MODULE - 4
Matrix Theory

SECTION: 2 -Matrix Eigenvalue Problems

Determinants

A square matrix possesses an associated determinant. If

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

it's associated determinant

$$det(C) = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = c_{11}c_{22} - c_{21}c_{12}$$

$$det(C) = \begin{vmatrix} 4 & 6 \\ 3 & 1 \end{vmatrix} = 4 \cdot 1 - 6 \cdot 3 = -14$$

Determinants

A three by three matrix has an associated determinant

$$det(C) = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}$$

$$det(C) = c_{11} \begin{vmatrix} c_{22} & c_{23} \\ c_{32} & c_{33} \end{vmatrix} - c_{12} \begin{vmatrix} c_{21} & c_{23} \\ c_{31} & c_{33} \end{vmatrix} + c_{13} \begin{vmatrix} c_{21} & c_{22} \\ c_{31} & c_{32} \end{vmatrix}$$

$$det(C) = \begin{vmatrix} 6 & 5 & 4 \\ 2 & -1 & 7 \\ -3 & 2 & 0 \end{vmatrix} = 6 \begin{vmatrix} -1 & 7 \\ 2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & 7 \\ -3 & 0 \end{vmatrix} + 4 \begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix}$$

$$det(C) = 6(-14) - 5(21) + 4(1) = -185$$

Matrix Eigenvalue Problems

- Consider a given square matrix A. If X is a column vector and λ is a scalar (a number) then the relation

$$AX = \lambda X \tag{1}$$

  is called an eigenvalue problem.
- Since $X = 0$ is always a solution for any λ and thus not interesting, we only admit solutions with $X \neq 0$.
- The λ's that satisfy (1) are called eigenvalues of A and the corresponding nonzero X's that also satisfy (1) are called eigenvectors of A.
- Eigenvalues also known as characteristic roots, characteristic values , proper values, or latent roots.

Eigenvalue Problems

- Take all unknowns to one side :

$$(A - \lambda I)X = 0 \tag{2}$$

where I is a unit matrix with the same dimensions as A. (Note that $AX - \lambda X = 0$ does not simplify to $(A - \lambda)X = 0$ as you cannot subtract a scalar λ from a matrix A).

- This equation (2) is a homogeneous system of equations. For such a system we know that non-trivial solutions will only exist if the determinant of the coefficient matrix is zero:

$$det(A - \lambda I) = 0 \tag{3}$$

- Equation (3) is called the characteristic equation of the eigenvalue problem.

Eigenvalue Problems

- The characteristic equation only involves one unknown λ. The characteristic equation is generally a polynomial in λ, with degree being the same as the order of A (so if A is 2×2 the characteristic equation is a quadratic, if A is a 3×3 it is a cubic equation, and so on).
- For each value of λ that is obtained the corresponding value of X is obtained by solving the original equations (1). These X's are called eigenvectors.

Eigenvalue Problems - Example 1

Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

The eigenvalues and eigenvectors are found by solving the eigenvalue probelm

$$AX = \lambda X \qquad i.e. \quad (A - \lambda I)X = 0$$

Nontrivial solutions will exist if $det(A - \lambda I) = 0$ that is

$$det \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} = 0$$

Eigenvalue Problems - Example 1

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0 \implies (1-\lambda)(2-\lambda) = 0$$

$$\implies \lambda = 1, \lambda = 2$$

When $\lambda = 1$ (smaller eigenvalue)

$$(A - \lambda I)X = 0 \implies (A - I)X = 0$$

that is

$$\left[\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4}$$

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2 \qquad \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Take $y = t$

$$x + y = 0 \implies x = -y = -t$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}_{t=1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Note: The eigenvectors corresponding to eigenvalue $\lambda = 1$ are all proportional to

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

When $\lambda = 2$ (larger eigenvalue)

$$(A - \lambda I)X = 0 \implies (A - 2I)X = 0$$

Eigenvalue Problems - Example 1

$$\left[\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5}$$

$$T = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \; R_2 \to R_2 + R_1 \qquad \sim \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

Take $y = t$

$$-x = 0 \implies x = 0$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}_{t=1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda = 1, X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \lambda = 2, X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigenvalue Problems: Method-II -Procedure

1. Characteristic equation is given by

   (i) A is a 2×2 matrix:

$$\lambda^2 - trace(A)\lambda + det(A) = 0$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{aligned} trace(A) &= \text{sum of the main diagonal elements of } A \\ &= a_{11} + a_{22} \\ det(A) &= a_{11}a_{22} - a_{12}a_{21} \end{aligned}$$

   (ii) A is a 3×3 matrix:

$$\lambda^3 - trace(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - det(A) = 0$$

Eigenvalue Problems: Method-II -Procedure

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} trace(A) &= \text{sum of the main diagonal elements of } A \\ &= a_{11} + a_{22} + a_{33} \end{aligned}$$

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

2. Find the eigenvalues λ by solving the characteristic equation.
3. For each λ, find the eigenvectors $X \neq 0$ by solving the homogeneous system of equation $(A - \lambda I)X = 0$

Eigenvalue Problems: Method-II -Procedure

Eigen vectors can be easily find out using Cramer's rule.

(i) A is a 2×2 matrix:

$$a_1 x + a_2 y = 0$$

by using Cramer's rule

$$x = a_2, \quad y = -a_1$$

(ii) A is a 2×2 matrix: Suppose

$$a_1 x + a_2 y + a_3 z = 0$$

and

$$b_1 x + b_2 y + b_3 z = 0$$

are two linearly independent equations then by using Cramer's rule

$$x = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \quad y = -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \quad z = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Method-II - Example 1

Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

The eigenvalues and eigenvectors are found by solving the eigenvalue probelm

$$AX = \lambda X$$

Characteristic equation is given by

$$det(A - \lambda I) = 0$$

that is

$$\lambda^2 - trace(A)\lambda + det(A) = 0 \tag{6}$$

Method-II - Example 1

$$trace(A) = \text{sum of the main diagonal elements of } A = 1 + 2 = 3$$

$$det(A) = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 1(2) - 1(0) = 2$$

$$(6) \implies \lambda^2 - 3\lambda + 2 = 0$$

$$\implies \lambda = 1, 2$$

When $\lambda = 1$

$$(A - \lambda I)X = 0 \implies (A - I)X = 0$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out a linearly independent equation.The row which contains zero only is linearly dependant. Taking the second row:

$$\begin{matrix} x & y \\ 1 & 1 \end{matrix} \quad x = 1 \quad y = -1 \quad X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Eigenvalue Problems - Example 1

When $\lambda = 2$

$$(A - \lambda I)X = 0 \implies (A - 2I)X = 0$$

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out a linearly independent equation. Taking the first row:

$$\begin{matrix} x & y \\ -1 & 0 \end{matrix} \quad x = 0 \quad y = -(-1) = 1 \quad X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda = 1, X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \lambda = 2, X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Eigenvalue Problem - Example 2

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix}$$

The eigenvalues and eigenvectors are found by solving the eigenvalue probelm

$$AX = \lambda X$$

Characteristic equation is given by

$$det(A - \lambda I) = 0$$

that is

$$\lambda^3 - trace(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - det(A) = 0$$

Eigenvalue Problem - Example 2

$trace(A)$ = sum of the main diagonal elements of $A = 6+5+7 = 18$

$$M_{11} = \begin{vmatrix} 5 & 0 \\ 0 & 7 \end{vmatrix} = 35, \quad M_{22} = \begin{vmatrix} 6 & -2 \\ -2 & 7 \end{vmatrix} = 38, \quad M_{33} = \begin{vmatrix} 6 & 2 \\ 2 & 5 \end{vmatrix} = 26$$

$$M_{11} + M_{22} + M_{33} = 35 + 38 + 26 = 99$$

$$det(A) = \begin{vmatrix} 6 & 2 & -2 \\ 2 & 5 & 0 \\ -2 & 0 & 7 \end{vmatrix} = 6 \begin{vmatrix} 5 & 0 \\ 0 & 7 \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 \\ -2 & 7 \end{vmatrix} - 2 \begin{vmatrix} 2 & 5 \\ -2 & 0 \end{vmatrix} = 162$$

$\therefore$ Characteristic equation is

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$$

$$\implies \lambda = 3, 6, 9$$

$\therefore$ the eigenvalues are 3,6,9

Eigenvalue Problem - Example 2

When $\lambda = 3$ (smaller eigenvalue)

$$(A - \lambda I)X = 0 \implies (A + 3I)X = 0$$

$$\begin{bmatrix} 3 & 2 & -2 \\ 2 & 2 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out two linearly independent equations. Taking the first two rows:

$$\begin{array}{ccc} x & y & z \\ 3 & 2 & -2 \\ 2 & 2 & 0 \end{array}$$

$$x = \begin{vmatrix} 2 & -2 \\ 2 & 0 \end{vmatrix} = 4, \quad y = -\begin{vmatrix} 3 & -2 \\ 2 & 0 \end{vmatrix} = -4, \quad z = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2$$

Eigenvalue Problem - Example 2

$\therefore$ the eigenvector corresponding to $\lambda = 3$ is given by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

When $\lambda = 6$

$$(A - \lambda I)X = 0 \implies (A - 6I)X = 0$$

$$\begin{bmatrix} 0 & 2 & -2 \\ 2 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out two linearly independent equations. Taking the first two rows:

Eigenvalue Problem - Example 2

$$\begin{array}{ccc} x & y & z \\ 0 & 2 & -2 \\ 2 & -1 & 0 \end{array}$$

$$x = \begin{vmatrix} 2 & -2 \\ -1 & 0 \end{vmatrix} = -2, \quad y = -\begin{vmatrix} 0 & -2 \\ 2 & 0 \end{vmatrix} = -4, \quad z = \begin{vmatrix} 0 & 2 \\ 2 & -1 \end{vmatrix} = -4$$

$\therefore$ the eigenvector corresponding to $\lambda = 6$ is given by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

When $\lambda = 9$

$$(A - \lambda I)X = 0 \implies (A - 9I)X = 0$$

Eigenvalue Problem - Example 2

$$\begin{bmatrix} -3 & 2 & -2 \\ 2 & -4 & 0 \\ -2 & 0 & -2 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{ccc} x & y & z \\ -3 & 2 & -2 \\ 2 & -4 & 0 \end{array}$$

$$x = \begin{vmatrix} 2 & -2 \\ -4 & 0 \end{vmatrix} = -8, \quad y = - \begin{vmatrix} -3 & -2 \\ 2 & 0 \end{vmatrix} = -4, \quad z = \begin{vmatrix} -3 & 2 \\ 2 & -4 \end{vmatrix} = 8$$

$\therefore$ the eigenvector corresponding to $\lambda = 9$ is given by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Eigenvalue Problem - Example 3

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The eigenvalues and eigenvectors are found by solving the eigenvalue probelm

$$AX = \lambda X$$

Characteristic equation is given by

$$det(A - \lambda I) = 0$$

that is

$$\lambda^3 - trace(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - det(A) = 0$$

Eigenvalue Problem - Example 3

$$\begin{aligned} trace(A) &= \text{sum of the main diagonal elements of } A \\ &= -2 + 1 + -2 = -3 \end{aligned}$$

$$M_{11} = \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} = -2, \quad M_{22} = \begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix} = 4, \; M_{33} = \begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix} = -2$$

$$M_{11} + M_{22} + M_{33} = -2 + 4 - 2 = 0$$

$$det(A) = \begin{vmatrix} -2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -2 \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} - 0 + 1 \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} = 4$$

$\therefore$ Characteristic equation is

$$\lambda^3 + 3\lambda^2 - 4 = 0$$

$$\implies \lambda = 1, -2, -2$$

$\therefore$ the eigenvalues are 1,-2,-2

Eigenvalue Problem - Example 3

When $\lambda = 1$

$$(A - \lambda I)X = 0 \implies (A - I)X = 0$$

$$\begin{bmatrix} 3 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out two linearly independent equations. Taking the last two two rows:

$$\begin{array}{ccc} x & y & z \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{array}$$

$$x = \begin{vmatrix} 0 & 0 \\ 0 & -3 \end{vmatrix} = 0, \quad y = -\begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = -3, \quad z = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

$\therefore$ the eigenvector corresponding to $\lambda = 1$ is given by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

When $\lambda = -2$

$$(A - \lambda I)X = 0 \implies (A + 2I)X = 0$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out two linearly independent equations. Taking the first two rows:

Eigenvalue Problem - Example 3

$$\begin{array}{ccc} x & y & z \\ 0 & 0 & 1 \\ 1 & 3 & 0 \end{array}$$

$$x = \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} = -3, \quad y = -\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1, \quad z = \begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix} = 0$$

$\therefore$ the eigenvector corresponding to $\lambda = 6$ is given by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvalue Problem - Example 4

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

The eigenvalues and eigenvectors are found by solving the eigenvalue probelm

$$AX = \lambda X$$

Characteristic equation is given by

$$det(A - \lambda I) = 0$$

that is

$$\lambda^3 - trace(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - det(A) = 0$$

Eigenvalue Problem - Example 4

$$\begin{aligned} trace(A) &= \text{sum of the main diagonal elements of } A \\ &= -2+1+0=-1 \end{aligned}$$

$$M_{11}=\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix}=-12,\, M_{22}=\begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix}=-3,\, M_{33}=\begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}=-6$$

$$M_{11}+M_{22}+M_{33}=-12+-3-6=-21$$

$$|A|=\begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}=-2\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix}-2\begin{vmatrix} 2 & -6 \\ -1 & 0 \end{vmatrix}-3\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}=45$$

$\therefore$ Characteristic equation is

$$\lambda^3+\lambda^2-21\lambda-45=0$$

$$\implies \lambda=5,-3,-3$$

$\therefore$ the eigenvalues are 5,-3,-3

Eigenvalue Problem - Example 4

While solving cubic degree equations using calculator, the distinct roots are only displayed when the roots are repeating. We can identify the repeating roots by the following methods:

1. Sum of the eigenvalue=trace
2. Product of the eigenvalue=determinant

Let x be the third eigenvalue

$$5 - 3 + x = -1 \implies 2 + x = -1 \implies x = -3$$

So the eigenvalues are 5,-3,-3

Eigenvalue Problem - Example 4

When $\lambda = 5$

$$(A - \lambda I)X = 0 \implies (A - 5I)X = 0$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out two linearly independent equations. Taking the first two rows.

$$\begin{array}{ccc} x & y & z \\ -7 & 2 & -3 \\ 2 & -4 & -6 \end{array}$$

$$x = \begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix} = -24, y = - \begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix} = -48, z = \begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix} = 24$$

Eigenvalue Problem - Example 4

$\therefore$ the eigenvector corresponding to $\lambda = 5$ is given by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -24 \\ -48 \\ 24 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

When $\lambda = -3$

$$(A - \lambda I)X = 0 \implies (A + 2I)X = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ 1 & 2 & -3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out two linearly independent equations. Here $R_2 = 2R_1, R_3 = R_1$, so we can not have two linearly independent rows. Hence we have to solve the homogeneous system of linear equations.

Eigenvalue Problem - Example 4

$$T = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ 1 & 2 & -3 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$n - r = 3 - 1 = 2$, hence we have to assign arbitrary values for two variables. Take $y = s, z = t$

$$x + 2y - 3z = 0 \implies x = 3z - 2y = 3t - 2s$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3t - 2s \\ s \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ the eigenvectors corresponding to $\lambda = -3$ are given by

$$\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvalue Problem - Example 5

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$$

The eigenvalues and eigenvectors are found by solving the eigenvalue probelm

$$AX = \lambda X$$

Characteristic equation is given by

$$det(A - \lambda I) = 0$$

that is

$$\lambda^3 - trace(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - det(A) = 0$$

Eigenvalue Problem - Example 5

$$\begin{aligned} trace(A) &= \text{sum of the main diagonal elements of } A \\ &= -3+4+2=3 \end{aligned}$$

$$M_{11} = \begin{vmatrix} 4 & 3 \\ 2 & 2 \end{vmatrix} = 2, M_{22} = \begin{vmatrix} -3 & -5 \\ 1 & 2 \end{vmatrix} = -1, M_{33} = \begin{vmatrix} -3 & -7 \\ 2 & 4 \end{vmatrix} = 2$$

$$M_{11} + M_{22} + M_{33} = 2 - 1 + 2 = 3$$

$$|A| = \begin{vmatrix} 3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{vmatrix} = 1$$

$\therefore$ Characteristic equation is

$$\begin{aligned} \lambda^3 - 3\lambda^2 + 3\lambda - 1 &= 0 \\ \implies \lambda &= 1, 1, 1 \end{aligned}$$

$\therefore$ the eigenvalues are 1,1,1

Eigenvalue Problem - Example 5

While solving cubic degree equations using calculator, the distinct roots are only displayed when the roots are repeating. We can identify the repeating roots by the following methods:

1. Sum of the eigenvalue=trace
2. Product of the eigenvalue=determinant

Let x,y be the remaining two eigenvalues

$$1 + x + y = 3 \implies x = 2 - y$$

$$1 \cdot x \cdot y = 1 \implies (2 - y)y = 1 \implies y^2 - 2y + 1 = 0 \implies y = 1$$

So the eigenvalues are 1,1,1

Eigenvalue Problem - Example 5

When $\lambda = 1$

$$(A - \lambda I)X = 0 \implies (A - I)X = 0$$

$$\begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out two linearly independent equations. Taking the first two rows.

$$\begin{array}{ccc} x & y & z \\ -4 & -7 & -5 \\ 2 & 3 & 3 \end{array}$$

$$x = \begin{vmatrix} -7 & -5 \\ 3 & 3 \end{vmatrix} = -6, y = - \begin{vmatrix} -4 & -5 \\ 2 & 3 \end{vmatrix} = 2, z = \begin{vmatrix} -4 & -7 \\ 2 & 3 \end{vmatrix} = 2$$

Eigenvalue Problem - Example 5

$\therefore$ the eigenvector corresponding to $\lambda = 1$ is given by

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

Properties of Eigenvalues

1. Eigenvalues of A and A^T are same
2. If λ is an eigenvalue of A then λ^n is an eigenvalue of A^n
3. If λ is an eigenvalue of A then $k\lambda$ is an eigenvalue of kA
4. If λ is an eigenvalue of A then $\lambda - k$ is an eigenvalue of $A - kI$
5. If λ is an eigenvalue of A then $\dfrac{1}{\lambda}$ is an eigenvalue of A^{-1}
6. If λ is an eigenvalue of A then $\dfrac{|A|}{\lambda}$ is an eigenvalue of adjoint of A
7. The sum of eigenvalues of a matrix is equal to the sum of main diagonals of the matrix
8. The product of eigenvalues of a matrix is equal to the determinant of the matrix

Properties of Eigenvalues: Example 1

If 1,2,3 are the eigenvalues of the matrix A find the eigenvalues of A^T, A^3, $3A$, $A + 5I$, A^{-1} and $adj(A)$.

i) Eigenvalues of A and A^T are same.

$\therefore$ eigenvalues of A^T are 1,2,3.

ii) If λ is an eigenvalue of A then λ^n is an eigenvalue of A^n. That is λ^3 is an eigenvalue of A^3.

$\therefore$ eigenvalues of A^3 are $1, 2^3$, $3^3 = 1, 8, 27$.

iii) If λ is an eigenvalue of A then $k\lambda$ is an eigenvalue of kA. That is 3λ is an eigenvalue of $3A$.

$\therefore$ eigenvalues of $3A$ are 3,6,9.

Properties of Eigenvalues: Example 1

iv) If λ is an eigenvalue of A then $\lambda - k$ is an eigenvalue of $A - kI$. That is $\lambda + 5$ is an eigenvalue of $A + 5I$.

$\therefore$ eigenvalues of $A + 5I$ are 6,7,8

v) If λ is an eigenvalue of A then $\dfrac{1}{\lambda}$ is an eigenvalue of A^{-1}.

$\therefore$ eigenvalues of A^{-1} are $1, \dfrac{1}{2}, \dfrac{1}{3}$.

vi) If λ is an eigenvalue of A then $\dfrac{|A|}{\lambda}$ is an eigenvalue of adjoint of A.

$|A| =$ product of the eigenvalue $= 1 \cdot 2 \cdot 3 = 6$

$\therefore$ eigenvalues of adjoint of A is $\dfrac{6}{1} = 6, \quad \dfrac{6}{2} = 3, \quad \dfrac{6}{3} = 2$

Properties of Eigenvalues: Example 2

If the sum of two eigenvalues of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 5 \\ 5 & 3 & -1 \end{bmatrix} \text{ is 4, find the third eigenvalue?}$$

sum of eigenvalues = trace=sum of main diagonals=1+3+-1=3.

Suppose λ_1,λ_2,λ_3 are the three eigenvalues, $\implies \lambda_1+\lambda_2+\lambda_3 = 3.$

Given that sum of two eigenvalues of the matrix =4 $\implies \lambda_1+\lambda_2$=4

$\therefore$ third eigenvalue, $\lambda_3 = 3 - 4 = -1$

Properties of Eigenvalues: Example 3

If the product of two eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix} \text{ is 4, find the third eigenvalue?}$$

We know that product of eigenvalues of a matrix is equal to the determinant of the matrix.

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix} = 12$$

Suppose λ_1,λ_2,λ_3 are the three eigenvalues, $\implies \lambda_1\lambda_2\lambda_3 = 12$.

Given that product of two eigenvalues $= 4 \implies \lambda_1\lambda_2 = 4$

$\therefore$ third eigenvalue, $\lambda_3 = \frac{12}{4} = 3$