Consider $A(z) = a_0 + a_1 z^{-1} + \dots a_p z^{-p}$

$$\delta[n] \longrightarrow \boxed{\frac{1}{A(z)}} \longrightarrow x[n]$$

Find $\{a_k\}_{k=1,\dots p}$ so that ~~satisfy~~ the filter output is the data that I have.

The filter DE is

$$a_0 x[n] + a_1 x[n-1] + \dots + a_p x[n-p] = \delta[n] \qquad (1)$$

Let $x[n] = 2^{-n} u[n] = \{\dots 0, 0, 0, \underset{\uparrow n=0}{1}, 1/2, 1/4, \frac{1}{8}, \dots\}$

and $p = 3$, $a_0 = 1$

Write eqn. (1) for any set of n.

I'll choose $n = 1, 2, 3$.

This exercise is only useful to show the problems with the formulation so don't worry.

$n=1 \quad x[1] + a_1 x[0] + a_2 x[-1] + a_3 x[-2] = 0$

$n=2 \quad x[2] + a_1 x[1] + a_2 x[0] + a_3 x[-1] = 0$

$n=3 \quad x[3] + a_1 x[2] + a_2 x[1] + a_3 x[0] = 0$

I have 3 unknowns (a_1, a_2, a_3) and 3 eqs. Can I solve?

Rewrite the above in matrix form.

$$\underbrace{\begin{bmatrix} x[1] \\ x[2] \\ x[3] \end{bmatrix}}_{\underline{x}} = - \underbrace{\begin{bmatrix} x[0] & x[-1] & x[-2] \\ x[1] & x[0] & x[-1] \\ x[2] & x[1] & x[0] \end{bmatrix}}_{\underline{X}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\underline{a}}$$

or

$$\underline{x} = - \underline{X} \cdot \underline{a}$$

$$\underline{a} = - \underline{X}^{-1} \cdot \underline{x} \qquad (?)$$

The proposed solu

$$\underline{a} = -\underline{X}^{-1} \cdot \underline{x} \qquad (2)$$

requires that X have an inverse.
Let's assume it does & we find $\underline{a}$.

What about $n = 4, 5, 6$?

$$\begin{bmatrix} x[4] \\ x[5] \\ x[6] \end{bmatrix} = -\begin{bmatrix} x[3] & x[2] & x[1] \\ x[4] & x[3] & x[2] \\ x[5] & x[4] & x[1] \end{bmatrix} \underline{a} \qquad (3)$$

or $n = 15, 35, 82$? any n ?

will the $\underline{a}$ ~~found~~ given by (2) satisfy (3) ?

You can check these with the "toy" signal $x[n] = \left(\frac{1}{2}\right)^n u[n]$. To find if a matrix has an inverse, check its rank

If $\operatorname{rank}(\underline{X}) < 3$, then the answer is NO.

Also use other data, e.g. TONE.wav.

Take any 3 values of n, form $\underline{X}$ + $\bar{x}$.

- Check if $\underline{X}^{-1}$ exists

- If it does, for one set of n, does your soln. satisfy eq (1) for other n.?

The answers will be no most of the time.

That's why we need to relax some requirements.

Rewrite (1) replacing $\delta[n]$ by $e[n]$.

$$\sum_{k=0}^{p} a_k \, x[n-k] = e[n]$$

and proceed in the same way as above.

What is $e[n]$?

What we have is a case of more eqs. than unknowns, as well as X^{-1} not guaranteed,

with $a_0 = 1$

$$x[n] + a_1 x[n-1] + \cdots + a_p x[n-1] = e[n] \qquad (4)$$

can be represented by

$$x[n] \longrightarrow \boxed{A(z)} \longrightarrow e[n]$$

so we want $e[n]$ to some how represent

$\delta[n] = 1$ $n = 0$ & 0 for other n.

<u>MSE to the rescue.</u>

Let $\xi = \sum_n e^2[n]$

Choose a_k s.t. ξ is minimized.

$$\mathcal{E} = \sum_n \left(\sum_{k=0}^{p} a_k \, x[n-k] \right)^2$$

$$\frac{\partial \mathcal{E}}{\partial a_i} = 0 \qquad i = 1, 2, \ldots p$$

$$\frac{\partial \mathcal{E}}{\partial a_i} = 2 \sum_n \sum_k a_k \, x[n-k] \quad x[n-i]$$

$$= 2 \sum_{k=0}^{p} a_k \underbrace{\sum_n x[n-k] \, x[n-i]}_{c_{xx}[k,i]}$$

$$\underset{(1+p)\times 1}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}} = \begin{bmatrix} c_{xx}[0,0] & c_{xx}[0,1] & \cdots \\ c_{xx}[1, & & \\ & \underline{C}_{(p+1)\times(p+1)} & \\ & & \end{bmatrix} \underset{(1+p)\times 1}{\begin{bmatrix} a_0 = 1 \\ a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}} \qquad (5)$$

$$\underline{C} = \begin{bmatrix} c_{xx}[0,1] & c_{xx}[1,1] & c_{xx}[2,1] & \cdots & c_{xx}[p,1] \\ c_{xx}[0,2] & c_{xx}[1,2] & \cdots & & c_{xx}[p,2] \\ \vdots & & & & \\ c_{xx}[0,p] & c_{xx}[1,p] & \cdots & & c_{xx}[p,p] \end{bmatrix}$$

$\underline{C}$ is called the auto correlation or auto covariance matrix.

- Explore its properties
- Since $a_0 = 1$, re arrang Eq. 5

$$\underset{p \times 1}{\begin{bmatrix} c_{xx}[0,1] \\ c_{xx}[0,2] \\ \vdots \\ c_{xx}[0,p] \end{bmatrix}} = \underset{p \times p}{\overline{C}} \cdot \underset{p \times 1}{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}}$$

What is $\overline{C}$?