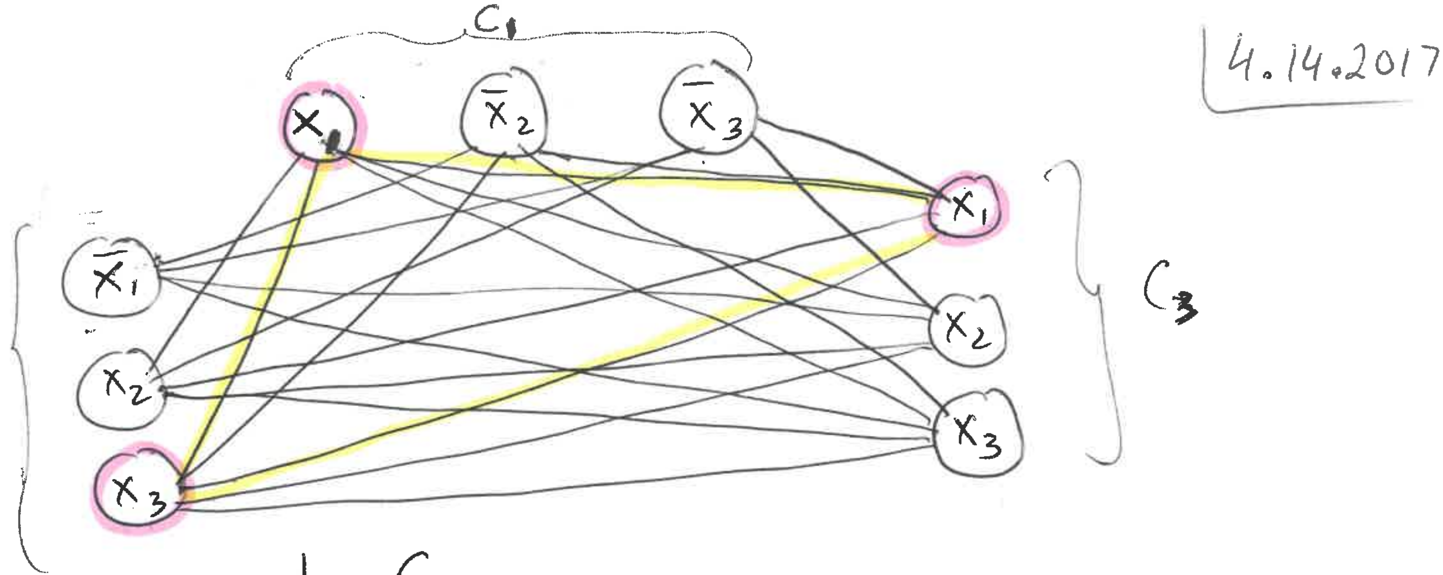


graph G

input of the reduction alg is ϕ in 3-CNF, with K clauses

output of the reduction alg is an instance of the CLIQUE problem $\langle G, K \rangle$

- show that the alg. is a reduction:

ϕ is 3-CNF-SAT iff G has a clique of size K

$\boxed{\Rightarrow}$

suppose ϕ has a satisfying assignment. Show that G has a clique of size K.

ϕ has a satisfying assignment $\Rightarrow$ for this assignment, each clause evaluates to 1, thus each clause has at least one "true" literal

- select one "true" literal from each clause
- let V' – set of vertices corresponding to the "true" literals selected previously
- then V' forms a CLIQUE of size K in G

(vertices in V' correspond to literals which are example consistent, thus they will be conneted by edges

satisfying assignment $\langle x_1 = 1, x_2 = 1, x_3 = 1 \rangle$

C_1: x_1

C_2: x_3

C_3: x_1

---

satisfying assignment $\langle x_1 = 1, x_2 = 0, x_3 = 1 \rangle$

$C_1 = \bar{x}_2$

C_2: x_3

C_3: x_1

$\boxed{\Leftarrow}$ Suppose that G has a clique of size K. Show that ϕ has a satisfying assignment.

- each vertex from the clique must be in a different triplet (corresponding to a different clause)
- set the literals corresponding to the vertices in the clique to 1
- assign arbitrary values 0/1 to the other literals

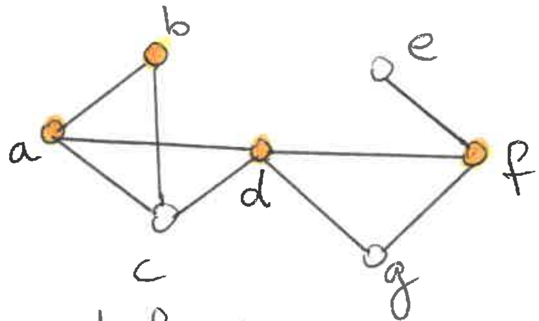
$\Rightarrow$ this is a satisfying assignment for ϕ

* the reduction alg. takes polynomial time

The Vertex-Cover (VC) problem

Let $G(V,E)$- undirected

A vertex-cover (VC) is a subset $V' \subseteq V$ such that for every edge $(u,v) \in E$ either $u \in V'$ or $v \in V'$ or both $u,v \in V$.



$V' = \{a, b, d, f\}$

$|V'| = 4$

Problem definition

- optimization problem : Given a graph $G(V,E)$ undirected, find a VC of minimum size.
- decision problem : Given a graph $G(V,E)$ undirected and a value k, does G have a VC of size k?

Theorem

VC is NP-complete.

proof

- $VC \in NP$

- a certificate contains the set of vertices V'

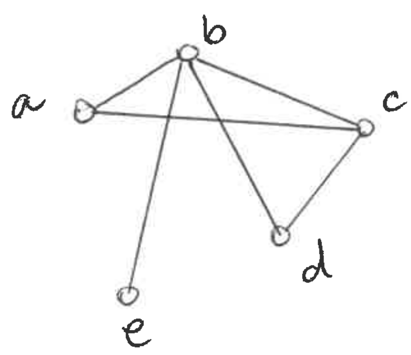
verification algorithm (polynomial time)

- for each edge (u,v), check if at least one vertex u or v is in V'
- check that all the vertices in V' are distinct and that $V' \subseteq V$

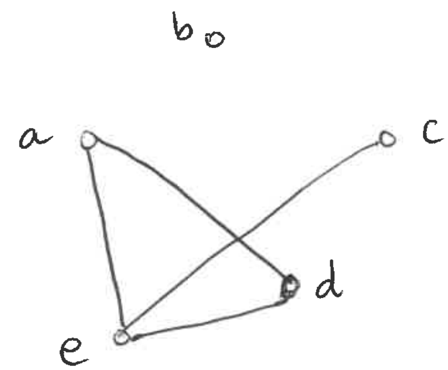
- <u>VC is NP-hard</u>

$\text{CLIQUE} \le_p \text{VC}$

Given a graph $G(V,E)$ undirected we define the <u>complement graph</u> $\bar{G}(V,\bar{E})$ where $\bar{E} = \{(u,v) : (u,v) \notin E\}$



<u>graph G</u>



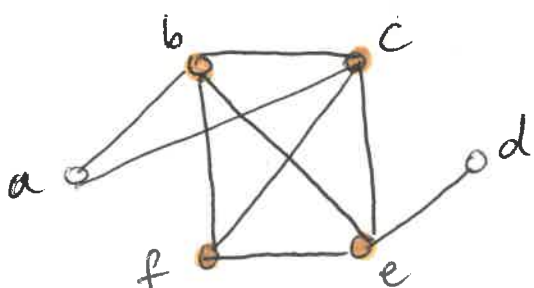
<u>graph $\bar{G}$</u>

<u>reduction algorithm</u>

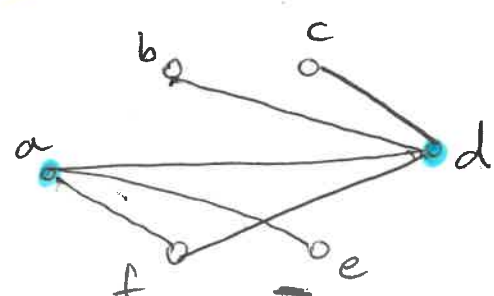
- takes an instance $\langle G, K\rangle$ of the CLIQUE pb. and outputs an instance $(\bar{G}, |V|-K)$ of the VC problem
  - take $\langle G, K\rangle$ an instance of the clique problem
  - compute $\bar{G}$
  - output is $\langle \bar{G}, |V|-K\rangle$, an instance of the VC problem

show that the alg. is a reduction:

G has a clique of size K iff $\bar{G}$ has a VC of size $|V|-K$



<u>graph G</u>
clique $\{b, c, e, f\}$



<u>graph $\bar{G}$</u>
$\{a, d\}$ is a VC

$\boxed{\Rightarrow}$ suppose G has a clique of size K. Show that $\bar{G}$ has a VC of size $|V|-K$

- let V' be a clique of size K in G

let $(u,v) \in \bar{E}$ $\Rightarrow$ $(u,v) \notin E$ $\Rightarrow$ we cannot have both u,v in the clique V' $\Rightarrow$ at least one of these vertices u,v ~~are~~ are not in V' $\Rightarrow$ at least one $u,v \in V-V'$ $\Rightarrow$

$\Rightarrow$ $V-V'$ is a VC in $\bar{G}$, of size $|V|-K$

$\boxed{\Leftarrow}$ suppose $\bar{G}$ has a VC $V' \subseteq V$ of size $|V|-K$. Show that G has a clique of size K.

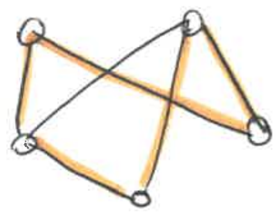
- we'll show that $V-V'$ is a clique in G
- let $u,v \in V-V'$ $\Rightarrow$ $(u,v) \notin \bar{E}$ $\Rightarrow$ $(u,v) \in E$

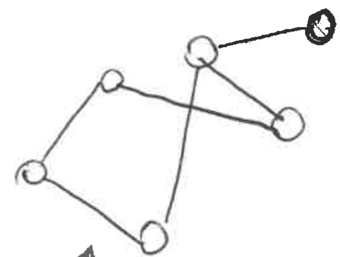
$\Rightarrow$ $V-V'$ is a clique in G of size K

* reduction alg. takes polynomial time

The Hamiltonian-Cycle (HC) problem

- Optimization problem: Given a graph $G(V,E)$ undirected, find a hamiltonian cycle (a simple cycle that contains all vertices in V).
- Decision problem: Given a graph $G(V,E)$ undirected, does G have a hamiltonian cycle?





does NOT have a HC

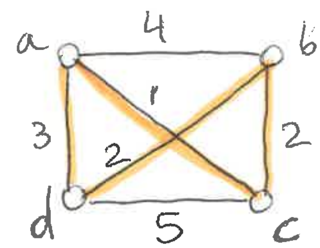
Theorem

HC is NP-complete.

The Traveling-Salesman Problem (TSP)

Given:
- graph (G,E) complete
- cost $c(i,j)$ to travel from city i to city j

Optimization problem: find a minimum cost tour which visits each city exactly once.



Decision problem $\langle G, c, k \rangle$

Given a graph G - complete, undirected
cost function $c : V \times V \to \mathbb{Z}$
bound value $k \in \mathbb{Z}$,
does G have a tour with cost at most k?

Theorem

TSP is NP-complete.

proof

- $TSP \in NP$

certificate: n-vertices in the tour $\langle v_1, v_2, \dots, v_{|V|} \rangle$

<u>verification alg (polynomial)</u>

- check if all vertices are distinct and in V
- sum of the cost on the edges of the tour is $\leq k$

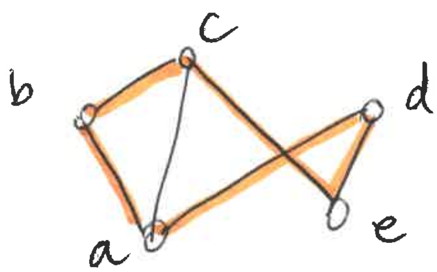
- <u>TSP is NP-hard</u>

$$HC \leq_p TSP$$

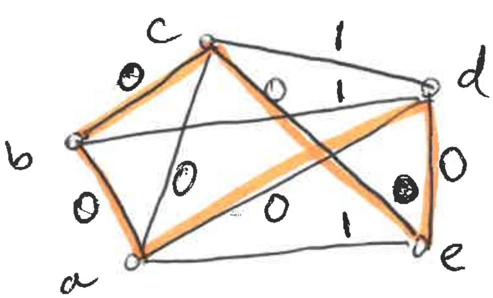
<u>Reduction alg.</u> - takes an instance of the HC problem, and outputs an instance of the TSP

- let $G(V,E)$ undirected by an instance of HC problem
- construct $\langle G', c, 0 \rangle$ an instance of TSP

$G'(V, E')$ - <u>complete</u> graph, undirected



<u>graph G</u>



<u>graph G'</u>

define cost function c:

$$c(i,j) = \begin{cases} 0 & \text{if } (i,j) \in E \\ 1 & \text{otherwise} \end{cases}$$

show that the alg. is a reduction:

G has a HC iff G' has a tour with cost at most 0.

$\boxed{\Rightarrow}$ suppose G has a HC. Show that G' has a tour with cost ≤ 0.

If G has a HC h, then in G' the tour along h has cost 0.

$\boxed{\Leftarrow}$ suppose G' has a tour with cost ≤ 0. ~~then~~ Show that G has a HC.

- if G' has a tour h' with cost ≤ 0 $\Rightarrow$ all edges on h' have cost 0 $\Rightarrow$ all these edges are in E, and they form a HC in G

* this reduction alg. takes polynomial time.