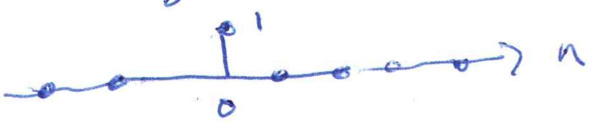
$\rightarrow$ DFT $\qquad$ fft

Properties.

$\rightarrow$ $x[n] \rightarrow \boxed{T\{x[n]\}} \rightarrow y[n]$ $\qquad$ LTI

$\delta[n]$ $\qquad$ $h[n]$



$\Big\}$ Impulse response.

Unit impulse sequence

LTI $\qquad y[n] = x[n] * h[n]$

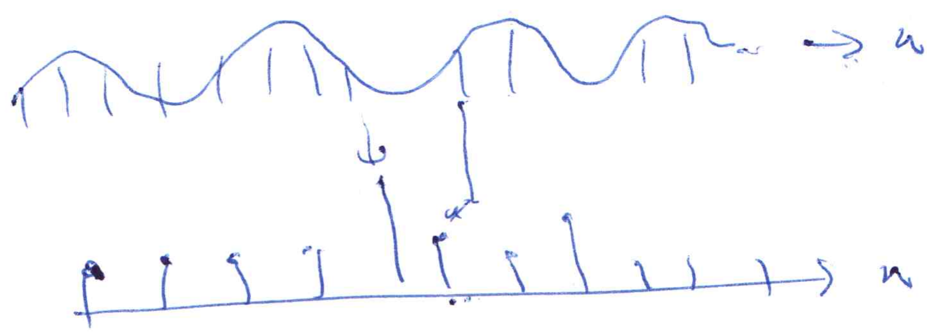
DFT $\qquad Y[k] = X[k]\,H[k]$ $\qquad$ (?)

DTFT $\qquad Y(e^{j\omega}) = X(e^{j\omega})\,H(e^{j\omega})$

$x[n] \overset{DTFT}{\longleftrightarrow}$ $\qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$

Intuitive $\qquad e^{-j\omega n} = \cos(\omega n) - j\sin(\omega n)$

$\cos(\omega n)$ or $\sin(\omega n)$



$X(e^{j\omega}) = \langle x[n], e^{+j\omega n} \rangle$

$$X(e^{j\omega}) = \sum_n x[n]\, e^{-j\omega n} = \langle x[n], e^{j\omega n} \rangle$$

Ex

Let $\omega_0 = \pi/6$ $\quad x[n] = e^{j\omega_0 n}$.

$$X(e^{j\omega})\Big|_{\omega=\omega_0=\pi/6} = \sum_n \underbrace{e^{j\omega_0 n}\, e^{-j\omega_0 n}} = \sum_{n=-\infty}^{\infty} 1 \to \infty$$

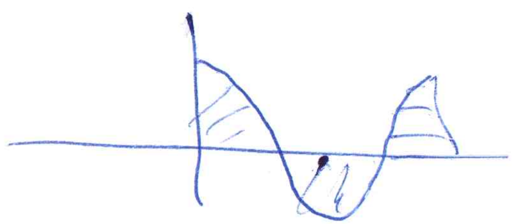
$e^{j\omega n}$ is a <u>kernel</u> or basis function.

Let $\omega_0 = \pi/6$ $\quad x[n] = e^{-j\omega_0 n}$

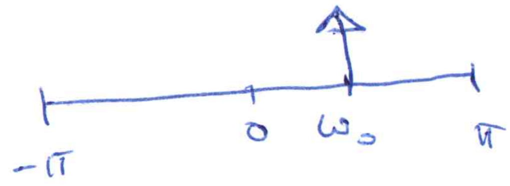
$$X(e^{j\omega})\Big|_{\omega_0} = \sum_n e^{-j\omega_0 n}\, e^{-j\omega_0 n} = \sum_n e^{-j2\omega_0 n}$$

$$= \sum_n \cos(2\omega_0 n) - j\sum_n \sin(2\omega_0 n)$$

$$= 0$$



Inner product measures the similarity between $x[n]$ and $e^{j\omega n}$.

$x[n] = e^{j\omega_0 n}$ $X(e^{j\omega}) =$ 

$X(e^{j\omega_0}) = \infty$

DTFT $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$

$X(e^{j\omega})$ relates to $X(z) = \sum x[n] z^{-n}$ @ $z = e^{j\omega}$ when exists.

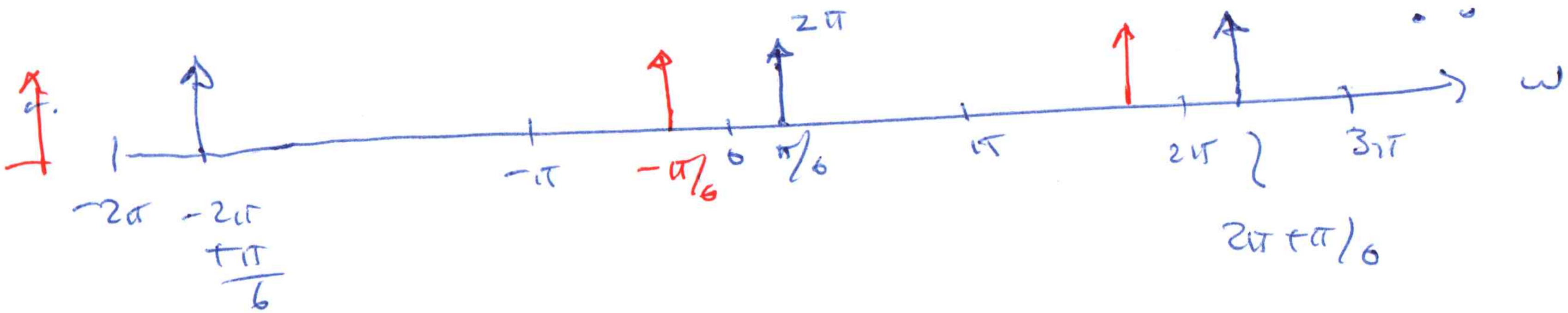
Properties

$X(e^{j\omega}) = \sum_n x[n] e^{-j\omega n}$

– $X(e^{j\omega})$ – is 2π - periodic.

Ex1. $x[n] = e^{j\pi/6\, n}$ $\qquad X(e^{j\omega}) = 2\pi\, \delta(\omega - \pi/6)$

& 2π - periodic



~~$x(t) = \cos(2\pi\, 120\, t)$~~

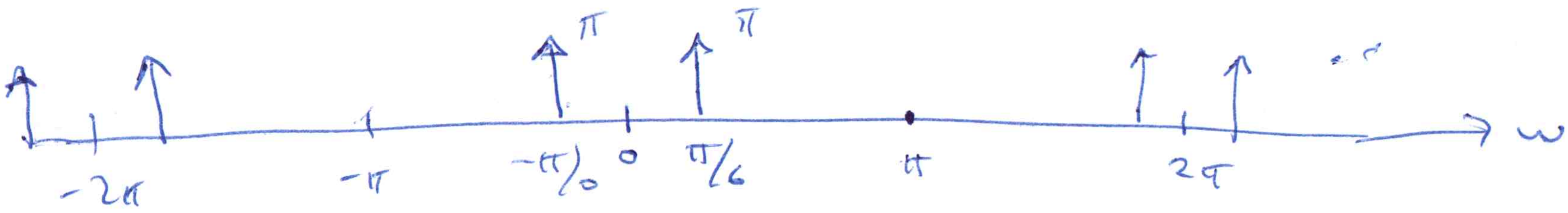
~~$x[n] =$~~

Ex2 $x[n] = e^{-j\pi/6\, n}$ $\qquad X(e^{j\omega}) = 2\pi\, (\delta(\omega + \pi/6))$

$+\ 2\pi$ pd.

Ex3 $x[n] = \cos(\frac{\pi}{6} n)$

$x[n] = \cos(\frac{\pi}{6} n)$



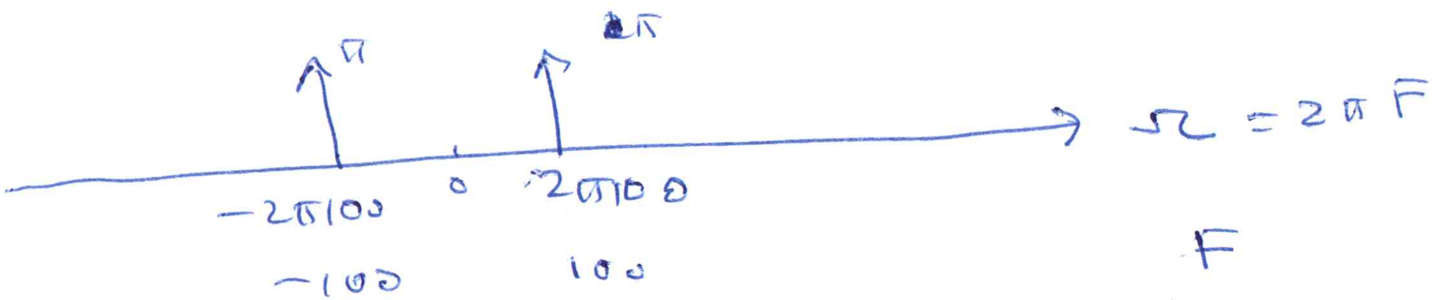
$F_s = 1200\,Hz.$

$x(t) = \cos(2\pi\, 100\, t)$

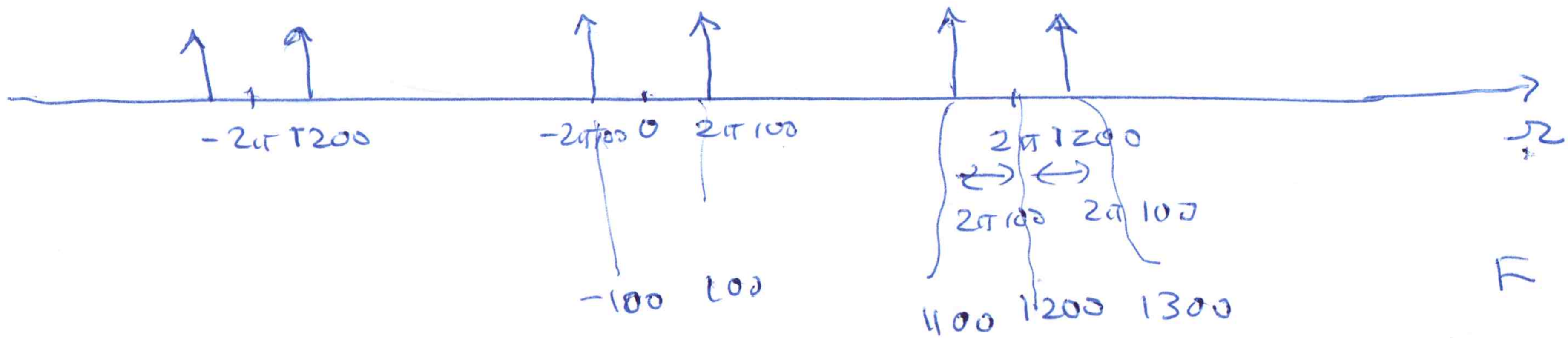
$x[n] = \cos(2\pi \frac{100}{1200} n) = \cos(\frac{\pi}{6} n)$

$x(t) \Leftrightarrow X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$ Fourier Trans.

~~If x(t)~~ $\cos(2\pi 100 t) \Leftrightarrow \pi\, \delta(\Omega - 2\pi 100) + \pi\, \delta(\Omega + 2\pi 100)$

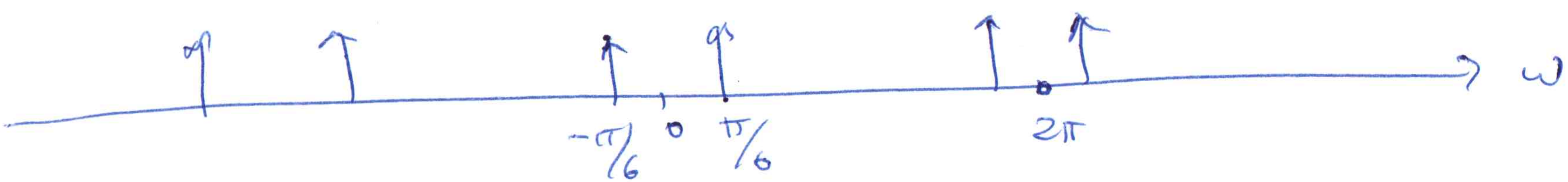


Sampled signal FT



$x[n] \Leftrightarrow X(e^{j\omega})$

$x[n] = x(t)\big|_{t = n/1200}$



$\omega = \dfrac{\Omega}{F_s}$ ω = normalized freq.

$\Omega = 2\pi F$ Analog freq.

F (Hz), Ω (rps)

$\omega = \frac{\Omega}{F_s}$ (Discrete-time) freq. normalized freq.

$\omega = 2\pi f$

$x(t) = \cos(2\pi\ 100\ t)$

$F_s = 1200$ Hz.

$x[n] = \cos(\frac{\pi}{6} n)$

Eg. $BW = 100$ Hz.

$F_N = 200$ Hz,

In practice.

$F_s > F_N$

$F_s = 60$ Hz. $\rightarrow$ Aliasing.

$x[n] = \cos(\underbrace{2\pi \frac{100}{60}}_{\omega_0} n)$

$\omega_0 = \frac{10}{3}\pi$

$X(e^{j\omega})$? must be 2π-periodic.

$= \frac{10}{5}\pi + \cancel{\Delta\omega_0}\ \Delta\omega_0$

$$\Delta\omega_0 = \frac{10}{3}\pi - \frac{10}{5}\pi$$

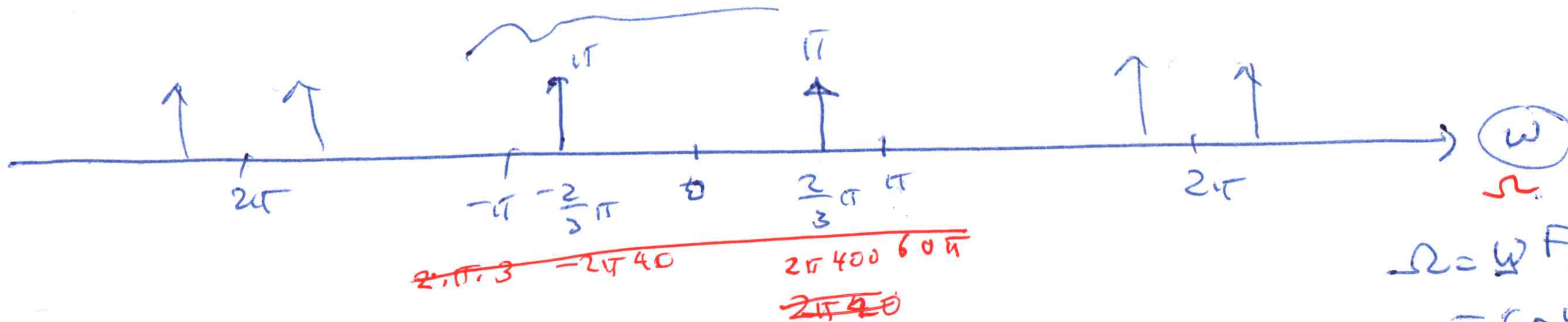
$$= 10\pi\left(\frac{2}{15}\right) = \frac{2}{3}\pi$$

$$\omega_0 = \frac{10}{3}\pi = 2\pi + \frac{2}{3}\pi \quad > 2\pi$$

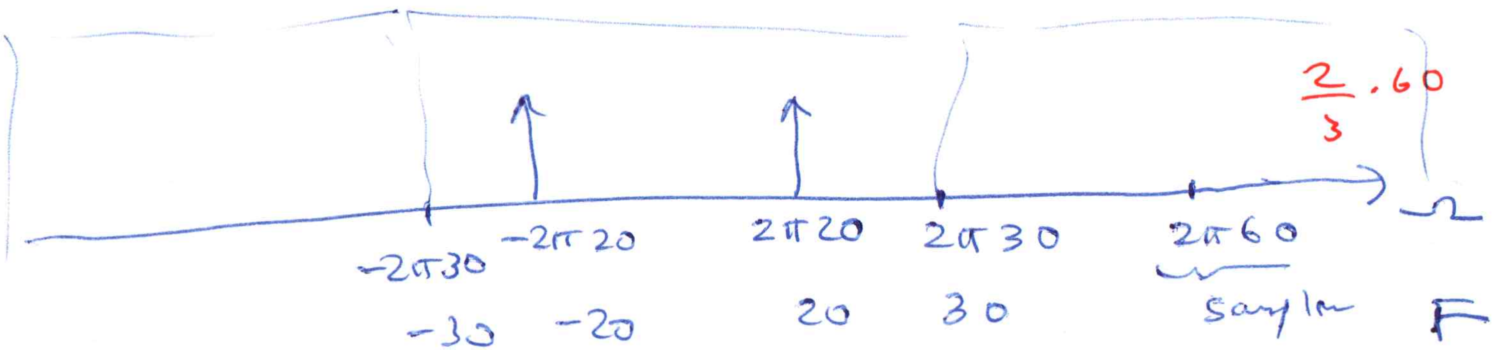
$$\rightarrow \cos(\omega_0 n) = \cos\left(\frac{10}{3}\pi n\right) = \cos\left(2\pi n + \frac{2}{3}\pi n\right)$$

$$= \cos\left(\frac{2}{3}\pi n\right)$$

$X(e^{j\omega})$ repeat @ 2π



$\Omega = \omega F_s$
$= 60\omega$



DTFT

$$X(e^{j\omega}) = \sum_n x[n]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

- Innerprod.
- Series fn. of ω.

$X(e^{j\omega})$ - is p dic.

Fourier series.

$T^{p} = 2\pi$ pd.

$\omega_p = 2\pi\nu = \frac{1}{T^p} = 1$ freq.

~~$X(e^{j\omega})$~~

F. Series

$$X(e^{j\omega}) = \sum_n c_n\, e^{-jn\omega_p\omega}$$

$$\boxed{X(e^{j\omega}) = \sum_n c_n\, e^{-jn\omega}}$$

$c_n = x[n]$

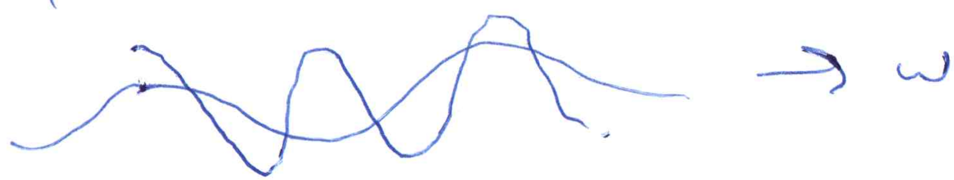
$x(t) = \sum c_n e^{-jn\omega_0 t}$

$$X(e^{j\omega}) = \sum_n x[n]\, e^{-j\omega n}$$

$x[n]$ are the FS coeffs. of a function (pdic) of ω, with $T_p(pd) = 2\pi$ & $\omega_p = 1$ (fundamental freq)

$$x[0] + \underline{x[1]\, e^{-j\omega}} + x[2]\, e^{-j2\omega}$$

$$= x[1]\cos(\omega) - j\,x[1]\sin(\omega)$$
$$+ x[2]\cos(2\omega) - j\,x[2]\sin(2\omega)$$
$$+ \text{etc.}$$



<u>DFT</u>

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk} \qquad (1)$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}nk} \qquad (2)$$

Relate the DFT to $X(e^{j\omega})$

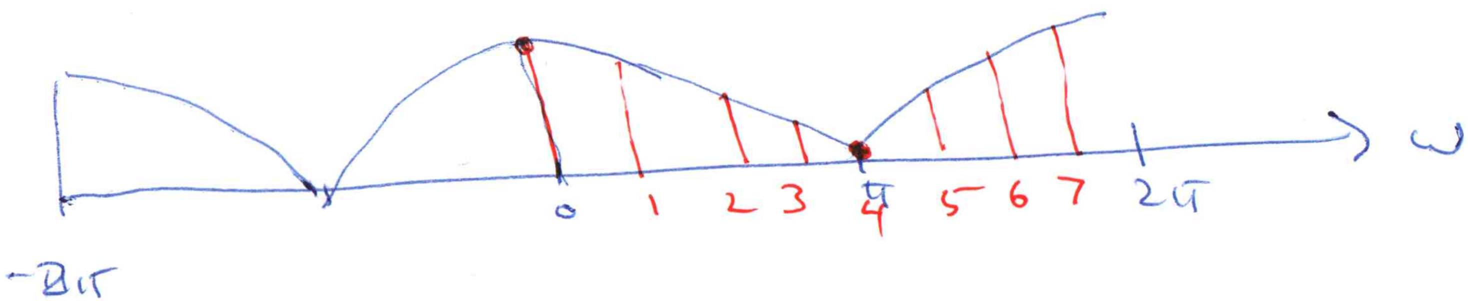
Let $x[n]$ $n = 0, 1, \dots N-1$ $x[n] = 0$ otherwise.

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]\, e^{-j\omega n} \qquad (3)$$

Compare to eq. (1)

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$$

$N = 8$



Practice

Signal $x[n]$ $n = 0 \ldots 7$

$x = [\, x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6 \; x_7 \; x_8 \,]$ MATLAB

$x(1)$ $x(2)$

DFT

$fx =$ fft(x, N)

N: # of samples betwen [0, and 2π)

$w = \frac{2\pi}{N} k$ $k = 0, 1, \ldots N-1$

$N \Longrightarrow$ sampling in freq.

N_{min} = length of x.

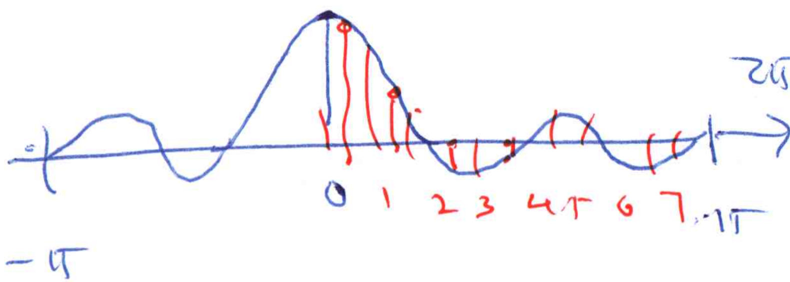
$$x[n] = \begin{cases} 1 & n = 0, 1, \ldots 7 \\ 0 & \text{otherwise} \end{cases}$$

$$X_8[k] \longrightarrow \quad \omega = \frac{2\pi}{8} k \qquad \text{8-pt DFT}$$

$$k = 0, \ldots 7$$

fx8 = fft(x, 8)

$$X(e^{j\omega}) = \frac{\sin(\omega \text{—})}{\sin(\omega \ldots)}$$



$$X_{16}[k] = \sum_{n=0}^{7} x[n] e^{-j\frac{2\pi}{16}k}$$

$$= \sum_{n=0}^{15} x[n] e^{-j\frac{2\pi}{16}k}$$

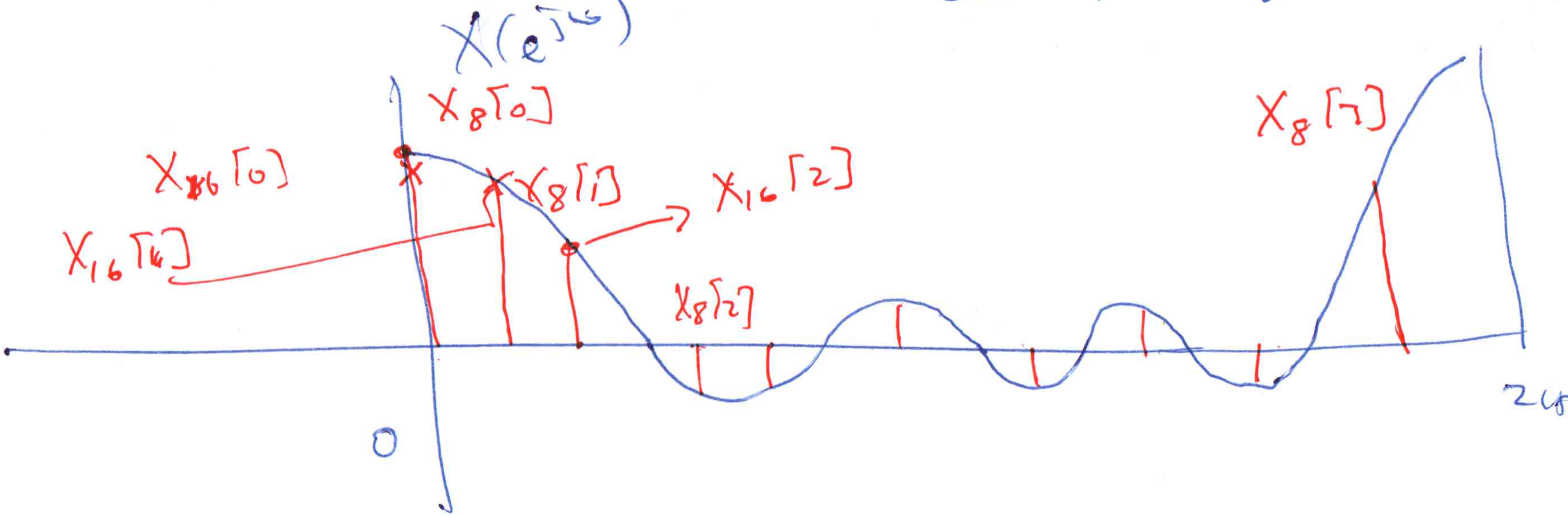
x[8], x[9], x[13]
?

16-pt DFT MATLAB

$x = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1]$

$fx16 = fft(x, 16)$

DFT $x = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \underbrace{0\ 0\ 0\ 0\ 0\ 0\ 0\ 0}_{\text{zero-padding.}}]$



$X_8[k] = \underline{X_8[0]}\ \ \underline{X_8[1]}\ - - - X_8[7]$

$X_{16}[k] = \underline{X_{16}[0]}\ X_{16}[1]\ \underline{X_{16}[2]}$

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$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}nk}$$

$$X[0] = \sum_{n=0}^{N-1} x[n] = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[7] \end{bmatrix}$$

$$X[1] = \sum x[n]\, e^{-j\frac{2\pi}{8}n} = \begin{bmatrix} 1 & e^{-j\frac{2\pi}{8}} & e^{-j\frac{2\pi}{8}2} & \cdots & e^{-j\frac{2\pi}{8}7} \end{bmatrix}$$

$N = 8$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[7] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \cdot & \cdot & \cdot & & & & \\ 1 & & e^{-j\frac{2\pi}{8}nk} & & & & & \\ 1 & & & & & & & \\ 1 & & & & & & & \\ 1 & & & & & & & \\ 1 & & & & & & & \\ 1 & & & & & & & \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[7] \end{bmatrix}$$

8×1 $\quad$ 8×8 $\quad$ 8×1