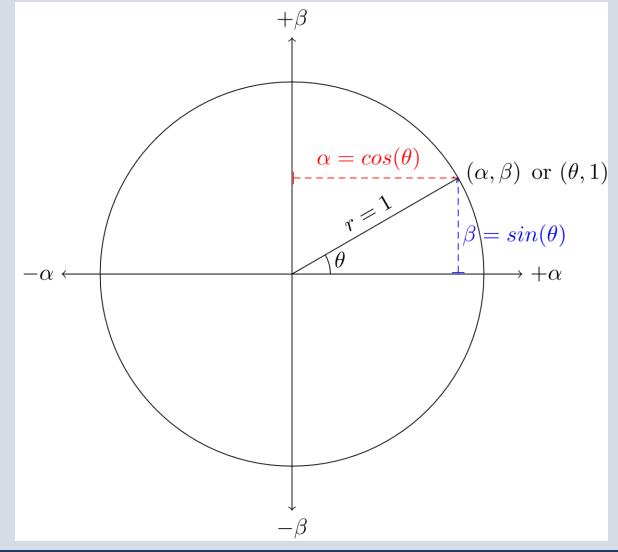
- point on the circle
- represents  $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$

$$(\theta, 1)$$

- point on the circle
- polar coordinate of (α,β)

Qubit: 
$$|\psi_{\theta}\rangle = cos(\theta)|0\rangle + sin(\theta)|1\rangle$$

- polar coordinate form
- $\alpha = cos(\theta)$
- $\beta = sin(\theta)$





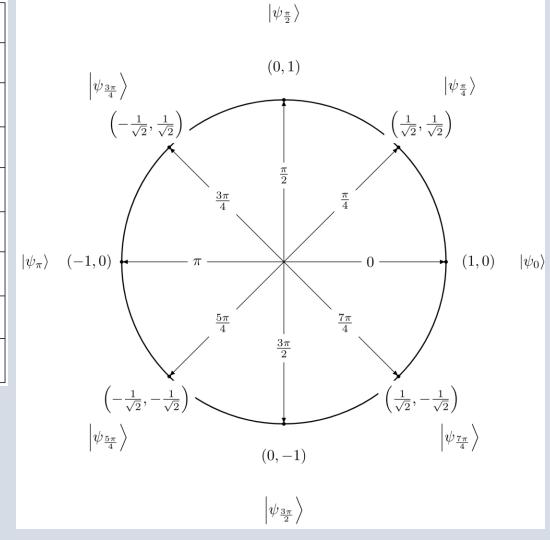


## Qubits as Points on a Circle: Examples



θ	$\alpha = cos(\theta)$	eta = sin( heta)	$ \psi_{ heta} angle = lpha  0 angle + eta  1 angle$
0	1	0	$ \psi_0 angle=1 0 angle+0 1 angle= 0 angle$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$ \psi_{rac{\pi}{4}} angle=rac{1}{\sqrt{2}} 0 angle+rac{1}{\sqrt{2}} 1 angle$
$\frac{\pi}{2}$	0	1	$ \psi_{rac{\pi}{2}} angle=0 0 angle+1 1 angle= 1 angle$
$\frac{3\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$ \psi_{rac{3\pi}{4}} angle = -rac{1}{\sqrt{2}} 0 angle + rac{1}{\sqrt{2}} 1 angle$
$\pi$	-1	0	$ \psi_\pi angle=-1 0 angle+0 1 angle=- 0 angle$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$ \psi_{rac{5\pi}{4}} angle = -rac{1}{\sqrt{2}} 0 angle -rac{1}{\sqrt{2}} 1 angle$
$\frac{3\pi}{2}$	0	-1	$ \psi_{rac{3\pi}{2}} angle=0 0 angle-1 1 angle=- 1 angle$
$\frac{7\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$ \psi_{rac{7\pi}{4}} angle=rac{1}{\sqrt{2}} 0 angle-rac{1}{\sqrt{2}} 1 angle$

Qubits that encode the **same probabilities** but have <u>different amplitudes</u> (different signs of amplitudes) are said to have <u>different phases</u>.







#### Qubits as Points on a Circle



Due to the <u>born rule</u> and our initial **restriction** that the <u>amplitudes are real numbers</u> (not yet complex numbers), we are able to define **any unique qubit** using a single parameter which is the angle  $\theta$ .

Qubit	$ \psi\rangle = \alpha  0\rangle + \beta  1\rangle$
Point (Rectangular Coordinate)	(α, β)
Point (Polar Coordinate)	( <del>0</del> , 1)
Qubit	$ \psi_{\theta}\rangle = cos(\theta) 0\rangle + sin(\theta) 1\rangle$

If you look at the <u>probability rule</u>  $p_0 + p_1 = 1$ , it can help you see that you only need 1 parameter to define a qubit with real amplitudes. If you specify the qubit's  $p_0$ , due to the probability rule, you can derive  $p_1 = (1-p_0)$  from  $p_0$ . They are not independent values. The same goes for the amplitudes. *i.e.*  $\beta = \sqrt{(1-\alpha^2)}$ 





## Operations on Single Qubits



- An operation U transforms a qubit by changing the qubit's amplitudes.
- The **operation U** should follow the <u>born rule</u>.

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle \longrightarrow U \longrightarrow |\psi'\rangle = \alpha' |0\rangle + \beta' |1\rangle$$

$$(\alpha, \beta) \longrightarrow U \longrightarrow (\alpha', \beta')$$

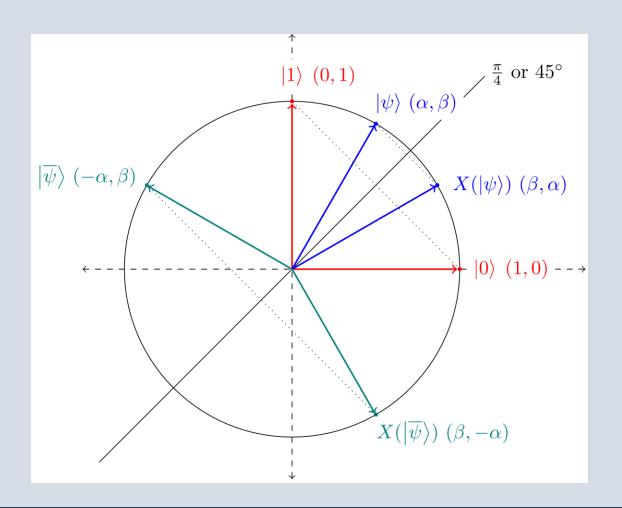
$$(\theta, 1) \longrightarrow U \longrightarrow (\theta', 1)$$

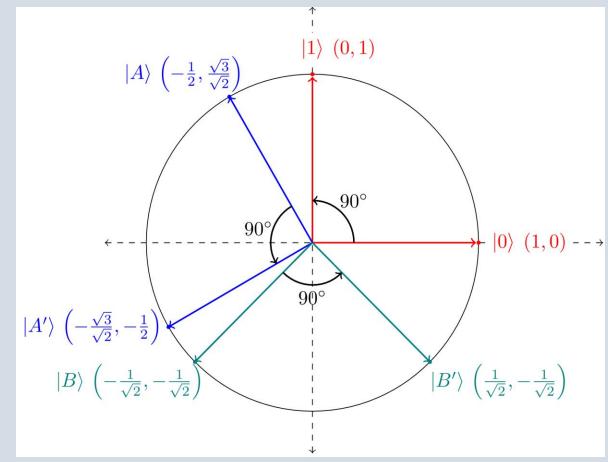


## Operations on Single Qubits



• Qubit **operations** can be viewed as <u>rotations</u> or <u>reflections</u>.







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## **Qubits with Complex Amplitudes**



• In general, the amplitudes,  $\alpha$  and  $\beta$ , of a qubit  $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$  are complex numbers.

#### **Real Amplitudes**

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

- $\alpha = x$
- β = y
- x, y are real numbers

#### **Complex Amplitudes**

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

- $\alpha = x + qi$  (complex)
- $\beta = y + ri$  (complex)
- x,q are real numbers
- **y, r** are real numbers

- When the amplitudes are real: two real numbers are needed to define a unique qubit
- When the amplitudes are complex: four real numbers are needed to define a unique qubit





## **Qubits with Complex Amplitudes**



The square of a complex number is also complex:

- $\alpha = x + qi$
- $\alpha^2 = (x + qi)^2 = (x + qi)(x + qi) = x^2 + 2xqi q^2 = (x^2 q^2) + (2xq)i$

This means we cannot use simple squaring to get <u>real probability</u> from the complex amplitude. The **square modulus**  $|\alpha|^2$  of a complex number  $\alpha = x + qi$  is a real non-negative and can be used as probability:

• 
$$|\alpha|^2 = (x + qi)(x - qi) = x^2 + (xqi - xqi) + q^2 = x^2 + q^2$$

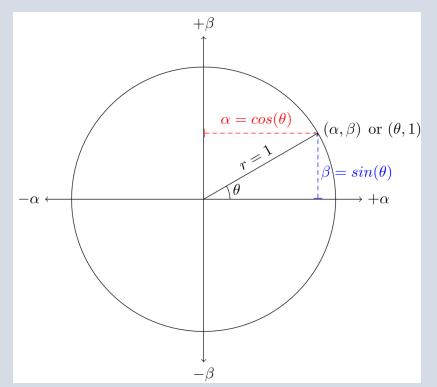


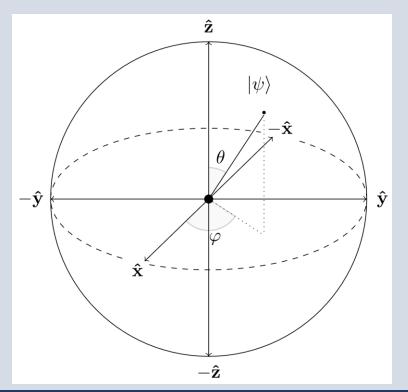


## Qubits with Complex Amplitudes + Born Rule



- Due to the **born rule**, when the <u>amplitudes</u>  $\alpha$  and  $\beta$  are <u>real</u>, you only need **1** parameter, the **angle**  $\vartheta$ , to define a unique qubit. From **2** <u>real parameters</u>  $\alpha$  and  $\beta$  to **1** parameter  $\vartheta$ .
- For the general qubit with <u>complex amplitudes</u>  $\alpha = x + qi$  and  $\beta = y + ri$ , using the born rule, the 4 <u>real parameters</u> x, q, y, r can be reduced 2 angle parameters  $\vartheta$  and  $\varphi$ .



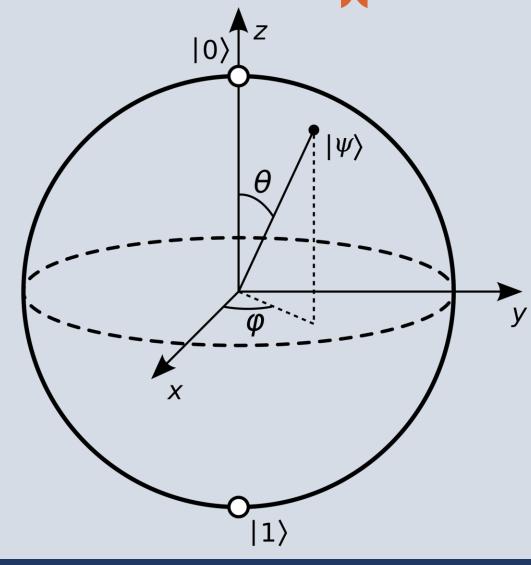




## Qubit & the Bloch Sphere

- A qubit with complex amplitudes can be represented a **point on a sphere** of radius 1. The sphere is called the *Bloch sphere*.
- In spherical coordinates, you only need two parameters, angle  $\vartheta$  and angle  $\varphi$ , to define a unique point on the sphere.
- The qubit specified by the two angles  $\vartheta$  and  $\varphi$ :

$$|\psi
angle = egin{array}{c} cos\left(rac{ heta}{2}
ight)|0
angle + e^{iarphi}sin\left(rac{ heta}{2}
ight)|1
angle \ egin{array}{c} egin{array}{c} eta \end{array}$$







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## Qubits can be Represented as Column Vectors



The qubit  $|0\rangle$  can be represented by the column vector:

$$|0
angle = egin{bmatrix} 1 \ 0 \end{bmatrix}$$

The qubit  $|1\rangle$  can be represented by the column vector:

$$|1
angle = egin{bmatrix} 0 \ 1 \end{bmatrix}$$

The generic qubit  $|\psi
angle=lpha|0
angle+eta|1
angle$  can be represented by column vector:

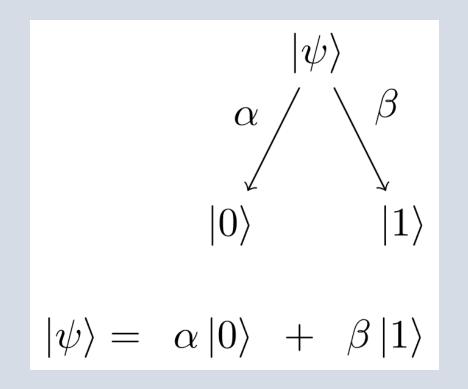
$$\ket{\psi} = egin{bmatrix} lpha \ eta \end{bmatrix} = lpha egin{bmatrix} 1 \ 0 \end{bmatrix} + eta egin{bmatrix} 0 \ 1 \end{bmatrix}$$

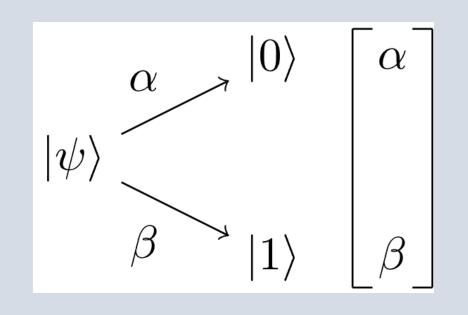




## Qubits drawn as Binary Trees









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#### Quantum State



- For quantum computing, a quantum state is simply a collection of qubits.
- If you have two qubits,  $|q_1\rangle$  and  $|q_2\rangle$ , the quantum state can be written as  $|q_1q_2\rangle$ .
- Example 1: If  $|q_1\rangle = |0\rangle$ ,  $|q_2\rangle = |0\rangle$ , then  $|q_1q_2\rangle = |00\rangle$ .
- Example 2: If  $|q_1\rangle = |0\rangle$ ,  $|q_2\rangle = |1\rangle$ , then  $|q_1q_2\rangle = |01\rangle$ .
- Example 3: If  $|q_1\rangle = |1\rangle$ ,  $|q_2\rangle = |0\rangle$ , then  $|q_1q_2\rangle = |10\rangle$ .
- Example 4: If  $|q_1\rangle = |1\rangle$ ,  $|q_2\rangle = |1\rangle$ , then  $|q_1q_2\rangle = |11\rangle$ .



#### Quantum State



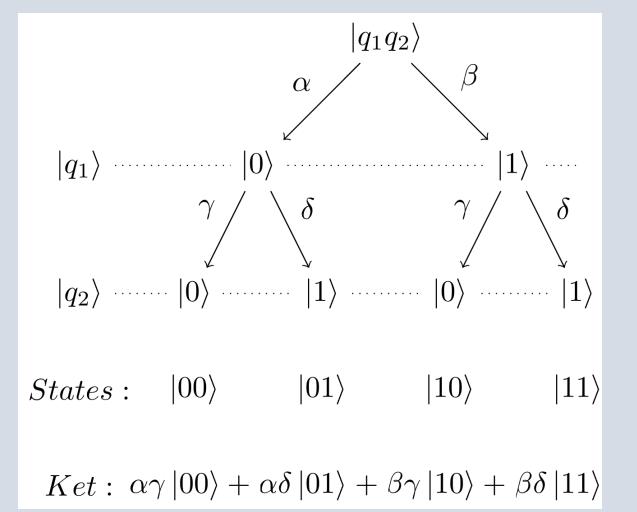
If you have the <u>independent</u> two qubits  $|\mathbf{q}_1\rangle = \alpha |0\rangle + \beta |1\rangle$  and  $|\mathbf{q}_2\rangle = \gamma |0\rangle + \delta |1\rangle$ , the quantum state can be written as:

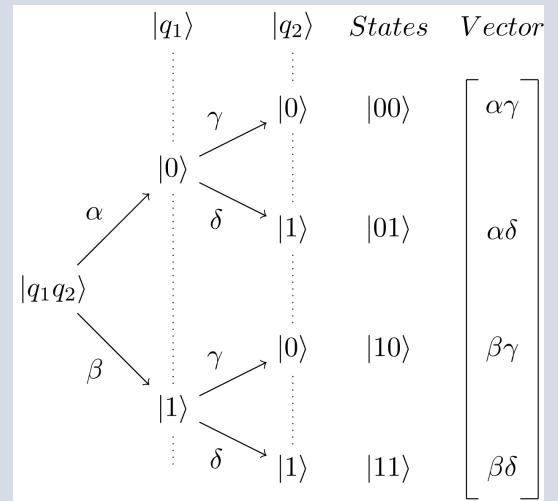
- $|q_1q_2\rangle = |q_1\rangle \otimes |q_2\rangle = \alpha \gamma |00\rangle + \alpha \delta |01\rangle + \beta \gamma |10\rangle + \beta \delta |11\rangle$
- $|q_1\rangle \otimes |q_2\rangle$  is called the <u>tensor product</u> of states/qubits  $|q_1\rangle$  and  $|q_2\rangle$ .
- $|q_1\rangle$  and  $|q_2\rangle$  both have the basis states:  $|0\rangle$  and  $|1\rangle$ .
- $|q_1\rangle \otimes |q_2\rangle$  has the basis states:  $|00\rangle$ ,  $|01\rangle$ ,  $|10\rangle$ , and  $|11\rangle$



## Two-Qubit Quantum State as Binary Trees











## **Entanglement of Multiple Qubits**



- Two qubits are **entangled** if their *amplitudes* (or probabilities) are <u>correlated</u>.
- You can entangle two qubits using 2-qubit entanglement operators.

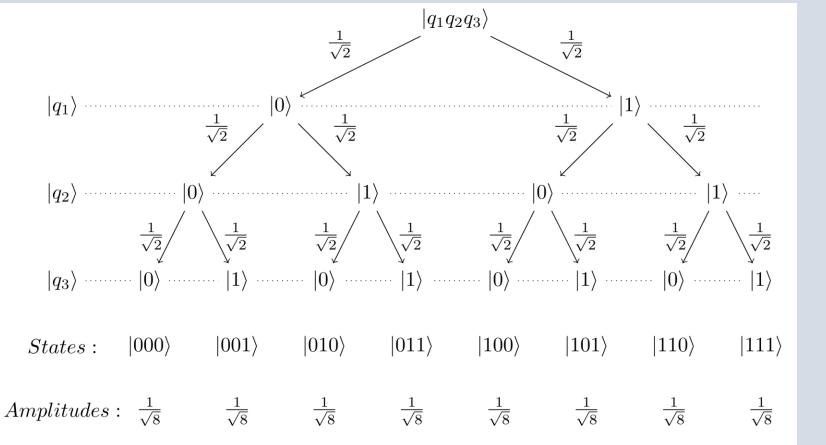
- $|q_1'\rangle$  and  $|q_2'\rangle$  are no longer independent but are <u>correlated</u>.
- The amplitudes  $\alpha'$ ,  $\beta'$  are <u>correlated</u> with amplitudes  $\gamma'$ ,  $\delta'$ .







**Example 0:** No entanglement. No amplitude correlation.



Initial qubit values:

$$|q_1\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

$$|q_2\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

$$|q_3\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

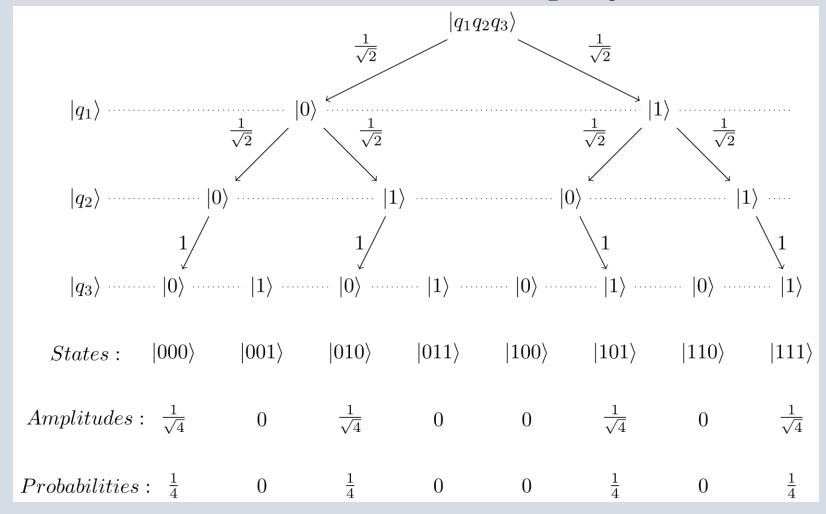


Probabilities:  $\frac{1}{8}$ 





#### **Example 1:** Entangled with correlation $q_1 = q_3$ .



Initial qubit values:

$$|q_1\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

$$|q_2\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

$$|q_3\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

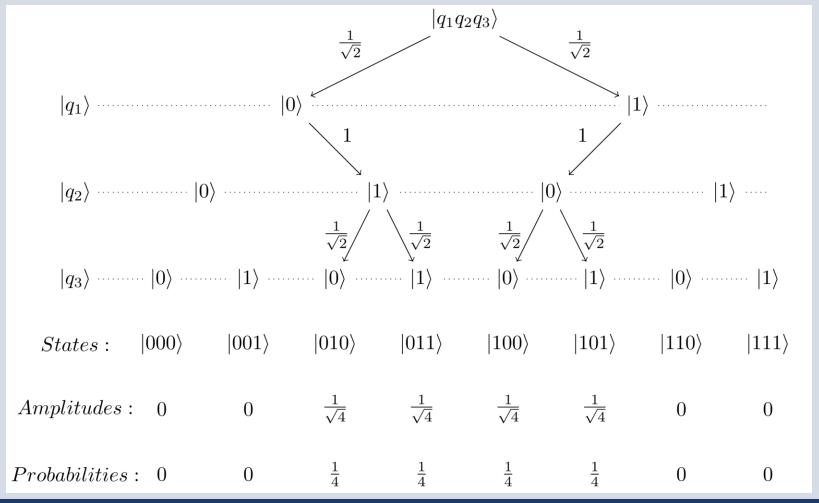
**Note:** *Edges with* <u>O amplitudes</u> are *omitted*.







**Example 2:** Entangled with correlation  $q_1 \neq q_2$ .



Initial qubit values:

$$|q_1\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

$$|q_2\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

$$|q_3\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

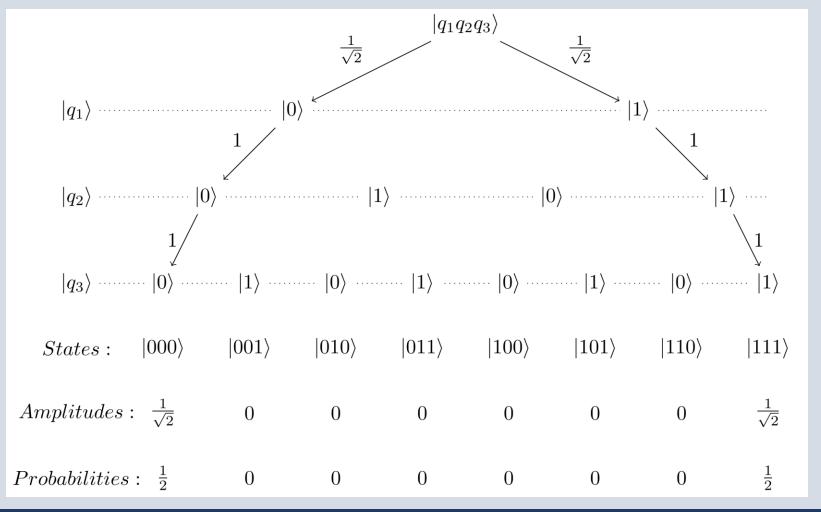
**Note:** Edges with <u>0 amplitudes</u> are **omitted**.







**Example 3:** Entangled with correlation  $q_1 = q_2 = q_3$ .



Initial qubit values:

$$|q_1\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

$$|q_2\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

$$|q_3\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

**Note:** Edges with <u>0 amplitudes</u> are **omitted**.









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(END)

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