Target parameter

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1 Setting/notation

We consider data as follows: $W \in \mathbb{R}^d$ are covariates, $A \in \{0, 1\}$ is an indicator of reporting chest pains when making the emergency call, $Z \in \{0, 1\}$ is an indicator of whether an ambulance was sent for the person, and $Y \in \{0, 1\}$ is an outcome indicator (30-day survival).

2 Counterfactuals

Let Z^1 denote the counterfactual mediator variable (indicator of ambulance) had their exposure been $A = 1$ (i.e., had the individual reported chest pains when making the emergency call). Equivalently, let Z^0 denote the counterfactual mediator variable (indicator of ambulance) had their exposure been $A = 0$ (i.e., had the individual not reported chest pains when making the emergency call).

Furthermore, let Y^{Z_1} be the counterfactual outcome observed had the mediator been distributed as if the individual reported chest pains when making the emergency call ($A = 1$), and let Y^{Z_0} be the counterfactual outcome observed had the mediator been distributed as if the individual had not reported chest pains when making the emergency call ($A = 0$).

3 Target parameter

Our question is if there is a difference between, on the one hand, the chance of 30-day survival among the individuals not reporting chest pains if the ambulance had been sent for them with the same probability as it was sent for the individuals reporting chest pains; and, on the other hand, the chance of 30-day survival if the ambulance had been sent for them with the probability as observed. These risks correspond to the two parameters $\Psi_1(P) = \mathbb{E}[Y^{Z_1} \mid A = 0]$ and $\Psi_0(P) = \mathbb{E}[Y^{Z_0} \mid A = 0] = \mathbb{E}[Y \mid A = 0]$, and our target parameter is the difference:

$$\Psi(P) = \Psi_1(P) - \Psi_0(P) = \mathbb{E}[Y^{Z_1} - Y^{Z_0} \mid A = 0].$$

4 Identification

We rewrite the target parameter as follows:

$$\begin{aligned}
\Psi(P) &= \mathbb{E}[Y^{Z_1} - Y^{Z_0} \mid A = 0] \\
&= \mathbb{E}\bigg[\sum_{z=0,1} Y^z \big(P(Z^1 = z \mid W) - P(Z^0 = z \mid W)\big) \,\bigg|\, A = 0 \bigg] \\
&= \mathbb{E}\bigg[\mathbb{E}\bigg[\sum_{z=0,1} Y^z \big(P(Z^1 = z \mid W) - P(Z^0 = z \mid W)\big) \,\bigg|\, A = 0, W \bigg] \,\bigg|\, A = 0 \bigg] \\
&= \mathbb{E}\bigg[\sum_{z=0,1} \mathbb{E}\big[Y^z \,\big|\, A = 0, W\big] \big(P(Z^1 = z \mid W) - P(Z^0 = z \mid W)\big) \,\bigg|\, A = 0 \bigg] \\
&\overset{(1)}{=} \mathbb{E}\bigg[\sum_{z=0,1} \mathbb{E}\big[Y \,\big|\, A = 0, Z = z, W\big] \big(P(Z^1 = z \mid W) - P(Z^0 = z \mid W)\big) \,\bigg|\, A = 0 \bigg] \\
&\overset{(2)}{=} \mathbb{E}\bigg[\sum_{z=0,1} \mathbb{E}\big[Y \,\big|\, A = 0, Z = z, W\big] \big(P(Z = z \mid A = 1, W) - P(Z = z \mid A = 0, W)\big) \,\bigg|\, A = 0 \bigg].
\end{aligned}$$

At (1) above we use that $Y^z \perp\!\!\!\perp Z \mid A, W$, and at (2) we use that $Z^a \perp\!\!\!\perp A \mid W$.

5 Estimation

Assume we are given $n \in \mathbb{N}$ iid observations of $O \sim P_0$ where $O = (W, A, Z, Y)$. For $a = 0, 1$ and $z = 0, 1$, let $Q(a, z, W) = \mathbb{E}[Y \mid A = a, Z = z, W]$ denote the conditional outcome regression. Furthermore, let $\gamma(z \mid a, W) = P(Z = z \mid A = a, W)$ denote the conditional distribution of the mediator Z. Finally, let $\pi(a \mid W) = P(A = a \mid W)$ denote the conditional exposure distribution and let μ be the distribution of covariates W.

The statistical target parameter is, c.f., Section 4,

$$\begin{aligned}
\Psi(P) &= \mathbb{E}\bigg[\sum_{z=0,1} \mathbb{E}\big[Y \,\big|\, A = 0, Z = z, W\big] \big(P(Z = z \mid A = 1, W) - P(Z = z \mid A = 0, W)\big) \,\bigg|\, A = 0 \bigg] \\
&= \int_{\mathcal{W}} \sum_{a=0,1} \sum_{z=0,1} \int_{\mathcal{Y}} y \, dP_Y(y \mid a, 0, w) \big(\gamma(z \mid 1, w) - \gamma(z \mid 0, w)\big) \frac{\mathbb{1}\{a = 0\}}{P(A = 0)} \pi(a \mid w) d\mu(w) \qquad . \\
&= \int_{\mathcal{W}} \sum_{z=0,1} Q(0, z, w) \big(\gamma(z \mid 1, w) - \gamma(z \mid 0, w)\big) \frac{\pi(0 \mid w)}{P(A = 0)} d\mu(w)
\end{aligned} \tag{1}$$

This tells us that we can define an estimator for $\psi_0 = \Psi(P_0)$ as follows:

$$\hat{\psi}_n = \frac{1}{\sum_{i=1}^n \mathbb{1}\{A_i = 0\}} \bigg(\sum_{i=1}^n \mathbb{1}\{A_i = 0\} \sum_{z=0,1} \Big(\hat{Q}_n(0, z, W_i) \big(\hat{\gamma}_n(z \mid 1, W_i) - \hat{\gamma}_n(z \mid 0, W_i)\big) \Big) \bigg),$$

substituting an estimator $\hat{\gamma}_n$ for γ and an estimator $\hat{Q}_n$ for Q.

6 Efficient influence function

Define $\bar{\pi} = P(A = 0)$. The efficient influence function for $\Psi_1(P)$ is given as follows:

$$\begin{aligned}\phi_1(P)(O) = & \frac{\gamma(Z \mid 1, W)}{\gamma(Z \mid 0, W)} \frac{\mathbb{1}\{A = 0\}}{\bar{\pi}} \Big(Y - Q(A, Z, W)\Big) \\ & + \frac{\mathbb{1}\{A = 1\}}{\pi(1 \mid W)} \frac{\pi(0 \mid W)}{\bar{\pi}} \Big(Q(0, Z, W) - \sum_{z=0,1} Q(0, z, W)\gamma(z \mid A, X)\Big) \\ & + \frac{\mathbb{1}\{A = 0\}}{\bar{\pi}} \Bigg(\sum_{z=0,1} Q(0, z, W)\gamma(z \mid 1, W) - \Psi_1(P)\Bigg),\end{aligned}$$

and for $\Psi_2(P)$ as follows:

$$\begin{aligned}\phi_2(P)(O) = & \frac{\mathbb{1}\{A = 0\}}{\bar{\pi}} \Big(Y - Q(A, Z, W)\Big) \\ & + \frac{\mathbb{1}\{A = 0\}}{\bar{\pi}} \Big(Q(0, Z, W) - \sum_{z=0,1} Q(0, z, W)\gamma(z \mid 0, W)\Big) \\ & + \frac{\mathbb{1}\{A = 0\}}{\bar{\pi}} \Bigg(\sum_{z=0,1} Q(0, z, W)\gamma(z \mid 0, W) - \Psi_2(P)\Bigg).\end{aligned}$$

The efficient influence function for $\Psi(P) = \Psi_1(P) - \Psi_2(P)$ is

$$\phi(P)(O) = \phi_1(P)(O) - \phi_2(P)(O).$$

The expressions above are derived in Section 7.

7 Deriving the efficient influence function

To derive the efficient influence function, we consider the two parameters $\Psi_0(P)$ and $\Psi_1(P)$ separately. Thus, for each $a^* = 0, 1$, we split up the functional derivative of Ψ_{a^*} as follows:

$$\frac{d}{d\varepsilon}\Psi_{a^*}(P_\varepsilon) = \frac{d}{d\varepsilon}\int_{\mathcal{W}}\sum_{z=0,1}\int_{\mathcal{Y}} y\, dP_{Y,\varepsilon}(y \mid z, a, w)\gamma_\varepsilon(z \mid a^*, w)\frac{\pi_\varepsilon(0 \mid w)}{P_\varepsilon(A=0)}d\mu_\varepsilon(w)$$

$$= \frac{d}{d\varepsilon}\int_{\mathcal{W}}\sum_{a=0,1}\sum_{z=0,1}\int_{\mathcal{Y}} y\, dP_{Y,\varepsilon}(y \mid z, a, w)\frac{\gamma(z \mid a^*, w)}{\gamma(z \mid a, w)}\frac{\mathbb{1}\{a=0\}}{P(A=0)}\gamma(z \mid a, w)\pi(a \mid w)d\mu(w) \tag{2}$$

$$+ \frac{d}{d\varepsilon}\int_{\mathcal{W}}\sum_{a=0,1}\sum_{z=0,1}\int_{\mathcal{Y}} y\, dP_Y(y \mid z, 0, w)\gamma_\varepsilon(z \mid a, w)\frac{\mathbb{1}\{a=a^*\}}{\pi(a \mid w)}\pi(a \mid w)\frac{\pi(0 \mid w)}{P(A=0)}d\mu(w) \tag{3}$$

$$+ \frac{d}{d\varepsilon}\int_{\mathcal{W}}\sum_{a=0,1}\sum_{z=0,1}\int_{\mathcal{Y}} y\, dP_Y(y \mid z, 0, w)\gamma(z \mid a^*, w)\frac{\pi_\varepsilon(a \mid w)}{P(A=0)}\mathbb{1}\{a=0\}d\mu(w) \tag{4}$$

$$+ \frac{d}{d\varepsilon}\int_{\mathcal{W}}\sum_{a=0,1}\sum_{z=0,1}\int_{\mathcal{Y}} y\, dP_Y(y \mid z, 0, w)\gamma(z \mid a^*, w)\frac{\pi(0 \mid w)}{P_\varepsilon(A=a)}\mathbb{1}\{a=0\}d\mu(w) \tag{5}$$

$$+ \frac{d}{d\varepsilon}\int_{\mathcal{W}}\sum_{z=0,1}\int_{\mathcal{Y}} y\, dP_Y(y \mid z, 0, w)\gamma(z \mid a^*, w)\frac{\pi(0 \mid w)}{P(A=0)}d\mu_\varepsilon(w). \tag{6}$$

Now we can consider each of the terms (2)–(6) one by one. First, look at (2):

$$\begin{aligned}
&\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\int_{\mathcal{W}}\sum_{z=0,1}\int_{\mathcal{Y}} y\, dP_{Y,\varepsilon}(y \mid z, a, w)\frac{\gamma(z \mid a^*, w)}{\gamma(z \mid a, w)}\frac{\mathbb{1}\{a=0\}}{P(A=0)}\gamma(z \mid a, w)\pi(a \mid w)d\mu(w) \\
&= \int_{\mathcal{W}}\sum_{z=0,1}\int_{\mathcal{Y}} y\Big(\mathbb{E}\big[S(O) \mid y, z, a, w\big] - \mathbb{E}\big[S(O) \mid z, a, w\big]\Big)dP_Y(y \mid z, a, w) \\
&\qquad\qquad \frac{\gamma(z \mid a^*, w)}{\gamma(z \mid a, w)}\frac{\mathbb{1}\{a=0\}}{P(A=0)}\gamma(z \mid a, w)\pi(a \mid w)d\mu(w) \\
&= \mathbb{E}\bigg[S(O)Y\frac{\gamma(Z \mid a^*, W)}{\gamma(Z \mid A, W)}\frac{\mathbb{1}\{A=0\}}{P(A=0)}\bigg] - \mathbb{E}\bigg[S(O)Q(Z, A, W)\frac{\gamma(Z \mid a^*, W)}{\gamma(Z \mid A, W)}\frac{\mathbb{1}\{A=0\}}{P(A=0)}\bigg],
\end{aligned}$$

and we see that

$$\phi^*_{Y,a^*}(P)(O) = \frac{\gamma(Z \mid a^*, W)}{\gamma(Z \mid A, W)}\frac{\mathbb{1}\{A=0\}}{P(A=0)}\Big(Y - Q(Z, A, W)\Big) = \begin{cases} \frac{\mathbb{1}\{A=0\}}{P(A=0)}\Big(Y - Q(Z, A, W)\Big) & a^* = 0, \\ \frac{\gamma(Z|1,W)}{\gamma(Z|0,W)}\frac{\mathbb{1}\{A=0\}}{P(A=0)}\Big(Y - Q(Z, A, W)\Big) & a^* = 1. \end{cases}$$

Second, consider (3):

$$\begin{aligned}
&\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\int_{\mathcal{W}}\sum_{a=0,1}\sum_{z=0,1}\int_{\mathcal{Y}} y\, dP_Y(y \mid z, 0, w)\gamma_\varepsilon(z \mid a, w)\frac{\mathbb{1}\{a=a^*\}}{\pi(a \mid w)}\frac{\pi(0 \mid w)}{P(A=0)}\pi(a \mid w)d\mu(w) \\
&= \int_{\mathcal{W}}\sum_{a=0,1}\sum_{z=0,1}\int_{\mathcal{Y}} y\, dP_Y(y \mid z, 0, w)\Big(\mathbb{E}\big[S(O) \mid z, a, w\big] - \mathbb{E}\big[S(O) \mid a, w\big]\Big)\gamma(z \mid a, w) \\
&\qquad\qquad \frac{\mathbb{1}\{a=a^*\}}{\pi(a \mid w)}\frac{\pi(0 \mid w)}{P(A=0)}\pi(a \mid w)d\mu(w) \\
&= \mathbb{E}\bigg[S(O)Q(Z, 0W)\frac{\mathbb{1}\{A=a^*\}}{\pi(A \mid W)}\frac{\pi(0 \mid W)}{P(A=0)}\bigg] - \mathbb{E}\bigg[S(O)\frac{\mathbb{1}\{A=a^*\}}{\pi(A \mid W)}\frac{\pi(0 \mid W)}{P(A=0)}\sum_{z=0,1}Q(z, 0, W)\gamma(z \mid A, W)\bigg],
\end{aligned}$$

from which we derive that

$$
\begin{aligned}
\phi^*_{\gamma,a^*}(P)(O) &= \frac{\mathbb{1}\{A=a^*\}}{\pi(A\mid W)}\frac{\pi(0\mid W)}{P(A=0)}\Big(Q(Z,0,W)-\sum_{z=0,1}Q(z,0,W)\gamma(z\mid A,W)\Big)\\
&= \begin{cases} \frac{\mathbb{1}\{A=0\}}{P(A=0)}\Big(Q(Z,0,W)-\sum_{z=0,1}Q(z,0,W)\gamma(z\mid A,W)\Big), & a^*=0,\\ \frac{\mathbb{1}\{A=1\}}{\pi(1|W)}\frac{\pi(0|W)}{P(A=0)}\Big(Q(Z,0,W)-\sum_{z=0,1}Q(z,0,W)\gamma(z\mid A,W)\Big), & a^*=1.\end{cases}
\end{aligned}
$$

Third, consider (4):

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{\mathcal{W}}\sum_{a=0,1}\sum_{z=0,1}\int_{\mathcal{Y}} y\,dP_Y(y\mid z,0,w)\gamma(z\mid a^*,w)\frac{\mathbb{1}\{a=0\}}{P(A=0)}\pi_\varepsilon(a\mid w)d\mu(w)\\
&=\int_{\mathcal{W}}\sum_{a=0,1}\sum_{z=0,1}\int_{\mathcal{Y}} y\,dP_Y(y\mid z,0,w)\gamma(z\mid a^*,w)\frac{\mathbb{1}\{a=0\}}{P(A=0)}\Big(\mathbb{E}\big[S(O)\mid a,w\big]-\mathbb{E}\big[S(O)\mid w\big]\Big)\pi(a\mid w)d\mu(w)\\
&=\mathbb{E}\Big[S(O)\frac{\mathbb{1}\{A=0\}}{P(A=0)}\sum_{z=0,1}Q(z,0,W)\gamma(z\mid a^*,W)\Big]-\mathbb{E}\Big[S(O)\frac{\pi(0\mid W)}{P(A=0)}\sum_{z=0,1}Q(z,0,W)\gamma(z\mid a^*,W)\Big],
\end{aligned}
$$

from which we see that

$$
\phi^*_{\pi,a^*}(P)(O)=\Big(\frac{\mathbb{1}\{A=0\}}{P(A=0)}-\frac{\pi(0\mid W)}{P(A=0)}\Big)\sum_{z=0,1}Q(z,0,W)\gamma(z\mid a^*,W),\qquad a^*=0,1.
$$

Next, consider (5):

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\int_{\mathcal{W}}\sum_{a=0,1}\sum_{z=0,1}\int_{\mathcal{Y}} y\,dP_Y(y\mid z,0,w)\gamma(z\mid a^*,w)\frac{\mathbb{1}\{a=0\}}{P_\varepsilon(A=a)}\pi(a\mid w)d\mu(w)\\
&=-\int_{\mathcal{W}}\sum_{a=0,1}\sum_{z=0,1}\int_{\mathcal{Y}} y\,dP_Y(y\mid z,0,w)\gamma(z\mid a^*,w)\mathbb{1}\{a=0\}\Big(\frac{\mathbb{E}[S(O)\mid a]-\mathbb{E}[S(O)]}{P(A=a)}\Big)\pi(a\mid w)d\mu(w)\\
&=-\Big(\mathbb{E}\Big[\mathbb{E}\big[S(O)\mid A\big]\frac{\mathbb{1}\{A=0\}}{P(A=0)}\mathbb{E}\Big[\sum_{z=0,1}Q(z,0,W)\gamma(z\mid a^*,W)\,\Big|\,A\Big]\Big]-\mathbb{E}\big[S(O)\Psi_{a^*}(P)\big]\Big)\\
&=-\Big(\mathbb{E}\Big[\mathbb{E}\big[S(O)\mid A\big]\frac{\mathbb{1}\{A=0\}}{P(A=0)}\mathbb{E}\Big[\sum_{z=0,1}Q(z,0,W)\gamma(z\mid a^*,W)\,\Big|\,A\Big]\Big]-\mathbb{E}\big[S(O)\Psi_{a^*}(P)\big]\Big)\\
&=-\Big(\mathbb{E}\Big[\mathbb{E}\big[S(O)\mid A\big]\frac{\mathbb{1}\{A=0\}}{P(A=0)}\mathbb{E}\Big[\sum_{z=0,1}Q(z,0,W)\gamma(z\mid a^*,W)\,\Big|\,A=0\Big]\Big]-\mathbb{E}\big[S(O)\Psi_{a^*}(P)\big]\Big)\\
&=-\Big(\mathbb{E}\Big[S(O)\frac{\mathbb{1}\{A=0\}}{P(A=0)}\Psi_{a^*}(P)\Big]-\mathbb{E}\big[S(O)\Psi_{a^*}(P)\big]\Big),
\end{aligned}
$$

i.e.,

$$
\phi^*_{P_A,a^*}(P)(O)=-\Big(\frac{\mathbb{1}\{A=0\}}{P(A=0)}-1\Big)\Psi_{a^*}(P),\qquad a^*=0,1,
$$

and, lastly, consider (6):

$$\begin{aligned}
&\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\mathcal{W}} \sum_{a=0,1} \sum_{z=0,1} \int_{\mathcal{Y}} y \, dP_Y(y \mid z, 0, w) \gamma(z \mid a^*, w) \frac{\pi(0 \mid w)}{P(A=0)} \mathbb{1}\{a=0\} d\mu_\varepsilon(w) \\
&\quad = \int_{\mathcal{W}} \sum_{a=0,1} \sum_{z=0,1} \int_{\mathcal{Y}} y \, dP_Y(y \mid z, 0, w) \gamma(z \mid a^*, w) \frac{\mathbb{1}\{a=0\}}{P(A=0)} \Big(\mathbb{E}[S(O) \mid w] - \mathbb{E}[S(O)] \Big) \pi(a \mid w) d\mu(w) \\
&\quad = \mathbb{E}\bigg[S(O) \frac{\pi(0 \mid W)}{P(A=0)} \sum_{z=0,1} Q(z, 0, W) \gamma(z \mid a^*, W) \bigg] - \mathbb{E}\big[S(O) \Psi_{a^*}(P) \big],
\end{aligned}$$

i.e.,

$$\phi^*_{w,a^*}(P)(O) = \frac{\pi(0 \mid W)}{P(A=0)} \sum_{z=0,1} Q(z, 0, W) \gamma(z \mid a^*, W) - \Psi_{a^*}(P). \qquad a^* = 0, 1,$$

If we collect the last three expressions, i.e., for $a^* = 0, 1$,

$$\begin{aligned}
\phi^*_{\pi,a^*}(P)(O) &= \bigg(\frac{\mathbb{1}\{A=0\}}{P(A=0)} - \frac{\pi(0 \mid W)}{P(A=0)} \bigg) \sum_{z=0,1} Q(z, 0, W) \gamma(z \mid a^*, W) \\
\phi^*_{P_A,a^*}(P)(O) &= -\bigg(\frac{\mathbb{1}\{A=0\}}{P(A=0)} - 1 \bigg) \Psi_{a^*}(P) \\
\phi^*_{w,a^*}(P)(O) &= \frac{\pi(0 \mid W)}{P(A=0)} \sum_{z=0,1} Q(z, 0, W) \gamma(z \mid a^*, W) - \Psi_{a^*}(P),
\end{aligned}$$

we immediately we see that some terms cancel out:

$$\phi^*_{\pi,a^*}(P)(O) + \phi^*_{P_A,a^*}(P)(O) + \phi^*_{w,a^*}(P)(O) = \frac{\mathbb{1}\{A=0\}}{P(A=0)} \bigg(\sum_{z=0,1} Q(z, 0, W) \gamma(z \mid a^*, W) - \Psi_{a^*}(P) \bigg)$$

Thus, we conclude that the efficient influence function for the parameter $\Psi_{a^*}(P)$ is:

$$\begin{aligned}
\phi_{a^*}(P)(O) &= \phi^*_{Y,a^*}(P)(O) + \phi^*_{\gamma,a^*}(P)(O) + \phi^*_{\pi,a^*}(P)(O) + \phi^*_{w,a^*}(P)(O) \\
&= \frac{\gamma(Z \mid a^*, W)}{\gamma(Z \mid 0, W)} \frac{\mathbb{1}\{A=0\}}{P(A=0)} \Big(Y - Q(Z, A, W) \Big) \\
&\quad + \frac{\mathbb{1}\{A=a^*\}}{\pi(A \mid W)} \frac{\pi(0 \mid W)}{P(A=0)} \bigg(Q(Z, 0, W) - \sum_{z=0,1} Q(z, 0, W) \gamma(z \mid A, W) \bigg) \\
&\quad + \frac{\mathbb{1}\{A=0\}}{P(A=0)} \bigg(\sum_{z=0,1} Q(z, 0, W) \gamma(z \mid a^*, W) - \Psi_{a^*}(P) \bigg)
\end{aligned}$$

References