

# *Control Charts for $\bar{x}$ and S*

## *The Shewhart Control Chart for Individual Measurements*

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# Control Charts for $\bar{x}$ and $s$

***Why  $\bar{x}$  and  $s$  Charts..?***

- Directly estimate process standard deviation ( $s$ )

***Preferable When***

- Sample size ( $n$ ) is moderately large ( $n > 10-12$ ).
- Sample size ( $n$ ) is variable.

# Construction and Operation of $\bar{x}$ and $s$ Charts

Setting up and operating control charts for  $\bar{x}$  and  $s$  requires about the same steps.

- The sample standard deviation is defined as

$$s = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1}}$$

- If the underlying distribution is normal with mean  $\mu$  and variance  $\sigma^2$ ,  $E(s) = c_4\sigma$ .
- The standard deviation of  $s$  is given by,  $SD(s) = \sigma\sqrt{1 - c_4^2}$ .

- The three-sigma control limits for  $s$  are then,

$$\text{UCL} = c_4\sigma + 3\sigma\sqrt{1 - c_4^2}$$

$$\text{CL} = c_4\sigma$$

$$\text{LCL} = c_4\sigma - 3\sigma\sqrt{1 - c_4^2}$$

It is customary to define the two constants

$$B_5 = c_4 - 3\sqrt{1 - c_4^2}, \quad B_6 = c_4 + 3\sqrt{1 - c_4^2}.$$

Then the parameters of the  $s$  chart with a standard value for  $\sigma$  given become,

### *s* Chart Control Limits (when $\sigma$ is known)

$$\text{UCL} = B_6\sigma$$

$$\text{CL} = c_4\sigma$$

$$\text{LCL} = B_5\sigma$$

- If no standard is given for  $\sigma$  it is estimated using  $\frac{\bar{s}}{c_4}$ ,

$$\text{UCL} = \bar{s} + 3 \frac{\bar{s}}{c_4} \sqrt{1 - c_4^2}$$

$$\text{CL} = \bar{s}$$

$$\text{LCL} = \bar{s} - 3 \frac{\bar{s}}{c_4} \sqrt{1 - c_4^2}$$

We usually define the constants

$$B_3 = 1 - \frac{3}{c_4} \sqrt{1 - c_4^2}, \quad B_4 = 1 + \frac{3}{c_4} \sqrt{1 - c_4^2}.$$

Then the parameters of the  $s$  chart without a standard value for  $\sigma$  given become,

### $s$ Chart Control Limits (when $\sigma$ is known)

$$\text{UCL} = B_4 \bar{s}$$

$$\text{CL} = \bar{s}$$

$$\text{LCL} = B_3 \bar{s}$$

- When  $\frac{\bar{s}}{c_4}$  is used to estimate  $\sigma$ , we may define the control limits on the corresponding  $\bar{x}$  chart as,

$$\text{UCL} = \bar{\bar{x}} + \frac{3\bar{s}}{c_n \sqrt{n}}$$

Center Line =  $\bar{\bar{x}}$

$$\text{LCL} = \bar{\bar{x}} - \frac{3\bar{s}}{c_n \sqrt{n}}$$

Let the constant  $A_3 = 3/c_n \sqrt{n}$ .

Then the  $\bar{x}$  chart parameters become,

### $\bar{x}$ Chart Control Limits (based on $s$ charts)

$$\text{UCL} = \bar{\bar{x}} + A_3 \bar{s}$$

Center Line =  $\bar{\bar{x}}$

$$\text{LCL} = \bar{\bar{x}} - A_3 \bar{s}$$

# Example: $\bar{x}$ and $s$ Charts for the Piston Ring Data

- This example shows how to construct and interpret:
  - $\bar{x}$  control chart (for process mean)
  - $s$  control chart (for process variability)
- The data are obtained from inside diameter measurements of forged automobile engine piston rings.
- Each sample (subgroup) consists of five piston rings.
- **Goal:** To determine whether the process is in statistical control.

Table 1: Inside Diameter Measurements (mm) for Automobile Engine Piston Rings

Sample	Obs1	Obs2	Obs3	Obs4	Obs5	$\bar{x}$	$s_i$
1	74.030	74.002	74.019	73.992	74.008	74.010	0.0148
2	73.995	73.992	74.001	74.011	74.004	74.001	0.0075
3	73.988	74.024	74.021	74.005	74.002	74.008	0.0147
4	74.002	73.996	73.993	74.015	74.009	74.003	0.0091
5	73.992	74.007	74.015	73.989	74.014	74.003	0.0122
6	74.009	73.994	73.997	73.985	73.993	73.996	0.0087
7	73.995	74.006	73.994	74.000	74.005	74.000	0.0055
8	73.985	74.003	73.993	74.015	73.988	73.997	0.0123
9	74.008	73.995	74.009	74.005	74.004	74.004	0.0055
10	73.998	74.000	73.990	74.007	73.995	73.998	0.0063
11	73.994	73.998	73.994	73.995	73.990	73.994	0.0029
12	74.004	74.000	74.007	74.000	73.996	74.001	0.0042
13	73.983	74.002	73.998	73.997	74.012	73.998	0.0105
14	74.006	73.967	73.994	74.000	73.984	73.990	0.0153
15	74.012	74.014	73.998	73.999	74.007	74.006	0.0073
16	74.000	73.984	74.005	73.998	73.996	73.997	0.0078
17	73.994	74.012	73.986	74.005	74.007	74.001	0.0106
18	74.006	74.010	74.018	74.003	74.000	74.007	0.0070
19	73.984	74.002	74.003	74.005	73.997	73.998	0.0085
20	74.000	74.010	74.013	74.020	74.003	74.009	0.0080
21	73.982	74.001	74.015	74.005	73.996	74.000	0.0122
22	74.004	73.999	73.990	74.006	74.009	74.002	0.0074
23	74.010	73.989	73.990	74.009	74.014	74.002	0.0119
24	74.015	74.008	73.993	74.000	74.010	74.005	0.0087
25	73.982	73.984	73.995	74.017	74.013	73.998	0.0162

$\bar{x} = 74.001 \quad \bar{s} = 0.009$



## Phase – Control Limits

- The grand average and the average standard deviation are:

$$\bar{\bar{x}} = \frac{1}{25} \sum_{i=1}^{25} \bar{x}_i = \frac{1}{25}(1850.028) = 74.001$$

$$\bar{s} = \frac{1}{25} \sum_{i=1}^{25} s_i = \frac{1}{25}(0.2351) = 0.0094$$

- Consequently, the parameters are:

$\bar{x}$  Chart:

$$\text{UCL} = \bar{\bar{x}} + A_3 \bar{x} = 74.014$$

$$\text{CL} = \bar{x} = 74.001$$

$$\text{LCL} = \bar{\bar{x}} - A_3 \bar{s} = 73.988$$

$s$  Chart:

$$\text{UCL} = B_4 \bar{s} = 0.0196$$

$$\text{CL} = \bar{s} = 0.0094$$

$$\text{LCL} = B_3 \bar{s} = 0$$

# Control Charts

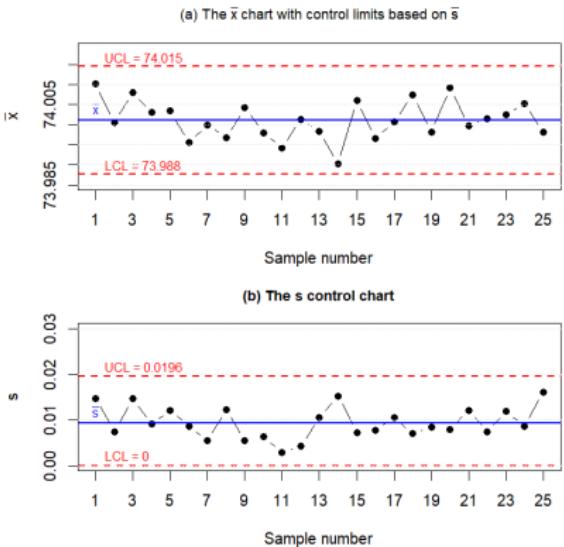


Figure 1:  $\bar{x}$ & $s$  charts for ring data

- There is no indication that the process is out of control, so those limits could be adopted to phase monitoring of the process

## Estimation of Process Standard Deviation

- $s/c_4$  is an unbiased estimator of the process standard deviation  $\sigma$ .
- For sample size  $n = 5$ ,  $c_4 = 0.9400$ .
- Therefore, our estimate of the process standard deviation is:

$$\hat{\sigma} = \frac{\bar{s}}{c_4} = \frac{0.0094}{0.9400} = 0.01$$

# The $\bar{x}$ and $s$ Control Charts with Variable Sample Size

## Weighted Mean for $\bar{x}$ Chart

Center Line of  $\bar{x}$  Chart

$$\bar{\bar{x}} = \frac{\sum_{i=1}^m n_i \bar{x}_i}{\sum_{i=1}^m n_i}$$

### Definitions

- $n_i$  = Sample size of  $i$ th subgroup
- $\bar{x}_i$  = Mean of  $i$ th subgroup

## Weighted Standard Deviation for $s$ Chart

Center Line of  $s$  Chart

$$\bar{s} = \left[ \frac{\sum_{i=1}^m (n_i - 1) s_i^2}{\sum_{i=1}^m n_i - m} \right]^{1/2}$$

### Definitions

- $n_i$  = Sample size of  $i$ th subgroup
- $s_i$  = Standard deviation of  $i$ th subgroup

## Control Limits for $\bar{x}$ Chart

$$\text{UCL} = \bar{\bar{x}} + A_3 \bar{s}$$

$$\text{Center Line} = \bar{\bar{x}}$$

$$\text{LCL} = \bar{\bar{x}} - A_3 \bar{s}$$

## Control Limits for $s$ Chart

$$\text{UCL} = B_4 \bar{s}$$

$$\text{CL} = \bar{s}$$

$$\text{LCL} = B_3 \bar{s}$$

$$B_3 = 1 - \frac{3}{c_4} \sqrt{1 - c_4^2}, \quad B_4 = 1 + \frac{3}{c_4} \sqrt{1 - c_4^2}$$

- These constants ( $A_3, B_3, B_4$ ) change for each samples.
- We can use average sample size to calculate CL.
- If average sample size value is a decimal number, we can use most common sample size to calculations.

# Example: $\bar{x}$ and $s$ Charts for the Piston Rings, Variable Sample Size

**Table 2:** Inside Diameter Measurements (mm) on Automobile Engine Piston Rings

Sample	Obs1	Obs2	Obs3	Obs4	Obs5	$\bar{x}_i$	$s_i$
1	74.030	74.002	74.019	73.992	74.008	74.010	0.0148
2	73.995	73.992	74.001	73.996		74.001	0.0046
3	73.988	74.024	74.021	74.005	74.002	74.008	0.0147
4	74.002	73.996	73.993	74.015	74.009	74.003	0.0091
5	73.992	74.007	74.015	73.989	74.014	74.003	0.0122
6	74.009	73.994	73.997	73.985		73.996	0.0099
7	73.995	74.006	73.994	74.000		73.999	0.0055
8	73.985	74.003	73.993	74.015	73.988	73.997	0.0123
9	74.008	73.995	74.009	74.005		74.004	0.0064
10	73.998	74.000	73.990	74.007	73.995	73.998	0.0063
11	73.994	73.998	73.994	73.995	73.990	73.994	0.0029
12	74.004	74.000	74.007	74.000	73.996	74.001	0.0042
13	73.983	74.002	73.998			73.994	0.0100
14	74.006	73.967	73.994	74.000	73.984	73.990	0.0153
15	74.012	74.014	73.998			74.008	0.0087
16	74.000	73.984	74.005	73.998	73.996	73.997	0.0078
17	73.994	74.012	73.986	74.005		73.999	0.0115
18	74.006	74.010	74.018	74.003	74.000	74.007	0.0070
19	73.984	74.002	74.003	74.005	73.997	73.998	0.0085
20	74.000	74.010	74.013			74.008	0.0068
21	73.982	74.001	74.015	74.005	73.996	74.000	0.0122
22	74.004	73.999	73.990	74.006	74.009	74.002	0.0074
23	74.010	73.989	73.990	74.009	74.014	74.002	0.0119
24	74.015	74.008	73.993	74.000	74.010	74.005	0.0087
25	73.982	73.984	73.995	74.017	74.013	73.998	0.0162

$$\begin{aligned}\bar{x} &= \frac{\sum_{i=1}^{25} n_i \bar{x}_i}{\sum_{i=1}^{25} n_i} = \frac{5(74.010) + 3(73.996) + \dots + 5(73.998)}{5 + 3 + \dots + 5} \\ &= \frac{8362.075}{113} = 74.001\end{aligned}$$

$$\begin{aligned}s &= \left[ \frac{\sum_{n=1}^{25} (n_i - 1)s_i^2}{\sum_{n=1}^{25} n_i - 25} \right]^{1/2} = \left[ \frac{4(0.0148)^2 + 2(0.0046)^2 + \dots + 4(0.0162)^2}{5 + 3 + \dots + 5 - 25} \right]^{1/2} \\ &= \left[ \frac{0.009324}{88} \right]^{1/2} = 0.0103\end{aligned}$$

Control limits for x-bar chart

$$\text{UCL} = 74.001 + 1.427 \times 0.0103 = 74.016$$

$$\text{CL} = 74.001$$

$$\text{LCL} = 74.001 - 1.427 \times 0.0103 = 73.986$$

Control limits for s chart

$$\text{UCL} = 2.089 \times 0.0103 = 0.022$$

$$\text{CL} = 0.0103$$

$$\text{LCL} = 0 \times 0.0103 = 0$$

# Control charts for piston-ring data with variable sample size

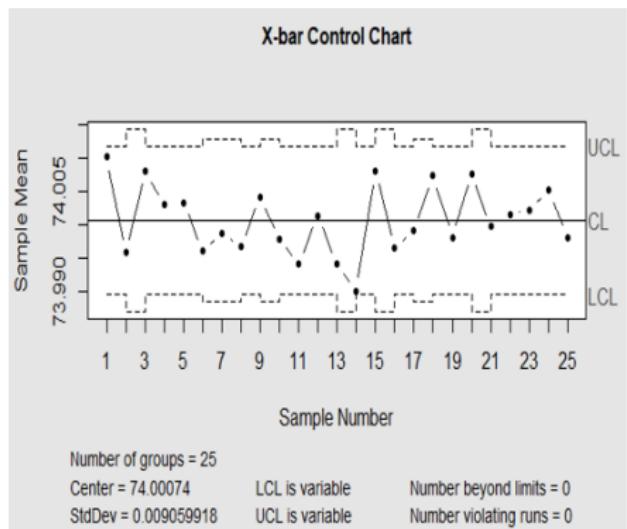


Figure 2:  $\bar{x}$  chart for variable sample sizes

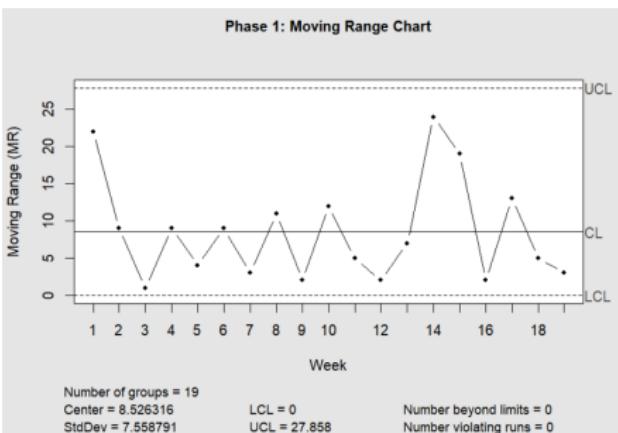


Figure 3: schart for variable sample sizes

**Table 3:** Inside Diameter Measurements (mm) on Automobile Engine Piston Rings

Sample	$n$	$\bar{x}$	$s$	$A_3$	$LCL_{\bar{x}}$	$UCL_{\bar{x}}$	$B_3$	$B_4$	$LCL_s$	$UCL_s$
1	5	74.010	0.0148	1.427	73.986	74.016	0	2.089	0	0.022
2	3	73.996	0.0046	1.954	73.981	74.021	0	2.568	0	0.026
3	5	74.008	0.0147	1.427	73.986	74.016	0	2.089	0	0.022
4	5	74.003	0.0091	1.427	73.986	74.016	0	2.089	0	0.022
5	5	74.003	0.0122	1.427	73.986	74.016	0	2.089	0	0.022
6	4	73.996	0.0099	1.628	73.984	74.018	0	2.266	0	0.023
7	4	73.999	0.0055	1.628	73.984	74.018	0	2.266	0	0.023
8	5	73.997	0.0123	1.427	73.986	74.016	0	2.089	0	0.022
9	4	74.004	0.0064	1.628	73.984	74.018	0	2.266	0	0.023
10	5	73.998	0.0063	1.427	73.986	74.016	0	2.089	0	0.022
11	5	73.994	0.0029	1.427	73.986	74.016	0	2.089	0	0.022
12	5	74.001	0.0042	1.427	73.986	74.016	0	2.089	0	0.022
13	3	73.994	0.0100	1.954	73.981	74.021	0	2.568	0	0.026
14	5	73.990	0.0153	1.427	73.986	74.016	0	2.089	0	0.022
15	3	74.008	0.0087	1.954	73.981	74.021	0	2.568	0	0.026
16	5	73.997	0.0078	1.427	73.986	74.016	0	2.089	0	0.022
17	4	73.999	0.0115	1.628	73.984	74.018	0	2.226	0	0.023
18	5	74.007	0.0070	1.427	73.986	74.016	0	2.089	0	0.022
19	5	73.998	0.0085	1.427	73.986	74.016	0	2.089	0	0.022
20	3	74.008	0.0068	1.954	73.981	74.021	0	2.568	0	0.026
21	5	74.000	0.0122	1.427	73.986	74.016	0	2.089	0	0.022
22	5	74.002	0.0074	1.427	73.986	74.016	0	2.089	0	0.022
23	5	74.002	0.0119	1.427	73.986	74.016	0	2.089	0	0.022
24	5	74.005	0.0087	1.427	73.986	74.016	0	2.089	0	0.022
25	5	73.998	0.0162	1.427	73.986	74.016	0	2.089	0	0.022

## Estimation of $\sigma$

- First, average all the values of  $s_i$  for which  $n_i=5$  (the most frequently occurring value of  $n_i$ ). This gives,

$$\bar{s} = \frac{\sum_{i=1}^N s_i}{N}$$

$$\bar{s} = \frac{0.1715}{17} = 0.0101$$

N – number of samples with same (model) sample size

- The estimate of the process  $\sigma$  is then

$$\hat{\sigma} = \frac{\bar{s}}{c_4} = \frac{0.0101}{0.9400} = 0.01$$

where the value of  $c_4$  (correction factor ) used is for samples of size  $n=5$ .

# The $s^2$ Control Chart

- Engineers use the R chart or the S chart to monitor process variability.
- The S chart is better than the R chart when the sample size is medium or large.
- Some people prefer a chart based directly on the sample variance, called the  $s^2$  control chart.

# Construction of $s^2$ Chart

- Suppose that a quality characteristic is normally distributed with mean  $\mu$  and standard deviation  $\sigma$ , where both  $\mu$  and  $\sigma$  are known. If  $x_1, x_2, \dots, x_n$  is a sample of size  $n$ , then the variance of this sample is,

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

- Then we know,

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

- Based on these information we have the probability statement,

$$P \left( \chi^2_{1-\alpha/2, n-1} \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{\alpha/2, n-1} \right) = 1 - \alpha$$

- By simplifying,

$$P \left( \chi_{1-\alpha/2, n-1}^2 \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi_{\alpha/2, n-1}^2 \right) = 1 - \alpha$$

$$P \left( \frac{\sigma^2}{(n-1)} \chi_{1-\alpha/2, n-1}^2 \leq S^2 \leq \frac{\sigma^2}{(n-1)} \chi_{\alpha/2, n-1}^2 \right) = 1 - \alpha$$

- According to that final control limit formulas for  $s^2$  charts are,

### $s^2$ Chart Control Limits (when $\sigma$ is known)

$$\text{UCL} = \frac{\sigma^2}{(n-1)} \chi_{\alpha/2, n-1}^2$$

$$\text{Center Line} = \sigma^2$$

$$\text{LCL} = \frac{\sigma^2}{(n-1)} \chi_{1-\alpha/2, n-1}^2$$

- Since most of the time the true variance  $\sigma^2$  is unknown, it is estimated using average sample variance  $\bar{s}^2$ .
- Then the control limit formulas for  $s^2$  charts are,

### $s^2$ Chart Control Limits (when $\sigma$ is unknown)

$$\text{UCL} = \frac{\bar{s}^2}{(n - 1)} \chi_{\alpha/2, n-1}^2$$

$$\text{Center Line} = \bar{s}^2$$

$$\text{LCL} = \frac{\bar{s}^2}{(n - 1)} \chi_{1-\alpha/2, n-1}^2$$

- Note that this control chart is defined with probability limits.

## Example: $s^2$ Chart for the Piston Ring Data

- For the above piston ring data average sample variance is,

$$\bar{s}^2 = \frac{1}{25} \sum_{i=1}^{25} s_i^2 = \frac{1}{25} 0.0025129 = 0.000101$$

- Then control limits for the  $s^2$  chart at 0.0027 significance level are,

$$\text{UCL} = \frac{0.000101}{4} \chi^2_{0.00135, 4} = 0.000447$$

$$\text{Center Line} = \bar{s}^2 = 0.000101$$

$$\text{LCL} = \frac{0.000101}{4} \chi^2_{0.99865, 4} = 0.000003$$

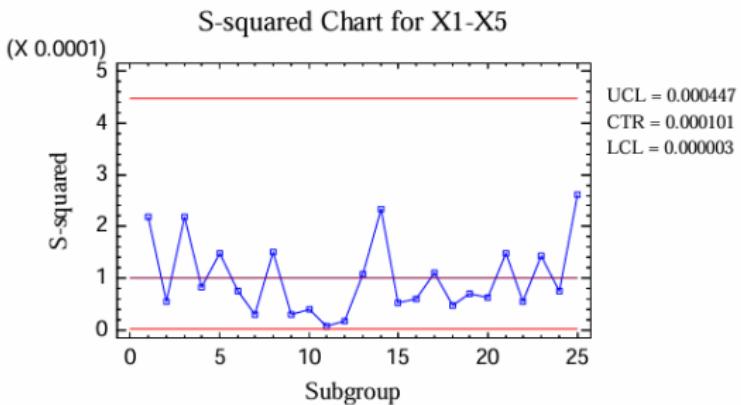
$s^2$  Chart for Phase I data

Figure 4:  $s^2$  control chart for ring data

- The  $s^2$  chart for the sample data shows no unusual signals.

# The Shewhart Control Chart for Individual Measurements

## When..?

- A Shewhart control chart for individual measurements is used when the sample size is 1 ( $n = 1$ ), meaning each data point comes from a single unit rather than a group.
- This happens in situations like:
  - Every unit is measured individually (e.g., automated inspection) and there's no natural way to make groups.
  - Data comes slowly, so it's not practical to wait for multiple items to form a sample.
- In these cases, using an individual measurement chart helps monitor the process and detect any unusual variation, even when we can't form larger samples.

## Moving Range Control Chart

- In a control chart for individual measurements, process variability is often estimated using the **moving range** of **two successive observations**.
- The moving range is defined as:

$$MR_i = |x_i - x_{i-1}|$$

- A **moving range control chart** can be created using these MR values.
- The procedure is illustrated in the following example.

# Example: Loan Processing Costs

- **Scenario:** A bank mortgage unit is monitoring the average weekly cost of processing loan applications.
- **Variable:** Average weekly loan processing cost

$$\text{Average cost} = \frac{\text{Total weekly cost}}{\text{Number of loans processed}}$$

- **Data:** Weekly average costs for the most recent 40 weeks
- **Goal:** Determine whether the process is in statistical control using:
  - ➊ Individuals Control Chart
  - ➋ Moving Range Control Chart

**Table 4:** Cost of Processing Mortgage Loan Applications (Complete dataset contains 40 weeks.  
Refer to table 6.7 Montgomery (2019), Statistical Quality Control)

Week	Cost ( $x$ )	Moving Range (MR)
1	310	-
2	288	22
3	297	9
4	298	1
5	307	9
6	303	4
7	294	9
8	297	3
9	308	11
10	306	2
11	294	12
12	299	5
13	297	2
14	299	2
15	314	15
16	295	19
17	293	2
18	306	13
19	301	5
20	304	3
$\bar{x} = 300.5$		$\overline{MR} = 7.79$

## Phase I - Establishing Trial Control Limits

### Individuals Control Chart

Monitor the stability of a process over time by plotting individual observations when subgrouping is not possible.

- **Center Line:** Sample average cost of the 20 observations,

$$\bar{x} = 300.5$$

- **Control Limits:**

$$\text{UCL} = \bar{x} + \frac{3\overline{MR}}{d_2} = 321.22$$

$$\text{LCL} = \bar{x} - \frac{3\overline{MR}}{d_2} = 279.78$$

### Moving Range Control Chart

Monitor short-term process variability by plotting the range between consecutive individual observations.

- **Center Line:** Average of the moving ranges of two observations,

$$\overline{MR} = 7.79$$

- **Control Limits:**

$$\text{UCL} = D_4 \overline{MR} = 25.45$$

$$\text{LCL} = D_3 \overline{MR} = 0$$

- Moving range of  $n = 2$  observations is used, therefore  $D_3 = 0$ ,  $D_4 = 3.267$ , and  $d_2 = 1.128$ .

## Control charts for individual observations on cost and for the moving range

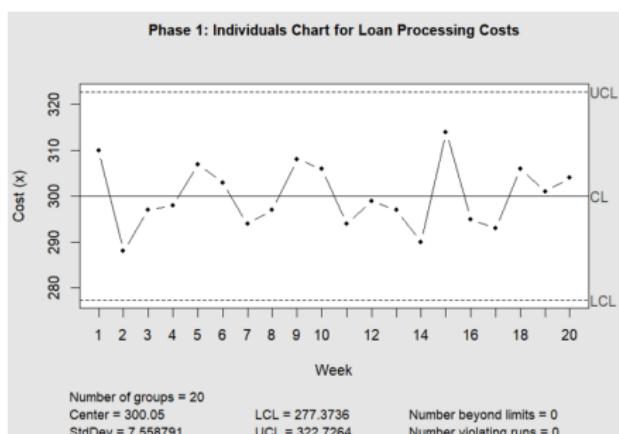


Figure 5: Control chart for individual observations on cost

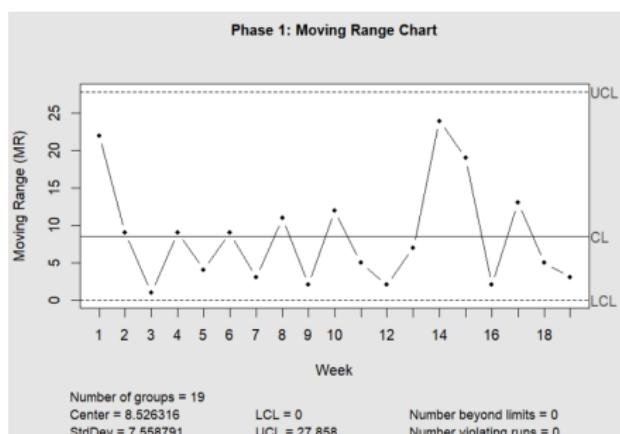
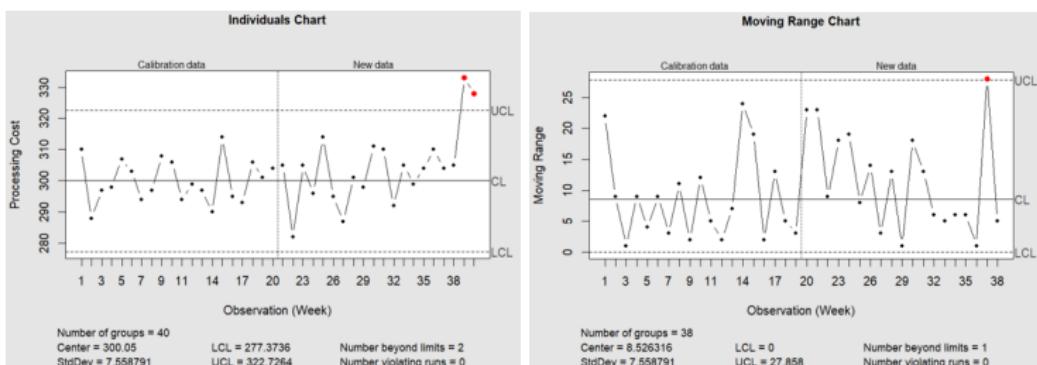


Figure 6: Control chart for the moving range

- All 20 points (weeks 1-20) plot within the limits, meaning the process is stable and these limits can be used for Phase II monitoring.

## Phase II - Process Monitoring & Interpretation

- This is a continuation of the Phase 1 control charts, monitoring Weeks 21–40 using the fixed control limits.



**Figure 7:** Continuation of the control chart for individuals and the moving range using the data for weeks 21–40

- As this chart makes clear, an upward shift in cost has occurred around week 39, followed by another out-of-control signal at week 40 on the chart for individuals.
- The Moving Range chart reinforces this finding with a significant spike at week 39, which helps to identify exactly where a process shift in the mean has occurred.
- An assignable cause should be investigated around Week 39.

## Statistical Limitations: Correlation and Detection Power

- The moving ranges are correlated because they share a common data point ( $MR_2$  uses  $x_1, x_2$ ;  $MR_3$  uses  $x_2, x_3$ ), this may often induce a pattern of runs or cycles, therefore analysts must exercise caution and not over interpret these patterns as process shifts.
- In contrast, the individual observations plotted on the individuals chart are assumed to be uncorrelated. However, The ability of the individuals control chart to detect small shifts is very poor.
- For an individuals control chart with conventional three-sigma limits, we can compute the following:

**Table 5: Shift Size vs. Average Run Length (ARL)**

Size of Shift	$\beta$	ARL <sub>1</sub>
$1\sigma$	0.9772	43.96
$2\sigma$	0.8413	6.30
$3\sigma$	0.5000	2.00

- If a shift in the process mean of about one standard deviation occurs, the information above tells us that it will take about 44 samples, on average, to detect the shift.

## Normality Assumption

- These control charts assume that observations are normally distributed; the bank's data passed a crude check via a Normal Probability Plot.

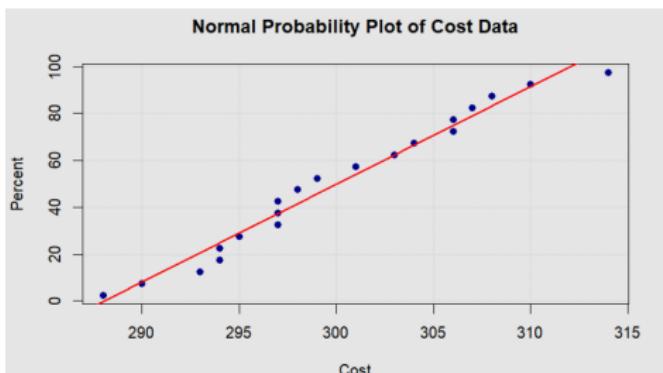


Figure 8: Normal probability plot of cost data

- If data is non-normal, It will dramatically reduce the value of  $ARL_0$  and increase the occurrence of the false alarms.
- Because of these limitations, It is suggested that Shewhart individuals chart be used with extreme caution, particularly in phase II monitoring.

# Example: The Use of Transformations

- Process: Single-wafer semiconductor deposition.
- Resistivity measurements of 25 silicon wafers.
- Construct Individuals (I) and Moving Range (MR) control charts.

**Table 6:** Resistivity Data with Natural Logarithm and Moving Range

Sample	Resistivity	$\ln(x_i)$	MR
1	216	5.37528	-
2	290	5.66988	0.29460
3	236	5.46383	0.20605
4	228	5.42935	0.03448
5	244	5.49717	0.06782
6	210	5.34711	0.15006
7	139	4.93447	0.41264
8	310	5.73657	0.80210
9	240	5.48064	0.25593
10	211	5.35186	0.12878
11	175	5.16479	0.18707
12	447	6.10256	0.93777
13	307	5.72685	0.37571

Sample	Resistivity	$\ln(x_i)$	MR
14	242	5.48894	0.23791
15	168	5.12396	0.36498
16	360	5.88610	0.76214
17	226	5.42053	0.46557
18	253	5.53339	0.11286
19	380	5.94017	0.40678
20	131	4.87520	1.06497
21	173	5.15329	0.27809
22	224	5.41165	0.25836
23	195	5.27300	0.13865
24	199	5.29330	0.02030
25	226	5.42053	0.12723
<b>Averages</b>		$\bar{\ln(x_i)} = 5.44402$	$\bar{MR} = 0.33712$

## Normality Check: Original Resistivity Data

- Normal probability plot of resistivity shows right skewness.
- Normality assumption is not satisfied.
- Individuals control chart on original data is not appropriate.

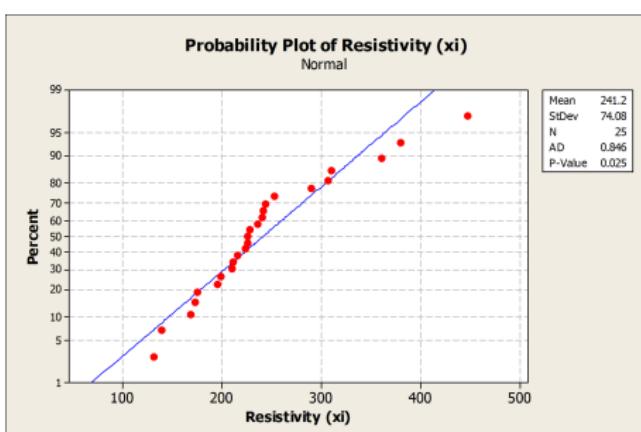


Figure 9: Normal probability plot of resistivity

## Log Transformation

- Applied natural logarithm.
- Log transformation reduces skewness.
- Transformed data approximately follow a normal distribution.

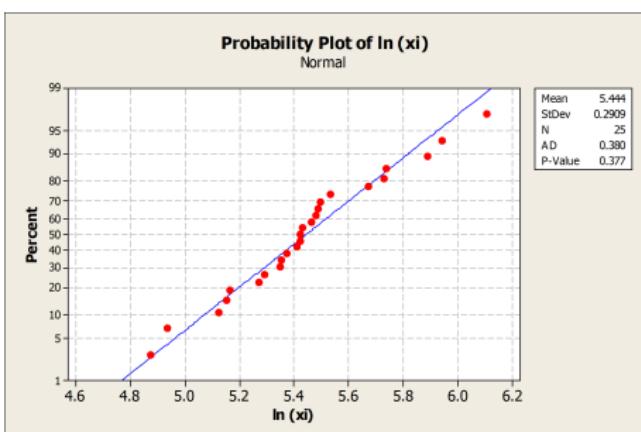


Figure 10: Normal probability plot of  $\ln(\text{resistivity})$

## Control limits for the Individuals chart

- UCL =  $\bar{\ln(x_i)} + 3\frac{\bar{MR}}{d_2} = 5.44402 + 3(0.2989) = 6.3406$
- CL =  $\bar{\ln(x_i)} = 5.44402$
- LCL =  $\bar{\ln(x_i)} - 3\frac{\bar{MR}}{d_2} = 5.44402 - 3(0.2989) = 4.5474$

Where  $MR = |\ln(x_i) - \ln(x_{i-1})|$ ,  $d_2 = 1.128$

## Control limits for the Moving Range chart

- UCL =  $D_4 \overline{MR} = 3.267 \times 0.33712 = 1.1014$
- CL =  $\overline{MR} = 0.33712$
- LCL =  $D_3 \overline{MR} = 0$

Where For moving ranges with span 2:  $D_3 = 0$ ,  $D_4 = 3.267$ .

# Individuals and moving range control charts on $\ln(\text{resistivity})$

- No points outside control limits.
- Process is statistically in control.
- Example shows importance of checking normality and using transformations.

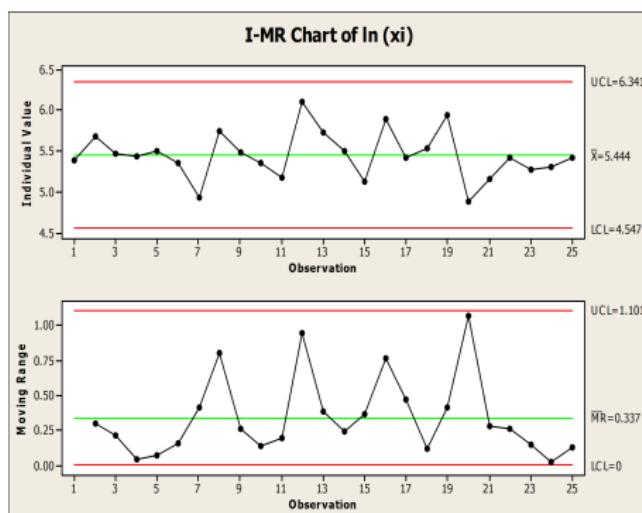


Figure 11: Individuals and moving range control charts on  $\ln(\text{resistivity})$

## Methods of Estimating

- From average moving range:

$$\hat{\sigma}_1 = \frac{\overline{MR}}{d_2} = 0.8865 \overline{MR}, \quad \text{where for span} = 2: d_2 = 1.128$$

- From sample standard deviation:

$$\hat{\sigma}_2 = \frac{s}{c_4}$$

- Both estimators are unbiased only if no assignable causes are present.

## Bias in $\sigma$ Estimation

- Assignable causes cause bias in  $\sigma$  estimates.
- A sustained shift in the process mean:
  - The sample standard deviation is greatly affected.
  - Moving range-based estimator is less affected since only one moving range changes.
- Hence, moving range estimation is preferred when shifts may exist.

## Median Moving Range & Statistic

- To reduce bias, use the median of moving ranges.

Median MR estimator:

$$\hat{\sigma}_3 = \frac{\overline{MR}}{d_4} = 1.047 \overline{MR}$$

- Comparison of  $\sigma$  estimators:

$$F^* = \left( \frac{\hat{\sigma}_1}{\hat{\sigma}_2} \right)^2$$

- Large values of  $F^*$  indicate that the process may not be in control.

## Span of the Moving Range

- Moving ranges can be computed with different spans.
- Increasing the span:
  - Increases bias in  $\sigma$  estimation.
  - Causes more moving ranges to be affected by one out-of-control point.
- Span = 2 is recommended for Individuals charts.
- span = 2 gives the most reliable estimate of  $\sigma$

# References

- ① Montgomery, D.C. (2009). Introduction to Statistical Quality Control. 6th ed. Hoboken, NJ: Wiley.
  
- ② StatPoint Technologies, Inc. (2013). X-Bar and S-Squared Charts. Rev. 9/16/2013. Statgraphics documentation.

# Thank You

## Thank You!

Questions and comments are welcome.

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