First Passage Time Distribution

dated: September 12, 2022

These notes are really meant as an accompaniment to this video which explains things in significantly more detail. However the basic outline is presented here as well.

Our goal is to explore a few simple cases in which one can calculate the exact distribution of first passage times for a stochastic process. We will study unbiased and biased Brownian motion, which can equivalently be described by the stochastic differential equation $\dot{x} = \xi(t)+v$, where $\xi(t)$ is noise with correlation function $\langle \xi(t)\xi(t')\rangle = 2D\delta(|t-t'|)$. v is a constant that can be positive or negative or 0 in the case of an unbiased walk. We would like to know: what is the distribution of times at which $x = 0$, given the initial condition $x(t = 0) = x_0$? First we will solve this for the unbiased walker $v = 0$, and then generalize this method to the biased walker.

We work with the Wiener process: a continuum limit of the discrete random walk, which says that the step taken by a random walker in an interval Δt is a gaussian random variable of variance $2D\Delta t$ and mean $u\Delta t$.

1 Unbiased Random Walk

Note that there is another common derivation of this first passage time distribution that considers the *survival* probability of the walker with an absorbing boundary condition at $x = 0$, and uses the method of images to solve for this quantity, which is related by a derivative to the rate of first hitting the origin. Here we will do things by a different (seemingly more laborious) method, which generalizes better to the case of the biased walker.

We know that the distribution of jumps in a interval of time Δt is given by

$$p(\Delta x|\Delta t) = \frac{1}{\sqrt{4\pi D\Delta t}} \exp\left[-\frac{(\Delta x)^2}{4D\Delta t}\right] \tag{1}$$

Let us say that the first passage time is a random variable T, which depends on x_0. By conditioning on the first step of the random walk, starting x_0, whose distribution we know, we can obtain the following backward equation for the distribution of T given x_0.

$$p(T|x_0) = \int p(\Delta x|\Delta t)p(T-\Delta t|x_0+\Delta x)d(\Delta x). \tag{2}$$

Then expanding to first order in Δt (which means second order in Δx) and taking expectation, we obtain the partial differential equation

$$\partial_T p = D\partial_{x_0}^2 p. \tag{3}$$

To solve this by Laplace transform, we need to solve the equation

$$s\hat{p}(s, x_0) = D\partial_{x_0}^2 \hat{p}(s, x_0). \tag{4}$$

The relevant solution which satisfies the correct boundary condition is

$$\hat{p}(s, x_0) = \exp\left[-\sqrt{\frac{s}{D}}x_0\right]. \tag{5}$$

To find the inverse Laplace transform of this we need to evaluate the integral

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \exp\left[-\sqrt{\frac{s}{D}}x_0 + sT\right] ds. \tag{6}$$

We can do this by a Bromwich contour — but there is an added complication from the fact that $\sqrt{s}$ is a multivalued function. So we choose a branch cut along $(-\infty, 0)$. Then we can close the contour, excluding this branch cut. We see that we should have

$$p(T|x_0) = -\frac{1}{2\pi i}\left[\int_{-\infty}^{0} \exp\left[-i\sqrt{\frac{|s|}{D}}x_0 + sT\right] ds + \int_{0}^{-\infty} \exp\left[i\sqrt{\frac{|s|}{D}}x_0 + sT\right] ds\right]. \tag{7}$$

We notice that the bit encircling the origin vanishes, since

$$\lim_{\epsilon\to 0}\int_{\pi}^{-\pi} \exp\left[-\sqrt{\frac{\epsilon}{D}}e^{i\theta/2}x_0 + \epsilon e^{i\theta}T\right]\epsilon i e^{i\theta}d\theta = 0. \tag{8}$$

In addition, the bits from the contour out at infinity vanish, and this can be checked by calculating their magnitude, which vanishes as $R \to \infty$ where R is the contour radius.

Manipulating our expression, we obtain

$$p(T|x_0) = \frac{1}{\pi} \int_0^\infty \sin\left[\sqrt{\frac{s}{D}}x_0\right] e^{-sT} ds = \frac{x_0}{2\sqrt{\pi D}} e^{-x_0^2/4DT} T^{-3/2}. \tag{9}$$

The last equality can be evaluated through the fact that

$$\int_0^\infty \sin(\alpha\sqrt{s})e^{-st} ds = \int_0^\infty \sin(\alpha u)e^{-u^2 t} 2u du = \text{Im} \int_0^\infty \exp\left[i\alpha u - u^2 t\right] 2u du. \tag{10}$$

Then we complete the square, to get

$$2e^{-\alpha^2/4t}\, \text{Im} \int_0^\infty u \exp\left[-\frac{(u - i\alpha/2t)^2}{1/t}\right] du. \tag{11}$$

Note that the real part of this is not in fact 0 (because the bounds of the integral are from 0 to ∞ instead of $-\infty$ to ∞), but since we only care about the imaginary part, we get the desired answer. In particular we know that the imaginary part of the integrand is an even function in u, and therefore that the integral from 0 to ∞ must be half of that from $-\infty$ to ∞, which is $e^{-\alpha^2/4t}\sqrt{\pi}\alpha t^{-3/2}$.

What can we notice about the distribution $p(T|x_0) = \frac{x_0}{2\sqrt{\pi D}} e^{-x_0^2/4DT} T^{-3/2}$? For large T this goes as $T^{-3/2}$. This shallow power law decay means that the mean first passage time diverges! This is a somewhat unexpected result: although the particle always returns to the origin (in fact it returns infinitely often), the mean time it takes to return is infinite.

2 Biased Random Walk

Here we solve the same problem, but with a constant bias, so that $\dot{x}(t) = \xi(t) + v$. Therefore now the distribution of jump sizes is

$$p(\Delta x|\Delta t) = \frac{1}{\sqrt{4\pi D\Delta t}} \exp\left[-\frac{(\Delta x - v\Delta t)^2}{4D\Delta t}\right]. \tag{12}$$

Therefore for our backward equation, we obtain (to first order in Δt and therefore second order Δx)

$$\partial_T p = D\partial_{x_0}^2 p + v\partial_{x_0} p. \tag{13}$$

Again by Laplace transform in T we obtain

$$s\hat{p}(s|x_0) = [D\partial_{x_0}^2 + v\partial_{x_0}]\hat{p}. \tag{14}$$

The relevant solution for this is

$$\hat{p}(s|x_0) = \exp\left[\frac{-v - \sqrt{v^2 + 4Ds}}{2D} x_0\right]. \tag{15}$$

Again we need to introduce a branch cut, but this time it is a semi-infinite line toward $-\infty$ starting at $-v^2/4D$. We end up needing to evaluate

$$\int_{-\infty}^{-v^2/4D} \exp\left[\frac{-v - i\sqrt{|v^2 + 4Ds|}}{2D} x_0 + sT\right] ds + \int_{-v^2/4D}^{-\infty} \exp\left[\frac{-v + i\sqrt{|v^2 + 4Ds|}}{2D} x_0 + sT\right] ds \tag{16}$$

Doing a substitution $u = s + v^2/4D$, we can manipulate this into the same form as in the unbiased case, with a prefactor. Namely,

$$p(T|x_0) = e^{-vx_0/2D - v^2T/4D} \frac{x_0}{2\sqrt{\pi D}} e^{-x_0^2/4DT} T^{-3/2}. \tag{17}$$

Note that this assumes $x_0 > 0$ but should be easily adaptable to $x_0 < 0$.

Also note that when $v > 0$ the integral of this distribution is less than 1. In this case is it possible that the particle never returns to the origin? Does the integral of the distribution then give the probability of returning to the origin at all?

3 References

For similar arguments in a different context, I found the supplement of this paper by Okada and Hallatschek to be very helpful.
See also this video that walks us through the calculation.