Priority Queues

Priority Queue

A priority queue (PQ) is an ADT that arranges data elements according to per-element keys ("priorities"): In a minimizing (maximizing, resp.) PQ the element with smallest (largest, resp.) overall key is served first.

- Keys need to belong to a totally ordered set.
- Standard operations for minimizing PQs:
    - **FindMin:** return element with smallest key
    - **DeleteMin:** return and remove element with smallest key from PQ
    - **Insert:** insert a new element
    - **DecreaseKey:** decrease the key of an element
    - **Remove:** remove an element from PQ
    - **Merge:** merge (aka meld) two PQs

Note: Standard implementation of PQ: Binary heap

Binomial Tree

A binomial tree is an ordered rooted binary tree is defined recursively as follows:

- A binomial tree of order 0 consists only of the root node
- For $k \in \mathbb{N}_0$, a binomial tree of order $k+1$ consists of two binomial trees of order k such that one binomial tree is the left-most substree of the other.

Lemma (112)

For $k \in \mathbb{N}_0$, a binomial tree of order k has k subtrees (from left to right) of orders $k-1$, $k-2$, ..., 1, 0

Proof:

Consider a binomial tree B_k of order k.

IB: Consider a binomial tree B_k. It's order is 1. Obviously, the tree comprises of 1 subtree of order 1.

IH: A binomial tree B_k consists of k subtrees of the orders $k-1$, $k-2$, ..., 1, 0.

IS: We consider a bionmial tree B_{k+1} of order $k+1$. According to the definition of a binomial tree it comprises of two binomial trees B_k. Applying the induction hypothesis to B_k tells us that the binomial tree comprises of subtrees of B_{k-1}, B_{k-2}, ..., 1, 0. Hence, we have B_k, B_{k-1}, B_{k-2}, ..., 1 in total.

Lemma (113)

For $k \in \mathbb{N}_0$, a binomial tree of order k has 2^k nodes and height k.

Proof:

IB: It's easy to see that a binomial tree B_0 consists of 1 node. (Definition)

IH: A binomial tree of order k has 2^k nodes and height k.

IS: Consider a binomial tree B_{k+1}. By definition a binomial tree comprises of two binomial trees B_k. Due to IH we know that both trees have 2^k nodes. Since $2^k + 2^k = 2^{k+1}$ we know that a binomial tree has B_{k+1} has 2^{k+1} nodes in total.

Lemma (114)

For $k \in \mathbb{N}_0$, a binomial tree of order k has $\binom{n}{d}$ nodes at depth d.

Binomial Heap

A binomial heap is a collection of binomial trees that satisfy the binomial heap property:

- Each binomial tree is a min heap, i.e., for all nodes v of the binomial tree, all keys of the children of v are greater than (or at most equal to) the key of v.
- For any $k \in \mathbb{N}_0$, there is at most one binomial tree of order k.
- The binomial trees are arranged in a right-to-left sorted sequence according to their orders, with the tree of smallest order being right-most.

Lemma (116)

For $n \in \mathbb{N}_0$ a binomial heap with a total of n nodes contains a binomial tree of order k if and only if the bit that corresponds to 2^k in the binary representation of n is 1.

Proof:

Every natural number can be uniquely represented as binary number. From lemma 113 we know that a binomial tree of order k has 2^k nodes. We can construct a binomial heap that comprises of different unique binomial trees B_k where the k-th bit is set to 1. Hence, the sum of all nodes in the binomial heap becomes:

$n = \sum_{i=0;i\text{-th bit is 1}}^{K} 2^i$

Merging Binomial Heaps

- We visit the binomial trees of both binomial heaps according to increasing order k, starting with $k := 0$.
  - If both heaps and the carry contain exactly...
    - no binomial tree of order k: Do nothing
    - one binomial tree B_1 of order k: Move B_1 to the result
    - two binomial trees B_1, B_2 of order k: Merge B_1 and B_2 into a tree B of order $k+1$ and move B to the carry.

- three binomial trees B_1, B_2, B_3 of order k: Merge B_1 and B_2 into a tree B of order $k+1$ and move B to the carry. Move B_3 to the result.
- Increment k after processing the bionomial trees of order k.

Lemma (117)

Merging two binomial heaps with a total of n nodes takes $O(log(n))$ time.

Proof:

The representation of any decimal number requires at most $\lfloor log(n) \rfloor + 1$ bits. Hence, lemma 116 tells us that we need at most $\lfloor log(n) \rfloor + 1$ binomial trees. Hence, we need to perform $O(log(n))$ trivial merges of two binomial trees of the same order. Each such merge takes $O(1)$ time.

Lemma (118)

A new element can be inserted into a binomial heap with a total of n nodes in $O(log(n))$ worst-case and $O(1)$ amortized time.

Proof:

- **Worst-case:** A new heap that contains only the new element and merge it with the old heap. Hence the worst-case complexity is $O(log(n))$.
- **Average Case: Aggregate method:** When does the i-th tree need to be changed?
  - The 1st tree gets added / removed every time
  - The 2nd tree gets added / removed every second time
  - The 3rd tree gets added / removed every fourth time

Hence, for a sequence of n inserts we get:

$\sum_{i=1}^{n} \lfloor \frac{n}{2^{i-1}} \rfloor \leq \sum_{i=0}^{n} n \sum_{i=1}^{n-1} \frac{1}{2^i} = 2n$

Therefore, the inserts can be done in amortized $O(1)$.

Lemma (119)

Finding the minimum element in a binomial heap with a total of n nodes takes $O(log(n))$ time.

Proof: It suffices to inspect the roots of all binomial trees. There are $\lfloor log(n) \rfloor + 1$ trees. Obviously, be keeping a pointer to the root with minimum key, this time can be reduced to $O(1)$.

Lemma (120)

The minimum element can be deleted from a binomial heap with a total of n nodes in $O(log(n))$ time.

Proof: The minimum can be found in $O(log(n))$ time. By removing the root / binomial tree we split one binomial tree into a sequence of subtrees which in turn are binomial heap. Now, we merge this new binomial heap with the rest of the original binomial heap. This can be done in $O(log(n))$ time.

Lemma (121)

The key of a known element of a binomial heap with a total of n nodes can be decreased in $O(log(n))$ time.

Proof:

After decreasing the key we may need to (repeatedly) exchange the corresponding node with its parent node if the min-heap property is violated. We need to put a bound on the height of the binomial tree.

The largest binomial tree as order $log(n)$. A binomial tree of order $log(n)$ has has size $log(n)$. Therefore, it can take $O(log(n))$ time.

Lemma (122)

An element can be deleted from a binomial heap with a total of n nodes in $O(log(n))$ time.

Proof:

We first decrease the key of the element to a value smaller than the minimum key. ($O(log(n))$) This causes the element to be the root of a tree. Deleting the element can therefore be done in $O(log(n))$ time.

Fibonacci Heaps

- The name is derived from the fact that the Fibonacci numbers show up in the complexity analysis of its operations.
- Similar to binomial heaps, but less rigid. Fibonacci heaps **lazily** defer all clean-up work after an Insert til the next DeleteMin

  For example, merge operation simply links two heaps, insert operation simply adds a new tree with single node. The operation extract minimum is the most complicated operation. It does delayed work of consolidating trees.

Fibonacci Heap

- Collection of min heaps (no binomial heap)
- Maintains pointer to element with minimum key
- Some nodes are "marked".

Representation

Heap representation:

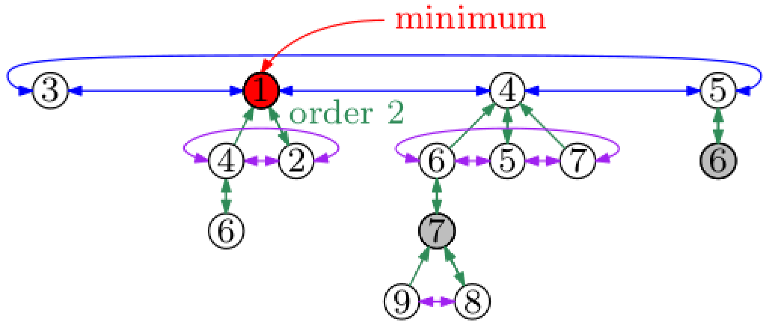
- Maintain root nodes in doubly-linked circular list
- Store pointer to root node with minimum key

Node representation:

- A pointer to its parent
- A pointer to one of its children
- The number of its children
- Pointers to its left and right siblings
- A binary flat that indicates whether the nodes is marked

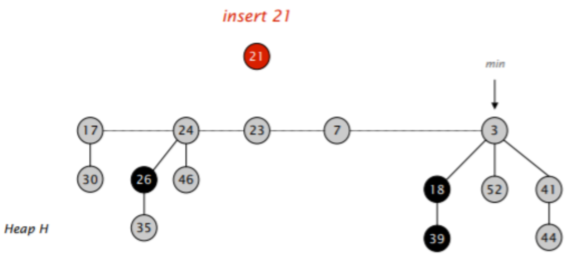
Marking of nodes:

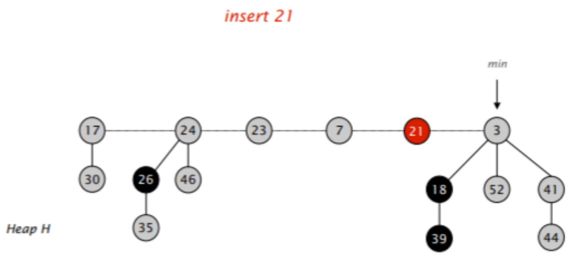
- **Unmarked:** The nodes has had no child cut
- **Marked:** The node has had once child cut
- Basic idea: When a child is cut from a marked parent node, then the parent node (together with its entire subtree) is cut, too, and moved to the root list.
- The marking of nodes ensures that Fibonacci heaps keep roughly the structure of binomial heaps after the deletion of nodes, thus ensuring amortized bounds
- The root node is always unmarked



Insert operation

- Create a new node and insert into the list of root nodes
- Update pointer to (new) minimum root node if required





Link operation

- If $r_1.key \geq r_2.key$ then make r_1 a child of $_2$, otherwise r_2 becomes a child of r_1.
- Update information on the order of r_2 (or r_1)

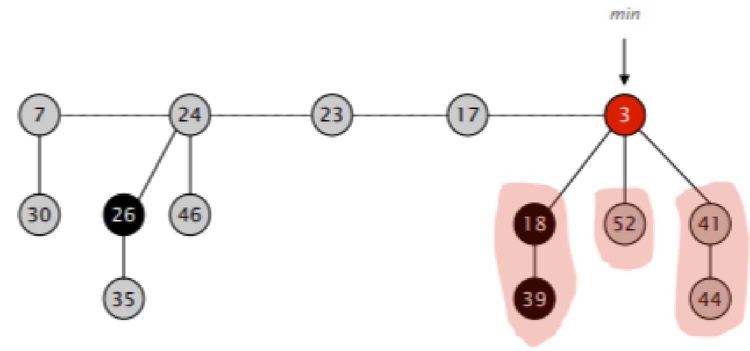
Cut operation

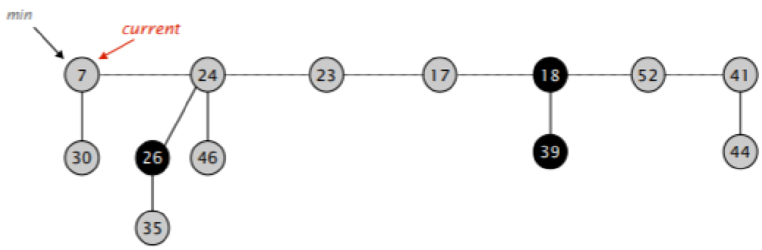
- Remove v from the child list of its parent p and insert it into the root list
- Update information on the order of p
- Mark p

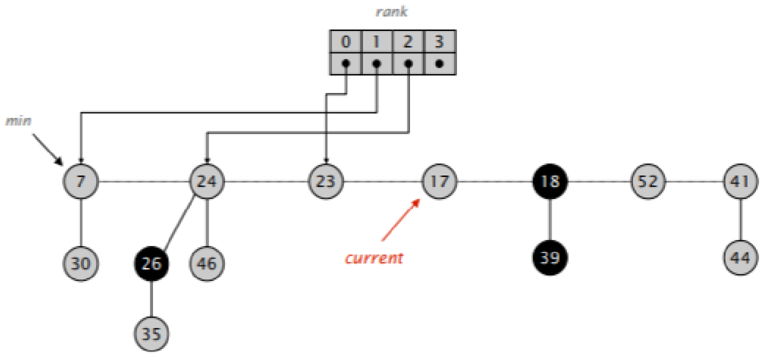
DeleteMin

- Delete the root node with the current minimum

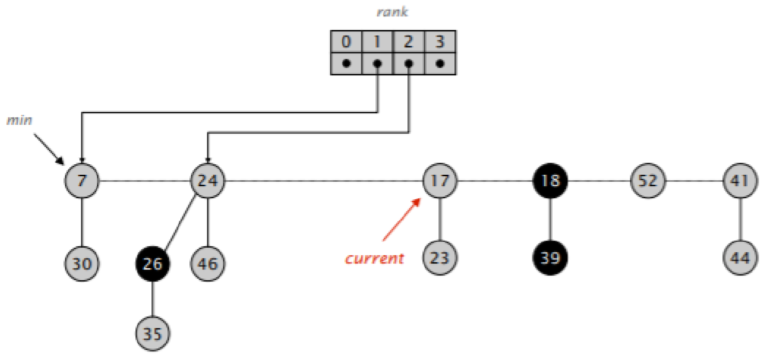
- Move its children as new root nodes into the list of root nodes
- Link trees until no pair of nodes has the same order
- Update pointer to minimum root



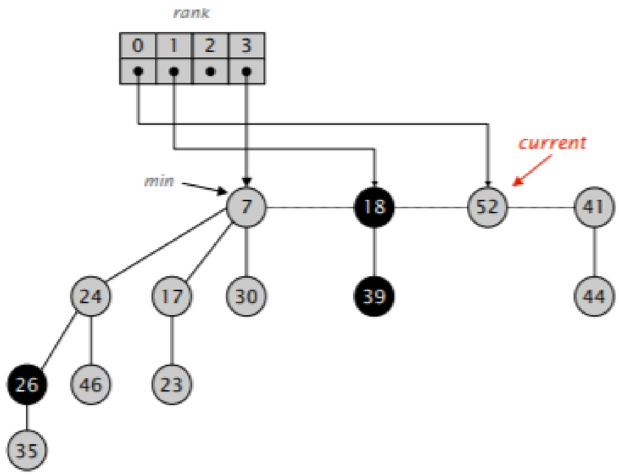


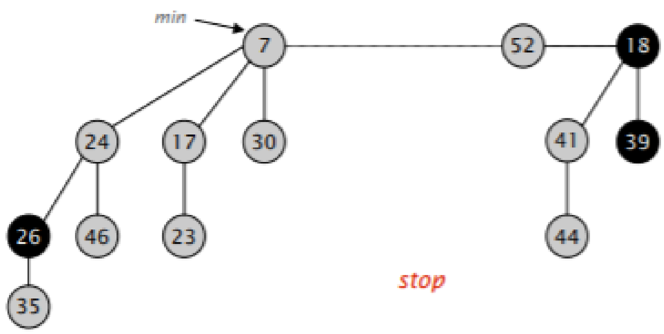


link 23 into 17



link 17 into 7

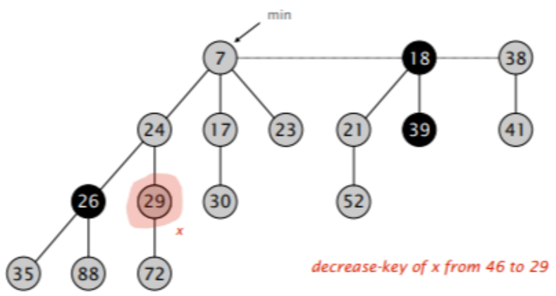


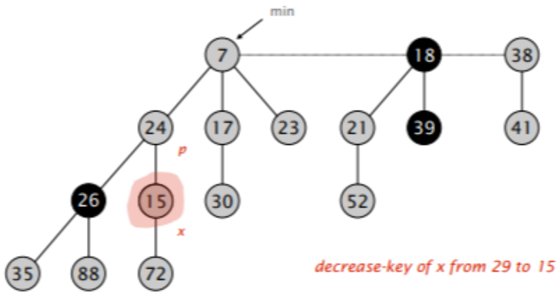


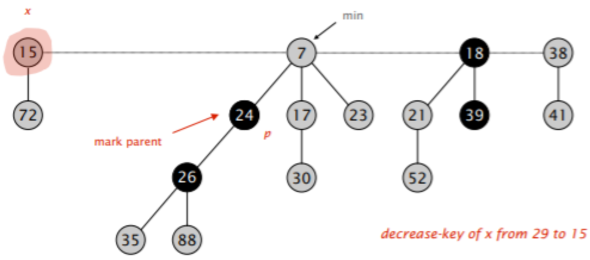
DecreaseKey

- If the new key of v is less than the key of the parent p then cut v and move it (with its subtree) to the root list.
- If p is not marked then mark p
- Else, cut p and move to root list and apply recursively to its parents
- Update pointer to minimum root

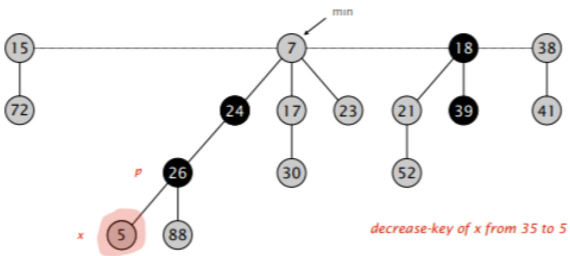
Case 1:

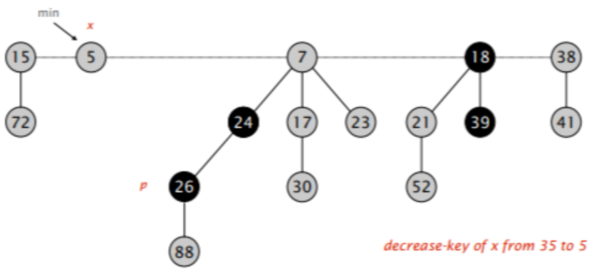


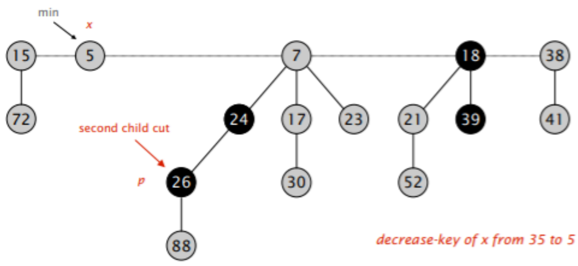

min
p
x
decrease-key of x from 29 to 15

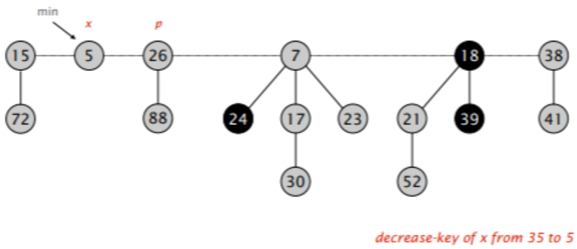

x
min
mark parent
p
decrease-key of x from 29 to 15

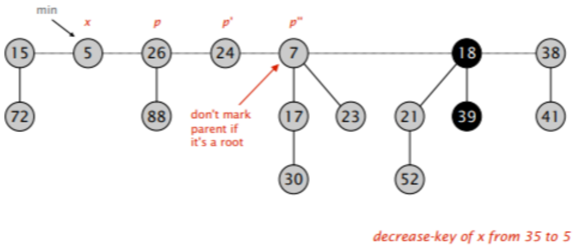
Case 2:


min
p
x
decrease-key of x from 35 to 5


min
x
p
decrease-key of x from 35 to 5


min
x
second child cut
p
decrease-key of x from 35 to 5


min
x
p
decrease-key of x from 35 to 5



Lemma (123)

If only Insert and DeleteMin operations are carried out, then a Fibonacci heap is a binomial heap after every DeleteMin operation.

Note: If no consolidation occurs (since no suitable DeleteMin operation is carried out) then a Fibonacci heap with n nodes may degenerate to one single tree, or even to an unsorted linked list (of n root nodes) or an "unary" tree of height $n - 1$.

Proof:

By induction: Every DeleteMin results in a consolidation phase during which pairs of trees which have root nodes of the same order are linked.

Lemma (124)

If a node of a tree in a Fibonacci heap has k children then it is the root of a subtree with at least F_{k+2} nodes.

Corollary (125)

Every node of a tree in a Fibonacci heap with a total of n nodes has at most $O(log(n))$ children.

Proof:

Let k be the number of children of a node v. By lemma 124, its subtree has F_{k+2} nodes. Hence,

$n \geq F_{k+2} \geq (\text{Lemma } 5)\ \phi^k$ implying $k \leq log_\phi n$

Theorem (126)

When starting from an initially empty heap, any sequence of a Insert, b DeleteMin and c DecreaseKey operations takes $O(a + b \cdot log(n) + c)$ worst-case time, where n is the maximum heap size.

- Hence, from a theoretical point of view, a Fibonacci heap is better than a binomial heap when c is smaller than b by a non-constant factor.
- A Fibonacci heap is also better than a binomial heap when frequent merging of heaps is required.
- However, the worst-case time for one DeleteMin or DecreaseKey operation is linear, which makes Fibonacci heaps less suitable for applications which cannot tolerate excessive running time for individual operation.
- Fibonacci heaps are sometimes reported to be slow in practice due to hidden constants.

Performance Summary

Performance Summary for Priority Queues with n Elements

Operation	Linked List	Binary Heap	Binomial Heap	Fibonacci Heap
Insert	$O(1)$	$O(\log n)$	$O(1)^*$	$O(1)$
FindMin	$O(n)$	$O(1)$	$O(\log n)$	$O(1)$
DeleteMin	$O(n)$	$O(\log n)$	$O(\log n)$	$O(\log n)^{**}$
DecreaseKey	$O(1)$	$O(\log n)$	$O(\log n)$	$O(1)^{**}$
Merge	$O(1)$	$O(n)$	$O(\log n)$	$O(1)$

*: amortized complexity; worst-case complexity is $O(\log n)$.
**: amortized complexity; worst-case complexity is $O(n)$.