Devoir N°1 Analyse II

1) $\int x\sqrt{\frac{x-2}{x+1}}\,dx$

on pose $\boxed{t=\sqrt{\frac{x-2}{x+1}}}$ $\iff t^2 = \frac{x+2}{x+1}$

$\iff x(t^2-1) = -2-t^2$

$\iff \boxed{x = \frac{2+t^2}{1-t^2}}$

$\iff dx = \frac{2t(1-t^2)+2t(2+t^2)}{(1-t^2)^2}\,dt$

$\iff dx = \frac{2t(\cancel{1}-\cancel{t^2}+2+\cancel{t^2})}{(1-t^2)^2}\,dt$

$\iff \boxed{dx = \frac{6t}{(1-t^2)^2}\,dt}$

$$\int x\sqrt{\frac{x-2}{x+1}}\,dx = \int \frac{2+t^2}{1-t^2}\cdot t\cdot\frac{6t}{(1-t^2)^2}\,dt$$

$$\boxed{\int x\sqrt{\frac{x-2}{x+1}}\,dx = \int\frac{6t^2(2+t^2)}{(1-t^2)^3}\,dt}$$

La décomposition en éléments simples :

$$F(t)=\frac{6t^2(2+t^2)}{(1-t^2)^3} = \frac{a_1}{1-t}+\frac{b_1}{(1-t)^2}+\frac{c_1}{(1-t)^3}+\frac{a_2}{1+t}+\frac{b_2}{(1+t)^2}+\frac{c_2}{(1+t)^3}$$

~~Donc~~ on a $F(-t)=F(t) \iff \frac{a_1}{1+t}+\frac{b_1}{(1+t)^2}+\frac{c_1}{(1+t)^3}+\frac{a_2}{1-t}+\frac{b_1}{(1-t)^2}+\frac{c_2}{(1+t)^3} = F(t)$

d'où $a_1=a_2$ et $b_1=b_2$ et $c_1=c_2$

donc $F(t)=\frac{6t^2(2+t^2)}{(1-t^2)^3} = \frac{a}{1+t}+\frac{b}{(1+t)^2}+\frac{c}{(1+t)^3}+\frac{a}{1-t}+\frac{b}{(1-t)^2}+\frac{c}{(1-t)^3}$

$\iff \frac{6t^2(2+t^2)}{(1+t)^3} = \frac{a}{1+t}(1-t)^3+\frac{b}{(1+t)^2}(1-t)^3+\frac{c(1-t)^3}{(1+t)^3}+\cancel{a}(1-t)^2+b(1+t)+c$

pour $t=1$ on a $\boxed{c=\frac{18}{8}}$

d'où $F(t) = \frac{6t^2(2+t^2)}{(1-t^2)^3} = \frac{a}{1+t} + \frac{b}{(1+t)^2} + \frac{18}{8(1+t)^3} + \frac{a}{1-t} + \frac{b}{(1-t)^2} + \frac{18}{8(1-t)^3}$

$F(0) = 0 = a + b + a + b + \frac{18}{4}$

$\Rightarrow \boxed{a + b = -\frac{18}{8}}$ (1)

et $F(2) = -\frac{16}{3} = \frac{a}{3} + a + \frac{b}{9} + b + \frac{18}{8 \times 3^3} + \frac{18}{8 \times (-1)}$

$-\frac{16}{3} = -\frac{2}{3}a + \frac{10}{9}b - \frac{13}{6}$

donc $-\frac{2}{3}a + \frac{10}{9}b = \frac{13}{6} - \frac{16}{3} = -\frac{19}{6}$

$\boxed{2a - \frac{10}{3}b = \frac{19}{2}}$ (2)

de (1) et (2) on a
$$\begin{cases} a + b = -\frac{18}{8} & (1) \\ 2a - \frac{10}{3}b = \frac{19}{2} & (2) \end{cases}$$

donc $2 \times (1) - 2 \Rightarrow 2b + \frac{10}{3}b = -\frac{18}{4} - \frac{19}{2}$

$\Rightarrow b = \frac{-\frac{18}{4} - \frac{19}{2}}{2 + \frac{10}{3}}$

$\boxed{b = -\frac{21}{8}}$

d'où $a - \frac{21}{8} = -\frac{18}{8}$

$a = -\frac{18}{8} + \frac{21}{8}$

$\boxed{a = \frac{3}{8}}$

donc $F(t) = \frac{3}{8} \times \frac{1}{1-t} + \frac{21}{8} \frac{1}{(1-t)^2} + \frac{18}{8} \times \frac{1}{(1-t)^3} + \frac{3}{8} \times \frac{1}{1+t} - \frac{21}{8} \frac{1}{(1+t)^2} + \frac{18}{8(1+t)^3}$

$$F(t) = \frac{1}{8}\left[\frac{3}{1+t} - \frac{3}{t-1} - \frac{21}{(t+1)^2} - \frac{21}{(t-1)^2} + \frac{18}{(t+1)^3} - \frac{18}{(t-1)^3}\right]$$

Alors

$$\int \frac{\cancel{6}\, t^2(2+t^2)}{(1-t^2)^3}\, dt = \frac{1}{8}\int \frac{3}{1+t} - \frac{3}{t-1} - \frac{21}{(t+1)^2} - \frac{21}{(t-1)^2} + \frac{18}{(t+1)^3} - \frac{18}{(t-1)^3}$$

$$= \frac{1}{8}\left(3\ln|t+1| - 3\ln|t-1| + \frac{21}{t-1} + \frac{21}{t+1} - \frac{9}{(t+1)^2} + \frac{9}{(t-1)^2}\right) + cte$$

remplaçons t par $\sqrt{\frac{x-2}{x+1}}$

$$\int x\sqrt{\frac{x-2}{x+1}}\, dx = \frac{1}{8}\left(3\ln\left(\sqrt{\frac{x-2}{x+1}}+1\right) - 3\ln\left(\sqrt{\frac{x-2}{x+1}}-1\right) + 21\left(\frac{1}{\sqrt{\frac{x-2}{x+1}}-1} + \frac{1}{\sqrt{\frac{x-2}{x+1}}+1}\right)\right.$$

$$\left. + 9\left(\frac{1}{\left(\sqrt{\frac{x-2}{x+1}}-1\right)^2} - \frac{1}{\left(\sqrt{\frac{x-2}{x+1}}+1\right)^2}\right)\right)$$

$$= \frac{3}{8}\ln\left(\frac{\sqrt{x-2}+\sqrt{x+1}}{\sqrt{x-2}-\sqrt{x+1}}\right) + \frac{21}{8}\left(\frac{\sqrt{x+1}}{\sqrt{x-2}-\sqrt{x+1}} + \frac{\sqrt{x+1}}{\sqrt{x-2}+\sqrt{x+1}}\right)$$

$$+ \frac{9}{8}\left(\frac{x+1}{(\sqrt{x-2}-\sqrt{x+1})^2} - \frac{x+1}{(\sqrt{x-2}+\sqrt{x+1})^2}\right)$$

$$= \frac{3}{8}\ln\left(\frac{x-2+x+1+2\sqrt{x^2+x-2x-2}}{x-2-x-1}\right) + \frac{21}{8}\left(\frac{\sqrt{x^2-2x+x-2}+x+1}{x-2-x+1}\right.$$

$$\left. + \frac{\sqrt{x^2-2x+x-2}-x-1}{x-2-x-1}\right) + \frac{9}{8}\left[(x+1)\left(\frac{x-2+x+1+2\sqrt{x^2-2x+x-2}}{(x-2-x-1)^2}\right.\right.$$

$$\left.\left. - \frac{x-2-x-1-2\sqrt{x^2-2x+x-2}}{(x-2-x-1)^2}\right)\right]$$

$$= \frac{3}{8}\ln\left(\frac{1}{3}\times\left(2\sqrt{x^2-x-2}+2x-1\right)\right)+\frac{21}{8}\left(-\frac{2}{3}\sqrt{x^2-x-2}\right)$$

$$+\ \frac{9}{8}(x+1)\left(\frac{2\sqrt{x^2-x-2}+2x-1}{3}+\frac{2\sqrt{x^2-x-2}+3}{3}\right)$$

$$= \frac{3}{8}\ln\left(2\sqrt{x^2-x-2}+2x-1\right)-\frac{3\ln(3)}{8}-\frac{7}{4}\sqrt{x^2-x-2}$$

$$+\ \frac{3}{8}(x+1)\left(4\sqrt{x^2-x-2}+2x+2\right)$$

d'où $\displaystyle\int x\sqrt{\frac{x-2}{x+1}}\,dx = \frac{(3x^2+4x-4)}{4}\sqrt{x^2-x-2}+\frac{3}{8}\ln\left(2\sqrt{x^2-x-2}+2x-1\right)$

$$-\frac{3\ln(3)}{8}+k\ ;\ k\in\mathbb{R}$$

2) $\displaystyle\int\frac{1}{(x+1)\sqrt{x^2+x+1}}\,dx$

On pose $t=\frac{1}{(x+1)} \iff x=\frac{1}{t}-1=\frac{1-t}{t}$

$$\iff x^2=\frac{(1-t)^2}{t^2}$$

et $dx = -\frac{\cancel{t}-1+\cancel{t}}{t^2}\,dt$

$$dx = -\frac{1}{t^2}\,dt$$

donc $\displaystyle\int\frac{1}{(x+1)\sqrt{x^2+x+1}}\,dx=\int -\frac{t}{t^2}\frac{dt}{\sqrt{\frac{(1-t)^2}{t^2}+\frac{1-t}{t}+1}}$

$$=\int -\frac{1}{\cancel{t}}\frac{dt}{\frac{\sqrt{1-2t+t^2+t-\cancel{t^2}+\cancel{t^2}}}{\cancel{t}}}$$

$$=-\int\frac{dt}{\sqrt{t^2-t+1}}$$

$$= -\int \frac{2(2\sqrt{t^2-t+1}+2t-1)}{2\sqrt{t^2-t+1}\,(2\sqrt{t^2-t+1}+2t-1)}\,dt$$

$$= -\int \frac{1+\frac{2(t-\frac{1}{2})}{2\sqrt{t^2-t+1}}}{t-\frac{1}{2}+\sqrt{t^2-t+1}}\,dt$$

$$= -\int \frac{\left(t-\frac{1}{2}+\sqrt{t^2-t+1}\right)'}{t-\frac{1}{2}+\sqrt{t^2-t+1}}\,dt$$

$$= -\ln\left|t-\frac{1}{2}+\sqrt{t^2-t+1}\right| + k \;;\; k \in \mathbb{R}$$

remplaçons t par $\frac{1}{x+1}$ on a

$$\int \frac{1}{(x+1)\sqrt{x^2+x+1}}\,dx = -\ln\left|\frac{1}{x+1}-\frac{1}{2}+\sqrt{\frac{1}{(x+1)^2}-\frac{1}{x+1}+1}\right| + k$$

$$= -\ln\left|\frac{1}{x+1}-\frac{1}{2}+\frac{\sqrt{1-x-1+x^2+2x+1}}{x+1}\right| + k$$

$$\boxed{\int \frac{1}{(x+1)\sqrt{x^2+x+1}}\,dx = -\ln\left|\frac{\sqrt{x^2+x+1}-x}{x+1}\right| + k}$$

3) $I=\int \sqrt{-x^2+4x+10}\,dx$

$$f'(x) = 1 \longrightarrow f(x) = x$$

$$g(x) = \sqrt{-x^2+4x+10} \longrightarrow g'(x) = \frac{-x+2}{\sqrt{-x^2+4x+10}}$$

~~par~~ intégration par partie on a

$$I = \int\sqrt{-x^2+4x+10}\,dx = x\sqrt{-x^2+4x+10} + \int \frac{x^2-2x}{\sqrt{-x^2+4x+10}}\,dx$$

On sait que : $\displaystyle\int \frac{x^2-2x}{\sqrt{-x^2+4x+10}}\,dx = \int \frac{(x^2-4x-10)+(2x+10)}{\sqrt{-x^2+4x+10}}\,dx$

$$= -\int \frac{-x^2+4x+10}{\sqrt{-x^2+4x+10}}\,dx + \int \frac{2x+10}{\sqrt{-x^2+4x+10}}$$

$$= -\int \sqrt{-x^2+4x+10}\,dx + \int \frac{2x+10}{\sqrt{-x^2+4x+10}}\,dx$$

d'où $\boxed{\displaystyle\int \frac{x^2-2x}{\sqrt{-x^2+4x+10}}\,dx = -I + \int \frac{2x+10}{\sqrt{-x^2+4x+10}}\,dx}$

donc $\displaystyle 2I = x\sqrt{-x^2+4x+10} + \int \frac{2x-4+14}{\sqrt{-x^2+4x+10}}\,dx$

$$2I = x\sqrt{-x^2+4x+10} - \int \frac{-2x+4}{\sqrt{-x^2+4x+10}}\,dx + 14\int \frac{dx}{\sqrt{-x^2+4x+10}}$$

d'où

$$\boxed{\int \sqrt{-x^2+4x+10}\,dx = \frac{x}{2}\sqrt{-x^2+4x+10} - \sqrt{-x^2+4x+10} + 7\arcsin\left(\frac{x-2}{\sqrt{14}}\right) + k}\,,\ k\in\mathbb{R}$$