



# Quantum States: Basic Quantum Mechanics Concepts for Quantum Computing

Ren Tristan A. de la Cruz  
Project Technical Specialist I  
Quantum Circuit Simulation Project  
DOST Advanced Science and Technology Institute  
2023 November



# Computation is Information Processing

A *model of computation* defines the following:

1. representation of **information**
2. how information is **processed** (valid operations).

	Classical Computing Model	Quantum Computing Model
Unit of Information	Bit	<i>Quantum Bit (Qubit)</i>
Info. Representation	Bit Strings (Binary Strings)	<i>Quantum State (Multi-Qubits)</i>
Operations	Binary String Operations	Quantum Operations

The focus of this presentation is the introduction of quantum mechanics concepts related to *quantum states*.



# Classical Two-State System: Storing One Bit of Information

- A **bit** **b** can either be **0** or **1**.
- Any system that have (at least) two states is needed. It is called **classical two-state system**.
- Let *state* **A** and *state* **B** be the two states of the system.
- Convention: **b = 0** if the system is in state **A**
- Convention: **b = 1** if the system is in state **B**



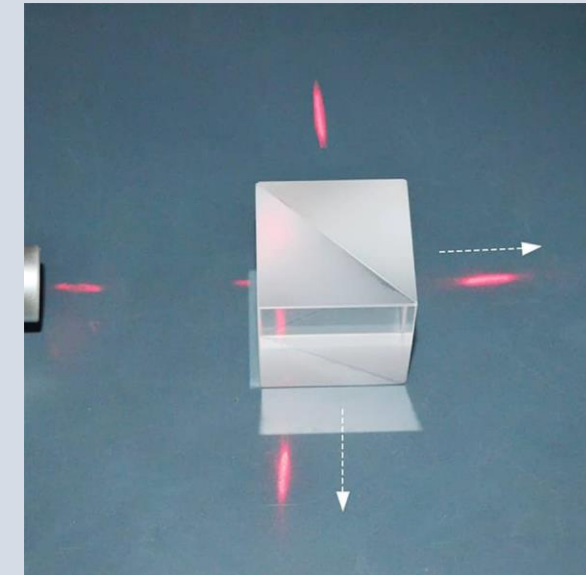
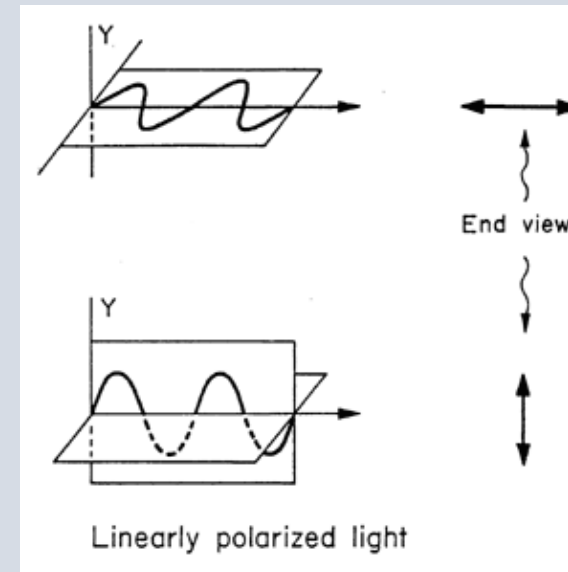
System	State <b>A</b>	State <b>B</b>
Toggle Switch	OFF	ON
Abacus Bead	Bead is down	Bead is up
Punched Tape	No Hole	Has Hole



# Quantum Two-State Systems

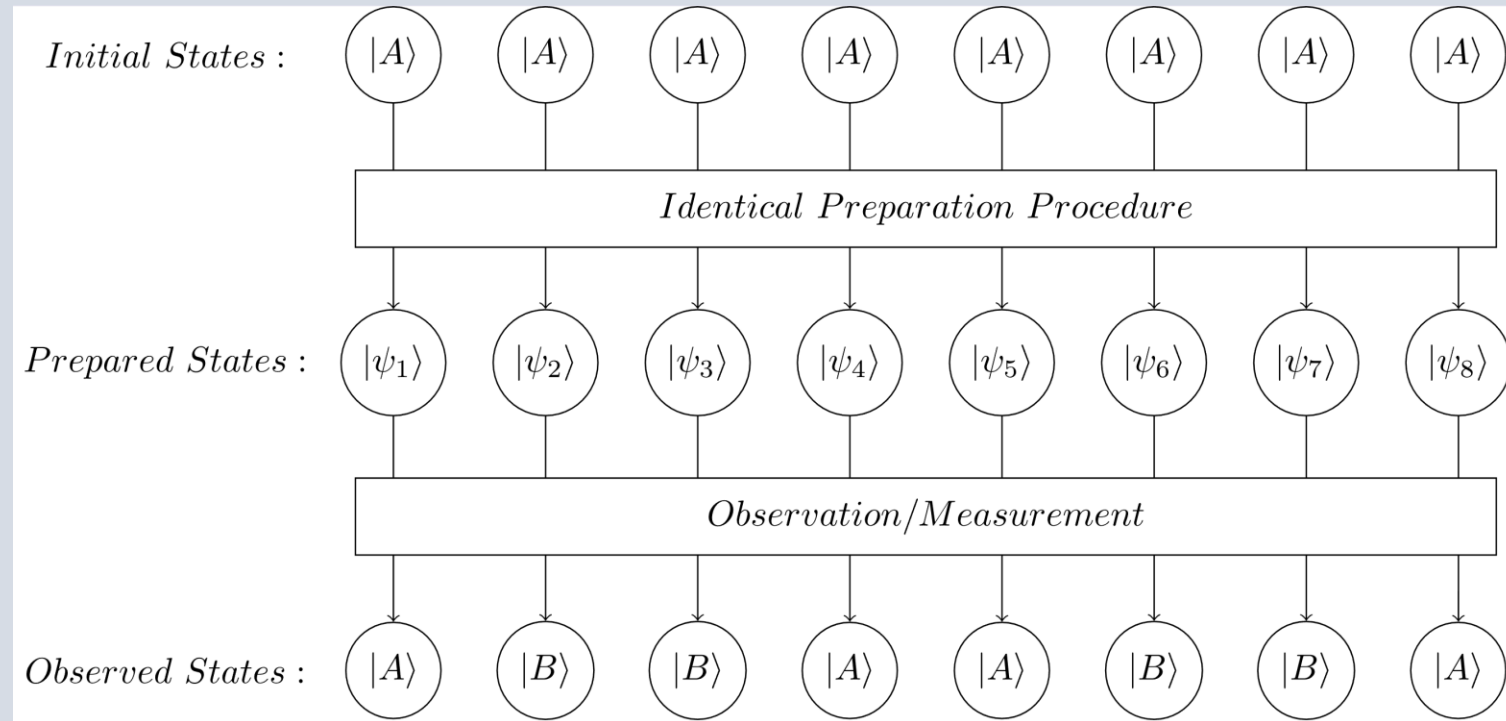
- A qubit  $q$  can be  $|0\rangle$  or  $|1\rangle$ .
- $|0\rangle$  - pronounced "ket 0",  $|1\rangle$  - pronounced "ket 1"
- In a **quantum two-state system**, there are also two observable states.
- Let the observable states be state  $|A\rangle$  and state  $|B\rangle$
- state  $|A\rangle$  - pronounced "ket A", state  $|B\rangle$  - pronounced "ket B"

System	State $ A\rangle$	State $ B\rangle$
Light Polarity	horizontal	vertical
Photon Direction	horizontal	vertical



## *Quantum Indeterminacy* Experiment Concept:

1. Prepare multiple quantum two-state systems. The system should be identical and have the same initial state  $|A\rangle$ .
2. Apply an identical preparation procedure  $U$  to all systems.
3. Observe the state of each system.



**Expectation:** The observed states should be identical.

**Results:** For quantum systems and certain procedures, the observed states are not guaranteed to be identical.

The experiment is formulated in a such a way as to eliminate external factors that can affect the state of the quantum system.

**Conclusion:** After applying the preparation procedure but before observation or measurement, the prepared states are not yet determined (*indeterminate*). Only after observation or measurement will a state be in one of the two definite states.

**Question:** Why even use quantum two-state systems to store information and do computations if there are situations where the state of the system is indeterminate?

**Answer:** The indeterminacy of a quantum systems is to **not totally random**. By extending the experiment you extract information about the indeterminate state.





# Quantum Indeterminacy & Probabilities

## *Quantum Indeterminacy* Experiment (extended):

Parameter:  $n$  – number of identical quantum systems

Parameter: initial state  $|\psi\rangle$  ( select one:  $|\psi\rangle = |A\rangle$  or  $|\psi\rangle = |B\rangle$  )

Parameter: procedure  $U$

Output:  $n_A$  – number of systems observed to be in state  $|A\rangle$

Output:  $n_B$  – number of systems observed to be in state  $|B\rangle$

Compute:  $p_A = (n_A/n)$  – portion of the systems observed to be in state  $|A\rangle$

Compute:  $p_B = (n_B/n)$  – portion of the systems observed to be in state  $|B\rangle$

**Task:** Increase  $n$  and observe what values  $p_A$  and  $p_B$  approach.

**Result:** For the given parameters,  $|\psi\rangle$  and  $U$ ,  $p_A$  and  $p_B$  approach some specific values.



# Probabilities, Superposition, and the Qubit

The prepared (aka process) quantum state can be **indeterminate** prior to observation. Knowing  $|\psi\rangle$  and  $U$ , we can still attach the information the following information to the quantum state:

$p_A$  – probability of observing  $|A\rangle$

$p_B$  – probability of observing  $|B\rangle$

$|\psi'\rangle$  – be prepared/processed state, the state of the system after applying procedure  $U$  to initial quantum state  $|\psi\rangle$

$|\psi'\rangle = \alpha|A\rangle + \beta|B\rangle$ , qubit  $|\psi'\rangle$  is a *superposition* of basis states  $|A\rangle$  and  $|B\rangle$

$\alpha$  – *amplitude* of state  $|A\rangle$ ,  $p_A = |\alpha|^2$

$\beta$  – *amplitude* of state  $|B\rangle$ ,  $p_B = |\beta|^2$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ , we will now use the logical basis states  $|0\rangle$  and  $|1\rangle$





# Some Examples of Qubits

Qubit	Ampl. of $ 0\rangle$	Ampl. of $ 1\rangle$	Prob. of $ 0\rangle$	Prob. of $ 1\rangle$
$ \psi_1\rangle = 1 0\rangle + 0 1\rangle =  0\rangle$	1	0	$1^2 = 1$	$0^2 = 0$
$ \psi_2\rangle = 0 0\rangle + 1 1\rangle =  1\rangle$	0	1	$0^2 = 0$	$1^2 = 1$
$ \psi_3\rangle = \frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 1\rangle$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$	$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$
$ \psi_4\rangle = \frac{1}{\sqrt{4}} 0\rangle + \frac{\sqrt{3}}{\sqrt{4}} 1\rangle$	$\frac{1}{\sqrt{4}}$	$\frac{\sqrt{3}}{\sqrt{4}}$	$\left(\frac{1}{\sqrt{4}}\right)^2 = \frac{1}{4}$	$\left(\frac{\sqrt{3}}{\sqrt{4}}\right)^2 = \frac{3}{4}$
$ \psi_5\rangle = \frac{\sqrt{5}}{\sqrt{8}} 0\rangle + \frac{\sqrt{3}}{\sqrt{8}} 1\rangle$	$\frac{\sqrt{5}}{\sqrt{8}}$	$\frac{\sqrt{3}}{\sqrt{8}}$	$\left(\frac{\sqrt{5}}{\sqrt{8}}\right)^2 = \frac{5}{8}$	$\left(\frac{\sqrt{3}}{\sqrt{8}}\right)^2 = \frac{3}{8}$



# Rule on Probabilities & the Born Rule

Qubit:  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\alpha$ , amplitude of state  $|0\rangle$

$\beta$ , amplitude of state  $|1\rangle$

$p_0 = |\alpha|^2$ , probability of observing state  $|0\rangle$

$p_1 = |\beta|^2$ , probability of observing state  $|1\rangle$

Probability Rule:  $p_0 + p_1 = 1$ , alternately:  $p_0 = (1 - p_1)$  or  $p_1 = (1 - p_0)$

Born Rule:  $|\alpha|^2 + |\beta|^2 = 1$ , or simply  $\alpha^2 + \beta^2 = 1$  if  $\alpha$  and  $\beta$  are real numbers.

The *Born Rule* is the probability rule expressed in terms of amplitudes.



# The Born Rule & Qubits as Points on a Circle

Qubit:  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

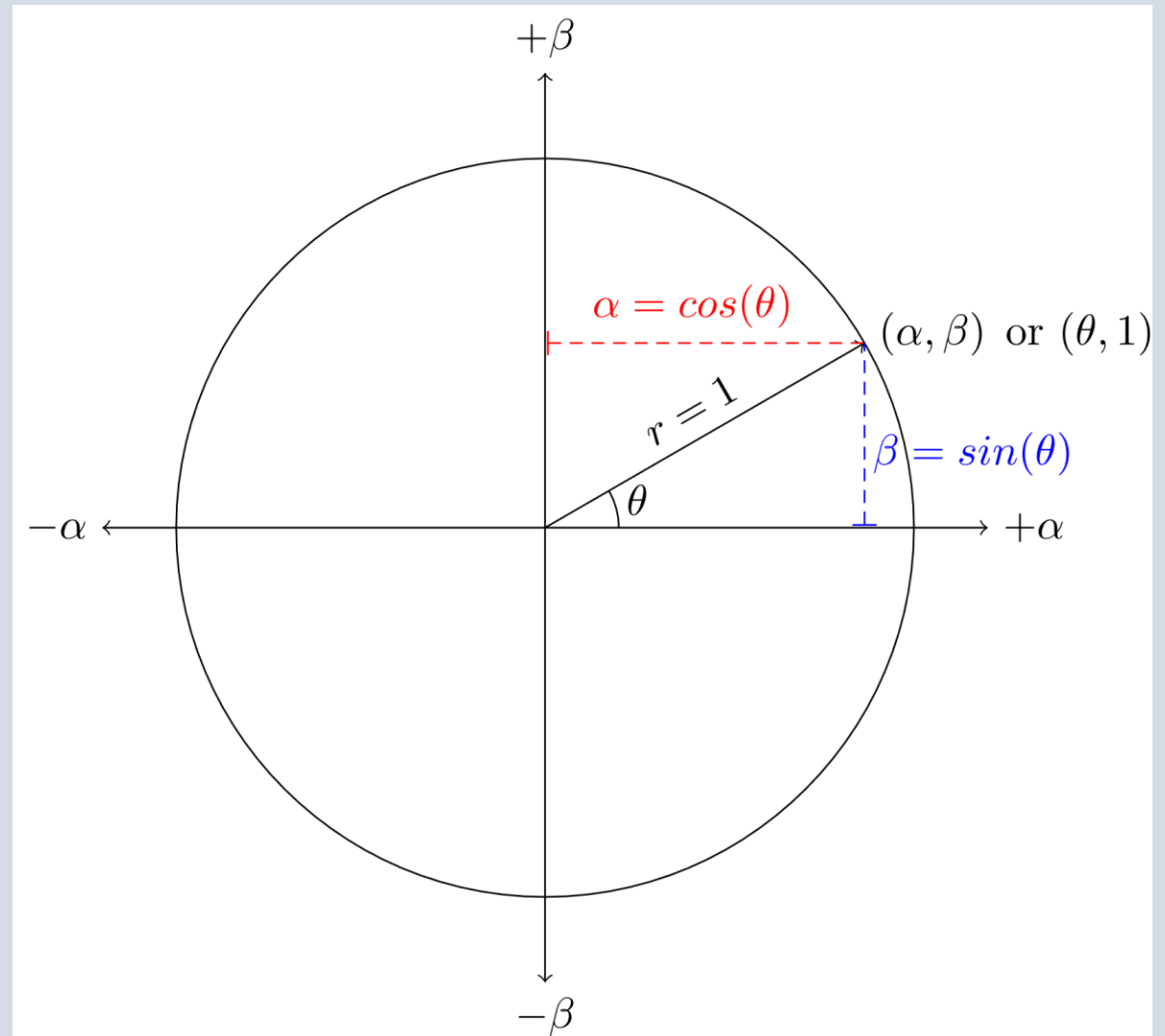
Born Rule: (for real numbers)

$$\alpha^2 + \beta^2 = 1$$

- circle on the  $\alpha\beta$ -plane
- center on  $(0,0)$
- radius = 1

$$x^2 + y^2 = 1$$

- circle on the  $xy$ -plane
- center on  $(0,0)$
- radius = 1



# Qubits as Points on a Circle

Qubit:  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$(\alpha, \beta)$

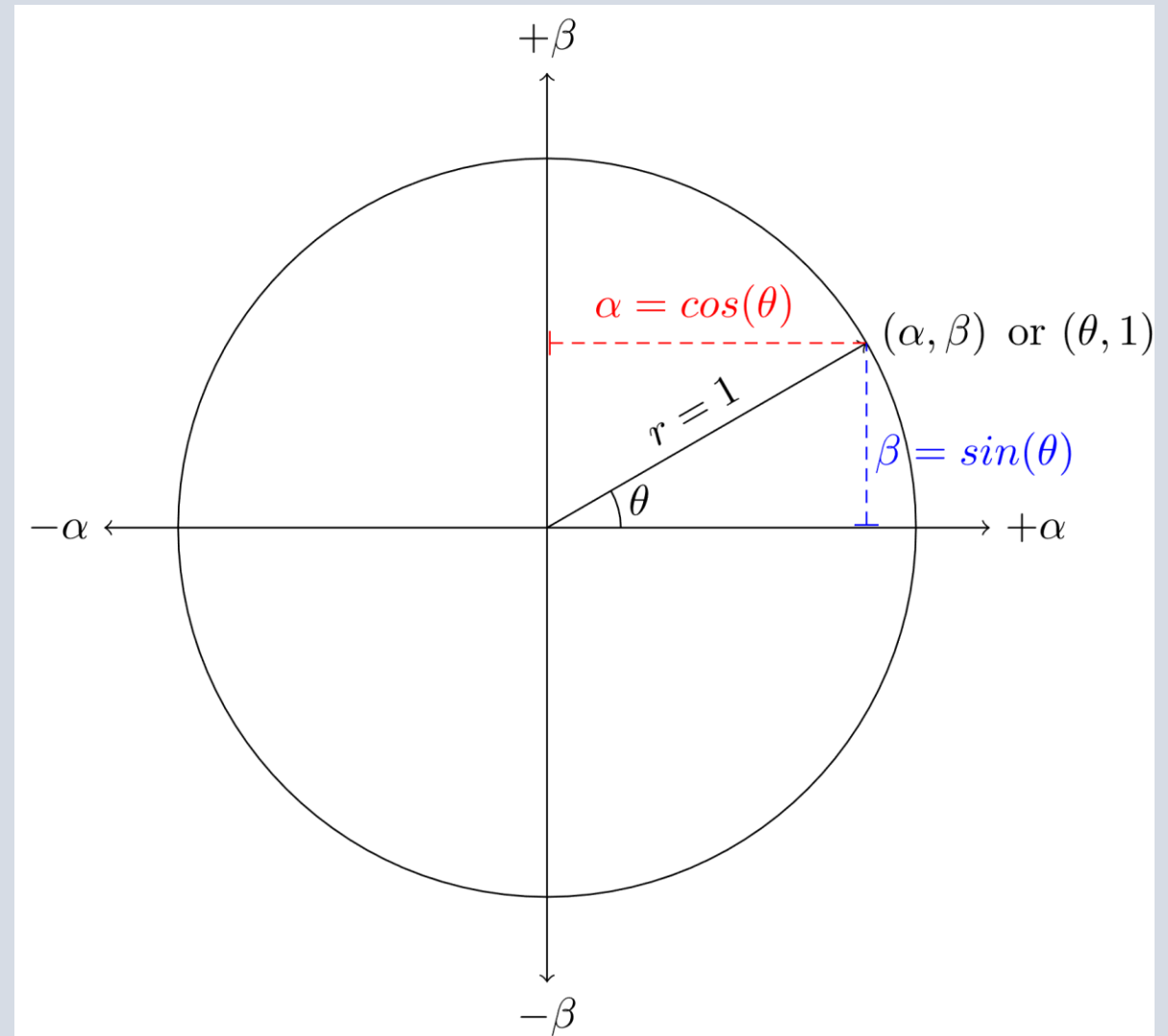
- point on the circle
- represents  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$(\theta, 1)$

- point on the circle
- polar coordinate of  $(\alpha, \beta)$

Qubit:  $|\psi_\theta\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$

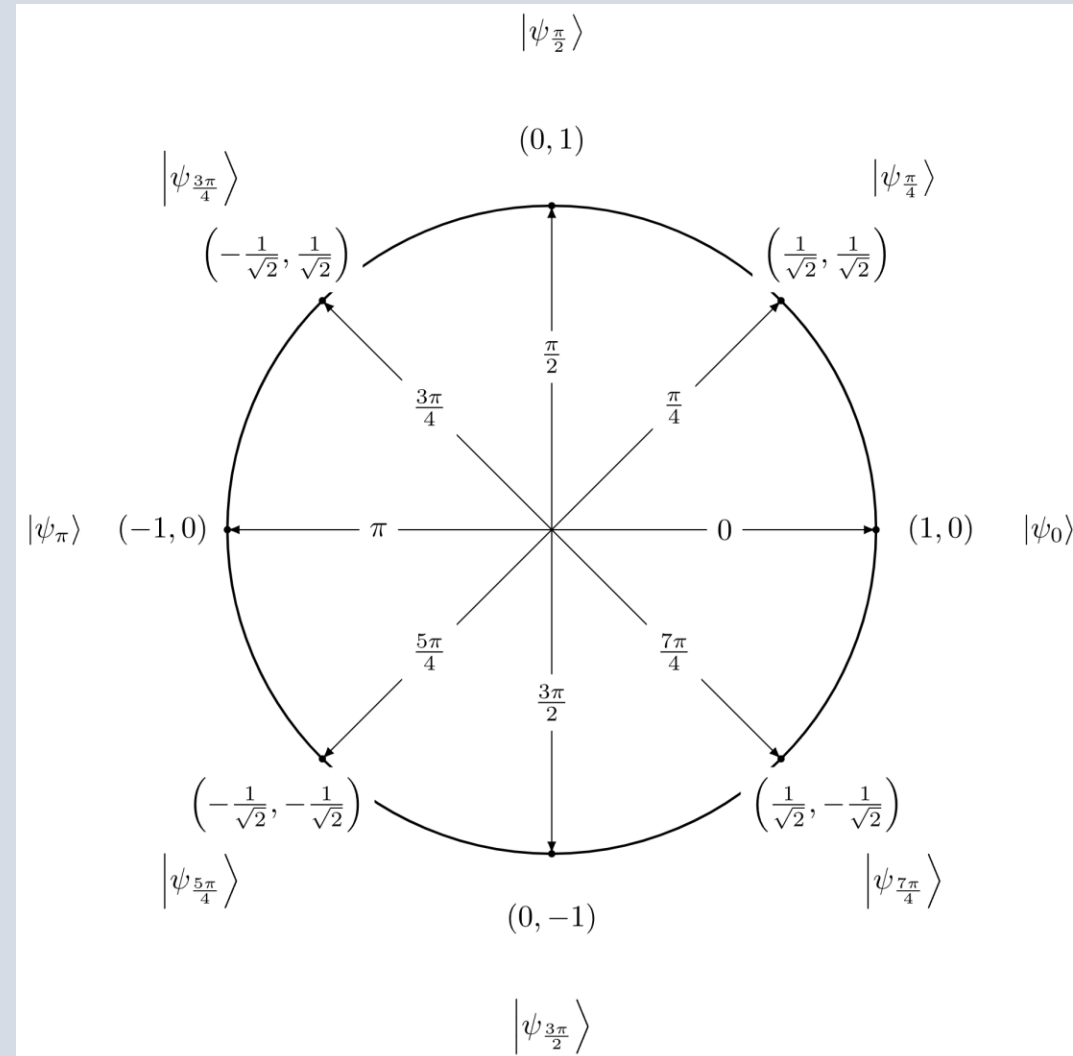
- polar coordinate form
- $\alpha = \cos(\theta)$
- $\beta = \sin(\theta)$



# Qubits as Points on a Circle: Examples

$\theta$	$\alpha = \cos(\theta)$	$\beta = \sin(\theta)$	$ \psi_\theta\rangle = \alpha 0\rangle + \beta 1\rangle$
0	1	0	$ \psi_0\rangle = 1 0\rangle + 0 1\rangle =  0\rangle$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$ \psi_{\frac{\pi}{4}}\rangle = \frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 1\rangle$
$\frac{\pi}{2}$	0	1	$ \psi_{\frac{\pi}{2}}\rangle = 0 0\rangle + 1 1\rangle =  1\rangle$
$\frac{3\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$ \psi_{\frac{3\pi}{4}}\rangle = -\frac{1}{\sqrt{2}} 0\rangle + \frac{1}{\sqrt{2}} 1\rangle$
$\pi$	-1	0	$ \psi_\pi\rangle = -1 0\rangle + 0 1\rangle = - 0\rangle$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$ \psi_{\frac{5\pi}{4}}\rangle = -\frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 1\rangle$
$\frac{3\pi}{2}$	0	-1	$ \psi_{\frac{3\pi}{2}}\rangle = 0 0\rangle - 1 1\rangle = - 1\rangle$
$\frac{7\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$ \psi_{\frac{7\pi}{4}}\rangle = \frac{1}{\sqrt{2}} 0\rangle - \frac{1}{\sqrt{2}} 1\rangle$

Qubits that encode the **same probabilities** but have different amplitudes (different signs of amplitudes) are said to have different phases.



# Qubits as Points on a Circle

Due to the born rule and our initial **restriction** that the amplitudes are real numbers (not yet complex numbers), we are able to define **any unique qubit** using a single parameter which is the angle  $\theta$ .

Qubit	$ \psi\rangle = \alpha 0\rangle + \beta 1\rangle$
Point (Rectangular Coordinate)	$(\alpha, \beta)$
Point (Polar Coordinate)	$(\theta, 1)$
Qubit	$ \psi_\theta\rangle = \cos(\theta) 0\rangle + \sin(\theta) 1\rangle$

If you look at the probability rule  $p_0 + p_1 = 1$ , it can help you see that you only need 1 parameter to define a qubit with real amplitudes. If you specify the qubit's  $p_0$ , due to the probability rule, you can derive  $p_1 = (1 - p_0)$  from  $p_0$ . They are not independent values. The same goes for the amplitudes. *i.e.*  $\beta = \sqrt{1 - \alpha^2}$





# Operations on Single Qubits

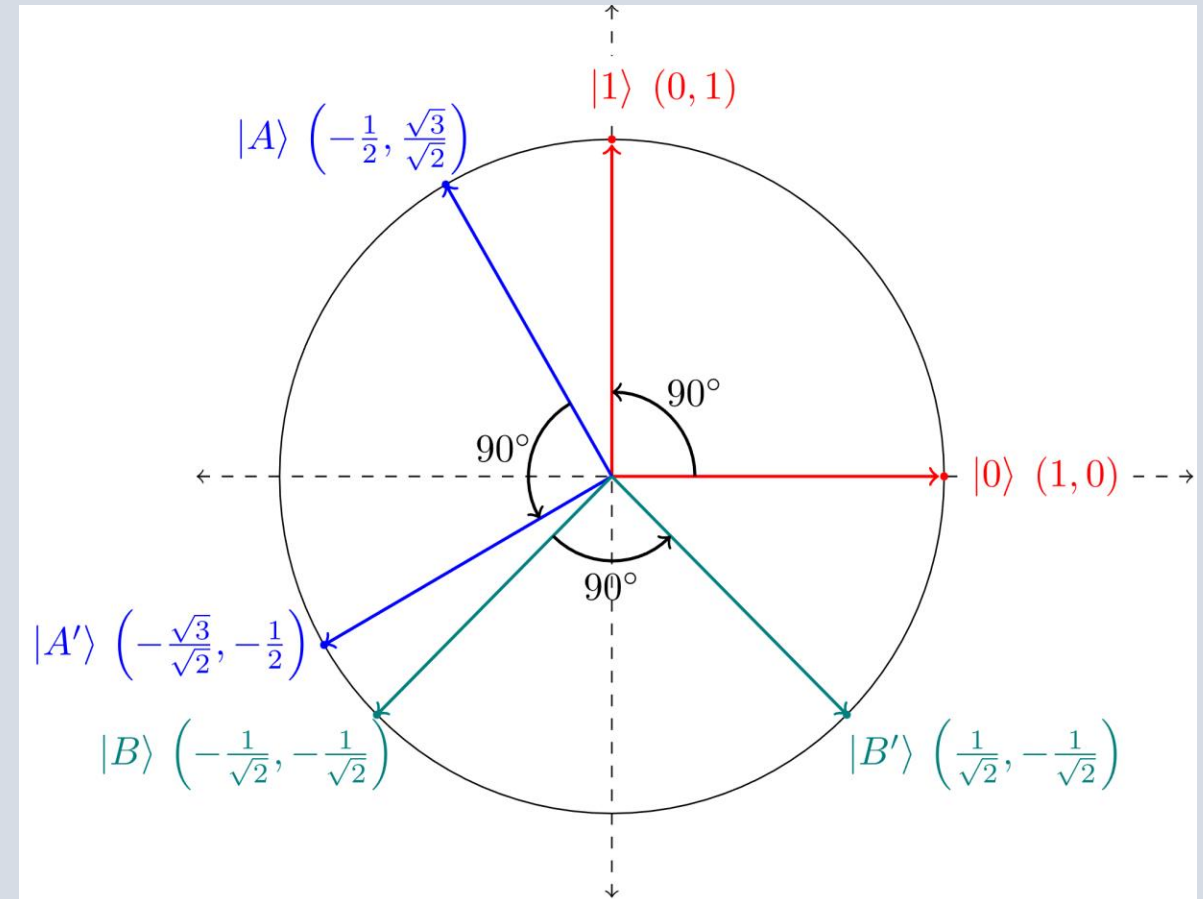
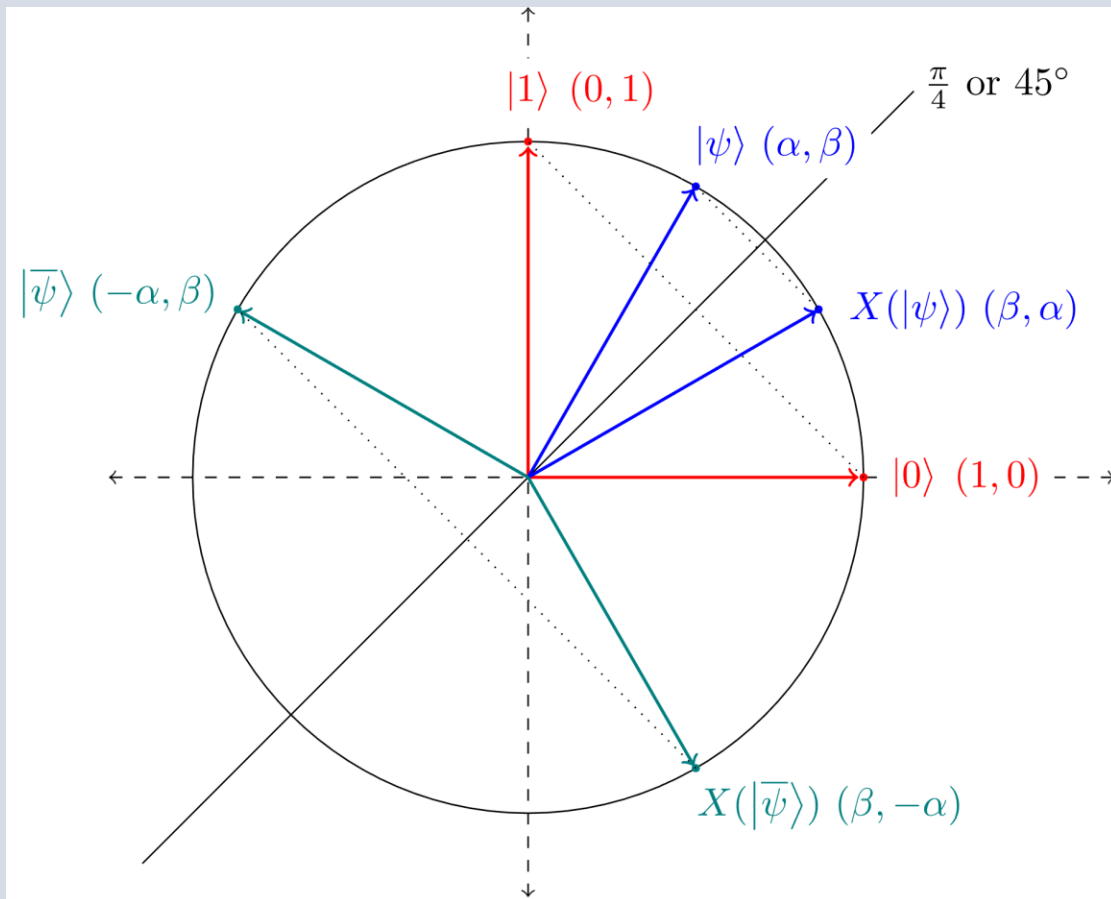
- An **operation**  $U$  transforms a qubit by changing the qubit's amplitudes.
- The **operation**  $U$  should follow the born rule.

$$\begin{aligned} |\psi\rangle = \alpha|0\rangle + \beta|1\rangle &\longrightarrow \boxed{U} \longrightarrow |\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle \\ (\alpha, \beta) &\longrightarrow \boxed{U} \longrightarrow (\alpha', \beta') \\ (\theta, 1) &\longrightarrow \boxed{U} \longrightarrow (\theta', 1) \end{aligned}$$



# Operations on Single Qubits

- Qubit **operations** can be viewed as rotations or reflections.



- In general, the amplitudes,  $\alpha$  and  $\beta$ , of a qubit  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$  are complex numbers.

## Real Amplitudes

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- $\alpha = x$
- $\beta = y$
- $x, y$  are real numbers

## Complex Amplitudes

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- $\alpha = x + qi$  (complex)
- $\beta = y + ri$  (complex)
- $x, q$  are real numbers
- $y, r$  are real numbers

- When the amplitudes are **real**: two real numbers are needed to define a unique qubit
- When the amplitudes are **complex**: four real numbers are needed to define a unique qubit



The square of a complex number is also complex:

- $\alpha = x + qi$
- $\alpha^2 = (x + qi)^2 = (x + qi)(x + qi) = x^2 + 2xqi - q^2 = (x^2 - q^2) + (2xq)i$

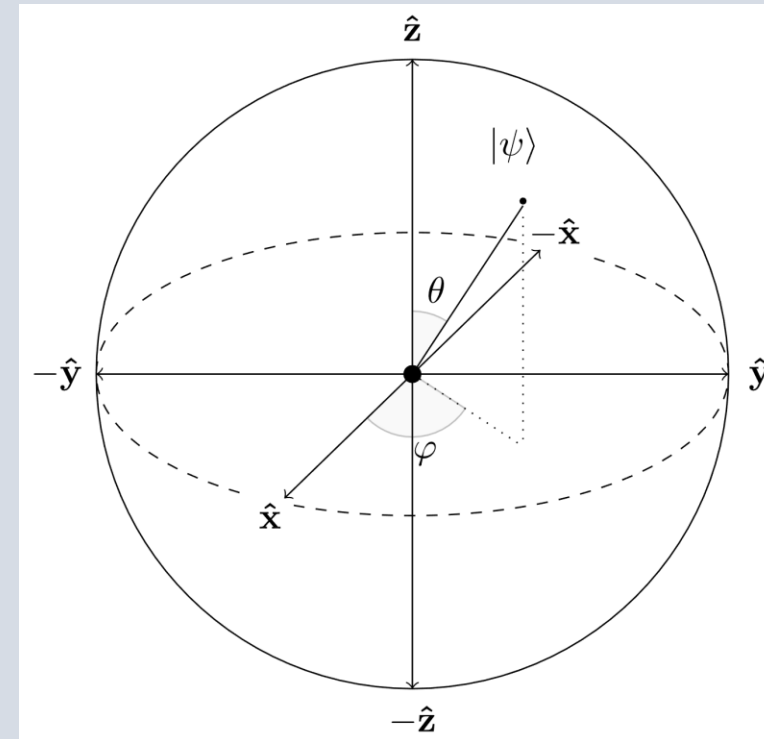
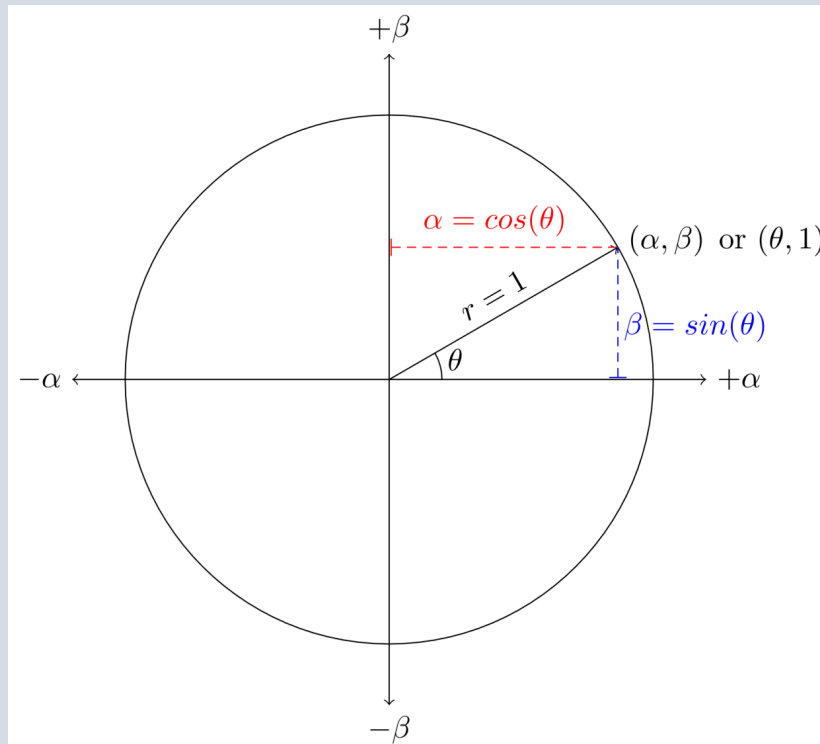
This means we cannot use simple squaring to get real probability from the complex amplitude. The **square modulus**  $|\alpha|^2$  of a complex number  $\alpha = x + qi$  is a real non-negative and can be used as probability:

- $|\alpha|^2 = (x + qi)(x - qi) = x^2 + (xqi - xqi) + q^2 = x^2 + q^2$



# Qubits with Complex Amplitudes + Born Rule

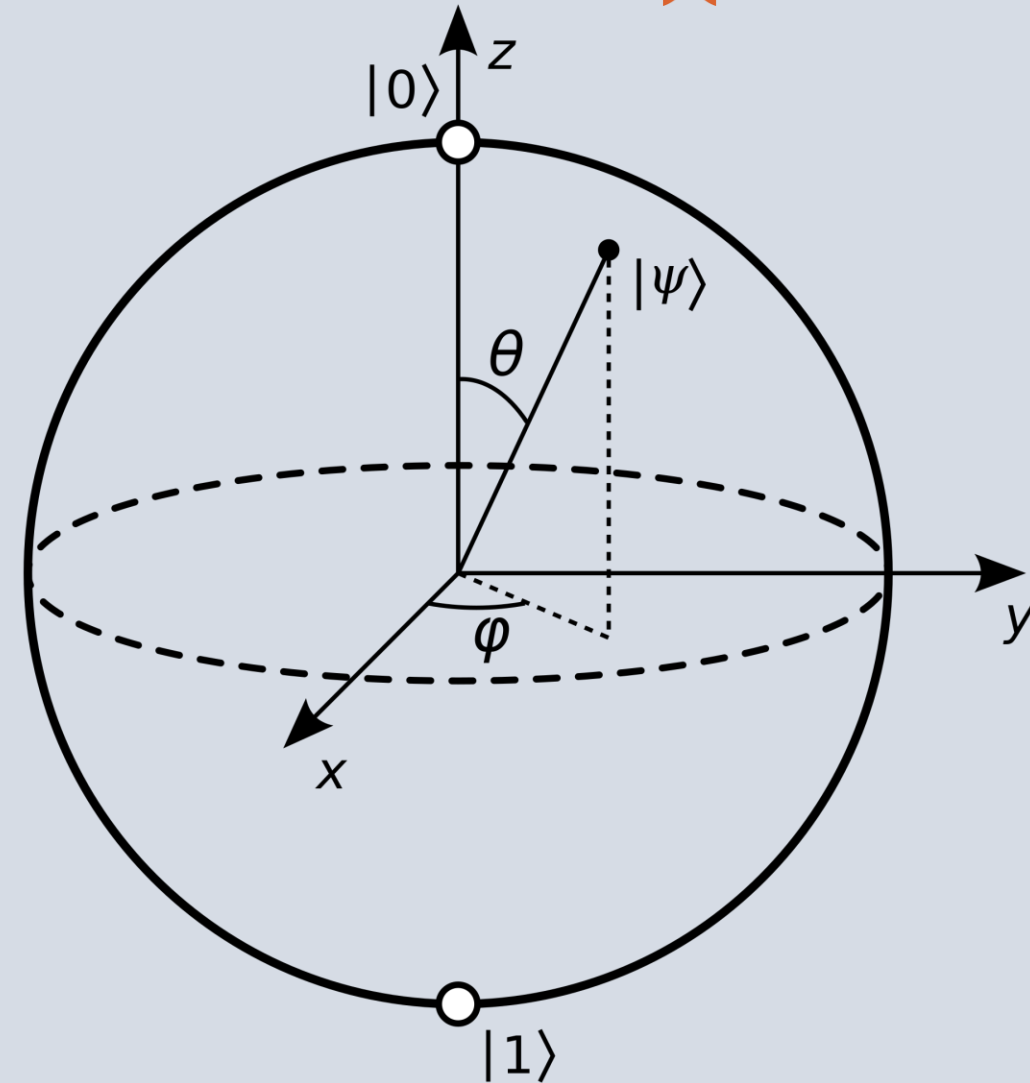
- Due to the **born rule**, when the amplitudes  $\alpha$  and  $\beta$  are real, you only need **1** parameter, the **angle**  $\theta$ , to define a unique qubit. From **2** real parameters  $\alpha$  and  $\beta$  to **1** parameter  $\theta$ .
- For the general qubit with complex amplitudes  $\alpha = x + qi$  and  $\beta = y + ri$ , using the born rule, the **4** real parameters  $x, q, y, r$  can be reduced **2** angle parameters  $\theta$  and  $\varphi$ .



# Qubit & the Bloch Sphere

- A qubit with complex amplitudes can be represented a **point on a sphere** of radius 1. The sphere is called the Bloch sphere.
- In spherical coordinates, you only need two parameters, angle  $\theta$  and angle  $\phi$ , to define a unique point on the sphere.
- The qubit specified by the two angles  $\theta$  and  $\phi$ :

$$|\psi\rangle = \underbrace{\cos\left(\frac{\theta}{2}\right)}_{\alpha} |0\rangle + \underbrace{e^{i\phi} \sin\left(\frac{\theta}{2}\right)}_{\beta} |1\rangle$$





# Qubits can be Represented as Column Vectors

The qubit  $|0\rangle$  can be represented by the column vector:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The qubit  $|1\rangle$  can be represented by the column vector:

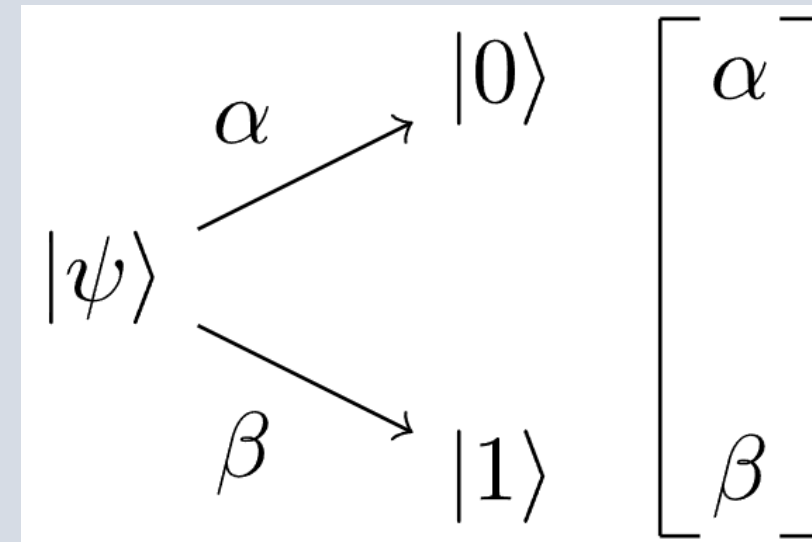
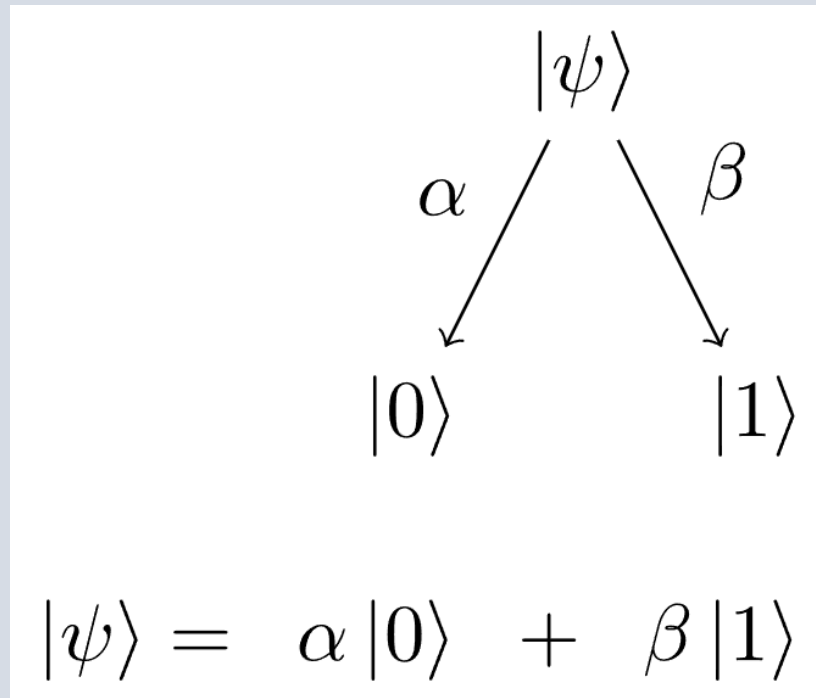
$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The generic qubit  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$  can be represented by column vector:

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



# Qubits drawn as Binary Trees



- For quantum computing, a **quantum state** is simply a collection of qubits.
- If you have two qubits,  $|q_1\rangle$  and  $|q_2\rangle$ , the quantum state can be written as  $|q_1q_2\rangle$ .
  - Example 1: If  $|q_1\rangle = |0\rangle$ ,  $|q_2\rangle = |0\rangle$ , then  $|q_1q_2\rangle = |00\rangle$ .
  - Example 2: If  $|q_1\rangle = |0\rangle$ ,  $|q_2\rangle = |1\rangle$ , then  $|q_1q_2\rangle = |01\rangle$ .
  - Example 3: If  $|q_1\rangle = |1\rangle$ ,  $|q_2\rangle = |0\rangle$ , then  $|q_1q_2\rangle = |10\rangle$ .
  - Example 4: If  $|q_1\rangle = |1\rangle$ ,  $|q_2\rangle = |1\rangle$ , then  $|q_1q_2\rangle = |11\rangle$ .

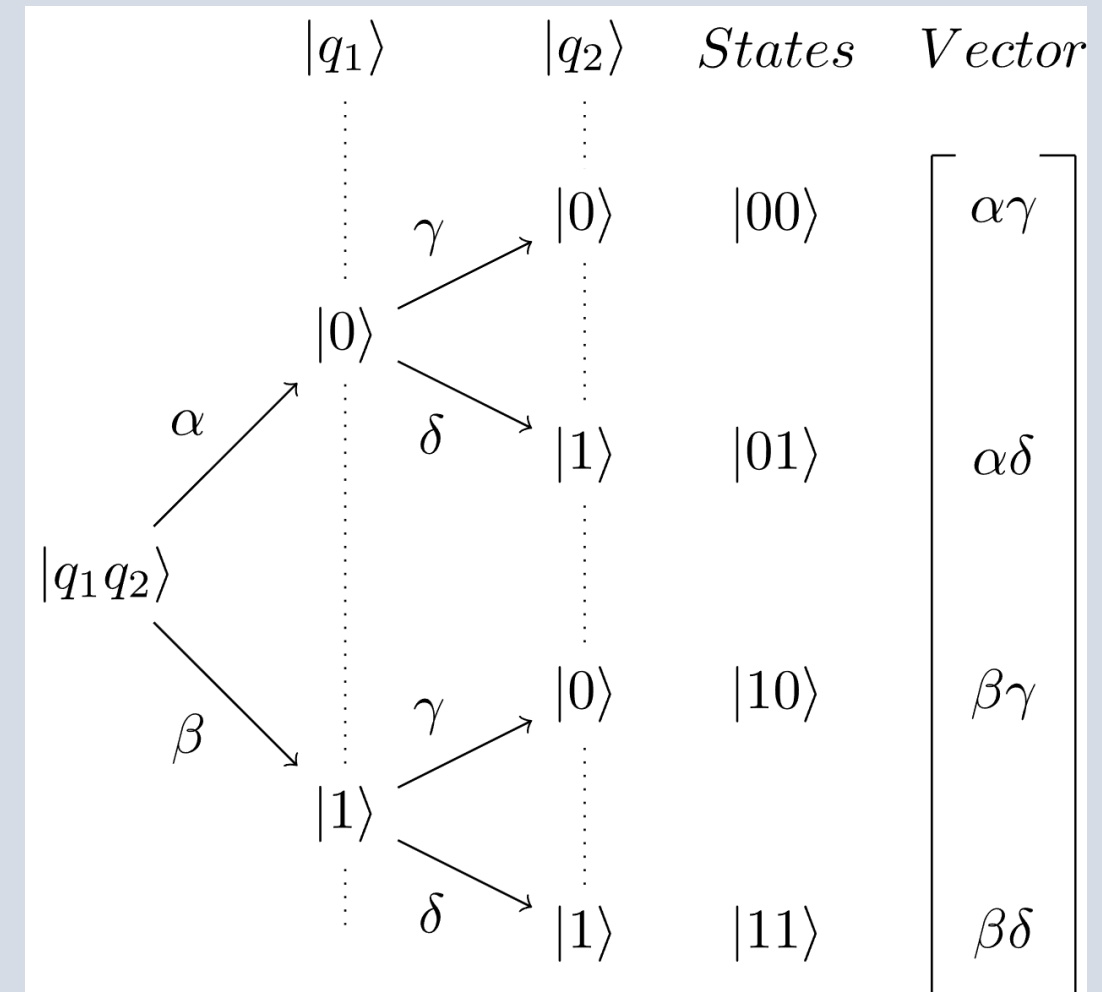
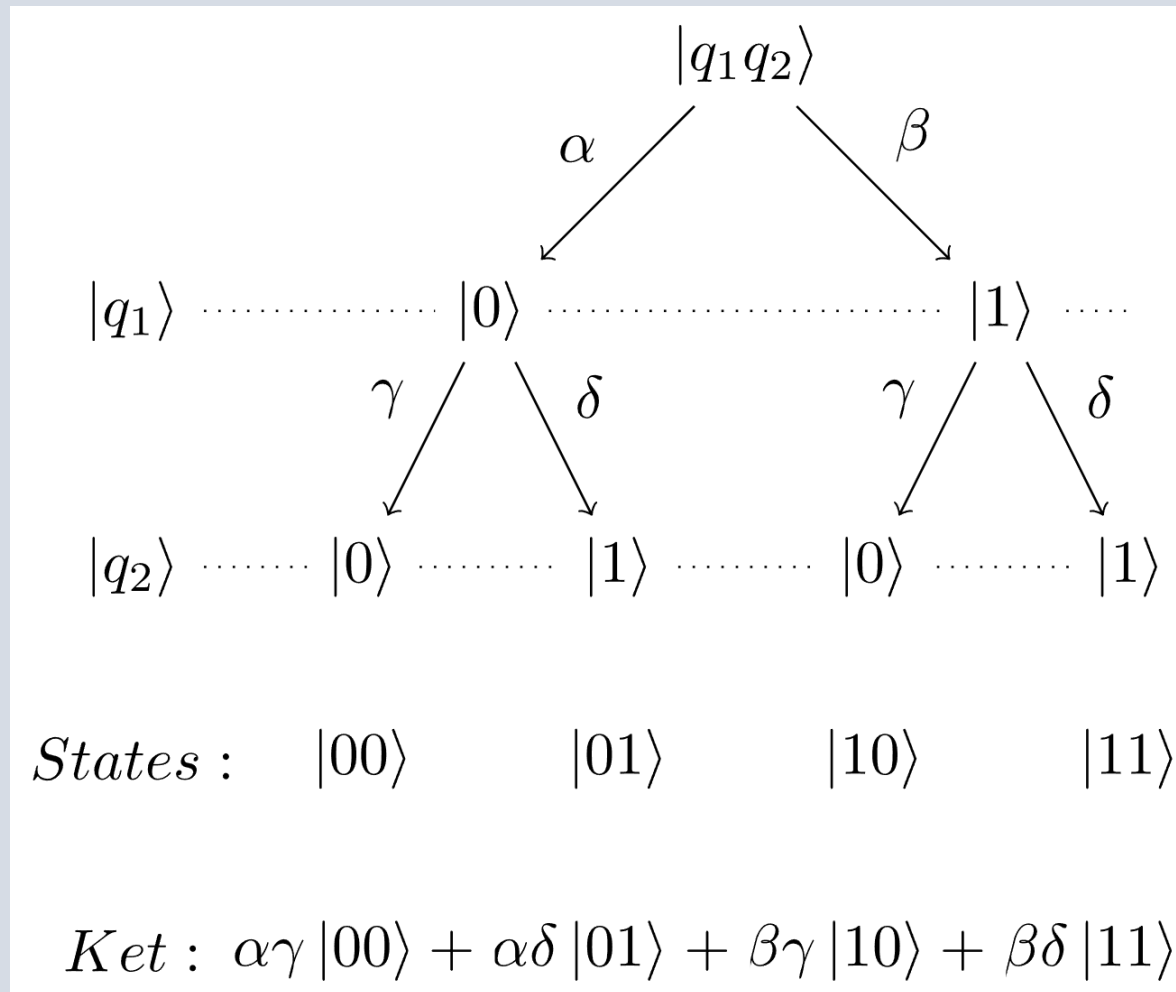


If you have the independent two qubits  $|q_1\rangle = \alpha|0\rangle + \beta|1\rangle$  and  $|q_2\rangle = \gamma|0\rangle + \delta|1\rangle$ , the quantum state can be written as:

- $|q_1q_2\rangle = |q_1\rangle \otimes |q_2\rangle = \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$
- $|q_1\rangle \otimes |q_2\rangle$  is called the tensor product of states/qubits  $|q_1\rangle$  and  $|q_2\rangle$ .
- $|q_1\rangle$  and  $|q_2\rangle$  both have the basis states:  $|0\rangle$  and  $|1\rangle$ .
- $|q_1\rangle \otimes |q_2\rangle$  has the basis states:  $|00\rangle$ ,  $|01\rangle$ ,  $|10\rangle$ , and  $|11\rangle$



# Two-Qubit Quantum State as Binary Trees



# Entanglement of Multiple Qubits

- Two qubits are **entangled** if their *amplitudes (or probabilities)* are correlated.
- You can entangle two qubits using 2-qubit *entanglement* operators.

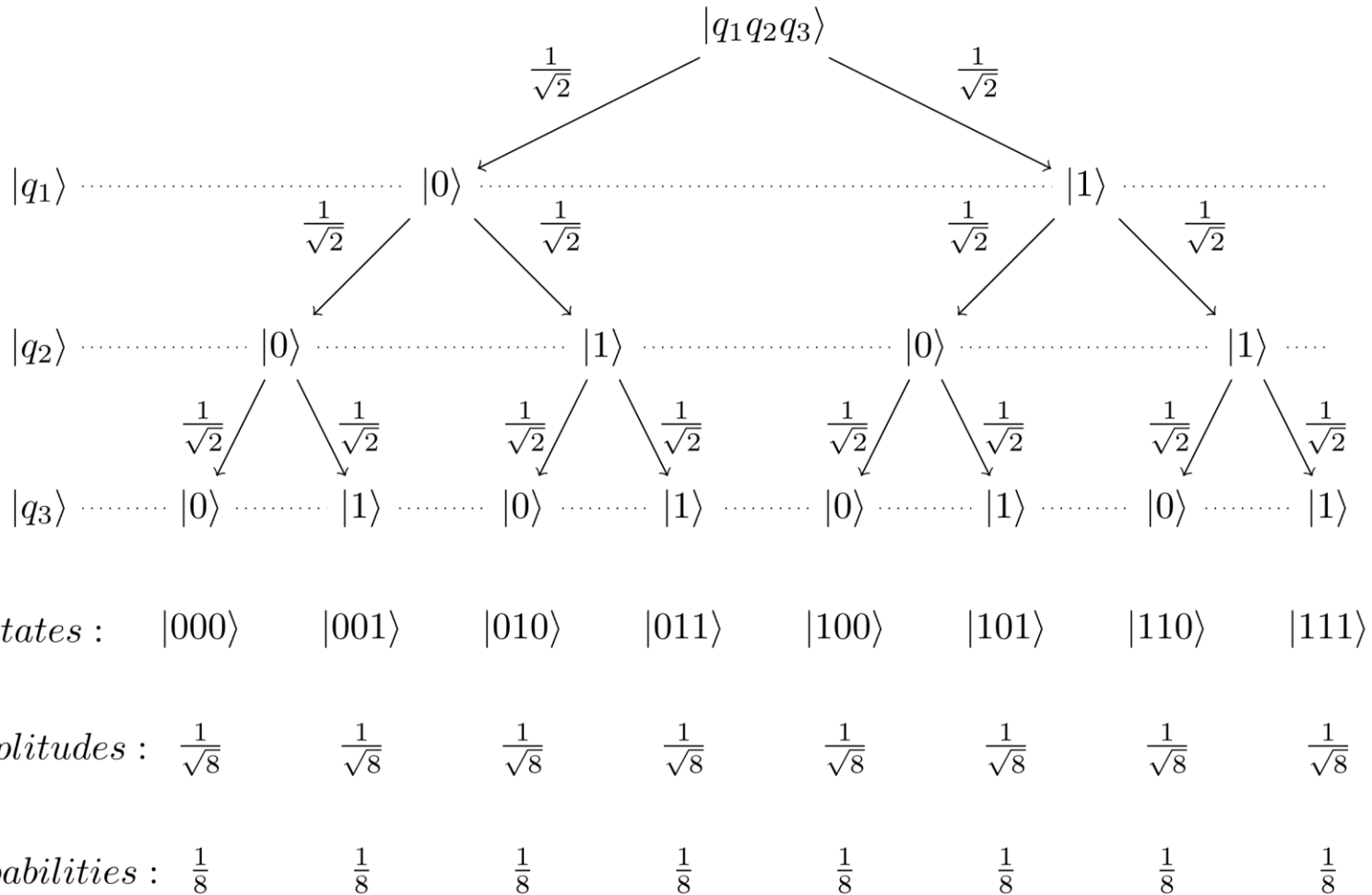
$$\begin{array}{lcl} |q_1\rangle = \alpha|0\rangle + \beta|1\rangle & \longrightarrow & \\ |q_2\rangle = \gamma|0\rangle + \delta|1\rangle & \longrightarrow & \end{array} \begin{array}{c} \boxed{U} \end{array} \begin{array}{lcl} \longrightarrow & & |q_1'\rangle = \alpha'|0\rangle + \beta'|1\rangle \\ \longrightarrow & & |q_2'\rangle = \gamma'|0\rangle + \delta'|1\rangle \end{array}$$

- $|q_1'\rangle$  and  $|q_2'\rangle$  are no longer independent but are correlated.
- The amplitudes  $\alpha', \beta'$  are correlated with amplitudes  $\gamma', \delta'$ .



# Example of an Entangled State

Example 0: No entanglement. No amplitude correlation.



Initial qubit values:

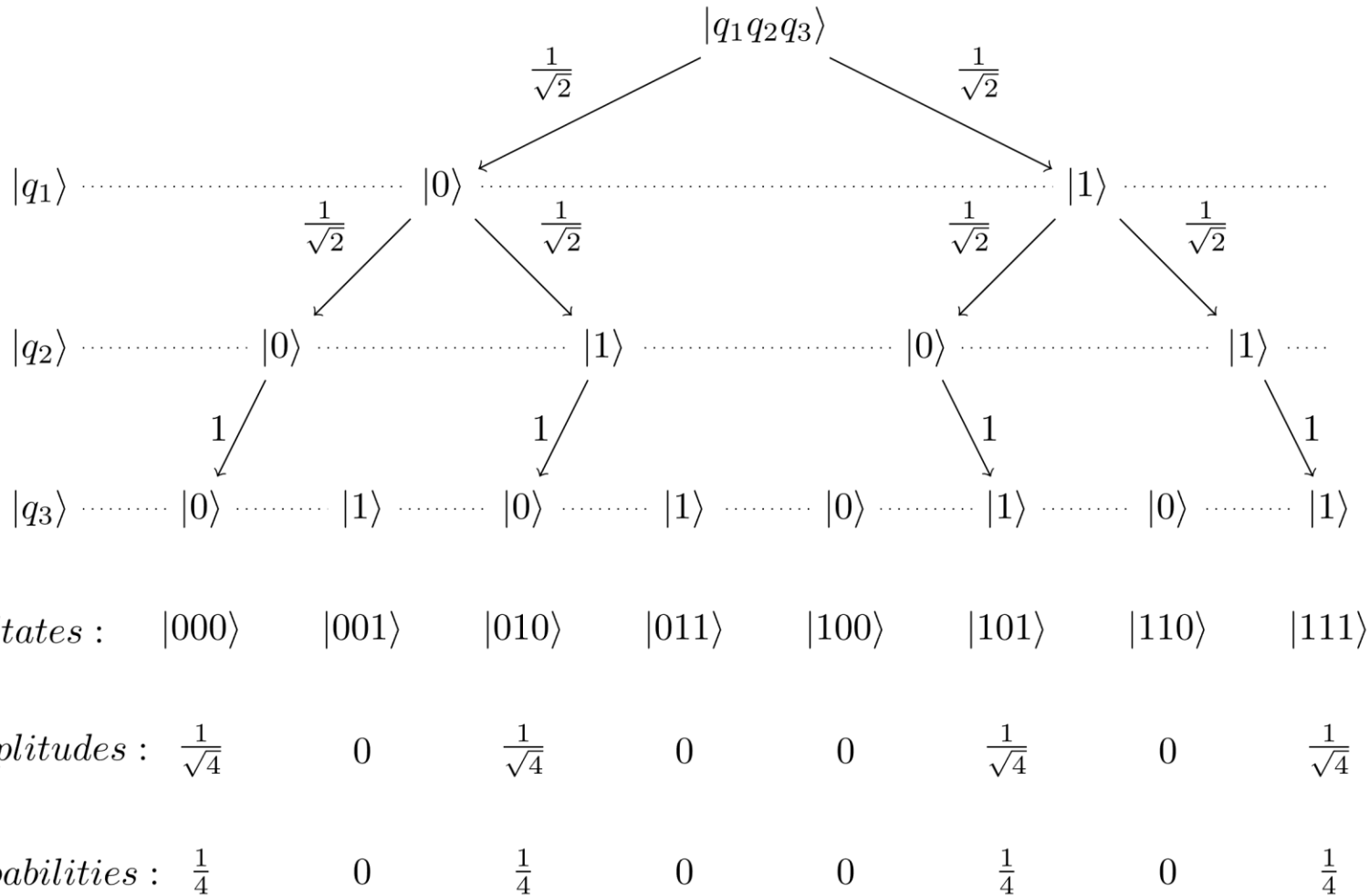
$$|q_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|q_2\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|q_3\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

# Example of an Entangled State

**Example 1:** Entangled with correlation  $q_1 = q_3$ .



Initial qubit values:

$$|q_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|q_2\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|q_3\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

**Note:** Edges with 0 amplitudes are omitted.



Republic of the Philippines

DEPARTMENT OF SCIENCE AND TECHNOLOGY

ADVANCED SCIENCE AND TECHNOLOGY INSTITUTE



# Example of an Entangled State

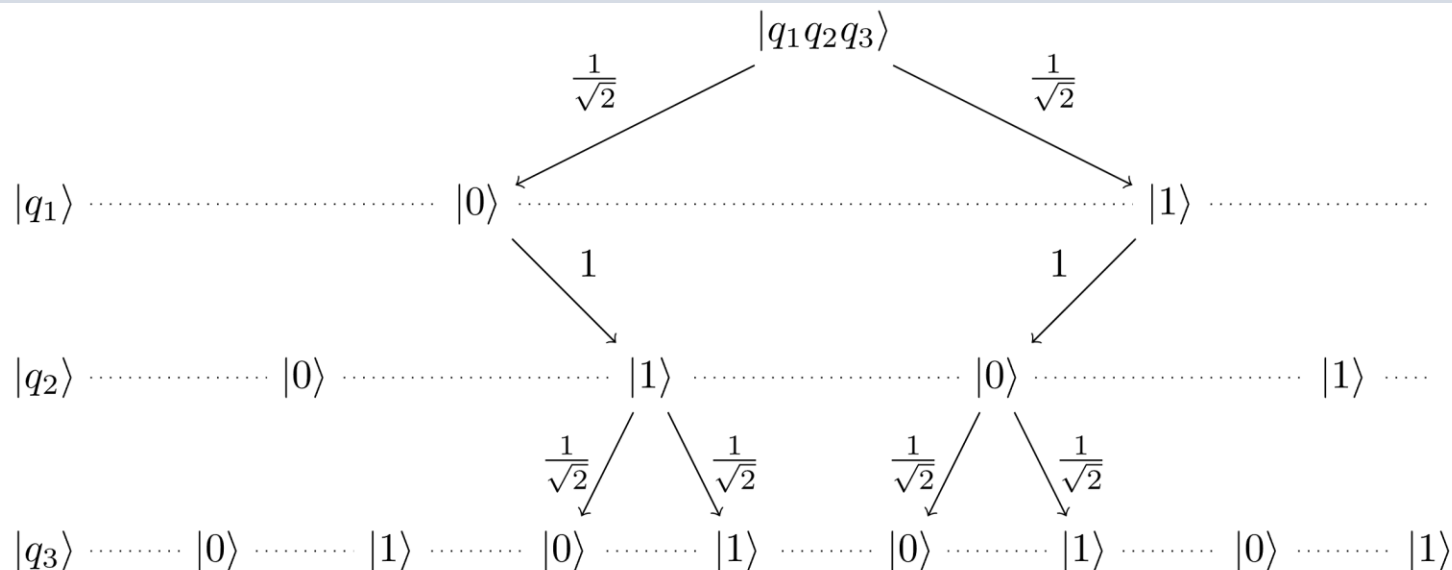
Example 2: Entangled with correlation  $q_1 \neq q_2$ .

Initial qubit values:

$$|q_1\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

$$|q_2\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

$$|q_3\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$



States :  $|000\rangle$   $|001\rangle$   $|010\rangle$   $|011\rangle$   $|100\rangle$   $|101\rangle$   $|110\rangle$   $|111\rangle$

Amplitudes : 0 0  $\frac{1}{\sqrt{4}}$   $\frac{1}{\sqrt{4}}$   $\frac{1}{\sqrt{4}}$   $\frac{1}{\sqrt{4}}$  0 0

Probabilities : 0 0  $\frac{1}{4}$   $\frac{1}{4}$   $\frac{1}{4}$   $\frac{1}{4}$  0 0

Note: Edges with 0 amplitudes are omitted.



Republic of the Philippines

DEPARTMENT OF SCIENCE AND TECHNOLOGY

ADVANCED SCIENCE AND TECHNOLOGY INSTITUTE



# Example of an Entangled State

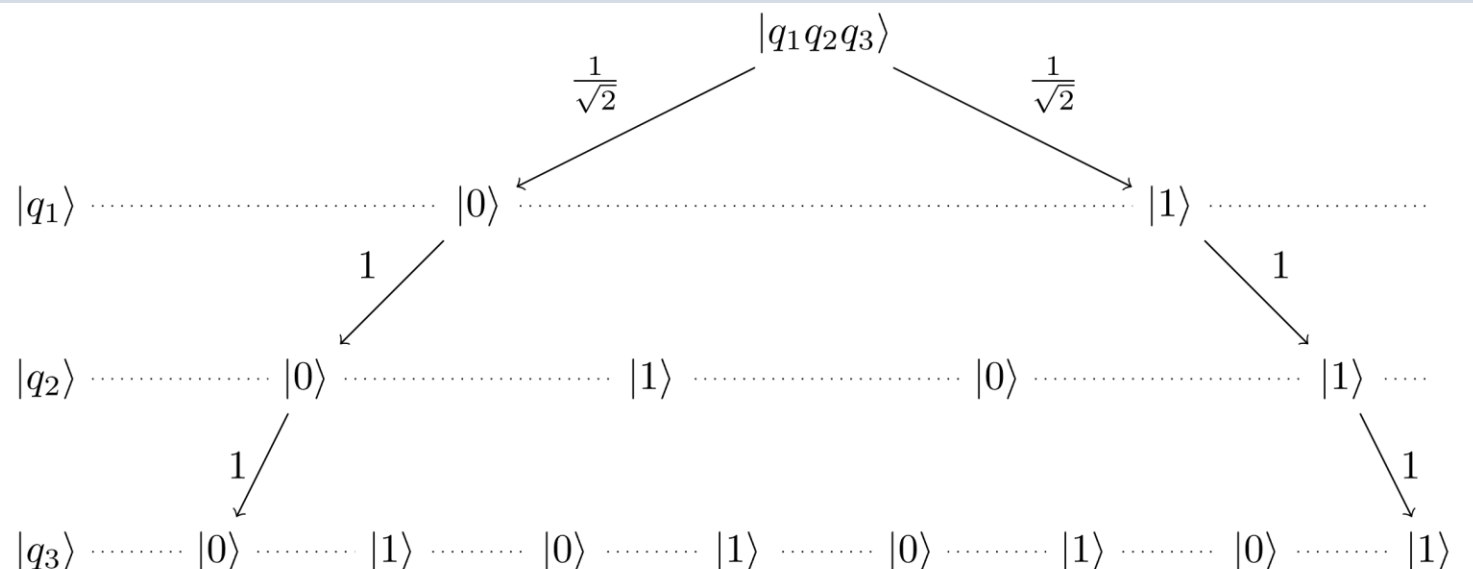
Example 3: Entangled with correlation  $q_1 = q_2 = q_3$ .

Initial qubit values:

$$|q_1\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

$$|q_2\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$

$$|q_3\rangle = (1/\sqrt{2})|0\rangle + (1/\sqrt{2})|1\rangle$$



States :  $|000\rangle$   $|001\rangle$   $|010\rangle$   $|011\rangle$   $|100\rangle$   $|101\rangle$   $|110\rangle$   $|111\rangle$

Amplitudes :  $\frac{1}{\sqrt{2}}$  0 0 0 0 0 0  $\frac{1}{\sqrt{2}}$

Probabilities :  $\frac{1}{2}$  0 0 0 0 0 0  $\frac{1}{2}$

Note: Edges with 0 amplitudes are omitted.



Republic of the Philippines

DEPARTMENT OF SCIENCE AND TECHNOLOGY

ADVANCED SCIENCE AND TECHNOLOGY INSTITUTE





# Quantum States: Basic Quantum Mechanics Concepts for Quantum Computing

(END)

Ren Tristan A. de la Cruz  
Project Technical Specialist I  
Quantum Circuit Simulation Project  
DOST Advanced Science and Technology Institute  
2023 November

