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Quantum States: Basic Quantum Mechanics Concepts for Quantum Computing

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Computation is Information Processing

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A ***model of computation*** defines the following:

1. representation of **information**
2. how information is **processed** (**valid operations**).

	Classical Computing Model	Quantum Computing Model
Unit of Information	Bit	***Quantum Bit (Qubit)***
Info. Representation	Bit Strings (Binary Strings)	***Quantum State (Multi-Qubits)***
Operations	Binary String Operations	Quantum Operations

The focus of this presentation is the introduction of quantum mechanics concepts related to *quantum states*.

Classical Two-State System: Storing One Bit of Information

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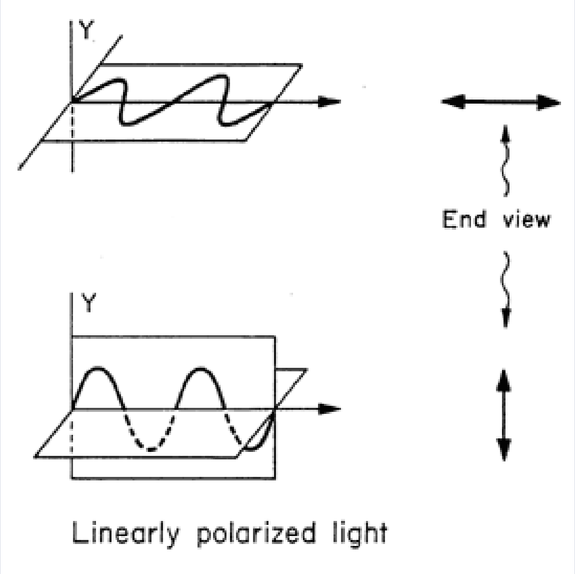
- A **bit b** can either be 0 or 1.
- Any system that have (at least) two states is needed. It is called **classical two-state system**.
- Let *state* A and *state* B be the two states of the system.
- Convention: b = 0 if the system is in state A
- Convention: b = 1 if the system is in state B

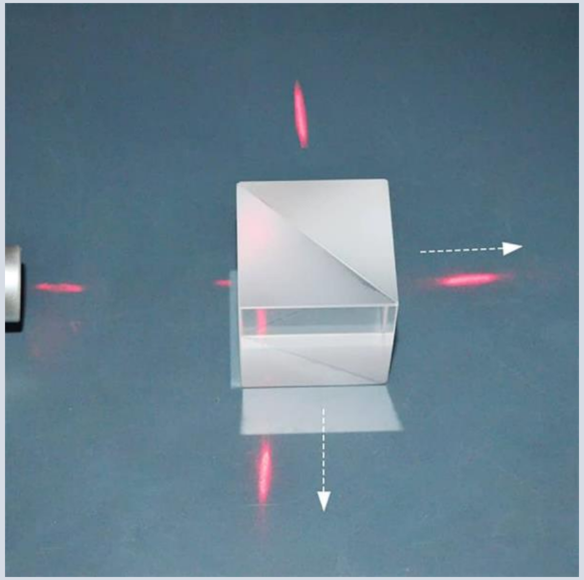
System	State A	State B
Toggle Switch	OFF	ON
Abacus Bead	Bead is down	Bead is up
Punched Tape	No Hole	Has Hole

Quantum Two-State Systems

- A **qubit q** can be $|0\rangle$ or $|1\rangle$.
- $|0\rangle$ - pronounced "*ket 0*", $|1\rangle$ - pronounced "*ket 1*"
- In a **quantum two-state system**, there are also two *observable* states.
- Let the observable states be *state* $|A\rangle$ and *state* $|B\rangle$
- *state* $|A\rangle$ - pronounced "*ket A*", *state* $|B\rangle$ - pronounced "*ket B*"

System	State $|A\rangle$	State $|B\rangle$
Light Polarity	horizontal	vertical
Photon Direction	horizontal	vertical





Quantum Indeterminacy & Quantum Two-State Systems

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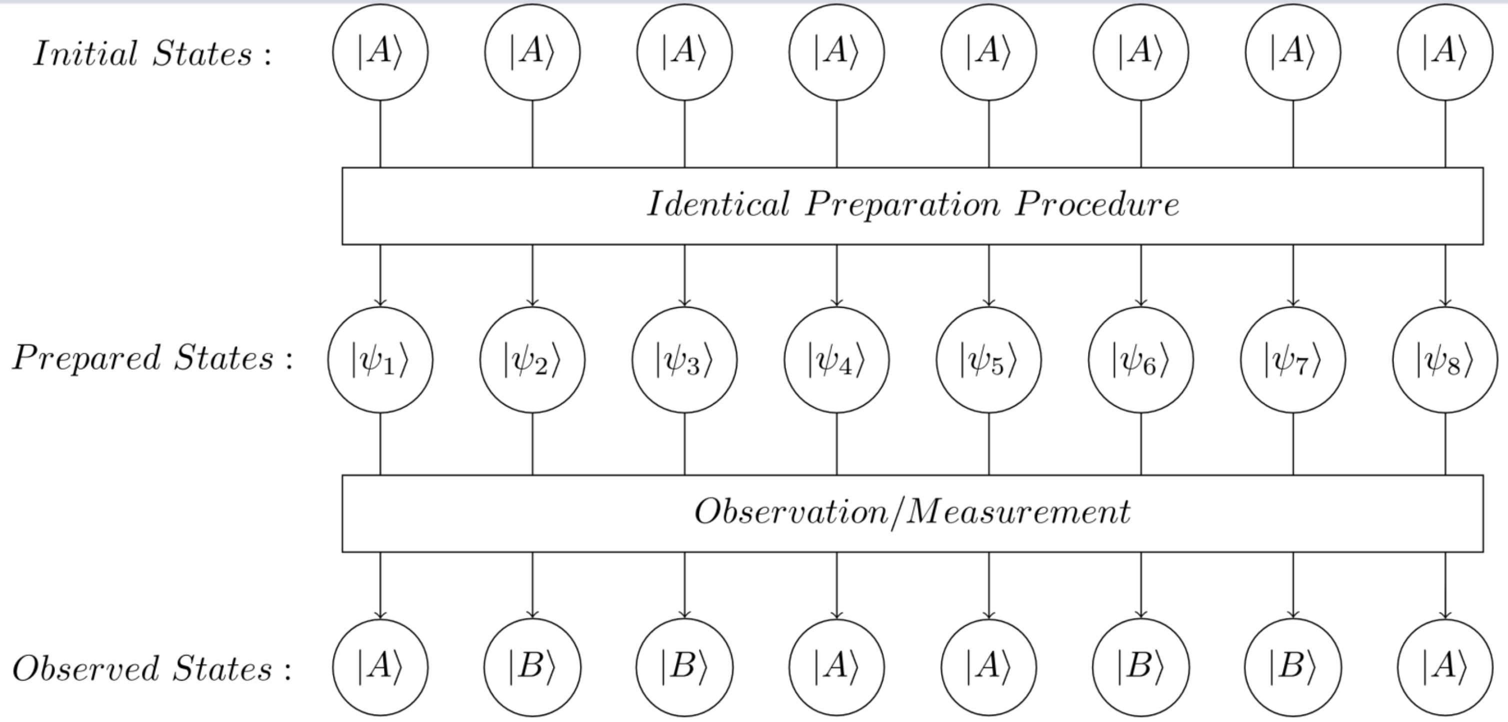
Quantum Indeterminacy Experiment Concept:

1. Prepare multiple quantum two-state systems. The system should be identical and have the same initial state $|A\rangle$.
2. Apply an identical preparation procedure U to all systems.
3. Observe the state of each system.



Expectation: The observed states should be identical.

Results: For quantum systems and certain procedures, the observed states are not guaranteed to be identical.

Quantum Indeterminacy & Quantum Two-State Systems

The experiment is formulated in a such a way as to <u>eliminate external factors that can affect the state</u> of the quantum system.

Conclusion: After applying the preparation procedure but before observation or measurement, the <u>prepared states are not yet determined</u> (***indeterminate***). Only after observation or measurement will a state be in one of the two definite states.

Question: Why even use quantum two-state systems to store information and do computations if there are situations where the state of the system is indeterminate?

Answer: The indeterminacy of a quantum systems is to **not totally random**. By extending the experiment you extract information about the indeterminate state.

Quantum Indeterminacy & Probabilities

Quantum Indeterminacy Experiment (extended):

Parameter: **n** – number of identical quantum systems
Parameter: initial state $|\psi\rangle$ (select one: $|\psi\rangle = |A\rangle$ or $|\psi\rangle = |B\rangle$)
Parameter: procedure **U**

Output: n_A – number of systems observed to be in state $|A\rangle$
Output: n_B – number of systems observed to be in state $|B\rangle$

Compute: $p_A = (n_A/n)$ – portion of the systems observed to be in state $|A\rangle$
Compute: $p_B = (n_B/n)$ – portion of the systems observed to be in state $|B\rangle$

Task: Increase **n** and observe what values p_A and p_B approach.
Result: For the given parameters, $|\psi\rangle$ and U, p_A and p_B approach some specific values.

Probabilities, Superposition, and the Qubit

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The prepared (aka process) quantum state can be **indeterminate** prior to observation. Knowing $|\psi\rangle$ and **U**, we can still attach the information the following information to the quantum state:

p_A – probability of observing $|A\rangle$

p_B – probability of observing $|B\rangle$

$|\psi'\rangle$ – be prepared/processed state, the state of the system after applying procedure **U** to initial quantum state $|\psi\rangle$

$|\psi'\rangle = \alpha|A\rangle + \beta|B\rangle$, **qubit** $|\psi'\rangle$ is a ***superposition*** of *basis states* $|A\rangle$ and $|B\rangle$

α – *amplitude* of state $|A\rangle$, $p_A = |\alpha|^2$

β – *amplitude* of state $|B\rangle$, $p_B = |\beta|^2$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, we will now use the *logical basis states* $|0\rangle$ and $|1\rangle$

Some Examples of Qubits

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Qubit	Ampl. of $\|0\rangle$	Ampl. of $\|1\rangle$	Prob. of $\|0\rangle$	Prob. of $\|1\rangle$
$\|\psi_1\rangle = 1\|0\rangle + 0\|1\rangle = \|0\rangle$	1	0	$1^2 = 1$	$0^2 = 0$
$\|\psi_2\rangle = 0\|0\rangle + 1\|1\rangle = \|1\rangle$	0	1	$0^2 = 0$	$1^2 = 1$
$\|\psi_3\rangle = \frac{1}{\sqrt{2}}\|0\rangle + \frac{1}{\sqrt{2}}\|1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$	$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$
$\|\psi_4\rangle = \frac{1}{\sqrt{4}}\|0\rangle + \frac{\sqrt{3}}{\sqrt{4}}\|1\rangle$	$\frac{1}{\sqrt{4}}$	$\frac{\sqrt{3}}{\sqrt{4}}$	$\left(\frac{1}{\sqrt{4}}\right)^2 = \frac{1}{4}$	$\left(\frac{\sqrt{3}}{\sqrt{4}}\right)^2 = \frac{3}{4}$
$\|\psi_5\rangle = \frac{\sqrt{5}}{\sqrt{8}}\|0\rangle + \frac{\sqrt{3}}{\sqrt{8}}\|1\rangle$	$\frac{\sqrt{5}}{\sqrt{8}}$	$\frac{\sqrt{3}}{\sqrt{8}}$	$\left(\frac{\sqrt{5}}{\sqrt{8}}\right)^2 = \frac{5}{8}$	$\left(\frac{\sqrt{3}}{\sqrt{8}}\right)^2 = \frac{3}{8}$

Rule on Probabilities & the Born Rule

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Qubit: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

α, *amplitude* of state $|0\rangle$
β, *amplitude* of state $|1\rangle$

$p_0 = |\alpha|^2$, *probability* of observing state $|0\rangle$
$p_1 = |\beta|^2$, *probability* of observing state $|1\rangle$

Probability Rule: $p_0 + p_1 = 1$, alternately: $p_0 = (1-p_1)$ or $p_1 = (1-p_0)$
Born Rule: $|\alpha|^2 + |\beta|^2 = 1$, or simply $\alpha^2 + \beta^2 = 1$ if α and β are real numbers.

The *Born Rule* is the probability rule expressed in terms of amplitudes.

The Born Rule & Qubits as Points on a Circle

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Qubit: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

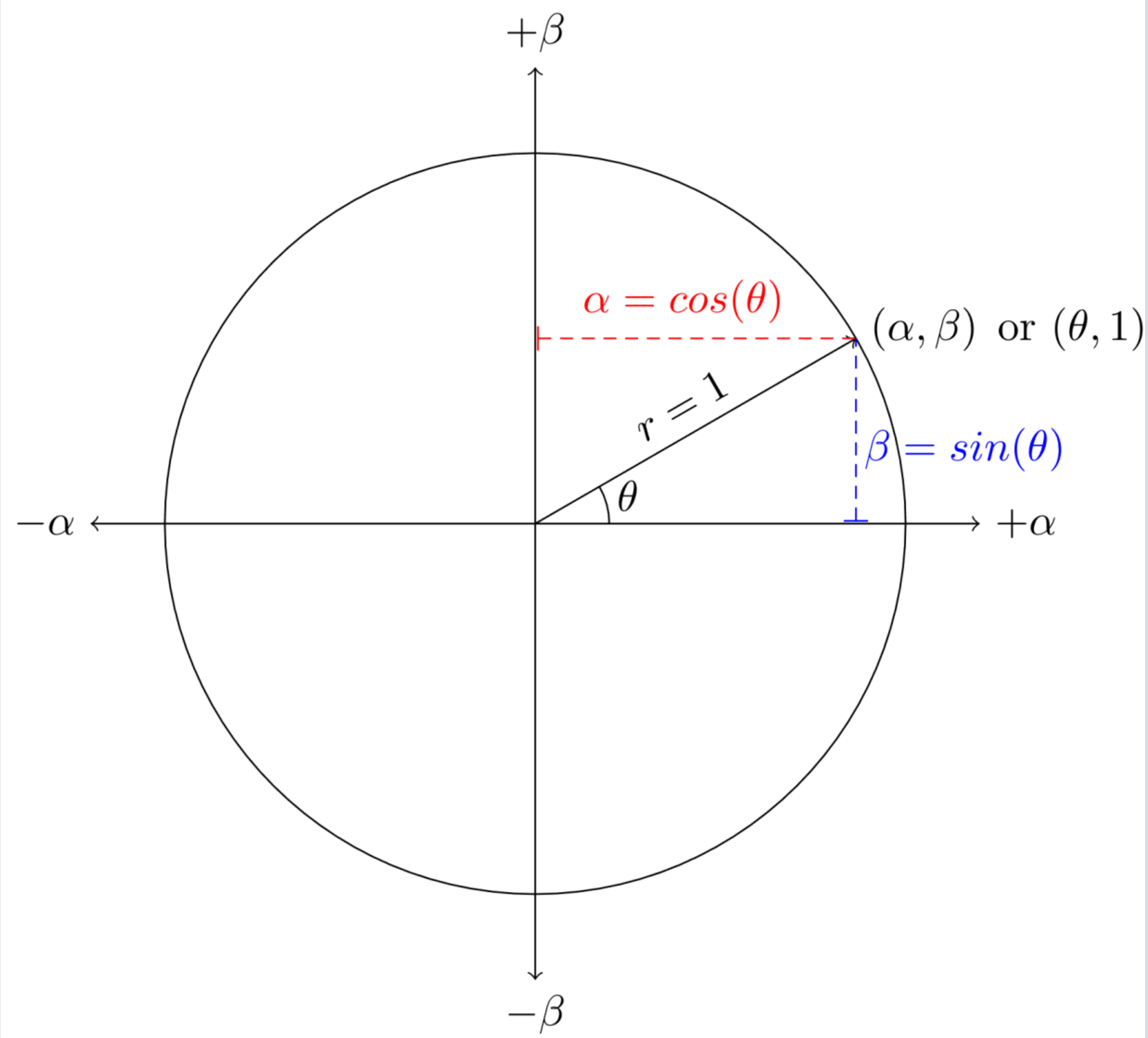
Born Rule: (for real numbers)

$\alpha^2 + \beta^2 = 1$

- circle on the $\alpha\beta$-plane
- center on (0,0)
- radius = 1

$x^2 + y^2 = 1$

- circle on the xy-plane
- center on (0,0)
- radius = 1



Qubits as Points on a Circle

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<u>Qubit</u>: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

(α, β)

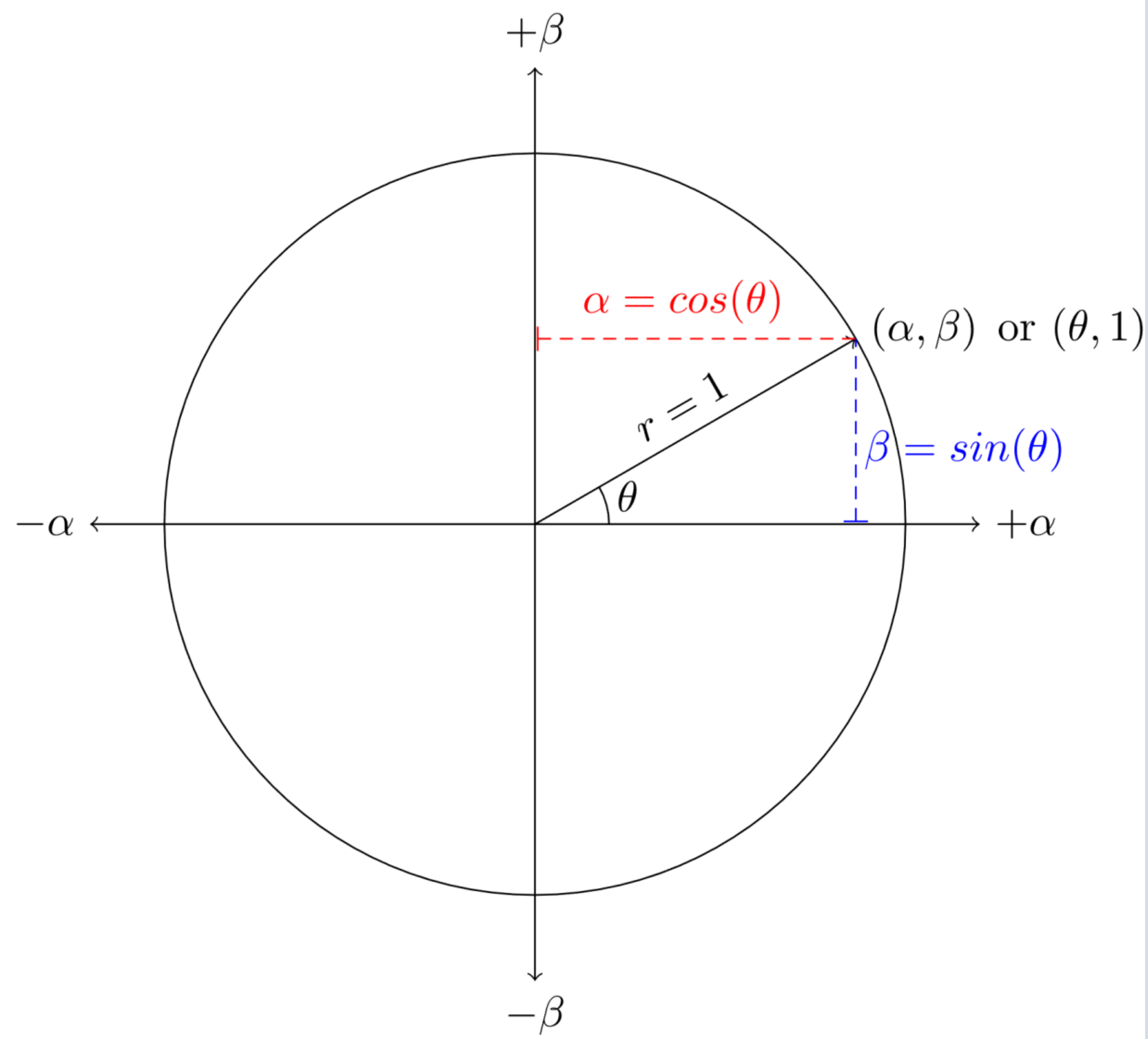
- point on the circle
- represents $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$(\theta, 1)$

- point on the circle
- polar coordinate of (α, β)

<u>Qubit</u>: $|\psi_\theta\rangle = cos(\theta)|0\rangle + sin(\theta)|1\rangle$

- polar coordinate form
- $\alpha = cos(\theta)$
- $\beta = sin(\theta)$



Qubits as Points on a Circle: Examples

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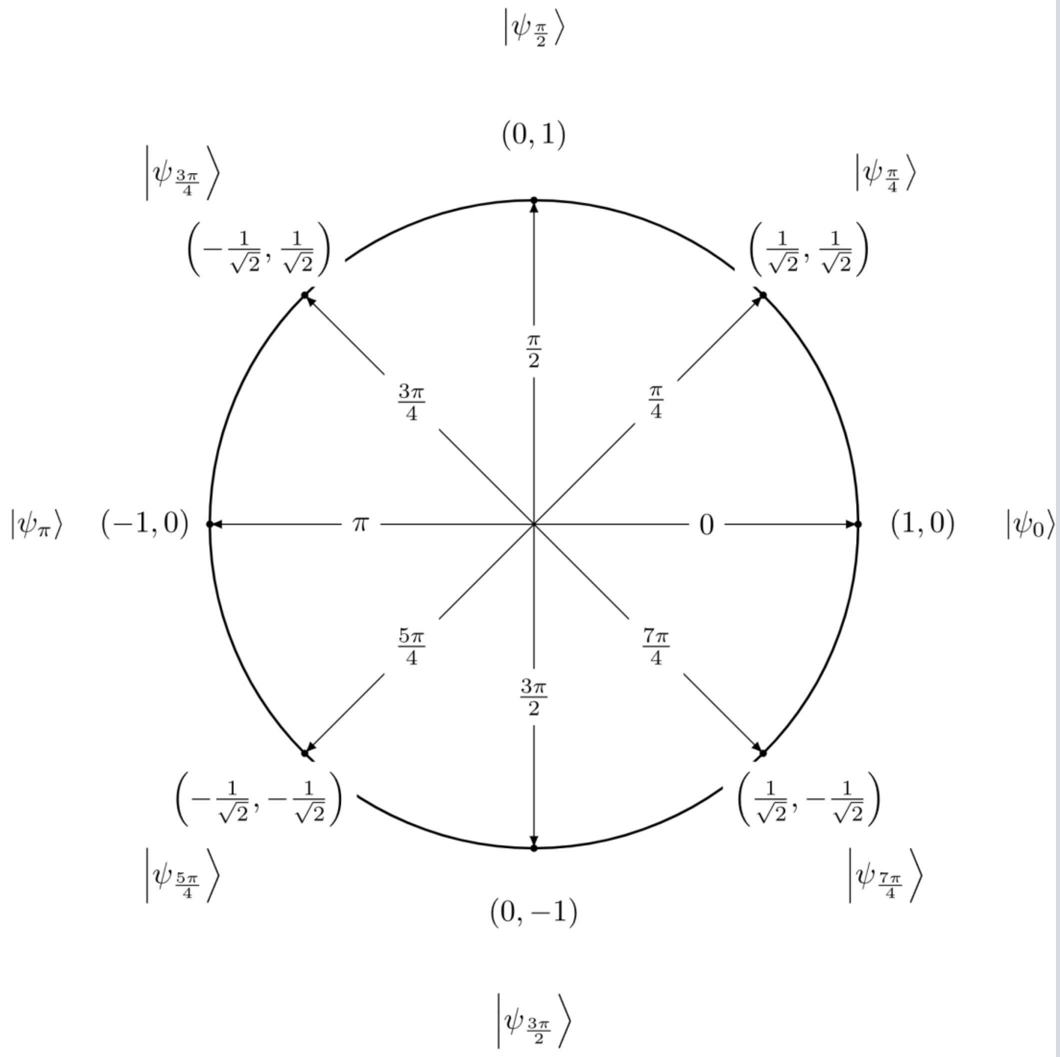
θ	$\alpha = cos(\theta)$	$\beta = sin(\theta)$	$\vert\psi_\theta\rangle = \alpha\vert 0\rangle + \beta\vert 1\rangle$
0	1	0	$\vert\psi_0\rangle = 1\vert 0\rangle + 0\vert 1\rangle = \vert 0\rangle$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\vert\psi_{\frac{\pi}{4}}\rangle = \frac{1}{\sqrt{2}}\vert 0\rangle + \frac{1}{\sqrt{2}}\vert 1\rangle$
$\frac{\pi}{2}$	0	1	$\vert\psi_{\frac{\pi}{2}}\rangle = 0\vert 0\rangle + 1\vert 1\rangle = \vert 1\rangle$
$\frac{3\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\vert\psi_{\frac{3\pi}{4}}\rangle = -\frac{1}{\sqrt{2}}\vert 0\rangle + \frac{1}{\sqrt{2}}\vert 1\rangle$
π	-1	0	$\vert\psi_\pi\rangle = -1\vert 0\rangle + 0\vert 1\rangle = -\vert 0\rangle$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\vert\psi_{\frac{5\pi}{4}}\rangle = -\frac{1}{\sqrt{2}}\vert 0\rangle - \frac{1}{\sqrt{2}}\vert 1\rangle$
$\frac{3\pi}{2}$	0	-1	$\vert\psi_{\frac{3\pi}{2}}\rangle = 0\vert 0\rangle - 1\vert 1\rangle = -\vert 1\rangle$
$\frac{7\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\vert\psi_{\frac{7\pi}{4}}\rangle = \frac{1}{\sqrt{2}}\vert 0\rangle - \frac{1}{\sqrt{2}}\vert 1\rangle$



Qubits that encode the **same probabilities** but have *different amplitudes* (different signs of amplitudes) are said to have *different phases*.

Qubits as Points on a Circle

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Due to the born rule and our initial **restriction** that the amplitudes are real numbers (not yet complex numbers), we are able to define **any unique qubit** using a single parameter which is the angle θ.

Qubit	$\vert\psi\rangle = \alpha\vert 0\rangle + \beta\vert 1\rangle$
Point (Rectangular Coordinate)	(α, β)
Point (Polar Coordinate)	$(\theta, 1)$
Qubit	$\vert\psi_\theta\rangle = \cos(\theta)\vert 0\rangle + \sin(\theta)\vert 1\rangle$

If you look at the probability rule $p_0 + p_1 = 1$, it can help you see that you only need 1 parameter to define a qubit with real amplitudes. If you specify the qubit's p_0, due to the probability rule, you can derive $p_1 = (1-p_0)$ from p_0. They are not independent values. The same goes for the amplitudes. *i.e.* $\beta = \sqrt{(1-\alpha^2)}$

Operations on Single Qubits

- An **operation U** transforms a qubit by changing the qubit's amplitudes.
- The **operation U** should follow the born rule.

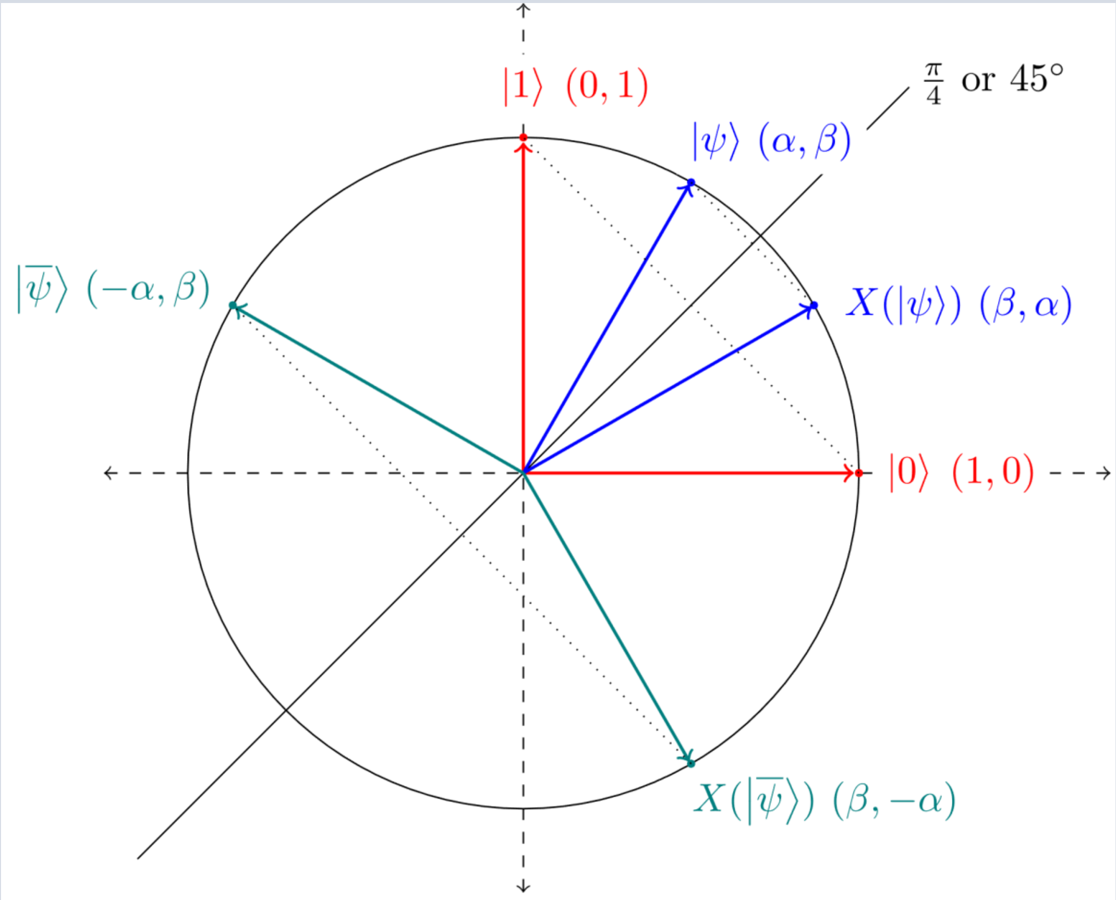
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow \boxed{U} \longrightarrow |\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$$

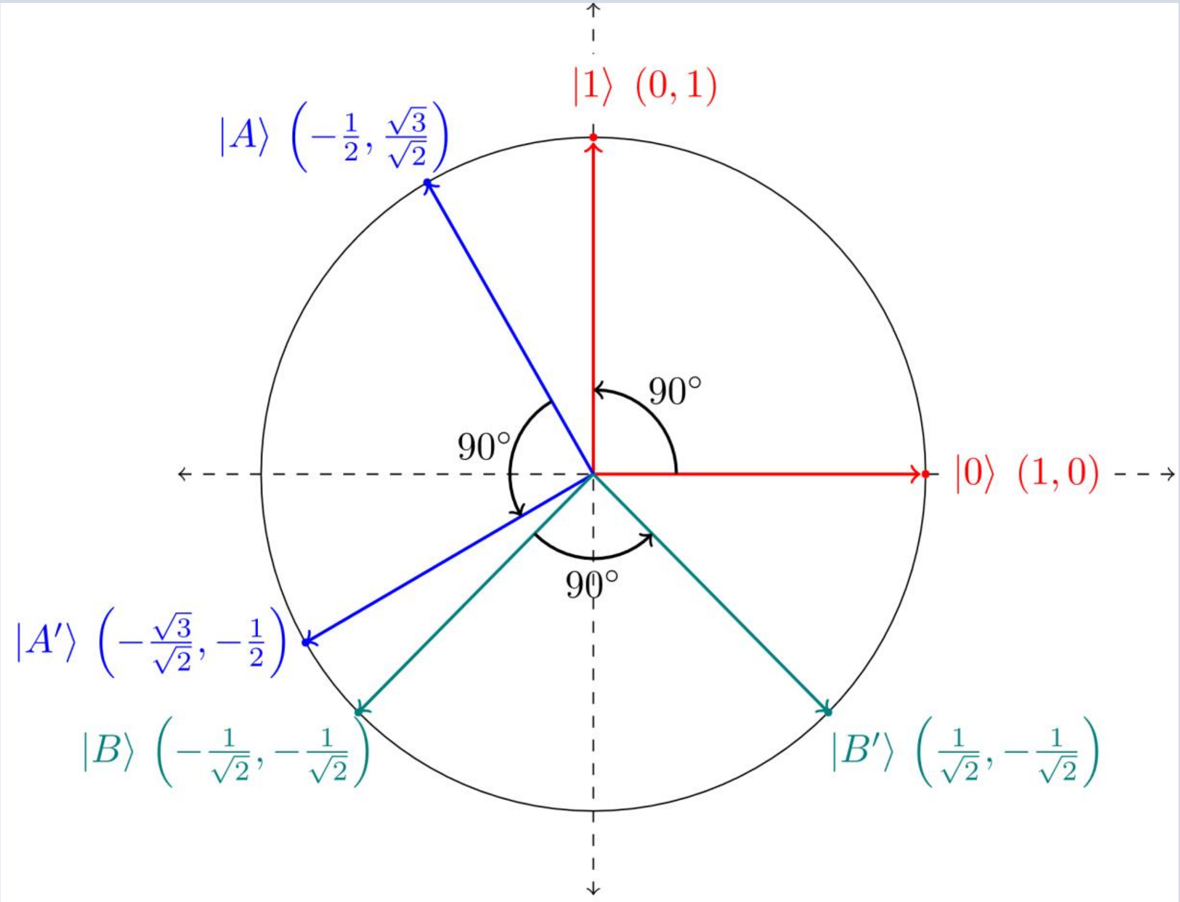
$$(\alpha, \beta) \longrightarrow \boxed{U} \longrightarrow (\alpha', \beta')$$

$$(\theta, 1) \longrightarrow \boxed{U} \longrightarrow (\theta', 1)$$

Operations on Single Qubits

- Qubit **operations** can be viewed as <u>rotations</u> or <u>reflections</u>.





Qubits with Complex Amplitudes

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- In general, the amplitudes, **α** and **β**, of a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ are complex numbers.

Real Amplitudes

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- $\alpha = x$
- $\beta = y$
- x, y are real numbers

Complex Amplitudes

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- $\alpha = x + qi$ (complex)
- $\beta = y + ri$ (complex)
- x, q are real numbers
- y, r are real numbers

- When the amplitudes are **real**: **two** real numbers are needed to define a unique qubit
- When the amplitudes are **complex**: **four** real numbers are needed to define a unique qubit

Qubits with Complex Amplitudes

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The square of a complex number is also complex:

- $\alpha = x + qi$
- $\alpha^2 = (x + qi)^2 = (x + qi)(x + qi) = x^2 + 2xqi - q^2 = (x^2 - q^2) + (2xq)i$

This means we cannot use simple squaring to get real probability from the complex amplitude. The **square modulus** $|\alpha|^2$ of a complex number $\alpha = x + qi$ is a real non-negative and can be used as probability:

- $|\alpha|^2 = (x + qi)(x - qi) = x^2 + (xqi - xqi) + q^2 = x^2 + q^2$

Qubits with Complex Amplitudes + Born Rule

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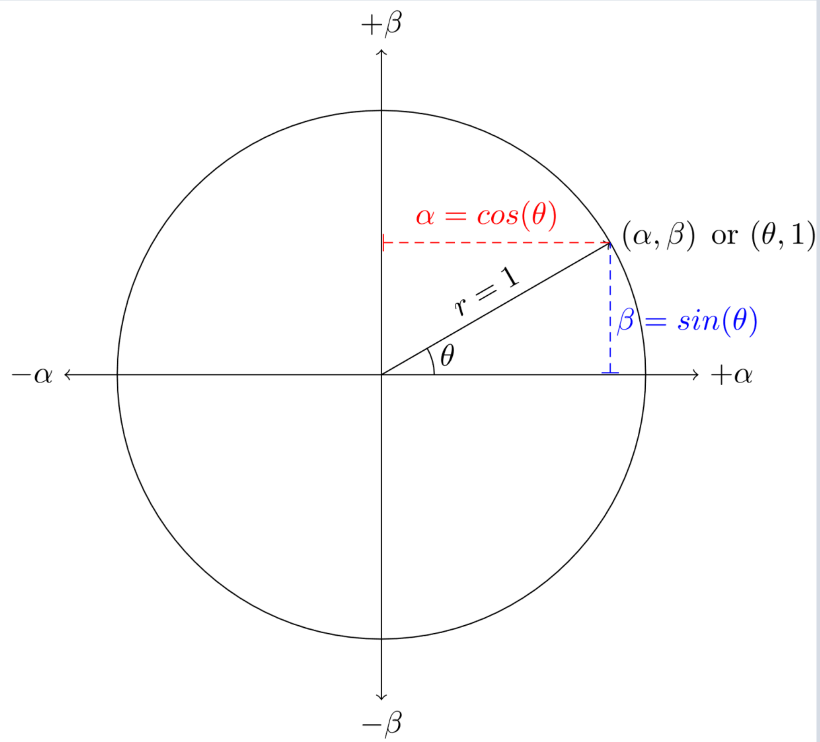
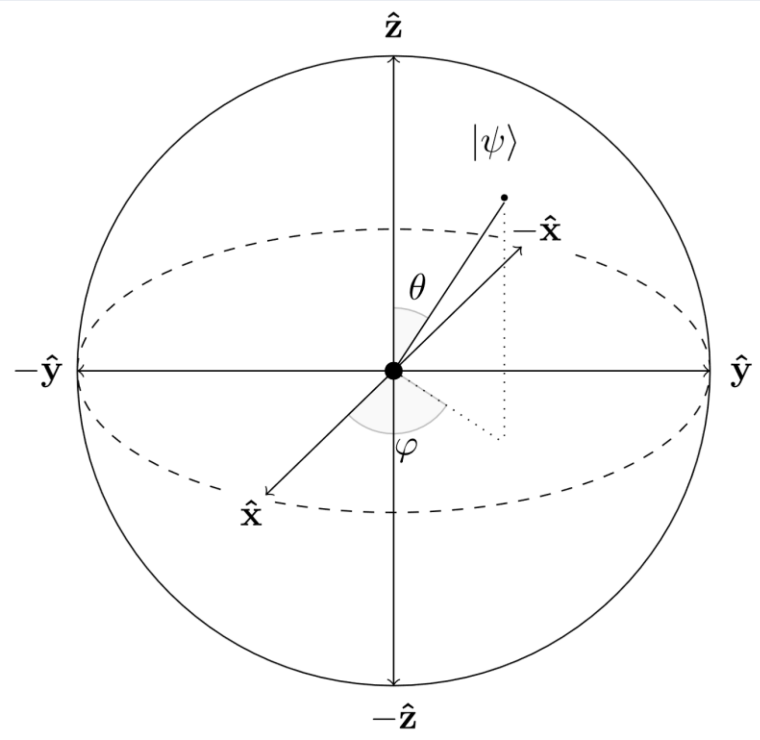
- Due to the **born rule**, when the amplitudes α and β are real, you only need **1** parameter, the ***angle*** θ, to define a unique qubit. From **2** real parameters **α** and **β** to **1** parameter **θ.**
- For the general qubit with complex amplitudes α = x + q*i* and β = y + r*i*, using the born rule, the **4** real parameters x, q, y, r can be reduced **2** angle parameters θ and *φ*.





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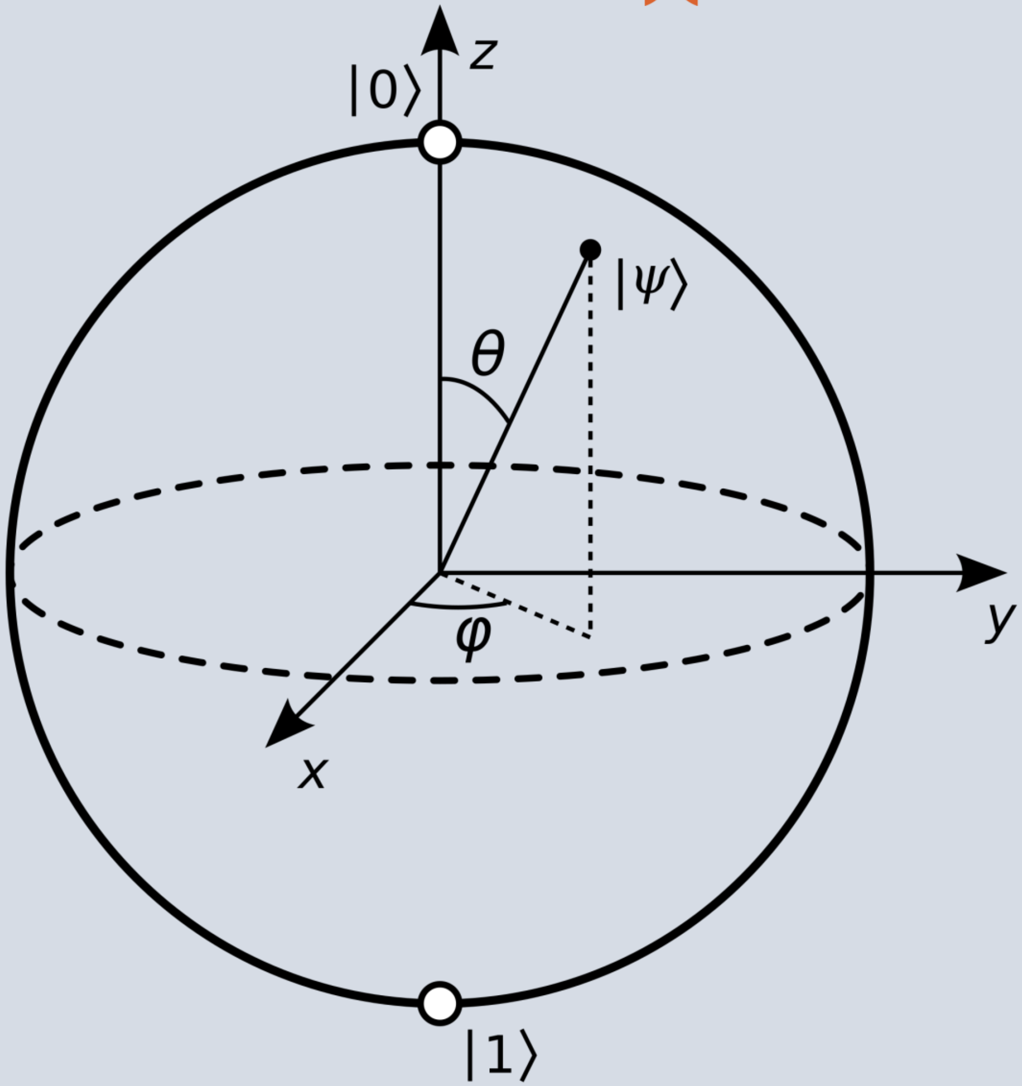
Qubit & the Bloch Sphere

- A qubit with complex amplitudes can be represented a **point on a sphere** of radius 1. The sphere is called the *Bloch sphere*.

- In spherical coordinates, you only need two parameters, angle **θ** and angle φ, to define a unique point on the sphere.

- The qubit specified by the two angles **θ** and φ:

$$|\psi\rangle = \underbrace{\boxed{cos\left(\frac{\theta}{2}\right)}}_{\alpha}|0\rangle + \underbrace{\boxed{e^{i\varphi}sin\left(\frac{\theta}{2}\right)}}_{\beta}|1\rangle$$



Qubits can be Represented as Column Vectors

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The qubit $|0\rangle$ can be represented by the column vector:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

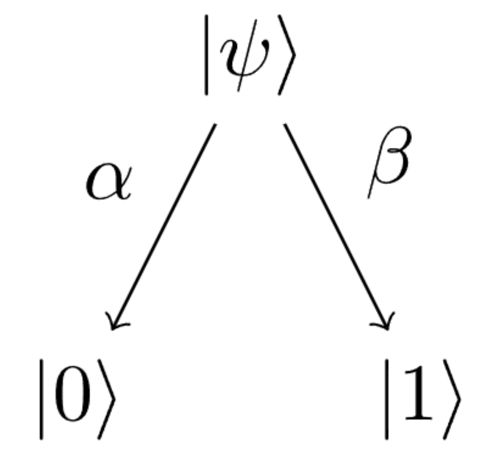
The qubit $|1\rangle$ can be represented by the column vector:

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

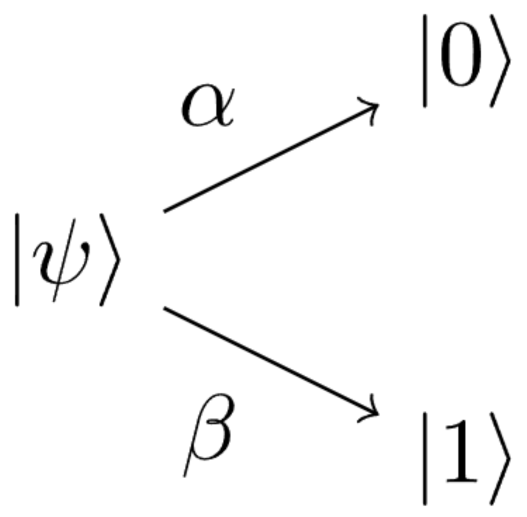
The generic qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ can be represented by column vector:

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Qubits drawn as Binary Trees



$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$



$$\begin{bmatrix} \alpha \\ \\ \beta \end{bmatrix}$$

Quantum State

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- For quantum computing, a **quantum state** is simply a *<u>collection of qubits</u>*.

- If you have two qubits, $|q_1\rangle$ and $|q_2\rangle$, the quantum state can be written as $|q_1q_2\rangle$.

- Example 1: If $|q_1\rangle = |0\rangle$, $|q_2\rangle = |0\rangle$, then $|q_1q_2\rangle = |00\rangle$.
- Example 2: If $|q_1\rangle = |0\rangle$, $|q_2\rangle = |1\rangle$, then $|q_1q_2\rangle = |01\rangle$.
- Example 3: If $|q_1\rangle = |1\rangle$, $|q_2\rangle = |0\rangle$, then $|q_1q_2\rangle = |10\rangle$.
- Example 4: If $|q_1\rangle = |1\rangle$, $|q_2\rangle = |1\rangle$, then $|q_1q_2\rangle = |11\rangle$.

Quantum State

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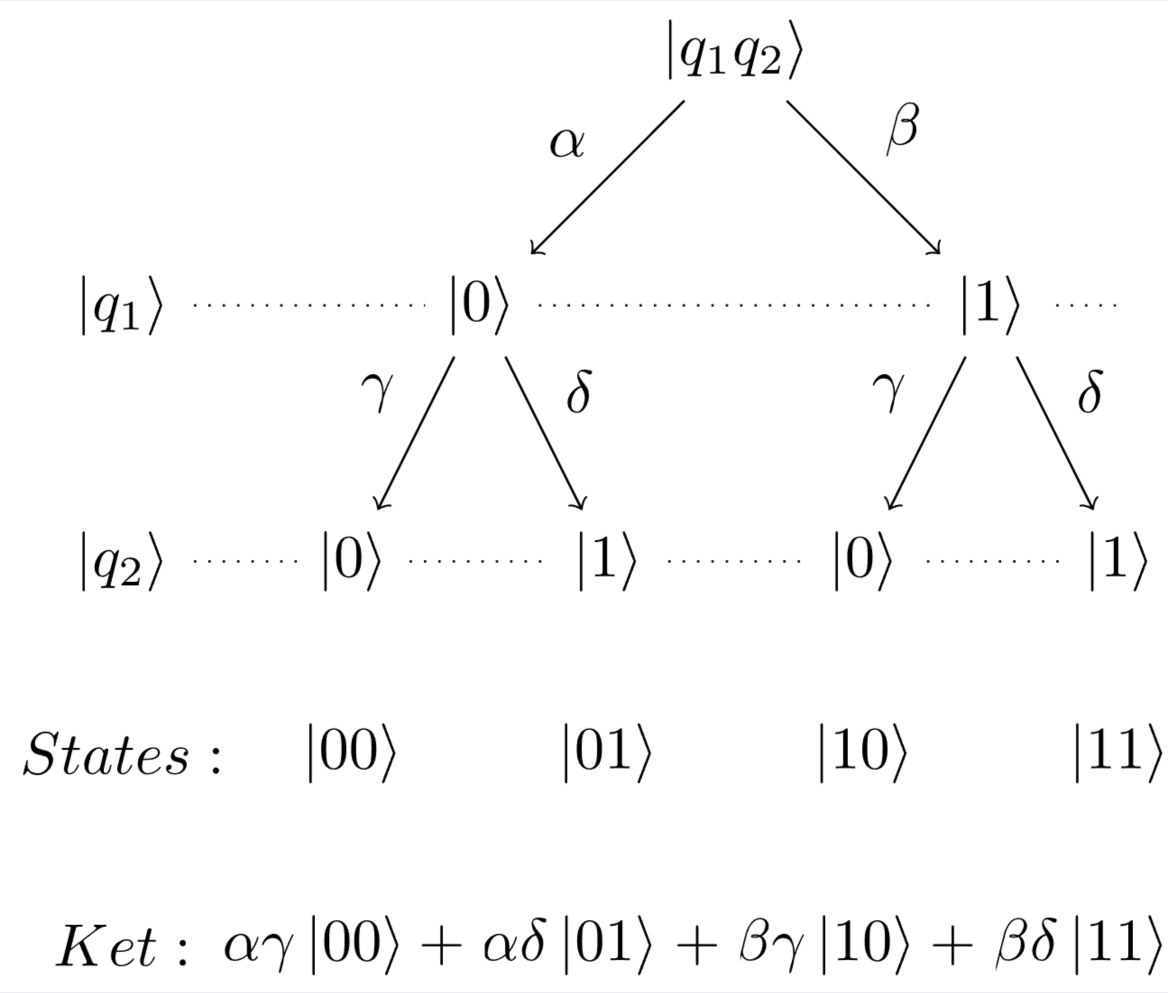
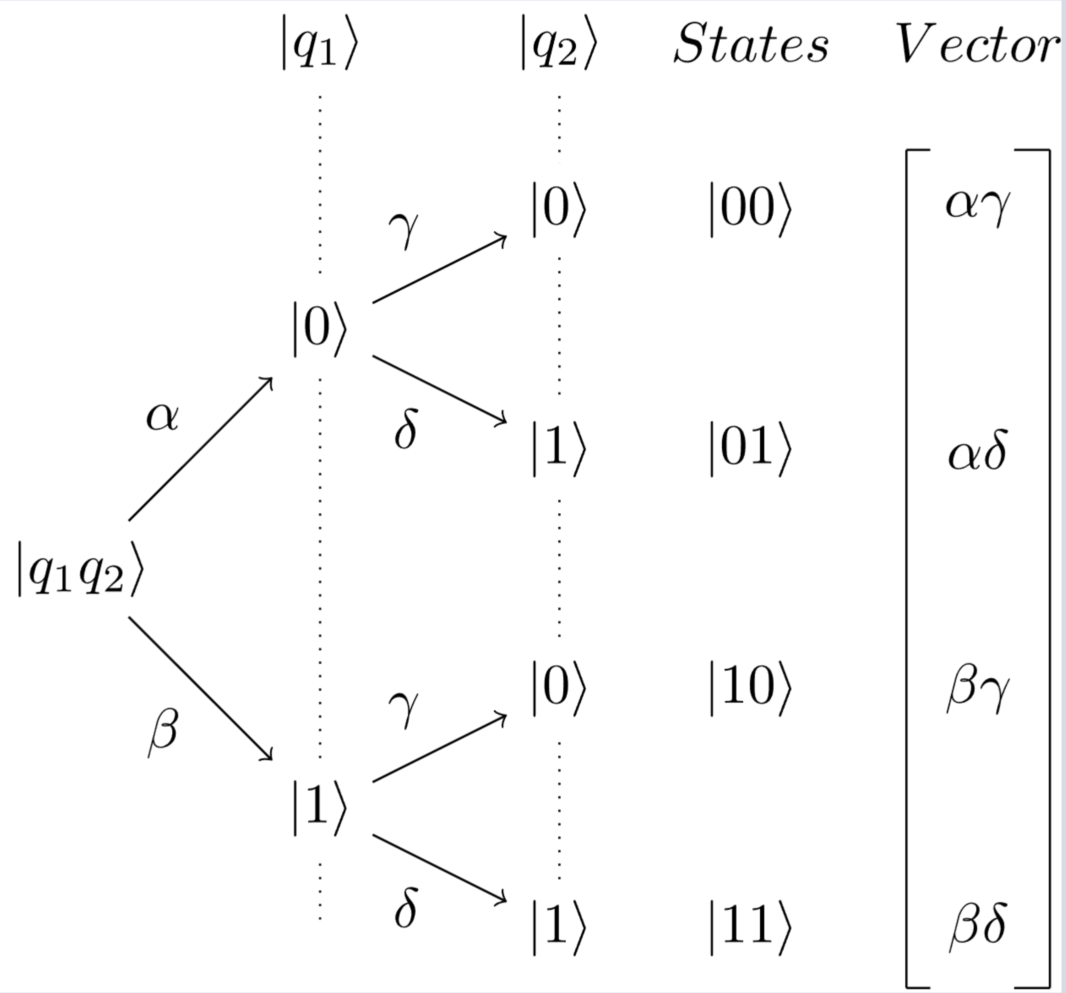
If you have the independent two qubits $|q_1\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|q_2\rangle = \gamma|0\rangle + \delta|1\rangle$, the quantum state can be written as:

- $|q_1q_2\rangle = |q_1\rangle \otimes |q_2\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$
- $|q_1\rangle \otimes |q_2\rangle$ is called the *tensor product* of states/qubits $|q_1\rangle$ and $|q_2\rangle$.
- $|q_1\rangle$ and $|q_2\rangle$ both have the basis states: $|0\rangle$ and $|1\rangle$.
- $|q_1\rangle \otimes |q_2\rangle$ has the basis states: $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$

Two-Qubit Quantum State as Binary Trees





Entanglement of Multiple Qubits

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- Two qubits are **entangled** if their *amplitudes (or probabilities) are correlated*.
- You can entangle two qubits using 2-qubit *entanglement* operators.

$$|q_1\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow \boxed{U} \longrightarrow |q_1'\rangle = \alpha'|0\rangle + \beta'|1\rangle$$

$$|q_2\rangle = \gamma|0\rangle + \delta|1\rangle \longrightarrow \boxed{U} \longrightarrow |q_2'\rangle = \gamma'|0\rangle + \delta'|1\rangle$$

- $|q_1'\rangle$ and $|q_2'\rangle$ are no longer independent but are correlated.
- The amplitudes α', β' are correlated with amplitudes γ', δ'.

Example of an Entangled State

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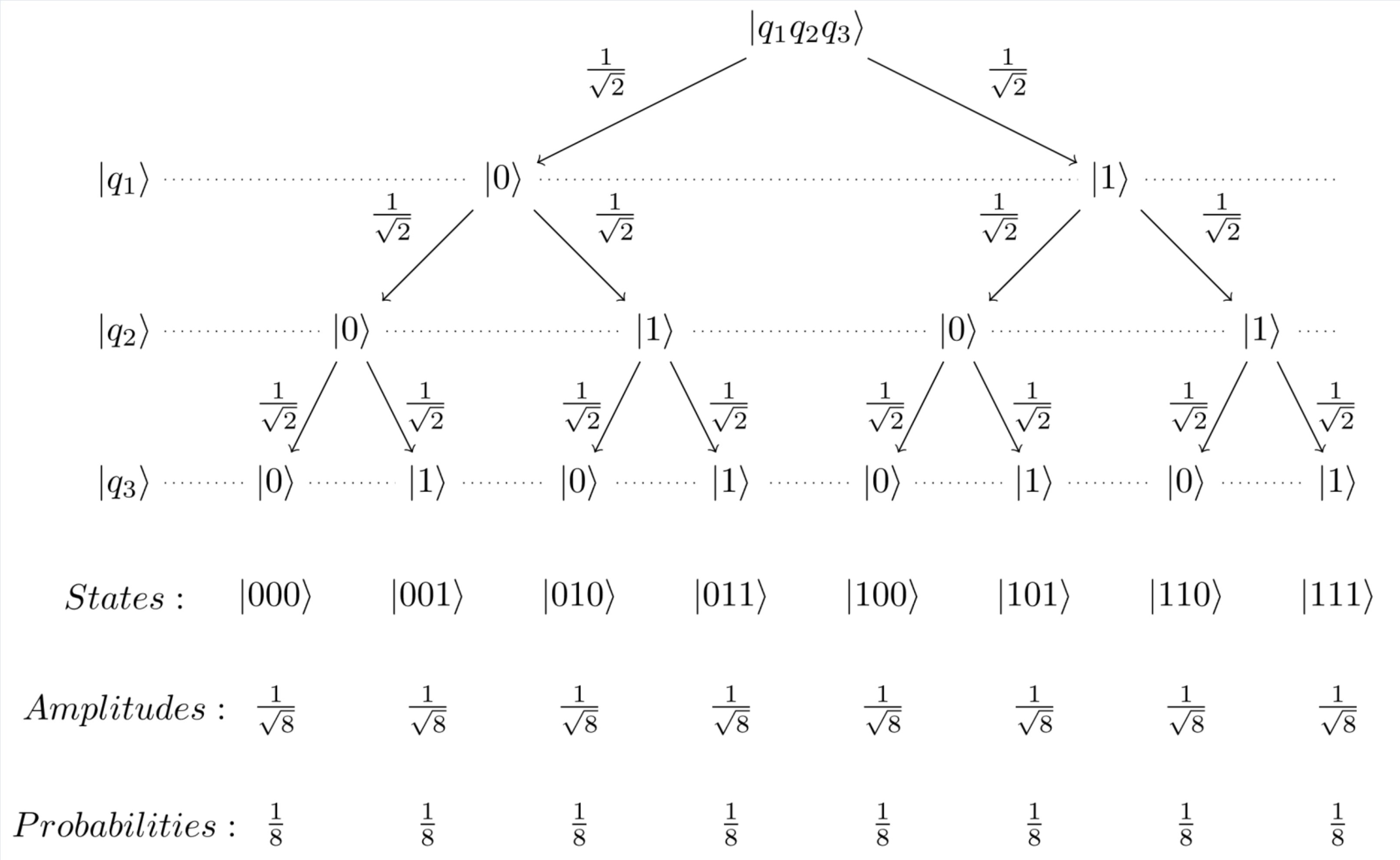
Example 0: No entanglement. No amplitude correlation.



Initial qubit values:

$|q_1\rangle = (1/\sqrt{2})|0\rangle + \mathbf{(1/\sqrt{2})}|1\rangle$

$|q_2\rangle = (1/\sqrt{2})|0\rangle + \mathbf{(1/\sqrt{2})}|1\rangle$

$|q_3\rangle = (1/\sqrt{2})|0\rangle + \mathbf{(1/\sqrt{2})}|1\rangle$

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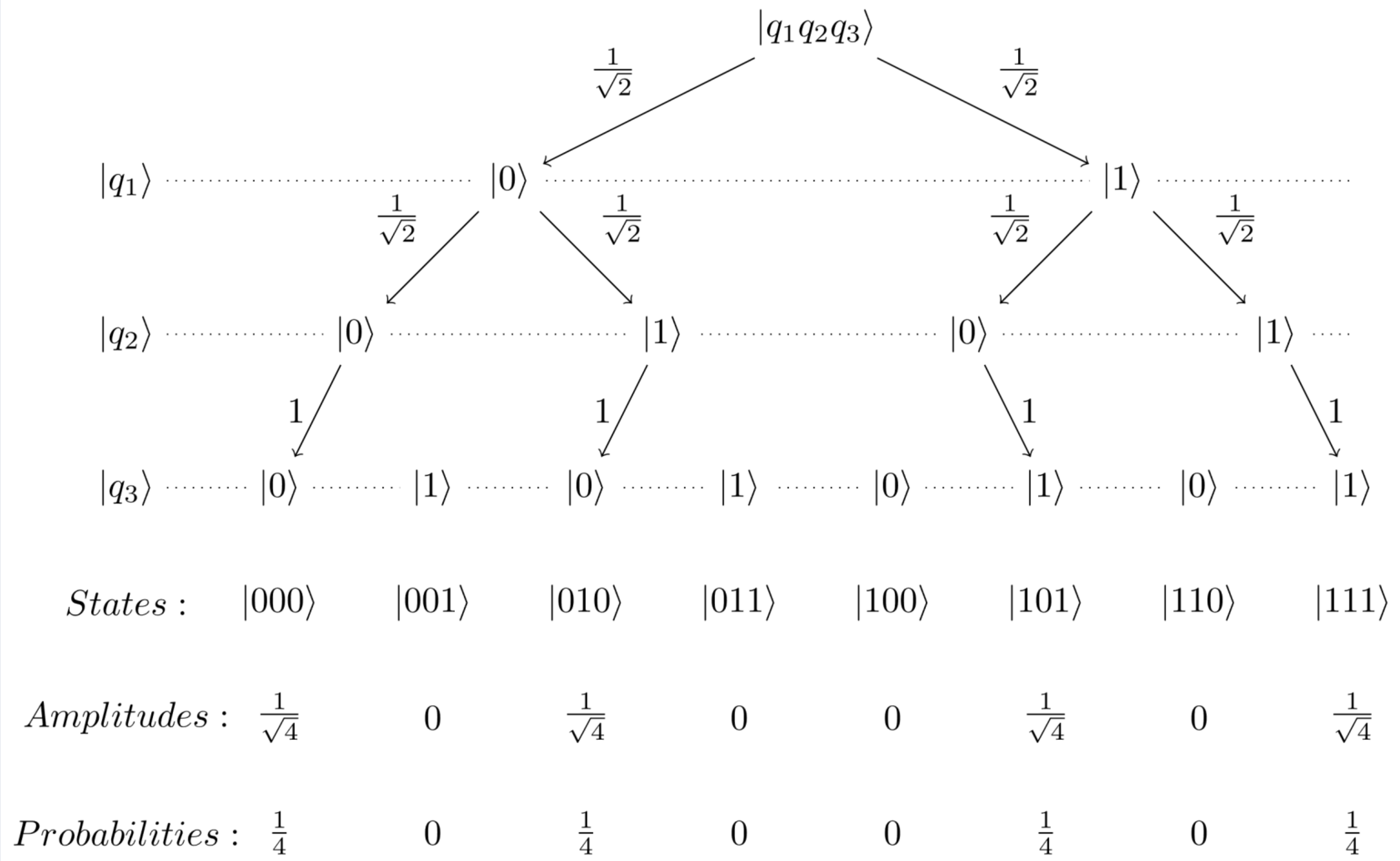
Example of an Entangled State

Example 1: Entangled with correlation $\mathbf{q_1 = q_3}$.



Initial qubit values:

$|q_1\rangle = (1/\sqrt{2})|0\rangle + \mathbf{(1/\sqrt{2})}|1\rangle$

$|q_2\rangle = (1/\sqrt{2})|0\rangle + \mathbf{(1/\sqrt{2})}|1\rangle$

$|q_3\rangle = (1/\sqrt{2})|0\rangle + \mathbf{(1/\sqrt{2})}|1\rangle$

Note: *Edges with 0 amplitudes are omitted.*

Example of an Entangled State

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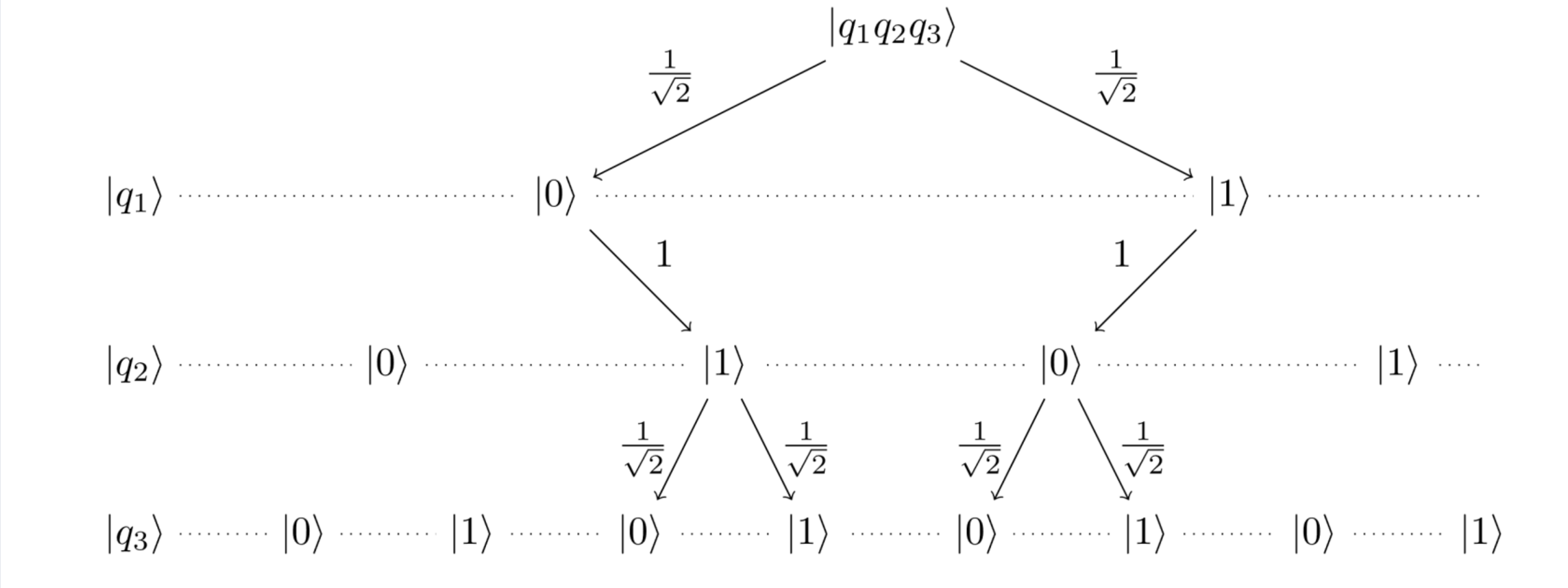
Example 2: Entangled with correlation $\mathbf{q_1 \neq q_2}$.



$States:$	$\lvert 000\rangle$	$\lvert 001\rangle$	$\lvert 010\rangle$	$\lvert 011\rangle$	$\lvert 100\rangle$	$\lvert 101\rangle$	$\lvert 110\rangle$	$\lvert 111\rangle$
$Amplitudes:$	0	0	$\frac{1}{\sqrt{4}}$	$\frac{1}{\sqrt{4}}$	$\frac{1}{\sqrt{4}}$	$\frac{1}{\sqrt{4}}$	0	0
$Probabilities:$	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0

Initial qubit values:

$|q_1\rangle = (1/\sqrt{2})|0\rangle + \mathbf{(1/\sqrt{2})}|1\rangle$

$|q_2\rangle = (1/\sqrt{2})|0\rangle + \mathbf{(1/\sqrt{2})}|1\rangle$

$|q_3\rangle = (1/\sqrt{2})|0\rangle + \mathbf{(1/\sqrt{2})}|1\rangle$

Note: *Edges with 0 amplitudes are omitted.*

Example of an Entangled State

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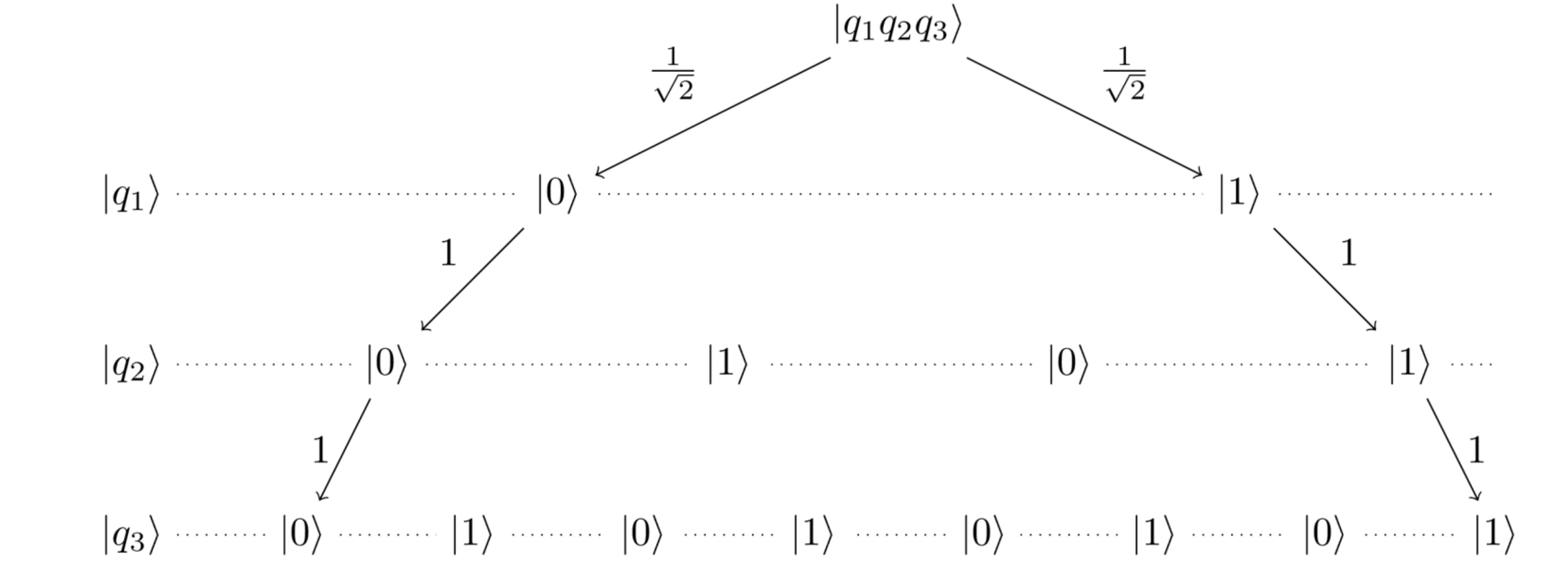
Example 3: Entangled with correlation $\mathbf{q_1 = q_2 = q_3}$.



States :	$\|000\rangle$	$\|001\rangle$	$\|010\rangle$	$\|011\rangle$	$\|100\rangle$	$\|101\rangle$	$\|110\rangle$	$\|111\rangle$
Amplitudes :	$\frac{1}{\sqrt{2}}$	0	0	0	0	0	0	$\frac{1}{\sqrt{2}}$
Probabilities :	$\frac{1}{2}$	0	0	0	0	0	0	$\frac{1}{2}$

Initial qubit values:

$|q_1\rangle = (1/\sqrt{2})|0\rangle + \mathbf{(1/\sqrt{2})}|1\rangle$

$|q_2\rangle = (1/\sqrt{2})|0\rangle + \mathbf{(1/\sqrt{2})}|1\rangle$

$|q_3\rangle = (1/\sqrt{2})|0\rangle + \mathbf{(1/\sqrt{2})}|1\rangle$

Note: *Edges with <u>0 amplitudes</u> are omitted.*

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Quantum States: Basic Quantum Mechanics Concepts for Quantum Computing

(END)

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Quantum Circuit Simulation Project
DOST Advanced Science and Technology Institute
2023 November