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P – 2484

Reg. No. :

Name :

Fifth Semester B.Sc. Degree Examination, December 2022

First Degree Programme under CBCSS

Mathematics

Core Course

MM 1541 : REAL ANALYSIS I

(2018 Admission Onwards)

Time : 3 Hours

Max. Marks : 80

SECTION I

Answer **all** the questions.

1. Define absolute value function $|x|$.
2. Define *least upper bound* for a set $A \subseteq \mathbb{R}$.
3. Find $\inf(A)$ if $A = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$
4. Find $\lim_{n \rightarrow \infty} \left(\frac{3n+2}{2n-2} \right)$.
5. Define the limit of a real sequence.
6. Write the harmonic series. Is it convergent?
7. Define Cauchy sequence.
8. Let $\epsilon = \frac{1}{2}$ and $a = 3$, then find the ϵ - neighborhood, $V_\epsilon(a)$ of a .
9. Define closed set in \mathbb{R} . Write an example for closed set.
10. Define perfect set in \mathbb{R} .

(10 × 1 = 10 Marks)

P.T.O.

SECTION II

Answer any **eight** questions

11. Show that, two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.
12. Prove that $\sqrt{3}$ is irrational.
13. Using triangle inequality prove the inequality $\|a\| - \|b\| \leq \|a - b\|$.
14. Let $A \subseteq \mathbb{R}$ be bounded above, and let $c \in \mathbb{R}$. Define the set $c + A$ by $c + A = \{c + a : a \in A\}$. Show that $\sup(c + A) = c + \sup A$.
15. Show that $\lim \left(\frac{n+1}{n} \right) = 1$.
16. Show that every convergent sequence is bounded.
17. Let $\lim a_n = a$, and $\lim b_n = b$, then prove that $\lim(a_n + b_n) = a + b$.
18. Show that if $(b_n) \rightarrow b$, then the sequence of absolute values $|b_n|$ converges to $|b|$.
19. Write the examples for
 - (a) Sequences (x_n) and (y_n) , which both diverge, but whose sum $(x_n + y_n)$ converges;
 - (b) A convergent sequence (b_n) with $b_n \neq 0$ for all n such that $(1/b_n)$ diverges.
20. Prove that the sequence defined by $x_1 = 3$ and $x_{n+1} = \frac{1}{4 - x_n}$ converges.
21. Show that, sub sequences of a convergent sequence converge to the same limit as the original sequence.
22. Show that, every convergent sequence is a Cauchy sequence.
23. Let $a \in A$. Prove that a is an isolated point of A if and only if there exists an ϵ -neighborhood $V_\epsilon(a)$ such that $V_\epsilon(a) \cap A = \{a\}$.
24. Show that if K is compact, then $\sup K$ and $\inf K$ both exist and are elements of K .
25. Prove that cantor set is a Compact set.
26. Let E is nowhere-dense in \mathbb{R} , then show that the complement of \bar{E} is dense in \mathbb{R} .

(8 × 2 = 16 Marks)

SECTION III

Answer any **six** questions

27. Assume $s \in \mathbb{R}$ is an upper bound for a set $A \subseteq \mathbb{R}$. Then, show that $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < 0$.
28. State and prove Nested Interval Property.
29. Show that the set of rational numbers, \mathbb{Q} is countable.
30. Given any set A , prove that there does not exist a function $f : A \rightarrow P(A)$ that is onto.
31. Assume $\lim a_n = a$ and $\lim b_n = b$, then show that
 - (a) If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.
 - (b) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.
32. Show that; if $x_n \leq y_n \leq z_n \forall n \in \mathbb{N}$, and if $\lim x_n = \lim z_n = l$, then $\lim y_n = l$.
33. If a sequence is monotone and bounded, then show that it is convergent.
34. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.
35. Prove that, a point x is a limit point of a set A if and only if $x = \lim a_n$ for some sequence (a_n) contained in A satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.
36. For any $A \subseteq \mathbb{R}$, Show that the closure \bar{A} is a closed set and is the smallest closed set containing A .
37. If $K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \dots$ is a nested sequence of nonempty compact sets, then show that, the intersection $\bigcap_{n=1}^{\infty} K_n$ is not empty.
38. Prove that the set of real numbers \mathbb{R} cannot be written as the countable union of nowhere-dense sets.

(6 × 4 = 24 Marks)

SECTION IV

Answer any **two** questions

39. Prove that
- (a) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.
 - (b) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $\frac{1}{n} < y$.
 - (c) For every two real numbers a and b with $a < b$, \exists a rational number r , satisfying $a < r < b$.
40. (a) Show that, there exists a real number $\alpha \in \mathbb{R}$ satisfying $\alpha^2 = 2$.
- (b) Find *suprema* and *infima* of the following sets.
- (i) $\{n \in \mathbb{N} : n^2 < 10\}$
 - (ii) $\{\frac{n}{m} : m, n \in \mathbb{N} \text{ with } m + n \leq 10\}$
41. (a) State and prove Bolzano-Weierstrass theorem.
- (b) Prove or disprove: Every Bounded sequence of real number are convergent.
42. Show that
- (a) Cauchy sequences are bounded.
 - (b) A sequence converges if and only if it is a Cauchy sequence.
43. (a) State and prove Cauchy Criterion for Series.
- (b) Using Alternating Series Test, test the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7+2n}$.
44. (a) Show that
- (i) The union of an arbitrary collection of open sets is open.
 - (ii) The intersection of a finite collection of open sets is open.
- (b) Let $B = \left\{ \frac{(-1)^n n}{n+1}, n \in \mathbb{N} \right\}$
- (i) Find the limit points of B .
 - (ii) Does B contain any isolated points?
 - (iii) Find closure of B , \bar{B} .

(2 × 15 = 30 Marks)