Probability Transformation

Amaires@May 2024

1 Distribution of $F(x)$

Suppose a univariate continuous random variable x's probability density function (PDF) is $f(x)$ and its cumulative distribution function (CDF) $F(x)$. $F(x)$ itself is also a random variable, so what is $F(x)$'s distribution?

$F(x)$ has a couple of properties.

- $F(x)$'s range is [0, 1].
- $F(\cdot)$ is a monotonically non-decreasing function, but all commonly seen $F(\cdot)$ monotonically increases. In this case, $F^{-1}(\cdot)$ exists and also monotonically increases.

Let $y = F(x)$. y's CDF is

$$\Pr(F(x) < y) = \Pr(F^{-1}(F(x)) < F^{-1}(y)) = \Pr(x < F^{-1}(y)) = F(F^{-1}(y)) = y$$

Take its derivative, we have $y = F(x) \sim U(0, 1)$.

2 Uniform Distribution to Other Continuous Distributions

Suppose $x \sim U(0, 1)$, what kind of transformation G can be applied to x such that $y = G(x)$ follows a given distribution with PDF $f(y)$ and CDF $F(y)$?

Section 1 explains the transformation of a random variable from a distribution to $U(0, 1)$, it seems to suggest the inverse transformation might work for this problem. That is, F^{-1} might be the G needed. Lets's do the math. $F^{-1}(x)$'s CDF is

$$\Pr(F^{-1}(x) < y) = Pr(F(F^{-1}(x)) < F(y)) = Pr(x < F(y)) = F(y)$$

so we are correct. $F^{-1}(x)$'s CDF is indeed $F(y)$ and its PDF $f(y)$.

3 One General Distribution to Another

Given a random variable x with PDF $f(x)$ and CDF $F(x)$, how can we transform it to another random variable y with PDF $g(y)$ and CDF $G(y)$?

Based on Section 1 and Section 2, $y = G^{-1}(F(x))$ is what we are looking for. The proof is obvious. Figure 1 may help one visualize and memorize the transformation steps.

$$\begin{tikzcd} x \sim f(x) \arrow[r, shift right, "F(x)"'] & U(0,1) \arrow[l, shift right, "F^{-1}(x)"'] \arrow[r, shift right, "G(y)"'] & y \sim g(y) \arrow[l, shift right, "G^{-1}(y)"'] \end{tikzcd}$$

Figure 1: Transform between two different distributions