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L – 1564

Reg. No. : ....

Name : ....

Sixth Semester B.Sc. Degree Examination, March 2021

First Degree Programme under CBCSS

Mathematics

Core Course

MM 1644 : ABSTRACT ALGEBRA – II

(2015-2017 Admission)

Time : 3 Hours

Max. Marks : 80

SECTION – I

All the first ten questions are compulsory. Each carries **1** mark.

1. Determine whether the map $\phi : GL(n, \mathbb{R}) \to \mathbb{R}$ given by $\phi(A) = tr(A)$ is a homomorphism, where, $GL(n,\mathbb{R})$ is the multiplicative group of all invertible $n \times n$ matrices.

2. Find the order of $(\mathbb{Z}_4 \times \mathbb{Z}_2)/\langle(2, 1)\rangle$

3. Find the solution of the congruence $36x \equiv 15 (\text{mod}\, 24)$, if it exists.

4. Find the order of the ring $M_2(\mathbb{Z}_2)$

5. Find a solution of the quadratic equation $x^2 + 2x + 4 = 0$ in the ring $\mathbb{Z}_6$

6. Find the number of zero divisors in the ring $\mathbb{Z}_4$

7. Compute the product (12)(16) in $\mathbb{Z}_{24}$

P.T.O.

8. State whether true or false: "$\mathbb{Z}$ is a subfield of $\mathbb{Q}$"

9. Find all ideals of $\mathbb{Z}_{12}$

10. Find the characteristic of the ring $\mathbb{Z}_6 \times \mathbb{Z}_{15}$

($10 \times 1 = 10$ Marks)

SECTION – II

Answer **any eight** questions from this Section. Each carries **2** marks.

11. Let $\phi : G \to G'$ be a group homomorphism of G onto G'. Prove that G' is abelian if G is abelian.

12. Find $\ker\phi$ and $\phi(3)$ for $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_{20}$ such that $\phi(1) = 8$.

13. Let $\phi : G \to G'$ be a group homomorphism, show that if $|G|$ is finite, then $|\phi[G]|$ is finite and is a divisor of $|G|$.

14. Find the order of $5 + \langle 4 \rangle$ in $\mathbb{Z}_{12}/\langle 4 \rangle$

15. Show that A_n is a normal subgroup of S_n and compute S_n / A_n

16. Prove that the factor group of a cyclic group is cyclic.

17. Compute the factor group $(\mathbb{Z}_4 \times \mathbb{Z}_6)/\langle 0, 2 \rangle$

18. Let H be a normal subgroup of an abelian group G. Then show that G/H is abelian.

19. Describe all ring homomorphisms of $\mathbb{Z}$ into $\mathbb{Z}$.

20. Find the remainder when 3^{47} is divided by 23.

21. Describe all units in the ring $\mathbb{Z} \times \mathbb{Z}$

22. Let R be a commutative ring with unity of characteristic 3. Compute and simplify $(a+b)^6$ for $a, b \in R$.

($8 \times 2 = 16$ Marks)

SECTION – III

Answer **any six** questions from this Section. Each carries **4** marks.

23. Prove that a group homomorphism $\phi : G \to G'$ is a one-one map if and only if $\ker \phi = \{e\}$

24. Let $\phi : G \to H$ be a group homomorphism. Show that $\phi[G]$ is abelian if and only if, for every $x, y \in G$ we have $xyx^{-1}y^{-1} \in \ker \phi$.

25. Show that arbitrary intersection of normal subgroups of a group G is again a normal subgroup.

26. Show that the characteristic of an integral domain must be either zero or a prime p.

27. Find the last two digits in the decimal representation of 3^{256}.

28. Show that for every integer n, the number $n^{33} - n$ is divisible by 15.

29. Let $d = \gcd(a, m,)$. Prove the congruence $ax \equiv b (\text{mod } m)$ has a solution if and only if d / b

30. Show that the group homomorphism $\phi : G \to G'$ where $|G|$ is prime must either be trivial or a one-one map

31. State and prove Fermat's Little Theorem.

($6 \times 4 = 24$ Marks)

SECTION – IV

Answer **any two** questions from this Section. Each carries **15** marks.

32. Let R be a ring that contains at least two elements. Suppose for each non-zero $a \in R$, there exists a unique $b \in R$ such that $aba = a$.
    (a) Show that R has no divisors of zero.
    (b) Show that $bab = b$.
    (c) Show that R has unity.
    (d) Show that R is a division ring.

33. (a) Show that all automorphisms of a group G form a group under function composition.

(b) Show that the inner automorphisms of a group G form a normal subgroup of the group of all automorphisms of G under function composition.

34. State and prove fundamental theorem of Ring Homomorphism.

35. Prove that

(a) Every field is an Integral Domain.

(b) Every finite integral domain is a field.

(c) If p is a prime, then $\mathbb{Z}_p$ has no divisors of zero.

(2 × 15 = 30 Marks)
