# 1 Scalar (Closed Economy) Model

**Firm Problem**: The baseline model features CES demand with constant marginal cost of production $w/\phi_g$. We assume firms set prices to maximize static profits $\left(p = \tilde{\gamma}\frac{w}{\phi_g}\right)$ and choose data workers $L_t$ and technique $a_t$ to maximize expected profits over time.[1]

**Quality:** $A = \bar{A} - (a_t - z_t - \epsilon_t^a)^2$ (1)

**Demand:** $x_t = A_t \times y\dfrac{p^{-\gamma}}{P^{1-\gamma}} = A_t \times y\dfrac{\left(\tilde{\gamma}\frac{w}{\phi_g}\right)^{-\gamma}}{P^{1-\gamma}} = A_t \times \phi_g^{\gamma}\bar{x} \quad \text{s.t.} \quad \bar{x} = y\dfrac{(\tilde{\gamma}w)^{-\gamma}}{P^{1-\gamma}}$ (2)

**Profit:** $\pi_t = x_t\left(p - \dfrac{w}{\phi_g}\right) - wL_t = A_t\dfrac{\phi_g^{\gamma-1}\bar{x}w}{(\gamma-1)} - wL_t = A_t\bar{\pi} - wL_t$ (3)

**State Variables**: for the firm are the current optimal technique $z_t$, its current technique $a_t$, it's guess of optimal technique $\hat{z}_t$, and its certainty about that guess $\Sigma_t$. These variables evolve according to:

$$
\begin{aligned}
&\hat{z}_t = \mathbb{E}[z_t|\mathcal{I}_t] \\
&\Sigma_t = \mathbb{E}\left[(\mathbb{E}[z_t \mid \mathcal{I}_t] - z_t)^2\right] \\
&\dot{z}_t = -\theta z_t + \omega && \text{s.t.} \quad \omega \sim (0, Q) \\
&y_t = z_t + v && \text{s.t.} \quad V \sim \left(0, R_t^{-1}\right) \quad \text{and} \quad R_t = \left[\phi_d L_t^{\alpha_1}\mathbb{E}[x_t]^{\alpha_2} + \sigma_a^{-2}\right] \\
&\dot{\hat{z}}_t = -\theta\hat{z}_t + K(y_t - \hat{z}_t) && \text{s.t.} \quad K = \Sigma_t R_t \\
&\dot{\Sigma}_t = -\Sigma_t^2 R_t - 2\theta\Sigma_t + Q \\
&\dot{a}_t = \mu_t
\end{aligned}
$$

Where we have used the result for the continuous time Kalman filter (see section 2 for general form). We can rewrite these expressions as a single stochastic differential equation:

$$
dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \quad \text{where} \quad X_t = (z_t, \hat{z}_t, \Sigma_t, a_t)' \quad \text{s.t.}
$$

$$
b(X_t) = \begin{pmatrix} -\theta z_t \\ K(z_t - \hat{z}_t) - \theta\hat{z}_t \\ -\Sigma_t^2 R_t - 2\theta\Sigma_t + Q \\ \mu_t \end{pmatrix}, \qquad \sigma(X_l) = \begin{pmatrix} \sqrt{Q} & 0 \\ 0 & KR_t^{-1/2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{Q} & 0 \\ 0 & \Sigma_t\sqrt{R_t} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}
$$

**Deriving Expected Quality**:

$$
\begin{aligned}
\mathbb{E}[A_t|\mathcal{I}_t] &= \mathbb{E}\left[\bar{A} - (a_t - z_t - \epsilon_t^a)^2 \mid \mathcal{I}_t\right] \\
&= \bar{A} - \mathbb{E}\left[(a_t - z_t)^2 + (\epsilon_t^a)^2 - 2(a_t - z_t)\,\epsilon_{i,j,t}^a \mid \mathcal{I}_t\right] = \bar{A} - \mathbb{E}\left[(a_t - z_t)^2 \mid \mathcal{I}_t\right] - \sigma_a^2 \\
&= \bar{A} - \mathbb{E}\left[[(a_t - \hat{z}_t) + (\hat{z}_t - z_t)]^2 \mid \mathcal{I}_t\right] - \sigma_a^2 \\
&= \bar{A} - \mathbb{E}\left[(a_t - \hat{z}_t)^2 + 2(a_t - \hat{z}_t)(\hat{z}_t - z_t) + (\hat{z}_t - z_t)^2 \mid \mathcal{I}_t\right] - \sigma_a^2 \\
&= \bar{A} - \left[(a_t - \hat{z}_t)^2 + 2(a_t - \hat{z}_t)\,\mathbb{E}[\hat{z}_t - z_t|\mathcal{I}_t] + \mathbb{E}\left[(\hat{z}_t - z_t)^2 \mid \mathcal{I}_t\right]\right] - \sigma_a^2 \\
\tilde{A}_t &= \bar{A} - (a_t - \hat{z}_t)^2 - \Sigma_t - \sigma_a^2 = \bar{A} - \Sigma_t - \sigma_a^2
\end{aligned}
$$

[1] The assumption on pricing is made to guarantee a closed form solution

Where in the last line we have used that at optimal the firm will always set its technique to maximize quality.

**Simplified Firm Problem**: We can now write the firm problem in terms of one state variable:

$$v(\Sigma_0) = \max_{\{L_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} \left( \tilde{A}(\Sigma_t)\bar{\pi} - wL_t \right) dt \quad \text{s.t.} \quad \dot{\Sigma}_t = -\Sigma_t^2 R_t - 2\theta\Sigma_t + Q$$

**HJB Equation:** $v(\Sigma) = \max_L \tilde{A}(\Sigma)\bar{\pi} - wL + v'(\Sigma)\dot{\Sigma}$ (4)

**Optimal Choice of $L$:**

Necessary Derivatives

$R = \phi_d L^{\alpha_1} \mathbb{E}[x]^{\alpha_2} + \sigma_a^{-2} \quad \rightarrow \quad \dfrac{\partial R}{\partial L} = \xi L^{\alpha_1 - 1} \quad \text{s.t.} \quad \xi = \alpha_1 \phi_d \mathbb{E}[x]^{\alpha_2}$

HJB components that are functions of L

$= -wL + v_\Sigma \left[ -\Sigma^2 R - 2\theta\Sigma + Q \right] \quad \rightarrow \quad wL + v_\Sigma \left[ \Sigma^2 R \right]$

FOC

$0 = -w - \xi v_\Sigma \Sigma^2 L^{\alpha_1 - 1}$

$L = \left( -w^{-1} \xi v_\Sigma \Sigma^2 \right)^{\frac{1}{1-\alpha_1}}$

**Choice of Data Workers to Ensure Zero Drift**

$0 = -\Sigma_t^2 R_0 - 2\theta\Sigma_t + Q$

$R_0 = \dfrac{(Q - 2\theta\Sigma_t)}{\Sigma_t^2}$

$L_0 = \left( \dfrac{Q - 2\theta\Sigma_t - \sigma_a^{-2}\Sigma_t^2}{\phi_d \mathbb{E}[x_t]^{\alpha_2} \Sigma_t^2} \right)^{1/\alpha_t}$

**$v_\Sigma^*$ necessary to incentivize zero drift**:

$L_0 = \left( w^{-1} \xi v_\Sigma^* \Sigma^2 \right)^{\frac{1}{1-\alpha_1}}$

$v_\Sigma^* = \dfrac{L_0^{1-\alpha_1} w}{\xi \Sigma^2}$

### 1.a GE Extension

## 2 Two Country Version

### Changes from Scalar Version

Instead of automatically operating in a single market, a firm must choose a network of countries $J \subseteq \{1, 2\}$ in which to operate.

- To enter a market the firm must pay $c_j^e$ and must pay fixed cost $c_j^f$ at all times or immediately exit.
- Each market has its own optimal technique $z_{j,t}$ where the correlation between the stochastic components of their drifts is given by $\tilde{\lambda}$.
- The noise term $\epsilon_t^a$ is uncorrelated across markets.
- Firms choose the data resources $L_{i,t}$ to devote to processing the raw data from each market $\mathbb{E}[x_{j,t}]$

### State Evolution

We are able to benefit from the previous results for the value of $\tilde{A}_{j,t}$ and as such the evolution of our state can be summarized by the multivariate analogue of the result from the previous section.

$$\dot{\Sigma}_t = D\Sigma_t + \Sigma D_t + Q - \Sigma_t R_t \Sigma_t \quad \text{s.t.} \quad D = \begin{pmatrix} -\theta_1 & 0 \\ 0 & -\theta_2 \end{pmatrix}$$
$$Q = \begin{pmatrix} \sigma_{z,1}^2 & \tilde{\lambda}\sigma_{z,1}\sigma_{z,2} \\ \tilde{\lambda}\sigma_{z,1}\sigma_{z,2} & \sigma_{z,2}^2 \end{pmatrix} \qquad \text{and} \quad R_t = \begin{pmatrix} 1\{1 \in J_t\} \left[\phi_d L_{1,t}^{\alpha_1} \mathbb{E}\left[x_{1,t}\right]^{\alpha_2} + \sigma_{a,1}^{-2}\right] & 0 \\ 0 & \cdots \end{pmatrix} \tag{5}$$

The indicator function is necessary given that the firm will only receive the Jovanovich signal if it actually operates in the market and can observe demand for its product.[2]

### Firm Problem

Moving forward, I employ Ben Moll's framework for handling discrete choice in continuous time. Under this framework, I restructure the firm's problem as an optimal stopping problem, where $\kappa$ represents the, potentially infinite, amount of time that a firm with network $J$ will choose to delay before switching to a different network $J' \in \mathcal{J}$ by either entering or exiting one or more markets.[3] The firm problem can therefore be represented by:

$$v(\Sigma_0, J_0) = \max_{\{\kappa, L_t\}_{t\geq 0}} \int_0^k e^{-\rho t} \left( \sum_{j \in J_0} \tilde{A}_j\left(\Sigma_t\right) \bar{\pi} - wL_{j,t} - c_j^f \right) + e^{-\rho\kappa} \max_{J' \in \mathcal{J} \setminus J} \left\{ v(\Sigma_\kappa, J_k) - c^e(J, J') \right\}$$

### Estimation

First note that in the absence of the entry and exit decision it would be possible to apply the standard results and describe the value function for a particular network with a Hamilton Jacobi Bellman equation:

$$\rho v\left(\Sigma, J\right) = \max_L \sum_{j \in J} \left( \tilde{A}_j\left(\Sigma_t\right) \bar{\pi} - wL_j - c_j^f \right) + \dot{\Sigma} \nabla v(\Sigma, J) = \tilde{\pi}(\Sigma, J) + \dot{\Sigma}\nabla v(\Sigma, J) \tag{6}$$

Given the presence of entry and exit, it is however necessary to modify these results. To do so, I note that for each network $J$, there exists a (potentially empty) inaction region $\Upsilon_J$ wherein the

[2] Note that this equates to saying that the signal received for markets the firm does not operate in has an infinite variance

[3] In the two country case, $\mathcal{J} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

firm finds it optimal to retain its current network of export destinations. In this region standard results hold and we can employ an HJB equation equation to describe the firm value function. Outside this region, the firm finds it optimal to pay the entry costs necessary to adopt a new network, and $v(\Sigma, J) = \max_{J' \neq J} \left\{ v(\Sigma_\kappa, J_k) - c^e(J, J') \right\} = S_{\Sigma,J}$. We can write these conditions:

$$\begin{array}{llcl}
\forall\, \Sigma \in \Upsilon_J : & v(\Sigma, J) \geq S(\Sigma, J) & \text{and} & \rho v(\Sigma, J) = \tilde{\pi}(\Sigma, J) + \dot{\Sigma} \nabla v(\Sigma, J) \\
\forall\, \Sigma \notin \Upsilon_J : & v(\Sigma, J) = S(\Sigma, J) & \text{and} & \rho v(\Sigma, J) \geq \tilde{\pi}(\Sigma, J) + \dot{\Sigma} \nabla v(\Sigma, J)
\end{array}$$

Noting that this must hold for all networks $J \in \mathcal{J}$, we can express the firm problem as a series of HJB variational inequalities:

$$0 = \min \left\{ \begin{array}{l} (v(\Sigma, J) - S(\Sigma, J), \\ \rho v(\Sigma, J) - \tilde{\pi}(\Sigma, J) - \dot{\Sigma} \nabla v(\Sigma, J) \end{array} \right\} \quad \forall J \in \mathcal{J} \tag{7}$$

To complete the estimation procedure, I discretize the state space with indeces $k$ such that (6) can be rewritten:

$$\rho v_k = \tilde{\pi}_k - \dot{\Sigma}_k \nabla v_k \quad \text{or in matrix form} \quad \rho v = \tilde{\pi} + \mathbf{A} v$$

where the discretized analogue of (7) is given by $\min\{\rho v - \tilde{\pi} - \mathbf{A} v, v - S\} = 0$, where the unsubscripted variables represent the quantities stacked across each export network. Denoting excess value $\varpi = v - S$ and $B = \rho I - \mathbf{A}$, this final condition can be re-expressed via the following three equations:

$$\varpi'(\mathbf{B} z + q) = 0, \qquad \varpi \geq 0, \qquad \mathbf{B} \varpi + q \geq 0$$

which is the standard form for a linear complimentary problem and that can be solved with readily available solvers.

### Optimal Behavior

The choice of network $J$ will be determined through the numeric approximation; however, the choice of $L$ can be solved in closed form within each inaction region.[4] For all $j \in J$, we can solve for $L_j$ by taking the FOC of the HJB.[5]

Conditional upon $j \in J$

$$R_{j,j} = \phi_d L_j^{\alpha_1} \mathbb{E}\left[x_j\right]^{\alpha_2} + \sigma_{a,j}^{-2} \quad \to \quad \frac{\partial R_{j,j}}{\partial L_j} = \xi_j L_j^{\alpha_1 - 1} \quad \text{s.t.} \quad \xi_j = \alpha_1 \phi_d \mathbb{E}[x_j]^{\alpha_2}$$

FOC

$$0 = -w - \frac{\partial R_{j,j}}{\partial L_j} \sum_{i,k \in \{1,2\}} \frac{\partial v}{\partial \Sigma_{i,k}} \Sigma_{i,j} \Sigma_{j,k}$$

$$0 = -w - \xi_j L^{\alpha_1 - 1} \left( \Sigma' \nabla v \Sigma \right)_{jj}$$

$$L_j = \left( -w^{-1} \xi_j \left( \Sigma' \nabla v \Sigma \right)_{jj} \right)^{\frac{1}{1-\alpha_1}}$$

where we have used that $R$ is diagonal and therefore $(\Sigma R \Sigma)_{i,k} = \sum_{j \in 1,2} \Sigma_{ij} R_{jj} \Sigma_{jk}$. Note that:
Higher $v_\Sigma \to$ Lower $L \to$ Lower $R \to$ Higher $\dot{\Sigma}$

[4] If a firm is not in the inaction region at a particular point in time it will simply change networks and choose $L$ according to the standard optimization procedure given the new network.

[5] $L_j = 0 \, \forall \, j \notin J$ as there is no raw data to process.