MODULE - 4
Matrix Theory

SECTION: 3 -Diagonalisation of a Matrix

Diagonalisation of a Matrix

A square matrix A is called diagonalizable if it is similar to a diagonal matrix, i.e., if there exists an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$

Suppose A is the given matrix.We can diagonalize the matrix A using the following steps:

1. Find out the eigenvalues and eigenvectors of the given matrix.
2. Form a matrix P by writing the eigenvectors as column vectors which is known as the modal matrix.
3. Find P^{-1}
4. Then $P^{-1}AP = D$ is the required diagonalized matrix.Here D is a diagonal matrix whose main diagonal entries are the eigenvalues of A.

Diagonalisation of a Matrix - Example 1

Diagonalize the matrix $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

The eigenvalues and eigenvectors are found by solving the eigenvalue probelm

$$AX = \lambda X$$

Characteristic equation is given by

$$\lambda^2 - trace(A)\lambda + det(A) = 0$$

$trace(A) =$ sum of the main diagonal elements of $A = 1 + 2 = 3$

$$det(A) = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 1(2) - 1(0) = 2$$

Diagonalisation of a Matrix - Example 1

$$\lambda^2 - 3\lambda + 2 = 0$$
$$\implies \lambda = 1, 2$$

When $\lambda = 1$

$$(A - \lambda I)X = 0 \implies (A - I)X = 0$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out a linearly independent equation.The row which contains zero only is linearly dependant. Taking the second row:

$$\begin{matrix} x & y \\ 1 & 1 \end{matrix} \quad x = 1 \quad y = -1 \quad X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Diagonalisation of a Matrix - Example 1

When $\lambda = 2$

$$(A - \lambda I)X = 0 \implies (A - 2I)X = 0$$

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out a linearly independent equation. Taking the first row:

$$\begin{matrix} x & y \\ -1 & 0 \end{matrix} \quad x = 0 \quad y = -(-1) = 1 \quad X_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Form a matrix P by writing the eigenvectors as column vectors which is known as the modal matrix

$$P = \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Diagonalisation of a Matrix - Example 1

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$D = P^{-1}AP$ is the required diagonalized matrix. Here D is a diagonal matrix whose main diagonal entries are the eigenvalues of A.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Diagonalisation of a Matrix - Example 2

Diagonalize the matrix $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

The eigenvalues and eigenvectors are found by solving the eigenvalue probelm

$$AX = \lambda X$$

Characteristic equation is given by

$$\lambda^3 - trace(A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - det(A) = 0$$

$$\begin{aligned} trace(A) &= \text{sum of the main diagonal elements of } A \\ &= -2 + 1 + 0 = -1 \end{aligned}$$

Diagonalisation of a Matrix - Example 2

$$M_{11} = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} = -12,\ M_{22} = \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} = -3,\ M_{33} = \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -6$$

$$M_{11} + M_{22} + M_{33} = -12 + -3 - 6 = -21$$

$$|A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = -2\begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & -6 \\ -1 & 0 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} = 45$$

$\therefore$ Characteristic equation is

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\implies \lambda = 5, -3, -3$$

$\therefore$ the eigenvalues are 5,-3,-3

Diagonalisation of a Matrix - Example 2

While solving cubic degree equations using calculator, the distinct roots are only displayed when the roots are repeating. We can identify the repeating roots by the following methods:

1. Sum of the eigenvalue=trace
2. Product of the eigenvalue=determinant

Let x be the third eigenvalue

$$5 - 3 + x = -1 \implies 2 + x = -1 \implies x = -3$$

So the eigenvalues are 5,-3,-3

Diagonalisation of a Matrix - Example 2

When $\lambda = 5$

$$(A - \lambda I)X = 0 \implies (A - 5I)X = 0$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out two linearly independent equations. Taking the first two rows.

$$\begin{array}{ccc} x & y & z \\ -7 & 2 & -3 \\ 2 & -4 & -6 \end{array}$$

$$x = \begin{vmatrix} 2 & -3 \\ -4 & -6 \end{vmatrix} = -24, y = -\begin{vmatrix} -7 & -3 \\ 2 & -6 \end{vmatrix} = -48, z = \begin{vmatrix} -7 & 2 \\ 2 & -4 \end{vmatrix} = 24$$

Diagonalisation of a Matrix - Example 2

$\therefore$ the eigenvector corresponding to $\lambda = 5$ is given by

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -24 \\ -48 \\ 24 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

When $\lambda = -3$

$$(A - \lambda I)X = 0 \implies (A + 2I)X = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ 1 & 2 & -3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

For finding the corresponding eigenvector we have to find out two linearly independent equations. Here $R_2 = 2R_1, R_3 = R_1$, so we can not have two linearly independent rows. Hence we have to solve the homogeneous system of linear equations.

Diagonalisation of a Matrix - Example 2

$$T = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ 1 & 2 & -3 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$n - r = 3 - 1 = 2$, hence we have to assign arbitrary values for two variables. Take $y = s, z = t$

$$x + 2y - 3z = 0 \implies x = 3z - 2y = 3t - 2s$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3t - 2s \\ s \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ the eigenvectors corresponding to $\lambda = -3$ are given by

$$X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Diagonalisation of a Matrix - Example 2

Form a matrix P by writing the eigenvectors as column vectors which is known as the modal matrix

$$P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$P^{-1} = \frac{1}{8} \begin{bmatrix} 1 & 2 & -3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{bmatrix}$$

$D = P^{-1}AP$ is the required diagonalized matrix. Here D is a diagonal matrix whose main diagonal entries are the eigenvalues of A.

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 1 & 2 & -3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{bmatrix} \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

Quadratic Forms

Consider the function $Q(x_1, x_2) : \mathbf{R}^2 \to \mathbf{R}$ where $Q(x_1, x_2) = a_{11}{x_1}^2 + a_{12}x_1x_2 + a_{22}{x_2}^2$. We call this a quadratic form in $\mathbf{R}^2$. This can be expressed in matrix form as

$$Q = X^T A X$$

$$Q = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where $X = (x_1, x_2)$, and A is unique and symmetric. The matrix A is called the matrix of the quadratic form.

Quadratic Forms

Represent the matrix $A = \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix}$ as a Quadratic form?

The matrix representation is given by $Q = X^T AX$

$$\begin{aligned} Q &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} x + 5y \\ 4x + 2y \end{bmatrix} \\ &= x(x + 5y) + y(4x + 2y) \\ &= x^2 + 5xy + 4xy + 2y^2 \\ &= x^2 + 9xy + 2y^2 \end{aligned}$$

Quadratic Forms

Find the matrix of the quadratic form $4x^2 + 3y^2 - 2z^2 + 6xy - 2xz + 10yz$

$$A = \begin{array}{c} \\ (x) \\ (y) \\ (z) \end{array} \begin{array}{c} \begin{array}{ccc} (x) & (y) & (z) \end{array} \\ \begin{pmatrix} 4 & 6/2 & -2/2 \\ 6/2 & 3 & 10/2 \\ -2/2 & 10/2 & -2 \end{pmatrix} \end{array}$$

$$A = \begin{bmatrix} 4 & 3 & -1 \\ 3 & 3 & 5 \\ -1 & 5 & -2 \end{bmatrix}$$

Nature of a Quadratic Form

A quadratic form Q is

(i) positive definite if $Q(X) > 0$ for all $X \neq 0$
(ii) negative definite if $Q(X) < 0$ for all $X \neq 0$
(iii) positive semidefinite, if $Q(X) \geq 0$ for all X
(iv) negative semidefinite, if $Q(X) \leq 0$ for all X
(v) indefinite if $Q(X)$ assumes both positive and negative values

Quadratic Forms and Eigenvalues

The quadratic form $Q = X^T AX$ is

(i) positive definite, if all eigenvalues of A are positive.
(ii) negative definite, if all eigenvalues of A are negative.
(iii) positive semidefinite, if atleast one eigenvalue is zero and all others are positive.
(iv) negative semidefinite, if atleast one eigenvalue is zero and all others are negative.
(v) indefinite, if some eigenvalues are positive and others are negative.

Signature of a Quadratic Form

Signature of a quadratic form is defined as the triplet $[n_0, n_+, n_-]$ where

- n_0 is the number of zero eigenvalues
- n_+ is the number of positive eigenvalues
- n_- is the number of negative eigenvalues

Signature of a Quadratic Form - Example -1

Find nature, rank and signature of the quadratic form $3x^2 + 5y^2 + 3z^2 - 2xy + 2xz - 2yz$.

The matrix of the quadratic form is

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

- Characteristic equation is $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$
- Eigenvalues are 2,3,6.
- Since all eigenvalues are positive, quadratic form is positive definite.
- Rank of the quadratic form = number of nonzero eigenvalues = 3.
- Signature of the quadratic form $= [n_0, n_+, n_-] = [0, 3, 0]$.

Signature of a Quadratic Form - Example -2

Find nature,rank and signature of the quadratic form $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$.

The matrix of the quadratic form is

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

- Characteristic equation is $\lambda^3 - 6\lambda^2 = 0$
- Eigenvalues are 0,0,6,
- Since two eigenvalues are zero and one is positive, quadratic form is positive semidefinite.
- Rank of the quadratic form is number of nonzero eigenvalues $= 1$.
- Signature of the quadratic form $= [n_0, n_+, n_-] = [2,1,0]$.

Signature of a Quadratic Form - Example -3

Find nature,rank and signature of the quadratic form $-3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$.

The matrix of the quadratic form is

$$A = \begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$$

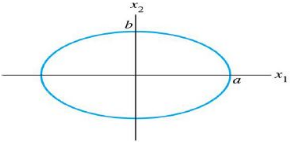
- Characteristic equation is $\lambda^3 - 9\lambda^2 + 24\lambda + 16 = 0$
- Eigenvalues are -1,-1,-4.
- Since all eigenvalues are negative, quadratic form is negative definite.
- Rank of the quadratic form is number of nonzero eigenvalues $= 3$.
- Signature of the quadratic form $= [n_0, n_+, n_-] =$ [0,0,3].

Canonical Form of a Quadratic Form

Any quadratic form $Q = X^T AX$ can be transformed into the form $Q = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$ where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of A and the form is called the principle axis form or canonical form.

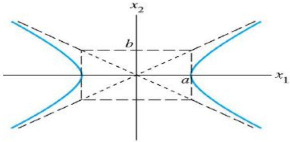
A Geometric View of Principal Axes

- Suppose $Q = X^T AX$, where A is an invertible symmetric matrix, and let C be a constant.
- We can show that the set of all X in $\mathbf{R^2}$ that satisfy $X^T AX = C$ either corresponds to an ellipse (or circle), a hyperbola, two intersecting lines, or a single point, or contains no points at all.
- If A is a diagonal matrix, the graph is in standard position, such as in the figure below.



$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \ a > b > 0$

ellipse



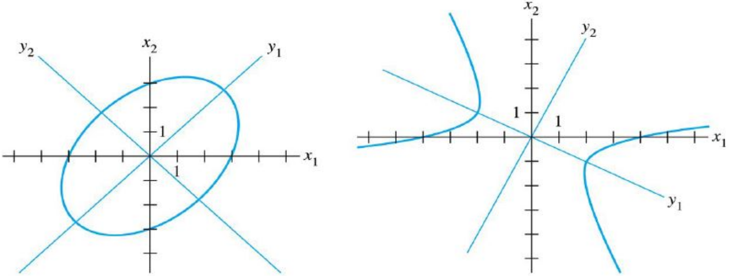
$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1, \ a > b > 0$

hyperbola

An ellipse and a hyperbola in standard position.

A Geometric View of Principal Axes

- If A is not a diagonal matrix, the graph of equation $X^T AX = C$ is rotated out of standard position, as in the figure below.



(a) $5x_1^2 - 4x_1x_2 + 5x_2^2 = 48$ (b) $x_1^2 - 8x_1x_2 - 5x_2^2 = 16$

An ellipse and a hyperbola *not* in standard position.

- Finding the principal axes (determined by the eigenvectors of A) amounts to finding a new coordinate system with respect to which the graph is in standard position.

Reducing a Quadratic form into a Canonical form

1). Find A, the matrix represented by the quadratic form.

2). Determine the eigenvalues and eigenvectors of A.

3). Find the Modal matrix

$$P = \begin{bmatrix} \frac{X_1}{||X_1||} & \frac{X_2}{||X_2||} & \frac{X_3}{||X_3||} \end{bmatrix}, \text{where } ||X_1|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

4). Introduce a new vector Y by the relation $X = PY$
Now $Q = X^T AX \implies$

$$Q = (PY)^T A(PY) = Y^T P^T APY = Y^T (P^T AP)Y = Y^T DY$$

where D is the diagonal matrix containing eigenvalues as the main diagonal entries and $Q = Y^T DY$ is the canonical form.

Canonical Form : Example - 1

Find the canonical form of the quadratic form $3x^2 + 2xy + 3y^2$ hence show that the equation $3x^2 + 2xy + 3y^2 - 8 = 0$ represents an ellipse. Also find the equation of major and minor axis of the ellipse?

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Characteristic equation is $\lambda^2 - 6\lambda + 8 = 0 \implies \lambda = 2, 4$
The eigenvector corresponding to $\lambda = 2$ is given by

$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$||X_1|| = \sqrt{x_1^2 + x_2^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

Canonical Form : Example - 1

The eigenvector corresponding to $\lambda = 4$ is given by

$$X_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$||X_2|| = \sqrt{x_1^2 + x_2^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

The modal matrix is given by $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

Define $X = PY$. Now $Q = X^T AX \implies$

$$Q = (PY)^T A(PY) = Y^T P^T APY = Y^T (P^T AP)Y = Y^T DY$$

Then

$$Q = Y^T DY = \lambda_1 y_1^2 + \lambda_2 y_2^2 = 2y_1^2 + 4y_2^2$$

Canonical Form : Example - 1

$$\therefore \text{the required canonical form is } Q = 2y_1^2 + 4y_2^2$$

We have $Q = 3x^2 + 2xy + 3y^2$
Given

$$\begin{aligned}
3x^2 + 2xy + 3y^2 - 8 &= 0 \\
Q - 8 &= 0 \\
Q &= 8 \\
2y_1^2 + 4y_2^2 &= 8 \\
\frac{2y_1^2}{8} + \frac{4y_2^2}{8} &= 1 \\
\frac{y_1^2}{4} + \frac{y_2^2}{2} &= 1 \quad \text{which represents an ellipse.}
\end{aligned}$$

Canonical Form : Example - 1

In the ellipse $\frac{y_1^2}{4} + \frac{y_2^2}{2} = 1$, major axis is y_1 and minor axis is y_2

$$X = PY \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Equation of major axis is $y_2 = 0$

$\implies (1/\sqrt{2})y_1 = x_1$ and $(-1/\sqrt{2})y_1 = x_2$

Solving we get $x_1 + x_2 = 0$, which is the equation of major axis.

Similarly equation of minor axis is $y_1 = 0$

$\implies (-1/\sqrt{2})y_2 = x_1$ and $(-1/\sqrt{2})y_2 = x_2$

Solving we get $x_1 - x_2 = 0$, which is the equation of minor axis.