So if we want to reach the swimmer in the least amount of time, we need to run all but the last 50 yards down the beach, and swim from there! This means we'll need to run:

$$\begin{aligned}\text{distance running: } &= 300 - x \\ &= 250 \text{ feet}\end{aligned}$$

and swim:

$$\begin{aligned}\text{distance swimming: } &= \sqrt{120^2 + x^2} \\ &= \sqrt{120^2 + 50^2} \\ &= 130 \text{ feet}\end{aligned}$$

Great. How long does it take us to get out to the swimmer? We know how to get there in the minimal amount of time, but *how minimal is this minimum*? Well, we already have a function for how long it takes us to reach the swimmer as a function of x, and now we have an x (i.e., 50), so we can just plug that in:

$$\begin{aligned}\text{time}(50) &= \frac{300-50}{13} + \frac{\sqrt{120^2+50^2}}{5} \\ &\approx \underbrace{19.231}_{\text{time running}} + \underbrace{26}_{\text{time swimming}} \\ &\approx 45.2 \text{ seconds}\end{aligned}$$

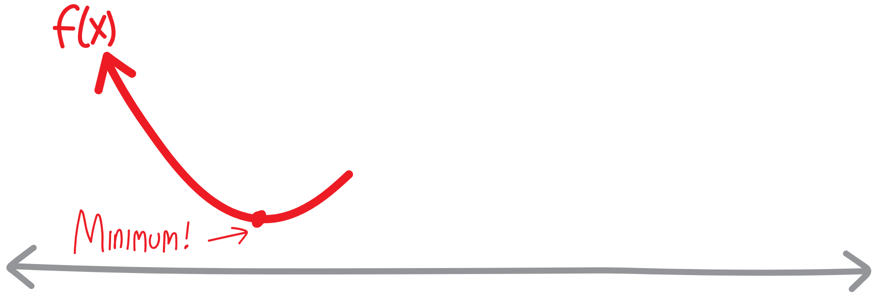
So it will take us about 45 seconds to reach the swimmer. Hopefully that's zippy enough to save them from the polar bear!

A video, not a photograph

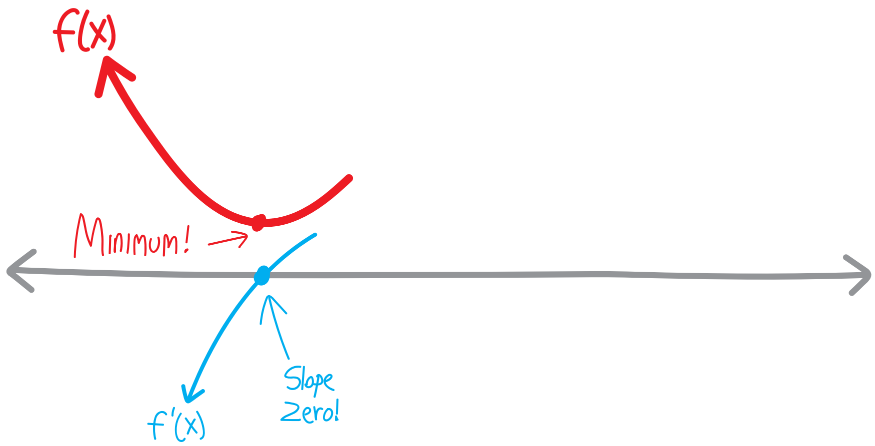
There's a major flaw in the reasoning we just used. Namely: how do we know we didn't *maximize* the amount of time it takes to rescue the guy? If all we're doing is finding where the slope is zero, that could be a minimum. Or it could be a maximum. Or it could have been one of those weird cubicky indecisive points!

That's a big difference. If we're trying to minimize the amount of time it takes to reach the guy, and we end up *maximizing* the amount of time it takes to reach the guy… then we're no longer going to have a rescue situation; we're going to have a recovery situation. If we're a junior analyst at Goldman Sachs and we end up *minimizing* profit rather than *maximizing* profit… we're going to also minimize our paycheck. If we're trying to do such-and-such, and we end up not only not doing such-and-such, but *doing the exact opposite of such-and-such*, that's pretty bad!

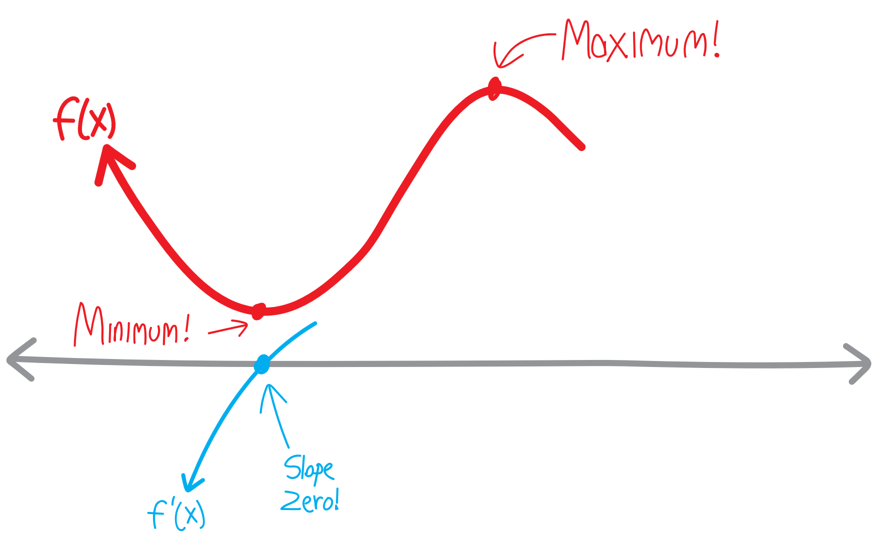
So we need more nuance. Let's think about how to find maxima and minima in more detail. Say we have a function, and it has a minimum:



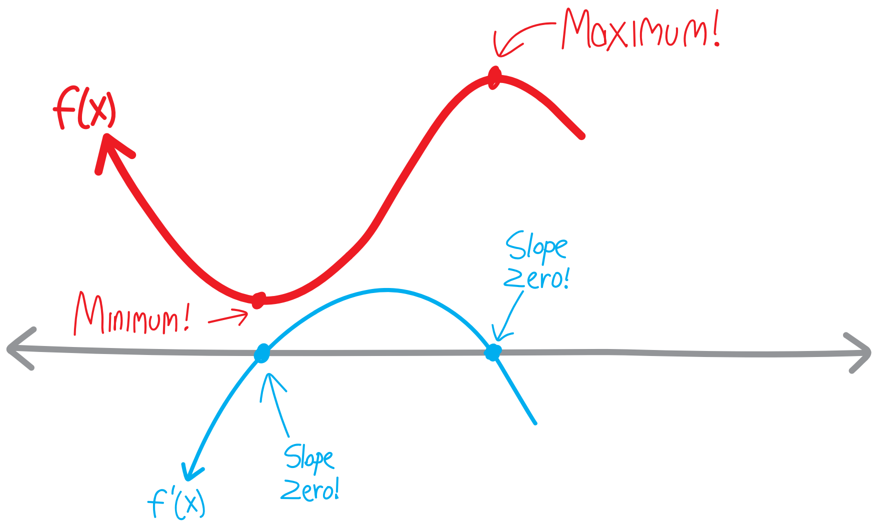
What’s the derivative of the function at that minimum? Zero! The function is flat. So its slope is zero! Let’s sketch the slope, too:



Now let’s say that function continues, and we have a maximum:

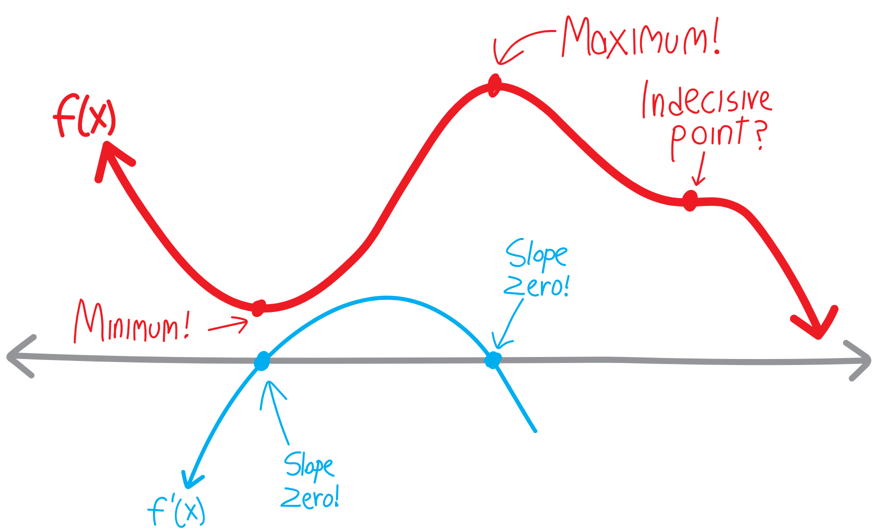


What’s the derivative at that maximum? Also zero! Let’s sketch that, too:

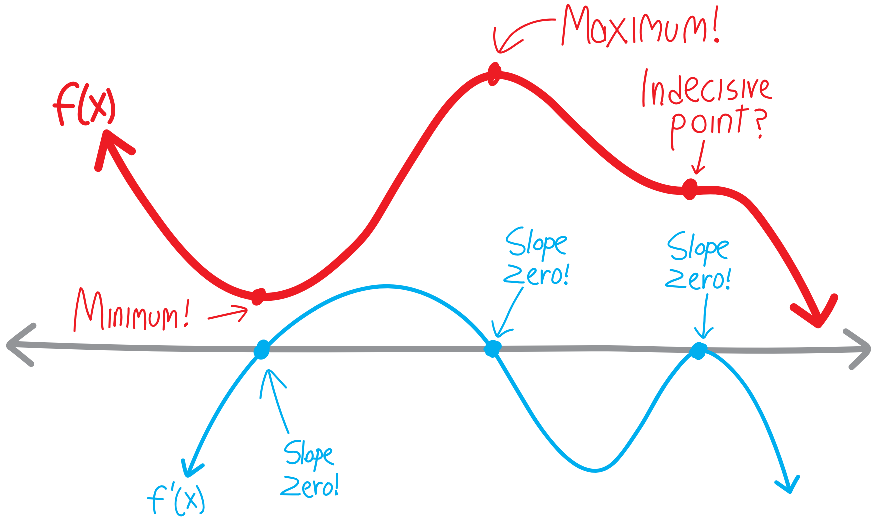


Hmm. Both minima *and* maxima have derivatives that are zero.

Let's say the function continues some more and we have one of those weird cubicky indecisive points, where it flattens out and *almost* becomes a minimum, but then changes its mind at the last moment and returns to going down:



What's the derivative at that point? Well, the function is basically always decreasing, so the derivative is always negative, except for that one moment where it's flat, where the derivative is zero. So the derivative *bounces* off the axis at that point:



So the derivative is zero at that cubicky indecisive point, too.

Eeek. All these points where the derivative is zero, corresponding to maxima and minima and neither!!! How do we tell the difference? How do we distinguish between maxima and minima and indecisive points!?! How do we figure that out, *algebraically* and *symbolically*, without needing a computer to graph things, only using *symbols on the page*? If we're just looking at places where the derivative is zero, we're lost in the fog of ambiguity. If the derivative is zero, we could have any of these three possibilities.

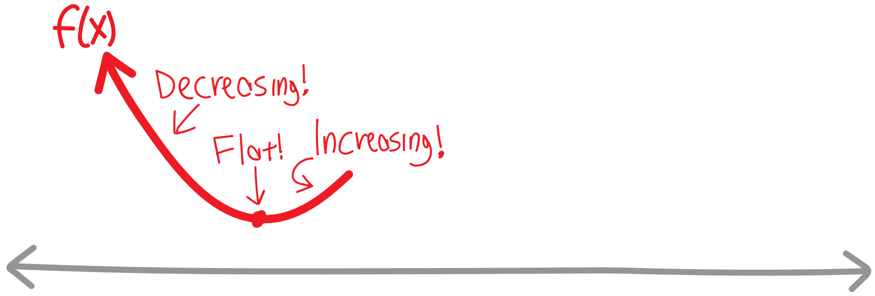
Here's the trick: we need to look at the *shape* of the derivative near where it's zero. We need to think about what's happening not *when* the derivative is zero—we know what's happening; it's zero; that's a tautology—but what's happening *near* when the derivative is zero. *We have a photograph, when what we need is a video*.

Let's look back at that drawing. The maximum and minimum happen where the derivative crosses the x-axis. But the indecisive point happens where the derivative *bounces* off the axis. So for a function to have an extremum, the derivative has to *cross through* the axis. It has to go from being positive to being negative, or from being negative to being positive. The derivative has to have not just a root, but a root of odd multiplicity, to use the language we learned last fall. If it bounces off the axis—if the derivative has a root of even multiplicity, to use our fancy throwback word—then we won't have an extremum. We'll have an indecisive point.

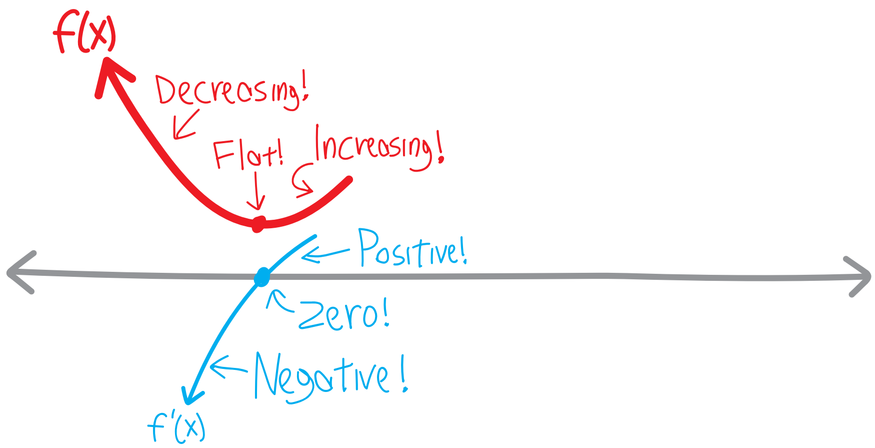
So, there's some clarity.

But how do we distinguish between a minimum and a maximum? It's not just that the derivative crosses the axis. If we look closely at that drawing, we can see that the minimum happens when the function goes from decreasing to flat to increasing, i.e. when its derivative goes from negative to zero to positive. That means that the derivative itself has a positive slope! But the maximum, by contrast, happens when the function goes from increasing to flat to decreasing, i.e. when its derivative goes from positive to zero to negative, which is like saying that the derivative itself has a negative slope.

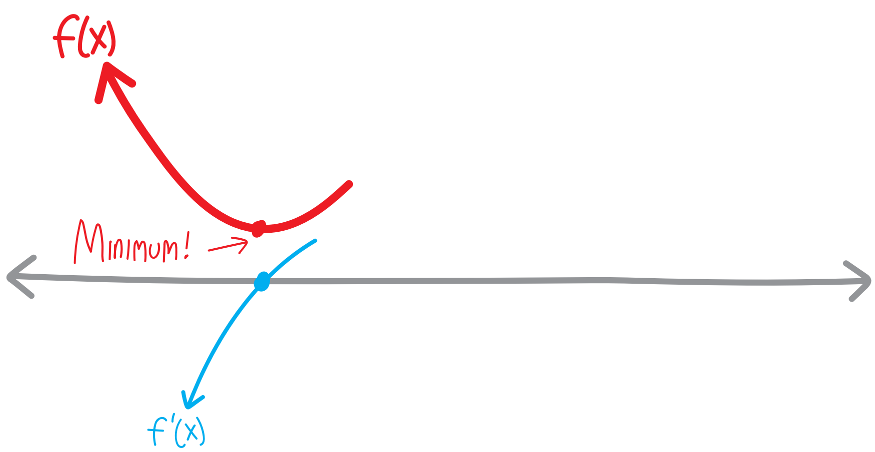
Let's argue this in more detail. Visually, if the function goes from decreasing to flat to increasing:



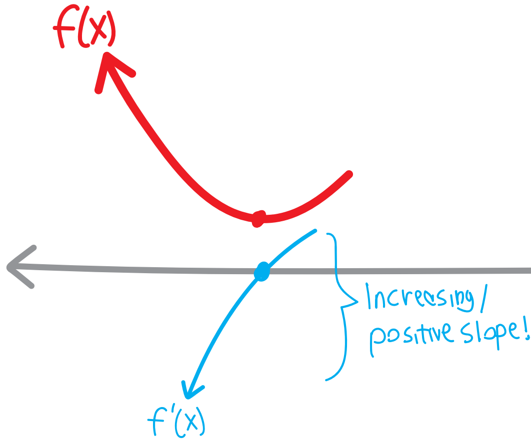
I.e., if the derivative goes from negative to zero to positive:



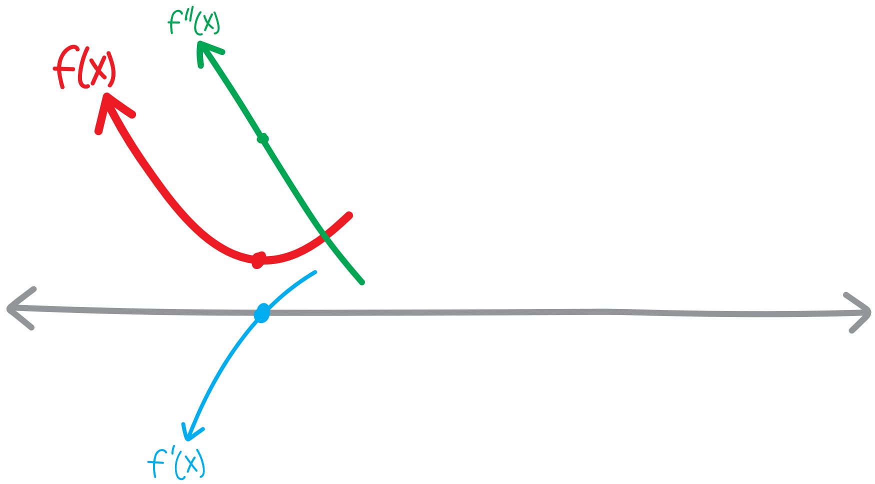
Then that point is a minimum!!!



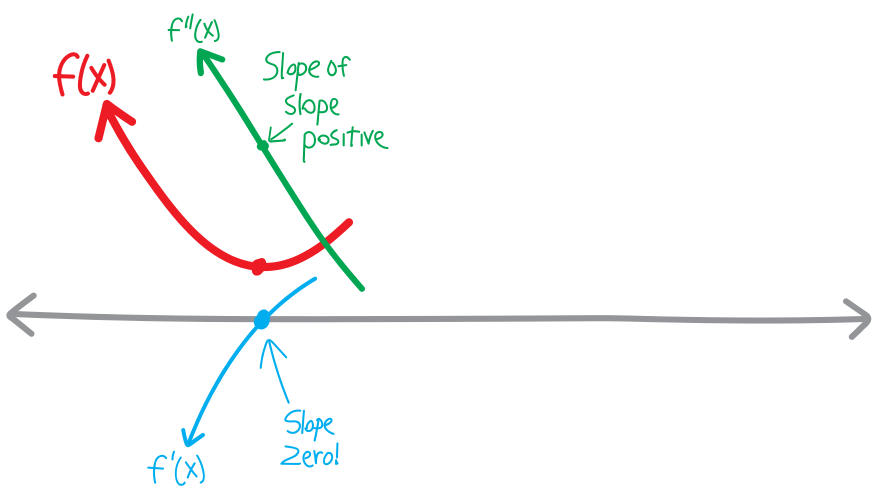
But hold up a second. This minimum happens when the function goes from decreasing to flat to increasing, i.e. when its derivative goes from negative to zero to positive… but if the derivative goes from negative to zero to positive, isn't that the same as saying that the derivative *itself* has a positive slope??? Like, the derivative is increasing?



But then we're talking about the derivative of the derivative… so really, we're just saying that the *second derivative* is positive!!! Let's sketch the second derivative:



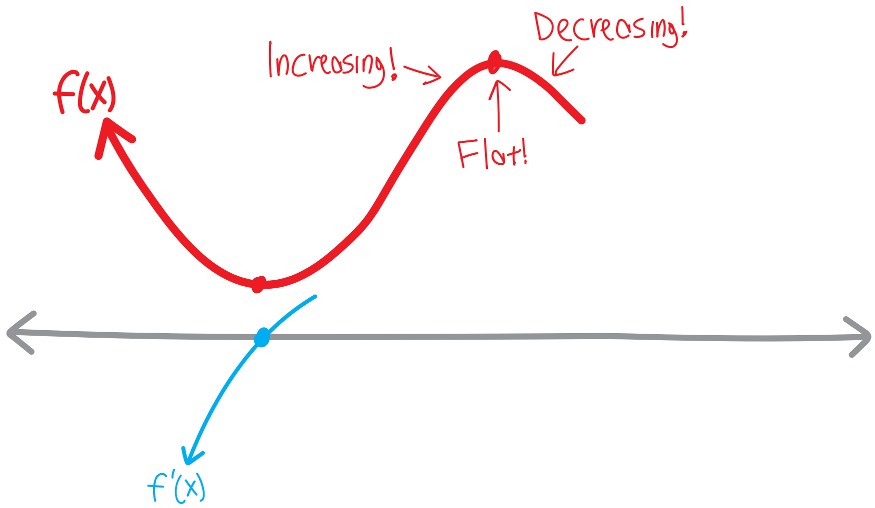
So at this point that's a minimum, the first derivative is zero, and the second derivative is positive!!! (It's the *value* of the second derivative, note, that's positive, not its own slope. Thought of as a function in its own right, the second derivative here has negative slope, but here we're interested in the slope of the slope, i.e. the *value* of the second derivative.)



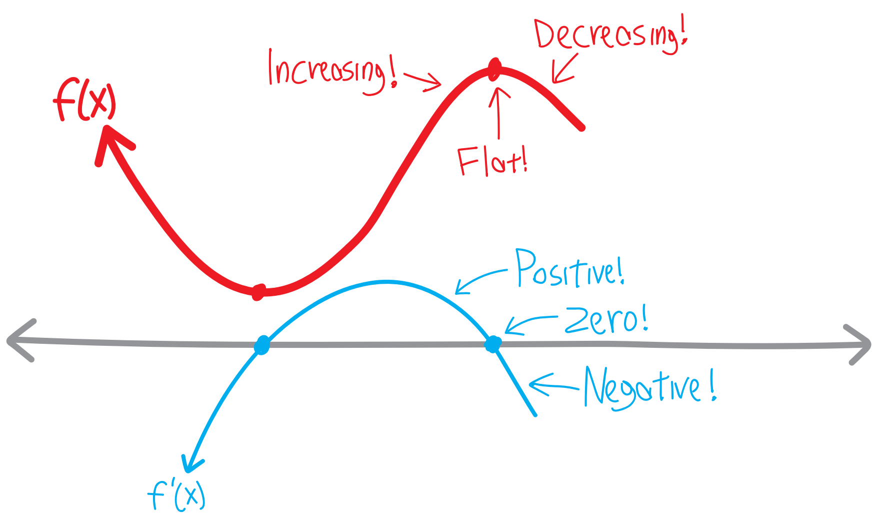
So in other words, if we have a point where the derivative is zero, AND the second derivative is positive, then that point must be a minimum!!!

$$\boxed{\begin{array}{c}\text{first derivative is zero}\\ \text{AND}\\ \text{second derivative is positive}\end{array} \implies \text{minimum}}$$

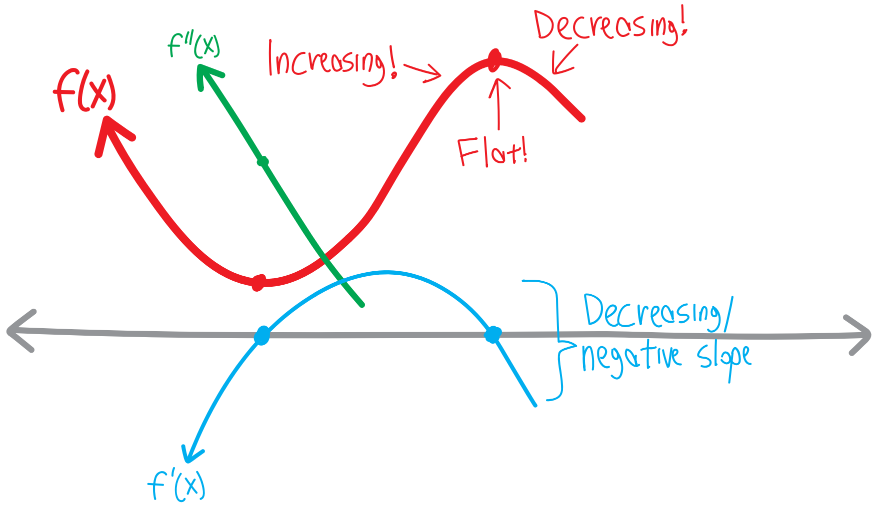
Meanwhile… what's going on with our maximum? At the maximum, the function is going from increasing to flat to decreasing:



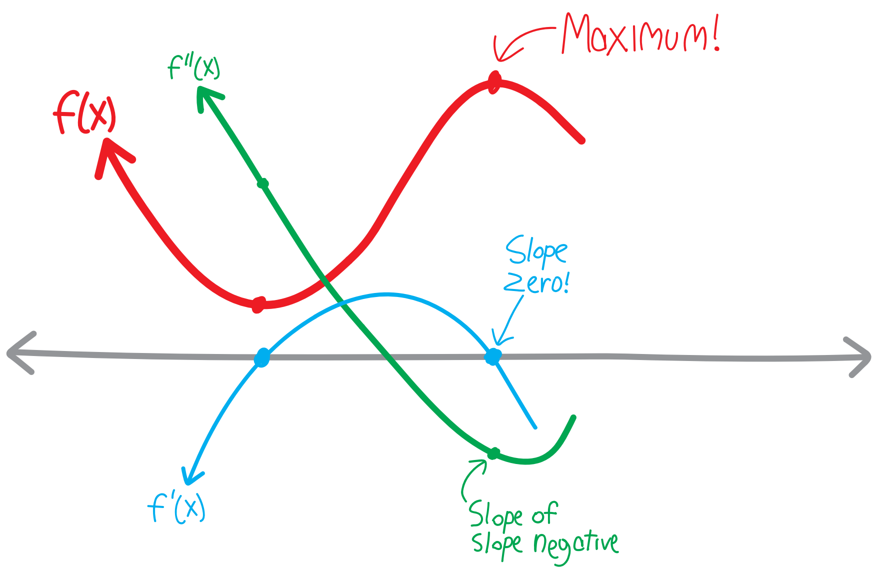
I.e., its derivative/slope goes from positive to zero to negative:



So, in other words, the derivative itself has negative slope:



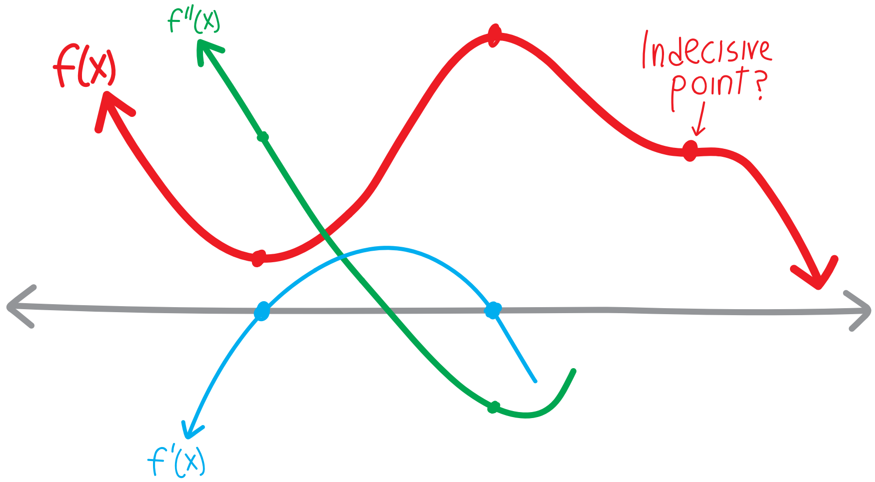
And that's the same as saying that the second derivative (the *value* of the second derivative) is negative at that point!!!



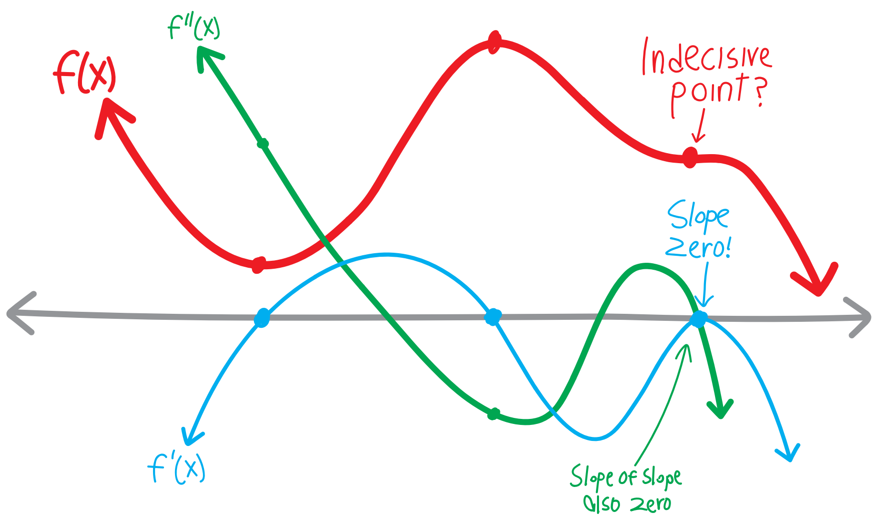
So then if we have a point where the derivative is zero, and the second derivative is negative, then we know we have a maximum!!!

$$\boxed{\begin{array}{c}\text{first derivative is zero}\\ \text{AND}\\ \text{second derivative is negative}\end{array} \implies \text{maximum}}$$

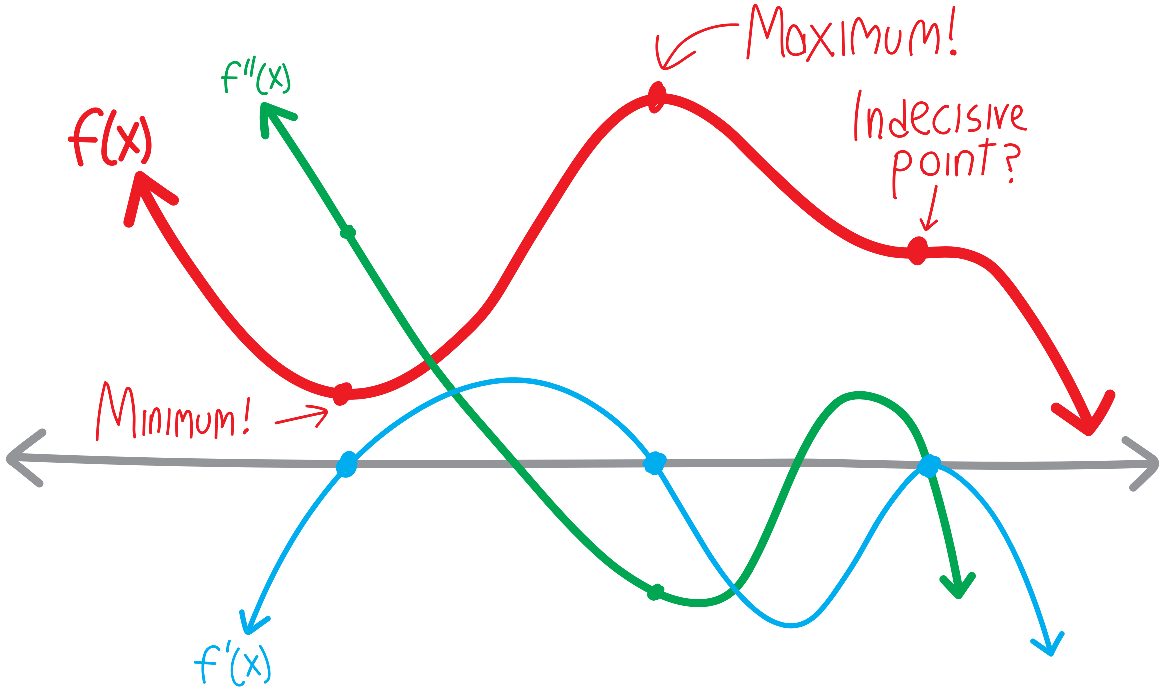
What about this weird ambiguous cubicky indecisive point?



Well… what's making it an indecisive point is the fact that, although the derivative has a root/zero/x-intercept, it doesn't actually cross the x-axis. It *touches*, but doesn't *cross*. It *bounces*. And so the derivative itself has an extremum there. So its own slope—the slope of the slope—is zero there.



Here's the full final diagram:



Textbooks call this stuff the **second derivative test**, but as always, that's not a word that's important, nor is it an algorithm you should try to memorize. Rather, we want to *feel* how these functions work. We want to understand functions and their curviness so deeply and thoroughly that it feels *impossible* for any of this to be otherwise. We want it to feel *impossible* that we never once understood all this. That's a big goal. And it takes a while, and some work.

more stuff

If it crosses the x-axis with a positive slope, then it's a min

if it crosses the x-axis with a negative slope, then it's a max

2nd deriv test

same buildup

In standard calculus textbooks, this gets called the **second derivative test**. Of course, if you actually understand what's going on, on a deep level, you probably don't need to call it anything!

but sometimes that doesn't work

x^8

prove second derivative test

so the derivative is zero

but it does change sides

the derivative

optimization, more broadly

so if the second derivartive is zero, do we have an inflection point?

let's consider x^4

we know it's got a min

but its second derivative is zero