The ϵ expansion for ϕ^4 theory

The goal of these notes is to review and recreate a famous calculation, first done by K.G. Wilson and M.E. Fisher (1972), which allows us, among other things, to systematically estimate the critical exponents of the 3 dimensional Ising model. It also provides a general calculational framework to approach critical phenomena below their upper critical dimension. We know that above the upper critical dimension, the results of mean field theory are exact at the critical point, i.e. we do not need to consider spatial variations in the free energy coming from Landau theory. However, below this dimension, which is often 4, (for example with the Ising model), scaling arguments show that the mean field theory will not be exact. Therefore in our word (spatial dimension 3) the results of mean field theory fall short and calculating critical properties of the Ising model is a difficult task.

Here we will approach the problem via the renormalization group. The idea here is to take our expression for the system's Hamiltonian and see how it changes when we integrate out some fraction of the degrees of freedom: in particular the small-lengthscale degrees of freedom. We will then recast the new expression of our Hamiltonian in terms of the old version, but with adjusted coupling constants. We will thereby obtain *flow equations* for how the couplings of our Hamiltonian change as we progressively integrate out more of the small-scale degrees of freedom. These equations provide a picture of how the effective parameters of the system change under coarse graining. As we will see, one is able to extract a significant amount of information about the critical point from these flow equations at criticality.

The machinery required to carry out this calculation looks rather intimidating at first sight. In particular we will employ diagrammatic techniques, which are at first difficult to understand but, once understood, make calculations significantly easier. We will not assume familiarity with the usage of diagrams and instead introduce them as needed to get to our desired result.

These notes draw heavily upon the similar passages in textbooks of Chaikin and Lubensky and of Nigel Goldenfeld. We are assuming familiarity with field theory and gaussian functional integrals; see these notes for a primer on the above.

1 Preliminaries

Below we introduce some mathematical preliminaries that will be important once we begin our calculation.

1.1 Wick's theorem

We will start by stating a theorem that will be of fundamental importance in our subsequent deployment of perturbation theory. In its most basic form it is a statement about the expectation of products of gaussian random variables. The statement is that the expectation of a product of zero-mean gaussians can be decomposed into the sum of the product of pairwise expectations, where each term in the sum corresponds to a pairing of the gaussian variables — therefore expectations of odd number of random variables will vanish. Writing this out for 4 random variables, we have

$$\mathrm{E}[X_1X_2X_3X_4] = \mathrm{E}[X_1X_2]\,\mathrm{E}[X_3X_4] + \mathrm{E}[X_1X_3]\,\mathrm{E}[X_2X_4] + \mathrm{E}[X_1X_4]\,\mathrm{E}[X_2X_3]. \tag{1}$$

This is naturally extended to higher order products by generating all pairings. For 4 random variables there are 3 pairings, for 6 random variables there will be 5×3, for 8 there will be $7 \times 5 \times 3$, and so on.

The statement can be proved by induction, or e.g. analyzing the moment generation function of the multivariate gaussian distribution.

1.2 Perturbation theory for non-gaussian integrals

We will discuss a toy problem that familiarizes us with the approach to approximating non-gaussian integrals, which in general cannot be done exactly. Our goal will be to evaluate the the integral

$$I(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\phi^2 - \frac{g}{4}\phi^4\right] d\phi. \tag{2}$$

If $g = 0$ then this is simply a gaussian integral, and evaluates to 1. For nonzero$g$, we can make perturbative progress if g is small compared to 1, by expanding $e^{-g\phi^4/4}$ as a power series in g, namely

$$e^{-g\phi^4/4} = 1 - g\frac{\phi^4}{4} + \frac{1}{2}g^2\frac{\phi^8}{16}\cdots = \sum_{k=0}^{\infty}\frac{1}{k!}(-g)^k\frac{\phi^{4k}}{4^k}. \tag{3}$$

Our original integral can then be written as

$$I(g) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-g)^k}{k!4^k} \int d\phi e^{-\phi^2/2} \phi^{4k}. \tag{4}$$

We therefore need to calculate moments of ϕ where it is gaussian distributed: luckily Wick's theorem gives us just the way to do this. In particular, we know that

$$\frac{1}{\sqrt{2\pi}} \int d\phi e^{-\phi^2/2} \phi^{4k} = (4k-1) \times (4k-3) \times \cdots \times 3 \times 1 \equiv (4k-1)!!, \tag{5}$$

where we have used the double factorial notation. Therefore

$$I(g) = \sum_{k=0}^{\infty} \frac{(-g)^k}{k!4^k} (4k-1)!!, \tag{6}$$

and we can calculate as many terms as we would like to generate the desired precision for nonzero g. In what follows below, calculating these terms will be quite laborious and we will only go to second order.

1.3 Cumulant expansion

Another important fact that we will use below is the idea of the *cumulant*, which is like a moment of a random variable, but with some extra nice properties. The cumulants are defined through a *cumulant generating function*, which is the log of the moment generating function:

$$K(t) = \log \mathrm{E}[e^{tX}]. \tag{7}$$

The cumulants are then defined as

$$\langle X^n \rangle_c \equiv \frac{d^n}{dt^n} K(t) \bigg|_{t=0}. \tag{8}$$

One can check that for $n = 1, 2$, the first two cumulants correspond to the mean and variance of X, respectively. However higher order cumulants are in general complicated expressions. The thing that makes them nice to work with — which can be seen from the definition above — and is the property we will exploit below — is that

$$\mathrm{E}[e^{f(X)}] = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n!} \langle f(X)^n \rangle_c \right]. \tag{9}$$

Therefore the log of $\mathrm{E}[e^{f(X)}]$ has a nice expression in terms of the cumulants of $f(X)$.

1.4 Two point functions

Here we quickly review some facts about gaussian functional integration and Fourier space that will be useful in the calculation to come. From techniques of gaussian functional integration reviewed in other notes, we know that, if

$$\mathcal{H}_0 = \int d^d x \left(r\phi(\mathbf{x})^2 + [\nabla \phi(\mathbf{x})]^2 \right), \tag{10}$$

then the correlation functions can be evaluated as gaussian integrals, giving (in real space)

$$\frac{1}{Z} \int \mathcal{D}\phi e^{-\mathcal{H}_0} \phi(\mathbf{x})\phi(\mathbf{x}') = G(\mathbf{x}, \mathbf{x}') = \int d^d q \frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}}{r + q^2}. \tag{11}$$

Using this fact, we can calculate in Fourier space that

$$\frac{1}{Z} \int \mathcal{D}\phi e^{-\mathcal{H}_0} \phi(\mathbf{q}_1)\phi(\mathbf{q}_2) = G(\mathbf{q}_1, \mathbf{q}_2) = \int d^d x d^d x' G(\mathbf{x}, \mathbf{x}') e^{i\mathbf{x}\cdot\mathbf{q}_1 + i\mathbf{x}'\cdot\mathbf{q}_2} = \frac{\delta(\mathbf{q}_1 + \mathbf{q}_2)}{r + q_1^2}. \tag{12}$$

2 The ϵ expansion

The central task will be to calculate how our Hamiltonian of interest changes when we integrate out some shell of the high-momentum degrees of freedom. In particular, this Hamiltonian has some number of coupling constants, which "flow" as we integrate out these high-momentum modes. We would like to understand how these coupling constants change as we progressively integrate out large momenta — this will give us insight into the large-scale low-energy properties of the theory we are studying. Our starting point will be the field-theoretic Hamiltonian for the Ising model, known also as the ϕ^4 scalar field theory. This is essentially a phenomenological Hamiltonian which can be obtained by including the first few terms which respect symmetries of the system. ϕ is the order parameter field — in this case the local magnetization. If we wish to write down a Hamiltonian which is symmetric under $\phi \mapsto -\phi$ and $x \mapsto -x$, then we are left with

$$\mathcal{H}[\phi] = r\phi^2 + u\phi^4. \tag{13}$$

This is precisely the form of the Landau free energy for the Ising model. Criticality occurs at $r = 0$, and from this form one can deduce the critical exponents of mean field theory. However, as discussed above, we wish to go beyond mean field theory and consider a *spatially-varying* order parameter field. Therefore the natural generalization of our Hamiltonian is to write

$$\mathcal{H}[\phi] = \int d^d x \left(r\phi(\mathbf{x})^2 + [\nabla\phi(\mathbf{x})]^2 + u\phi(\mathbf{x})^4 \right). \tag{14}$$

Due to the crucial presence of the ϕ^4 term, this Hamiltonian is often referred to as one of ϕ^4 theory.

As stated previously, we wish to coarse-grain this Hamiltonian by "integrating out the high momentum degrees of freedom." What does this mean? If we start with the Gibbs measure on field configurations specified by this Hamiltonian, and integrate (or marginalize) over the large Fourier components of $\phi(\mathbf{x})$, we will end up with some marginal measure over the field configurations which only depends on the lower momentum degrees of freedom: it is this marginalized measure which we are after. By then rescaling the momenta and repeating, we can obtain differential equations for how the Gibbs measure (and therefore the Hamiltonian) changes as we progressively integrate out more degrees of freedom.

How will we do these integrals? In general they are functional integrals over some non-gaussian measure. However, note that our Hamiltonian is almost harmonic but for the ϕ^4 term. If we denote the free Hamiltonian (with $u = 0$) by $\mathcal{H}_0$ and the non-harmonic part by $\mathcal{H}' = u \int d^d x \phi(\mathbf{x})^4$, then the partition function of this ϕ^4 Hamiltonian is

$$Z = \int \mathcal{D}\phi e^{-\beta\mathcal{H}_0} e^{-\beta\mathcal{H}'}. \tag{15}$$

We will carry out our functional integrals in perturbation theory, expanding $e^{-\mathcal{H}'}$ in a power series, and evaluating the average of the first few terms with respect to gaussian measure induced by $\mathcal{H}_0$.

2.1 Notation

The ϵ expansion uses a lot of symbols which look similar, but understanding their differences is important. Here we define our notation in a table for easy reference.

Symbol	Definition
$Z^>$	$\int \mathcal{D}\phi^> e^{-\mathcal{H}_0^>}$
$Z_{\Lambda/b}$	$\int \mathcal{D}\phi^> e^{-\mathcal{H}_\Lambda}$
$\mathcal{H}_{\Lambda/b}$	$-\log(Z_{\Lambda/b})$
$\phi^<(\mathbf{q})$	$\phi(\mathbf{q})\Theta(\Lambda/b - q)$
$\phi^>(\mathbf{q})$	$\phi(\mathbf{q})\Theta(\Lambda - q)\Theta(q - \Lambda/b)$
$\mathcal{H}_0^>$	$\frac{1}{2}\int d^dq(r+q^2)\lvert\phi^>(\mathbf{q})\rvert^2$
$\mathcal{H}_0^<$	$\frac{1}{2}\int d^dq(r+q^2)\lvert\phi^<(\mathbf{q})\rvert^2$
$\mathcal{H}'$	$u\int d^dq_1q_2q_3\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\phi(\mathbf{q}_3)\phi(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)$
$\langle\mathcal{A}[\phi(\mathbf{q})]\rangle$	$\frac{1}{Z^>}\int \mathcal{D}\phi^> e^{-\mathcal{H}_0^>} A[\phi(\mathbf{q})]$
$\int^> d^dq\mathcal{A}(\mathbf{q})$	$\int d^dq\Theta(q-\Lambda/b)\Theta(\Lambda-q)\mathcal{A}(\mathbf{q})$
Λ	original cutoff in q space
Λ/b	cutoff in q space after integrating (but before rescaling)

Table 1: Definition of notation.

2.2 Momentum space

Now we have the tools necessary to perform a partial average over the high-energy degrees of freedom that is useful to construct a renormalization group transformation. For convenience, we reproduce our Hamiltonian below:

$$\mathcal{H} = \int d^dx \left[\frac{1}{2}r\phi^2 + \frac{1}{2}(\nabla\phi)^2 + u\phi^4\right]. \tag{16}$$

Since we are integrating out momenta, it is useful to deal in terms of the field in momentum space $\phi(\mathbf{q})$. Let us take the Hamiltonian $\mathcal{H}_\Lambda$ to be the Hamiltonian of the system up to a cutoff scale Λ: this is the inverse of the smallest length in the system, and can be thought of as a microscopic length scale in the system, e.g. the lattice spacing. We want to integrate out the degrees of freedom corresponding to $\Lambda/b < q < \Lambda$, where $b > 1$, and thereby lower the cutoff scale and coarse-grain the Hamiltonian.

Note that by dual relationship between Fourier transform pairs,

$$\phi(\mathbf{q}) = \int d^dx e^{i\mathbf{q}\cdot\mathbf{x}}\phi(\mathbf{x}) \iff \phi(\mathbf{x}) = \int d^dq e^{-i\mathbf{q}\cdot\mathbf{x}}\phi(\mathbf{q}), \tag{17}$$

where we have omitted factors of 2π (and will continue to be similarly cavalier below). Therefore, in momentum space,

$$\mathcal{H} = \underbrace{\frac{1}{2}\int d^dq(r+q^2)|\phi(\mathbf{q})|^2}_{\mathcal{H}_0} + \underbrace{u\int d^dq_1q_2q_3\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\phi(\mathbf{q}_3)\phi(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3)}_{\mathcal{H}'}. \tag{18}$$

If we write $\phi(\mathbf{q}) = \phi^<(\mathbf{q}) + \phi^>(\mathbf{q})$, where $\phi^<$ is nonzero only for $0 < q < \Lambda/b$ and $\phi^>$ is nonzero only for $\Lambda/b < q < \Lambda$, then the integral we wish to perform is over the $\phi^>$ degrees of freedom. In particular, we want to know: what is the effective Hamiltonian of the system if we integrate out the large-q degrees of freedom? By effective Hamiltonian, we mean the one that gives the correct probability distribution over field configurations. We will find this effective Hamiltonian by calculating the distribution over field configurations first — by marginalizing out the large q degrees of freedom, and then by taking the log of the Boltzmann measure that we get, to find the effective Hamiltonian. In order to integrate out the large q degrees of freedom from the Boltzmann measure on field configurations, we need to calculate the coarse-grained partition function

$$Z_{\Lambda/b} = \int \mathcal{D}\phi^> e^{-\mathcal{H}_\Lambda}, \tag{19}$$

since the Boltzmann measure will be proportional to the value of this integral.

Note that $Z_{\Lambda/b}$ will still depend on the $\phi^<$ degrees of freedom. The fundamental realization is that if we take the negative logarithm of $Z_{\Lambda/b}$, we get a new Hamiltonian in terms of the coarse-grained degrees of freedom. At this point it will be convenient to define $Z^> = \int \mathcal{D}\phi^> e^{-\mathcal{H}_0^>}$. Noting that $\mathcal{H}_0$ decomposes neatly into a sum of $\mathcal{H}_0$ for $q < \Lambda/b$ and for $\Lambda/b < q < \Lambda$, denoted $\mathcal{H}_0^<$ and $\mathcal{H}_0^>$, we have

$$\frac{Z_{\Lambda/b}}{Z^>} = \frac{1}{Z^>} \int \mathcal{D}\phi^> e^{-\mathcal{H}_0^< - \mathcal{H}_0^> - \mathcal{H}'} = e^{-\mathcal{H}_0^<} \langle e^{-\mathcal{H}'} \rangle. \tag{20}$$

Here the angle brackets denote an average over the $\phi^>$ degrees of freedom, that is $\langle \mathcal{A}[\phi(\mathbf{q})] \rangle = \frac{1}{Z^>} \int \mathcal{D}\phi^> e^{-\mathcal{H}_0^>} A[\phi(\mathbf{q})]$. This partial partition function then gives us an expression for the renormalized Hamiltonian $\mathcal{H}_{\Lambda/b}$. Then, using the cumulant expansion (Equation 9 with $f(X) = -\mathcal{H}'$), we have

$$\mathcal{H}_{\Lambda/b} = -\log(Z_{\Lambda/b}) = \mathcal{H}_0^< - \log Z^> + \langle \mathcal{H}' \rangle - \frac{1}{2}(\langle \mathcal{H}'^2 \rangle - \langle \mathcal{H}' \rangle^2) + \cdots . \tag{21}$$

Therefore, in order to calculate our renormalized Hamiltonian, we must calculate the cumulants of $\mathcal{H}'$.

2.3 First order terms

The first order term is

$$\langle \mathcal{H}' \rangle = u \frac{1}{Z^>} \int \mathcal{D}\phi^> e^{-\mathcal{H}_0} \int d^d q_1 q_2 q_3 \phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\phi(\mathbf{q}_3)\phi(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \tag{22}$$

Recalling that $\phi(\mathbf{q}) = \phi^<(\mathbf{q}) + \phi^>(\mathbf{q})$, we note that if we were to expand out the product in this integral, we would get 16 terms. In order to calculate $\langle \mathcal{H}' \rangle$, we will first classify these terms, and then see how they can be represented in terms of diagrams.

Note that out of the 16 terms generated, we will have 4 terms which look like $\phi^<\phi^>\phi^>\phi^>$ and 4 terms of the form $\phi^>\phi^<\phi^<\phi^<$. These will vanish when we do the $\mathcal{D}\phi^>$ integral. We will also have 6 terms of form $\phi^>\phi^>\phi^<\phi^<$ and 1 term each of $\phi^>\phi^>\phi^>\phi^>$ and $\phi^<\phi^<\phi^<\phi^<$. Since we will be integrating over the $\mathbf{q}_i$, all the terms in each class are equivalent. Therefore

$$\langle \mathcal{H}' \rangle = u \frac{1}{Z^>} \int \mathcal{D}\phi^> e^{-\mathcal{H}_0} \int d^d q_1 q_2 q_3 \Big[6\phi^<(\mathbf{q}_3)\phi^<(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3)\phi^>(\mathbf{q}_1)\phi^>(\mathbf{q}_2) + \tag{23}$$

$$\phi^<(\mathbf{q}_1)\phi^<(\mathbf{q}_2)\phi^<(\mathbf{q}_3)\phi^<(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) + \tag{24}$$

$$\phi^>(\mathbf{q}_1)\phi^>(\mathbf{q}_2)\phi^>(\mathbf{q}_3)\phi^>(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \Big]. \tag{25}$$

Looking at the last term in this expansion, we see that when we do the $\mathcal{D}\phi^>$ integral, we can simplify it with Wick's theorem, since it is a fourth moment which can be decomposed into products of second moments, each of which is equivalent due again to the integration over the $\mathbf{q}_i$. Therefore we have

$$\frac{1}{Z^>} \int \mathcal{D}\phi^> e^{-\mathcal{H}_0} \phi^>(\mathbf{q}_1)\phi^>(\mathbf{q}_2)\phi^>(\mathbf{q}_3)\phi^>(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \tag{26}$$

$$= 3 \left[\frac{1}{Z^>} \int \mathcal{D}\phi^> e^{-\mathcal{H}_0} \phi^>(\mathbf{q}_1)\phi^>(\mathbf{q}_2) \right] \left[\frac{1}{Z^>} \int \mathcal{D}\phi^> e^{-\mathcal{H}_0} \phi^>(\mathbf{q}_3)\phi^>(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \right]. \tag{27}$$

We have now boiled $\langle \mathcal{H}' \rangle$ down to a bunch of two point correlation functions. Here let's use the notation

$$G^>(\mathbf{q}_1, \mathbf{q}_2) = \frac{1}{Z^>} \int \mathcal{D}\phi^> e^{-\mathcal{H}_0} \phi^>(\mathbf{q}_1)\phi^>(\mathbf{q}_2) = \frac{\delta(\mathbf{q}_1 + \mathbf{q}_2)}{r + q_1^2} \Theta(q_1 - \Lambda/b). \tag{28}$$

Then we have

$$\langle \mathcal{H}' \rangle = u \int d^d q_1 q_2 q_3 \Big[6\phi^<(\mathbf{q}_3)\phi^<(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) G^>(\mathbf{q}_1, \mathbf{q}_2) \tag{29}$$

$$+ \phi^<(\mathbf{q}_1)\phi^<(\mathbf{q}_2)\phi^<(\mathbf{q}_3)\phi^<(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \tag{30}$$

$$+ 3G^>(\mathbf{q}_1, \mathbf{q}_2) G^>(\mathbf{q}_3, -\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \Big]. \tag{31}$$

We can further simplify this as

$$\langle \mathcal{H}' \rangle = 6u \int d^d q |\phi^<(\mathbf{q})|^2 \int^> \frac{d^d q}{r+q^2} + u \int d^d q_1 q_2 q_3 \phi^<(\mathbf{q}_1)\phi^<(\mathbf{q}_2)\phi^<(\mathbf{q}_3)\phi^<(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) + 3u \left[\int^> \frac{d^d q}{r+q^2}\right]^2. \tag{32}$$

From this expression we can see how integrating out the large-q modes has changed the structure of our Hamiltonian. In particular it has added a constant term (which doesn't matter), and crucially it has changed the r coefficient, so that now we have

$$\frac{1}{2}(r^< + q^2) = \frac{1}{2}(r+q^2) + 6u \int^> \frac{d^d q}{r+q^2} \implies r^< = r + 12u \int^> \frac{d^d q}{r+q^2}. \tag{33}$$

However, we have not managed to renormalize u: for that we need to go to the next order in perturbation theory.

2.4 Diagrams

The rather messy algebra we have carried out above has a rather nice interpretation in terms of diagrams. To understand the diagrammatic representation, note that each term in the integrand of Equation 22 has 4 multiples of ϕ, each of which can be $\phi^<$ or $\phi^>$. We can represent each term as a point with four lines emerging from it, one for each multiple of ϕ. We will put a slash through the lines that represent $\phi^<$ to denote that we are not averaging over these lines; the $\phi^>$ lines will have no slash through them.

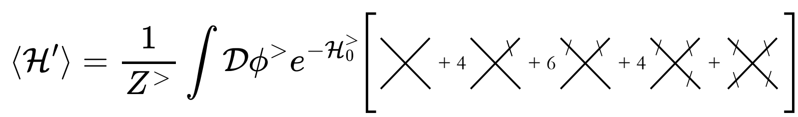

Figure 1: We can represent the various terms that arise in the perturbation expansion as diagrams. Integration over the momenta, $\int d^d q_1 q_2 q_3$ is not written, but is implied — this is the reason that each diagram with the same number of slashes is equivalent and we group them together with the appropriate combinatorial factor.

When we average over the $\phi^>$ degrees of freedom, we are essentially calculating gaussian moments of the external unslashed lines in each diagram. Since we know that

$$\langle \phi^>(\mathbf{q}_1)\phi^>(\mathbf{q}_2)\rangle = \frac{\delta(\mathbf{q}_1+\mathbf{q}_2)}{r+q_1^2}\Theta(q_1 - \Lambda/b), \tag{34}$$

this averaging is represented diagramatically by joining two free unslashed lines, and setting their momenta to be equal and opposite (this is the effect of the δ function). Note that if our diagram has an odd number of unslashed lines then it will vanish under averaging: this is because odd moments of gaussians vanish. In Figure 2 we show the result of this averaging — where again integration over the momenta q_1, q_2 and q_3 is implied. Note that we get a factor of three for the diagram with no slashed lines — this is a manifestation of Wick's theorem since it is possible to pair up (or "contract") the four lines in three different ways, each of which has the same contribution.

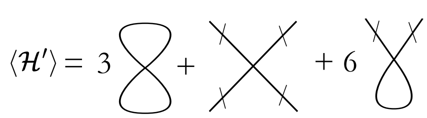

Figure 2: A joining of lines means that we have contracted two equal and opposite momenta, incurring a factor of $\frac{\delta(\mathbf{q}_1+\mathbf{q}_2)}{r+q_1^2}$ for momenta q_1 and q_2 carried by the lines in question. Lines with slashes are not affected by averaging over $\phi^>$, whereas diagrams with an odd number of un-slashed lines vanish under averaging. Note that the first diagram on the right hand side of the equation corresponds to the $\phi^>$-average of the first diagram in Figure 1, and obtains a multiplicity of 3 due to the 3 ways to contract pairs of lines: this is equivalent to Wick's theorem. Here, as in the previous figure, integration over the momenta is implied.

Note that the sum of momenta at each fourfold intersection of lines must vanish: this corresponds to locality in position space. In addition, the sum of momenta in a loop must vanish, and the factors of ϕ outside the $\mathcal{D}\phi^>$

average should be nonzero for some $q_1 < \Lambda/b$, otherwise the whole diagram vanishes (these are called single-particle reducible diagrams, which we will encounter in the second order term).

From our diagrams in Figure 2 we can reconstruct the algebraic representation of the same, shown in Equation 32. Note that each loop contributes a factor of $\int \frac{d^d q}{r+q^2}$, and each slashed line gives a factor of $\phi^<$. Since the sum of momenta in the four lines joining at the vertex is 0, we know for example that in the diagram with one loop, the momenta in the two free slashed lines must be equal and opposite.

2.5 Second order terms

Although we could (and did!) do the first order calculation without making use of the diagrams, they become very useful for messier calculations. We will now use them to calculate the second order contribution of u to the partition function, which is $\frac{1}{2}\langle \mathcal{H}'^2 \rangle_c$.

If we were to write this out in full, we would have

$$\frac{1}{2}\langle \mathcal{H}'^2 \rangle_c = u^2 \frac{1}{2Z^>} \int d^d q_1 q_2 q_3 d^d q_1' q_2' q_3' \int \mathcal{D}\phi^> e^{-\mathcal{H}_0} \Big[\phi(\mathbf{q}_1)\phi(\mathbf{q}_2)\phi(\mathbf{q}_3)\phi(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \tag{35}$$

$$\times \phi(\mathbf{q}_1')\phi(\mathbf{q}_2')\phi(\mathbf{q}_3')\phi(-\mathbf{q}_1' - \mathbf{q}_2' - \mathbf{q}_3') \Big] - \frac{1}{2}\langle \mathcal{H}' \rangle^2. \tag{36}$$

To represent this diagrammatically we need two nodes in our diagrams, since there are two combinations of 4 momenta which always sum to 0. There is another crucial fact to notice: since we are subtracting off $\langle \mathcal{H}' \rangle^2$ all *disconnected* diagrams, which are comprised of disparate components and can be written as the simple product of two fully contracted diagrams, will be subtracted away as well. Therefore we need only consider connected diagrams, which is a major simplification brought on by the cumulant expansion.

How should we go about calculating the second cumulant of $\mathcal{H}'$? In the first order case there were 5 diagrams before averaging over $\phi^>$, corresponding to anywhere from 0 to 4 slashed lines. Therefore now that we have two vertices, there will be $5 \times 5 = 25$ diagrams. However we only care about the diagrams which contribute to the renormalized u, since we already have the renormalized r to leading order in u, and don't need to keep higher order terms.

If one goes through all possible diagrams, keeping in mind the fact that only diagrams with an even number of unslashed legs can contribute under averaging, then the number of options is greatly reduced. If we then specify that we are looking for diagrams with exactly 4 unslashed legs — since they are the ones that will renormalize u — and we don't need higher order corrections to the r renormalization at the moment — then there are only two possible diagrams that can contribute. They are shown in Figure 3, with appropriate multiplicities.

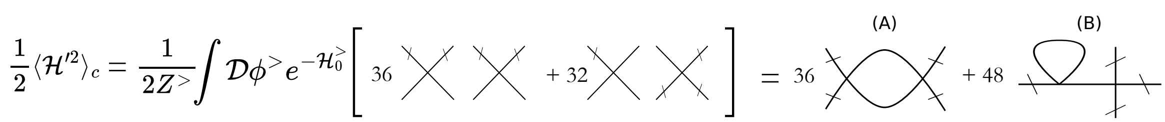

Figure 3: Diagrams for second order terms. As before, integration over q is implied. Note that diagram (B) vanishes, since its leftmost slashed leg is nonzero when it represents $\phi(q)$ for $q < \Lambda/b$, and the legs that join the two vertices necessarily represent $\phi^>(q)$ since they are joined by average. The loop has net momentum 0, and therefore we see that this diagram is zero for $q < \Lambda/b$ and for $q > \Lambda/b$, and therefore vanishes totally.

As explained in the caption of Figure 3, diagram (B) vanishes. We are therefore left with only one diagram that will enter the equation for renormalized u: diagram (A) from Figure 3.

A side note: when we evaluate contributions to the second order term, we generate a nonzero coefficient for a ϕ^6 term in our Hamiltonian, which comes from a diagram with 6 slashed legs. However we disregard this diagram because we know that ϕ^6 terms are irrelevant in 3 dimensions and higher — see an explanation here.

Therefore the relevant contribution to $\frac{1}{2}\langle \mathcal{H}'^2 \rangle_c$ will be $36u^2$ multiplied by the value of diagram (A) from Figure 3. We define the value of diagram (A) as R. Writing out the value of the diagram, we have

$$R = \int d^d q_1 q_2 q_3 d^d q_1' q_2' q_3' \phi^<(\mathbf{q}_1)\phi^<(\mathbf{q}_3)\phi^<(\mathbf{q}_2')\phi^<(-\mathbf{q}_1' - \mathbf{q}_2' - \mathbf{q}_3') G^>(\mathbf{q}_2, \mathbf{q}_1') G^>(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3, \mathbf{q}_3'), \tag{37}$$

where we have used the same rules as before: integration over q is implied, slashes indicate factors of $\phi^<$, and connected lines indicate two-point correlation functions. Plugging in Equation 28 for the two point functions in our expression for R, and integrating over q_2 and q_3', we find that

$$R = \int d^d q_1 q_3 d^d q_2' \phi^<(\mathbf{q}_1)\phi^<(\mathbf{q}_3)\phi^<(\mathbf{q}_2')\phi(-\mathbf{q}_1 - \mathbf{q}_3 - \mathbf{q}_2') \int^> \frac{d^d q_1'}{(r + q_1'^2)[r + (q_1 - q_1' + q_3)^2]}. \tag{38}$$

Note that the coupling of the quartic term is no longer exactly of the same form as it was for the original u vertex. However we assert that this difference is unimportant, and we expand the coupling around $q_1 = q_3 = 0$ and take the zeroth order term. We then obtain a quartic term which looks like

$$R \approx \int^> \frac{d^d q}{(r + q^2)^2} \int d^d q_1 q_2 q_3 \phi^<(\mathbf{q}_1)\phi^<(\mathbf{q}_2)\phi^<(\mathbf{q}_3)\phi(-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3). \tag{39}$$

We now have an expression for the renormalized u, namely

$$u^< = u - 36u^2 \int^> \frac{d^d q}{(r + q^2)^2}. \tag{40}$$

2.6 Recursion Relations

We now have a recursion relation for both r and u. We need to now rescale certain parts of our Hamiltonian to make it resemble our old Hamiltonian (before we integrated out the high q modes). Since our cutoff has been reduced from Λ to Λ/b, we should rescale all momenta by $q \mapsto bq$. We also can rescale our field ϕ by some factor ζ. We choose this value of ζ so that the coefficient of the $(\nabla\phi)^2$ term remains unity. Under rescaling this term obtains a factor of b^{d+2}. Therefore we should define $\zeta^2 = b^{d+2}$ so that this rescaling is taken care of automatically by the rescaling of ϕ, and the coefficient of this term remains unity.

Looking back to our Hamiltonian from Equation 18, we see that rescaling q will require us to absorb additional factors of b into our renormalized r and u. In particular we should have

$$r' = b^{-d}\zeta^2 r^< = b^2 r^<; \quad u' = b^{-3d}\zeta^4 u^< = b^{4-d} u^<. \tag{41}$$

Therefore we now have expressions for the renormalized (and rescaled) coefficients r and u, namely

$$r' = b^2 \left[r + 12u \int^> \frac{d^d q}{r + q^2} \right] \tag{42}$$

$$u' = b^{4-d} \left[u - 36u^2 \int^> \frac{d^d q}{(r + q^2)^2} \right]. \tag{43}$$

If we say that $b = 1 + \delta l$, then we have $r' = r + \delta l \frac{dr}{dl}$, and likewise for u. We then expand to leading order in δl and obtain

$$\frac{dr}{dl} = 2r + K_d \frac{12u}{1 + r} \tag{44}$$

$$\frac{du}{dl} = (4 - d)u - K_d \frac{36u^2}{(1 + r)^2}, \tag{45}$$

where K_d denotes a dimensionality-dependent constant that comes from integrating out a shell of width $\Lambda\delta l$ in momentum space, over which the integrand is approximately constant. What now? We will define $\epsilon \equiv 4 - d$; this is the parameter whose eponymous expansion we are carrying out! The flow equations have two fixed points: one at $r^* = u^* = 0$ and the other at $(r^*, u^*) = (-\frac{\epsilon}{6}, \frac{\epsilon}{36K_d})$ (to leading order in ϵ). For $\epsilon > 0$ the fixed point at nonzero r and u is a saddle and the one at $r^* = u^* = 0$ is unstable, whereas for $\epsilon \leq 0$ the fixed point at the origin is 0 and describes mean field solution, where the quartic coupling flows to 0 and the ϕ^4 term is irrelevant. The fixed point for $\epsilon < 0$ is unphysical since the model is not well defined if the coefficient of the quartic term is negative.

The flows around the nontrivial fixed point (sometimes called the Wilson-Fisher fixed point, sometimes the Heisenberg fixed point) tell us about the critical properties of the model. In particular, we are interested in the linearization of the flow around this fixed point. If we make a small perturbation around the fixed point in the r direction, the perturbation will grow as $\delta r \sim e^{\lambda_r l}$, where λ_r is the appropriate eigenvalue describing the instability in the r direction, and e^l is the rescaling factor. Keeping in mind the original interpretation of r as the *reduced temperature* $\tau = T/T_c - 1$ of the transition, which vanishes at criticality, we have that $e^l \sim \delta r^{1/\lambda_r}$. Now we must identify e^l with the inverse correlation length of the system: the logic of this comes from the fact that if we coarse grain by a factor of e^l then our lattice spacing goes from a to ae^l, and our correlation length measured in terms of this lattice spacing — which is the correlation length of the effective system described by the coarse-grained Hamiltonian — decreases by a factor of e^l. Therefore we realize we can interpret $1/\lambda_r$ as the correlation length exponent ν, defined as $\xi \sim \tau^{-\nu}$ for correlation length ξ.

If one works this out for the case above, one finds that for $\epsilon = 1$, in $d = 3$ dimensions, we have $\nu = 3/5$ as predicted by the RG. We still do not know how to calculate this number exactly; the current best estimate is very close to 0.63. We are able to get surprisingly close to this by just a first-order expansion in ϵ. However, more importantly, we are able to see *qualitatively* how spatial fluctuations contribute to the raising of ν from its mean field value of 1/2.

3 References

I highly recommend *Principles of Condensed Matter Physics* by Chaikin & Lubensky and *Lectures on phase transitions and the renormalization group* by Goldenfeld, for discussions of this calculation, and much of the necessary background.