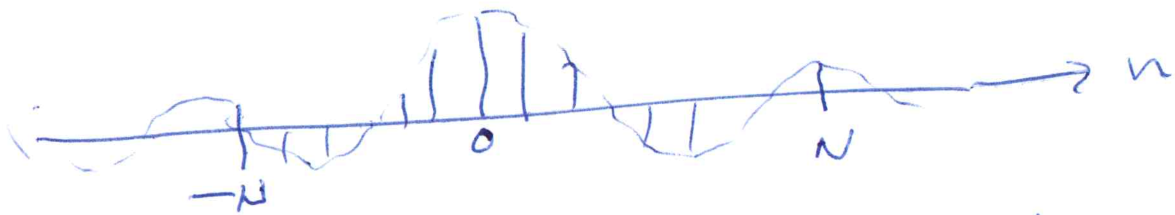


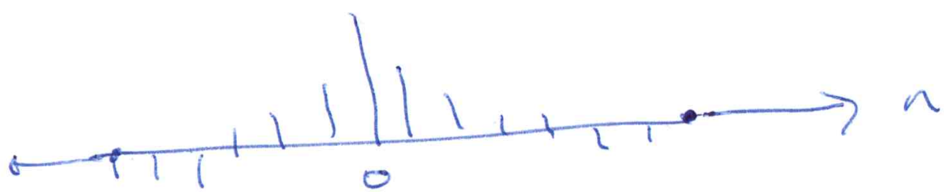
finite length, keep sym.

1 truncate → ~~$h_0[n] = h[n]$~~

2 trick 0 → $h_0[n] = 0 \quad n < -N,\ n > N$

$h_0[n]$ not causal

3

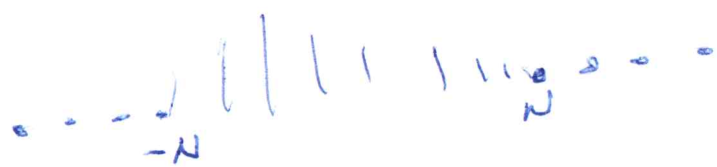


4 $h_1[n] = h_0[n-N]$ ← causal.

$h_0[n] = h[n] \cdot w[n]$

$w[n]$ = window

5



$h[n] - h_o[n] = \text{error} = e[n]$

$e[n]$ must be small $\Rightarrow$ measure. mse

~~mse~~

$$\mathcal{E} = \sum_{n=-\infty}^{\infty} e^2[n]$$ = total energy in $e[n]$

= l_2 norm of $e[n]$

$$\mathcal{E} = \sum_n (h[n] - h_o[n])^2$$

$$\mathcal{E} = \sum_n (h[n] - h[n]\,w[n])^2$$

$$\frac{\partial \mathcal{E}}{\partial w[k]} = 0 \qquad k = -N \text{ to } N$$

$$\cancel{\mathcal{E} = \sum h^2[n] - 2h}$$

5

$$\frac{\partial \bar{\varepsilon}}{\partial w[k]} = 0 \qquad k = -N, -N+1, \ldots N$$

$$\bar{\varepsilon} = \sum_{\substack{n \neq k \\ -N \leq n \leq N}} h^2[n] \left(1 - w[n]\right)^2 + h^2[k] \left(1 - w[k]\right)^2$$

$$\frac{\partial \varepsilon}{\partial w[k]} = \cancel{-2 h^2[k]} + -h^2[k]\, 2\left(1 - w[k]\right) = 0$$

$$1 - w[k] = 0 \qquad w[k] = 1$$

$$\mathcal{E} = \sum_n h^2[n]\,(1 - w[n])^2$$

$$= \sum_{n=-\infty}^{-N-1} h^2[n]\cdot 1 + \sum_{n=-N}^{N} h^2[n]\,(1-w[n])^2$$

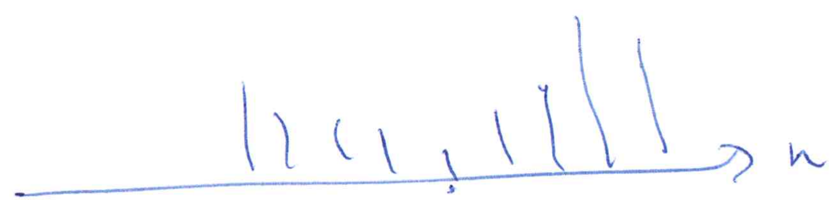
$$+ \sum_{n=N+1}^{\infty} h^2[n]$$

$$\bar{\mathcal{E}} = \sum_{-N}^{N} h^2[n]\,(1 - w[n])^2$$

min $\bar{\mathcal{E}}$ wrt $w[n]$ $n = -N$ to N

$\underline{w[n] = 1 \quad -N < n \le N}$ $\quad \bar{\mathcal{E}} = 0$

$e[n]$



$w[n]$

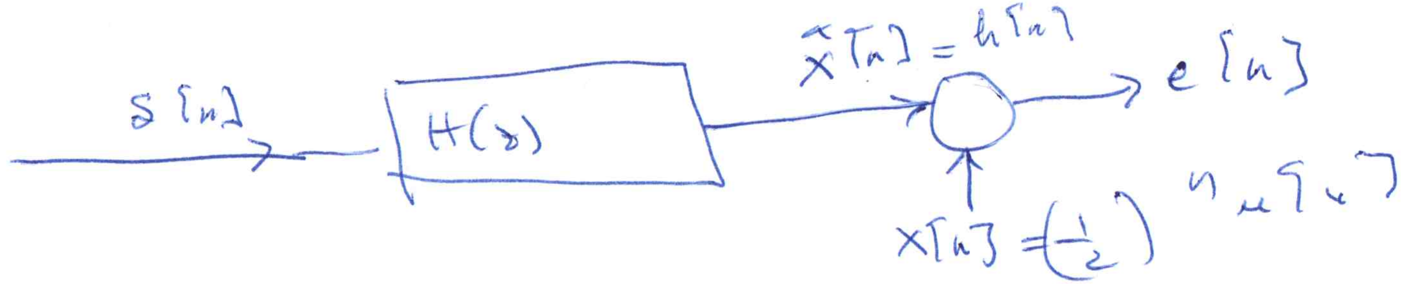
Let's ~~H(z)~~ choose $H(z)$ to be a ~~finite~~ FIR

$H(z)$ FIR, CAUSAL

$h[n]$ $n = 0, \ldots, N-1$ $\qquad H(z) = \sum_{n=0}^{N-1} h[n] z^{-n}$

$h[n] = 0$ other N

parameters $h[n]$ $n = 0, \ldots N-1$



$$\min_{h[n]} \quad \mathcal{E} = \sum_n e^2[n]$$

$$e[n] = \hat{x}[n] - x[n] = \begin{cases} h[n] - x[n] & n = 0, \ldots N-1 \\ -x[n] & \text{other } n \end{cases}$$

OR

$$\frac{\partial \bar{\varepsilon}}{\partial h[k]} = \frac{\partial}{\partial h[k]} \sum_{n=0}^{N-1} \left(h[n] - x[n] \right)^2$$

$$= \sum_{n=0}^{N-1} \frac{\partial}{\partial h[k]} \left(h[n] - x[n] \right)^2$$

$$= \sum_{\substack{n=0 \\ n \neq k}}^{N-1} 0 + \frac{\partial}{\partial h[k]} \left(h[k] - x[k] \right)^2$$

$$= 2 \left(h[k] - x[k] \right) = 0$$

$$h[k] = x[k]$$

$$\mathcal{E} = \underbrace{\sum_{n \notin [-N, N]} x^2[n]}_{\text{won't change w/ } h[n]} + \underbrace{\sum_{n=0}^{N-1} \left(h[n] - x[n]\right)^2}$$

$$\bar{\mathcal{E}} = \sum_{n=0}^{N-1} \left(h[n] - x[n]\right)^2 \qquad \min_{h[k]} \bar{\mathcal{E}}$$

$$\bar{\mathcal{E}} \geq 0 \quad \Rightarrow \quad \min \bar{\mathcal{E}} = 0$$

$$\Rightarrow \quad h[n] = x[n]$$

$x[n] = \left(\frac{1}{2}\right)^n u[n]$ using an FIR filter of length N, $n = 0, \ldots N-1$

$$\Rightarrow \quad h[n] = x[n] \qquad n = 0, \ldots N-1$$

$$= \begin{cases} \left(\frac{1}{2}\right)^n & n = 0, \ldots N-1 \\ 0 & \text{other } n \end{cases}$$

$x[n] = \quad 0\,0\,0,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32}\ \ldots$

$N = 3$

$h[n] \quad 0\,0\,0,\ 1,\ \frac{1}{2},\ \frac{1}{4}\quad 0\quad 0\quad 0\quad 0$

opt.

$$\varepsilon^* = \underbrace{\sum_{n=0}^{N-1} \left(h[n] - x[n]\right)^2}_{0} + \sum_{n=N}^{\infty} x^2[n]$$

$$\varepsilon^* = \text{minimum error (mse)} = \sum_{n=N}^{\infty} x^2[n]$$

$$= \sum_{n=0}^{\infty} x^2[n] - \sum_{n=0}^{N-1} x^2[n]$$

$$= \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n}}_{4/3} - \varepsilon_x[N]$$

$$= \frac{1 -}{1 - \frac{1}{4}} =$$

$N = 3$ $\qquad \varepsilon^* = \frac{4}{3} - \left(1^2 + \frac{1}{2}^2 + \frac{1}{4}^2\right)$

$$= \frac{4}{3} - \left(1 + \frac{1}{4} + \frac{1}{16}\right)$$

$$= \frac{4}{3} - \frac{16+4+1}{16}$$

$$= \frac{4}{3} - \frac{21}{16} = \frac{1}{48}$$

$\cancel{64}$

Good measure of how good ε^*

$$\frac{\varepsilon^*}{\varepsilon_x} = \frac{1/48}{4/3} = \frac{3}{4}\,\frac{1}{48}$$

$N = 3$ $\qquad 1, \frac{1}{2}, \frac{1}{4}, 0, 0, 0$

Not happy!

IIR $\quad H(z) = \dfrac{B(z)}{A(z)}$

Choose $\quad H(z) = \dfrac{1}{A(z)}$

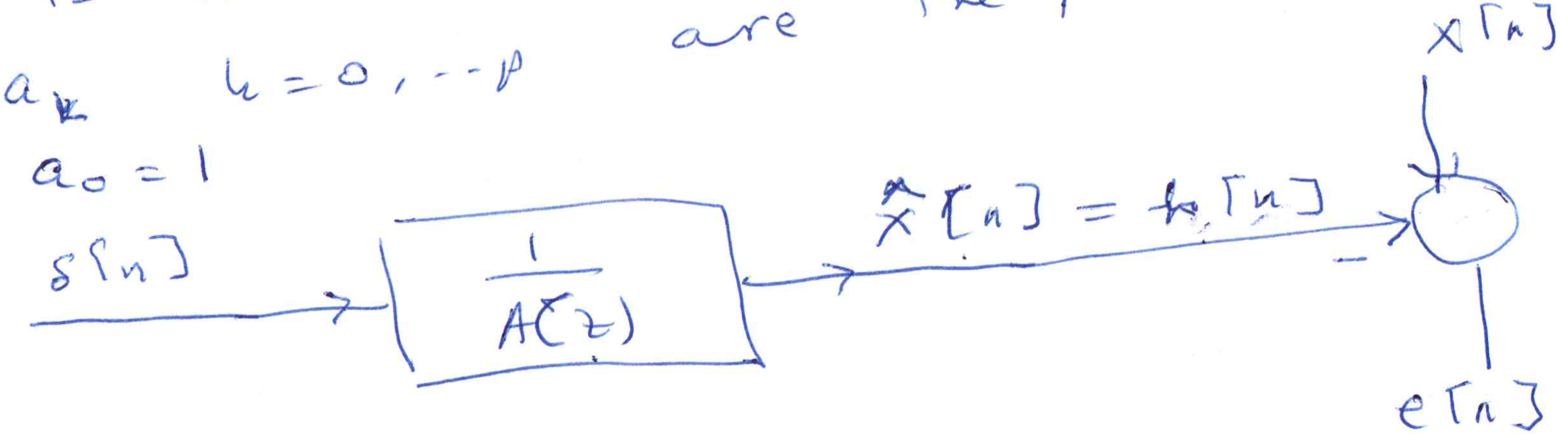
IIR
Easier to manipulate
than $\dfrac{B}{A}$

$$A(z) = a_0 + a_1 z^{-1} + \cdots a_p z^{-p}$$

p is the order of the filter.

$a_k \quad k = 0, \cdots p \quad$ are the parameters.

$a_0 = 1$



$$H(z) = \frac{1}{A(z)}$$

$$H(z)\,A(z) = 1 \implies h[n] * (a_0, a_1, \cdots a_p)$$

~~h[n]~~

$$\frac{1}{a_0 + a_1 z^{-1} + \cdots a_p z^{-p}} = H(z)$$

$$1 = a_0 H(z) + a_1 z^{-1} H(z) + a_2 z^{-2} H(z) + \cdots a_p z^{-p} H(z)$$

$$\delta[n] = \underbrace{a_0}_{=1} h[n] + a_1 h[n-1] + a_2 h[n-2] + a_p h[n-p]$$

$$e[n] = x[n] - h[n]$$

$$\mathcal{E} = \sum e^2[n] = \sum_{n=0}^{\infty} \left(x[n] - h[n]\right)^2$$

~~e[n]~~

$$h[n] = 1 - a_1 h[n-1] - a_2 h[n-2] \cdots - a_p h[n-p]$$

want $h[n] = x[n]$ $\quad p = 4$

$$x[n] = 1 - a_1 x[n-1] - a_2 x[n-2] - a_3 x[n-3] - a_4 x[n-4]$$

$x[n] \; n \geq 1$

$$1 = x[n] + a_1 x[n-1] + a_2 x[n-2] + \dots a_4 x[n-4]$$

~~n=0~~

~~$n=0 \quad 1 = x[0] + a$~~

$n=1 \quad 1 = x[1] + a_1 x[0]$

$n=2 \quad 1 = x[2] + a_1 x[1] + a_0 x[0]$

$n=3 \quad 1 = x[3] + a_1 x[2] + a_2 x[1] + a_3 x[0]$

$n=4 \quad 1 = x[4] \quad a_4 x[0]$

$n=5 \quad 1 = x[5] + a_1 x[4] + \quad a_4 x[1]$

$\vdots$

$n = 400$

∞ no of eqs
4 or p unknowns] over determined system.

$$\delta[n] \longrightarrow \boxed{\frac{1}{A(z)}} \longrightarrow x[n]$$

$$x[n] \longrightarrow \boxed{A(z)} \longrightarrow \delta[n]$$

Impossible to satisfy

Replace

$$x[n] \longrightarrow \boxed{A(z)} \longrightarrow e[n]$$

want $e[n]$ to be small.

$$\epsilon = \sum_{n=0}^{\infty} e^2[n] = \sum_{n=0}^{\infty}\left(x[n] + \sum_{k=1}^{p} a_k x[n-k]\right)^2$$

~~p3~~.

$$\frac{\partial \epsilon}{\partial a_i} = \sum_{n=0}^{\infty} 2\left(x[n] + \sum_{k=1}^{p} a_k x[n-k]\right) x[n-i]$$

$$= 0$$

$i = 1, 2, \ldots p$

$$0 = \cancel{2}\sum_{n=0}^{\infty} x[n]\, x[n-i]$$

$$+ \cancel{2} \sum_{n=0}^{\infty} \sum_{k=1}^{p} a_k \, x[n-k]\, x[n-i]$$

$$= c_{xx}[0, i] + \sum_{k=1}^{p} a_k \sum_n x[n-k]\, x[n-i]$$

$$0 = c_{xx}[i] + \sum_{k=1}^{p} a_k \, c_{xx}[k, i]$$

$$i = 1, \ldots p$$

p eqs., for p unknowns.

Solve $p = 2$

$p = 2$

$$x[n] = 2^{-n} u[n]$$

$$\frac{1}{1 - \frac{1}{2} z^{-1}} \longleftrightarrow \left(\frac{1}{2}\right)^n u[n] = x[n]$$

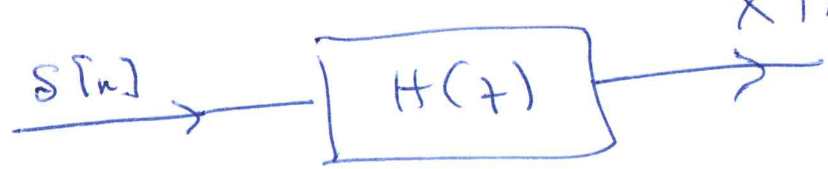
Optimal <u>filter</u>

Data $\quad x[n] = \left(\frac{1}{2}\right)^n u[n]$

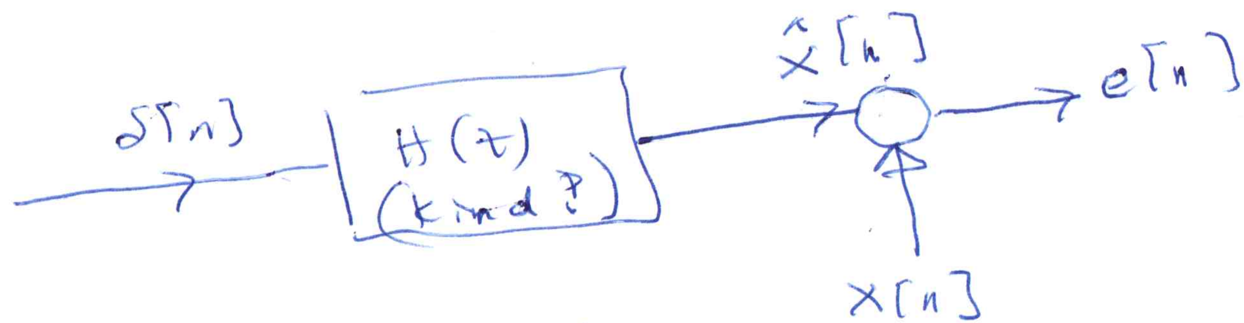
$$= \begin{cases} 0 & n < 0 \\ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots & n \geq 0 \end{cases}$$

Design a filter

Ideal



Reality

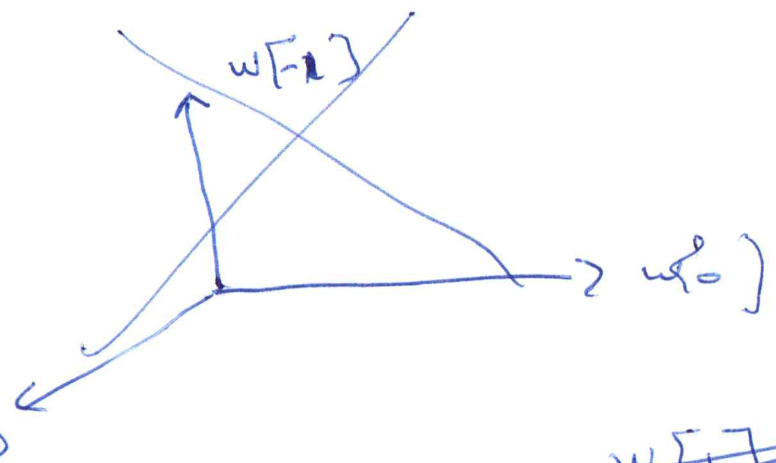


Parameters of $H(z)$?

$N = 1$

$$\bar{E} = h^2[-1]\left(1 - w[-1]\right)^2 + h^2[0]\left(1 - w[0]\right)^2 + h^2[1]\left(1 - w[1]\right)^2$$

Variables $w[0]$, $w[1]$ $w[-1]$



~~Hypothetical case~~ ~~$w[1] = w[-1]$~~

$N = 0$ $\bar{E} = h^2[0]\left(1 - w[0]\right)^2$