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Author(s): N. Macon and A. Spitzbart

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INVERSES OF VANDERMONDE MATRICES

N. MACON AND A. SPITZBART,* General Electric Company, Evendale

1. Introduction. In a recent paper [1], one finds explicit formulas for the derivatives of a polynomial $y=f(x)$ of degree n in terms of its values $y_i=f(x_i)$ at $n+1$ points defined by $x_i=x_0+ih$, ($i=0, \dots, n$). These results are used here to invert the Vandermonde matrix

$$V(x_1, \dots, x_n) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \cdot & \cdot & \dots & \cdot \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{bmatrix},$$

where the x_i are distinct, different from zero, but otherwise arbitrary.

The paper is in three parts. In the first we outline the results from [1] required later. In the second, these formulas are applied to the special Vandermonde matrix

$$V_{n+1}(x_0, h) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_0 + h & \dots & x_0 + nh \\ \cdot & \cdot & \dots & \cdot \\ x_0^n & (x_0 + h)^n & \dots & (x_0 + nh)^n \end{bmatrix};$$

and the elements of $V_{n+1}^{-1}(x_0, h)$ are obtained in terms of Stirling numbers. Finally, the methods of [1] are extended in such a way that the elements of $V^{-1}(x_1, \dots, x_n)$ can be expressed in terms of the elementary symmetric functions of the x 's. This last result is offered as an alternative to the derivation one would obtain from the classical formula for the values of Vandermonde determinants with missing powers ([2], p. 99).

2. Preliminary results. Let $y=f(x)$ be a polynomial of degree n , and $y_i=f(x_i)$, ($i=0, 1, \dots, n$), where $x_i=x_0+ih$. It was shown in [1] that if we write

$$(1) \quad h^k f^{(k)}(x) = \sum_{i=0}^n A_{mi}^k y_i,$$

then

$$(2) \quad A_{mi}^k = \sum_{j=k}^n \frac{(-1)^{i+j} \binom{j}{i} C_{mj}^k}{j!},$$

where

* Now at Alabama Polytechnic Institute and University of Wisconsin-Milwaukee, respectively.

$$\binom{j}{i} = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i > j, \end{cases}$$

$$(3) \quad C_{mj}^k = \sum_{r=k}^j p_k(r) S_j^r m^{r-k},$$

m is any real number, and $x = x_0 + mh$. In the above, $p_k(r)$ denotes the factorial polynomial of degree k , and the S_k^j are the Stirling numbers of the first kind. Thus, we have $p_k(x) = \sum_{j=1}^k S_k^j x^j$. A wide variety of classical numerical differentiation formulas are special cases of (1).

It was shown further that these results enable one to invert the Vandermonde matrix

$$M(m) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -m & 1-m & \cdots & n-m \\ \vdots & \vdots & \ddots & \vdots \\ (-m)^n & (1-m)^n & \cdots & (n-m)^n \end{bmatrix}.$$

If we write $M^{-1}(m) = \{a_{\lambda,\mu}^m\}$, $(\lambda, \mu = 1, \dots, n+1)$, then

$$(4) \quad a_{\lambda,\mu}^m = \frac{A_{m,\lambda-1}^{\mu-1}}{(\mu-1)!},$$

where $A_{m,\lambda-1}^{\mu-1}$ is given by (2).

3. Vandermonde Matrices for equally spaced points. It is easy to show that

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & h & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h^n \end{bmatrix} M(m) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ -mh & -mh+h & \cdots & -mh+nh \\ \vdots & \vdots & \ddots & \vdots \\ (-mh)^n & (-mh+h)^n & \cdots & (-mh+nh)^n \end{bmatrix}$$

If we write $x_0 = -mh$, it follows that

$$\begin{aligned} V_{n+1}(x_0, h) &\equiv \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_0+h & \cdots & x_0+nh \\ \vdots & \vdots & \ddots & \vdots \\ x_0^n & (x_0+h)^n & \cdots & (x_0+nh)^n \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & h & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h^n \end{bmatrix} M(-x_0/h), \end{aligned}$$

and so

$$V_{n+1}^{-1}(x_0, h) = M^{-1}(-x_0/h) \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & h^{-1} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & h^{-n} \end{bmatrix}.$$

Thus, by (4) the element in the λ th row and μ th column of $V_{n+1}^{-1}(x_0, h)$ is precisely

$$v_{\lambda\mu} = \frac{1}{h^{\mu-1}(\mu-1)!} A_{-x_0/h, \lambda-1}^{\mu-1}.$$

It is interesting to note that even though $V_{n+1}^{-1}(1, 1)$ can be obtained directly from the above, a slight modification enables one to express the elements of this inverse very compactly. For simplicity of notation, we apply the method to obtain $V_n^{-1}(1, 1)$.

Let us write

$$M(0) = \begin{bmatrix} 1 & Q_{1n} \\ R_{n1} & P_{nn} \end{bmatrix}, \quad M^{-1}(0) = \begin{bmatrix} b_{11} & S_{1n} \\ T_{n1} & U_{nn} \end{bmatrix},$$

where $Q_{1n} = (1, \cdots, 1)$, R_{n1} is a column vector of zeros, and

$$P_{nn} = \begin{bmatrix} 1 & 2 & \cdots & n \\ 1 & 4 & \cdots & n^2 \\ \cdot & \cdot & \cdots & \cdot \\ 1 & 2^n & \cdots & n^n \end{bmatrix}.$$

It suffices to invert the matrix P_{nn} , since its columns are scalar multiples of those of $V_n(1, 1)$. Now

$$M(0)M^{-1}(0) = \begin{bmatrix} b_{11} + Q_{1n}T_{n1} & S_{1n} + Q_{1n}U_{nn} \\ R_{n1}b_{11} + P_{nn}T_{n1} & R_{n1}S_{1n} + P_{nn}U_{nn} \end{bmatrix} = I,$$

and so $I_n = R_{n1}S_{1n} + P_{nn}U_{nn} = P_{nn}U_{nn}$. Hence $U_{nn} = P_{nn}^{-1}$. Denote the elements of U_{nn} by μ_{ik} . Since b_{11} contains only a single element, it follows that $\mu_{ik} = a_{i+1, k+1}^0$, and in turn, by (4), $\mu_{ik} = A_{0i}^k/k!$. We have from (2) that

$$A_{0, i}^k = \sum_{j=k}^n \frac{(-1)^{j+i} \binom{j}{i} C_{0j}^k}{j!} \quad (i, k = 1, \cdots, n).$$

From (3) $C_{0j}^k = p_k(k)S_j^k = k!S_j^k$. Finally, we have

$$A_{0i}^k = \sum_{j=k}^n \frac{(-1)^{j+i} \binom{j}{i} k! S_j^k}{j!},$$

and so, by (4),

$$(5) \quad \mu_{ik} = \sum_{j=k}^n \frac{(-1)^{j+i} \binom{j}{i} S_j^k}{j!}.$$

Since the k th column of V_{nn} $(1, 1)$ is k^{-1} times the k th column of P_{nn} , ($k=1, \dots, n$), it follows that $V_n^{-1}(1, 1) = \{i\mu_{ik}\}$, ($i, k=1, \dots, n$), where μ_{ik} is given by (5). As an illustration, the following inverses are given:

$$P_{44}^{-1} = \begin{bmatrix} 4 & -\frac{13}{3} & \frac{3}{2} & -\frac{1}{6} \\ -3 & \frac{19}{4} & -2 & \frac{1}{4} \\ \frac{4}{3} & -\frac{7}{3} & \frac{7}{6} & -\frac{1}{6} \\ -\frac{1}{4} & \frac{11}{24} & -\frac{1}{4} & \frac{1}{24} \end{bmatrix}, \quad V_4^{-1}(1, 1) = \begin{bmatrix} 4 & -\frac{13}{3} & \frac{3}{2} & -\frac{1}{6} \\ -6 & \frac{19}{2} & -4 & \frac{1}{2} \\ 4 & -7 & \frac{7}{2} & -\frac{1}{2} \\ -1 & \frac{11}{6} & -1 & \frac{1}{6} \end{bmatrix}.$$

These inverses may be checked by multiplication with the corresponding direct matrix. P_{44}^{-1} was obtained directly, by use of (5).

4. The inverse of the general Vandermonde matrix. Let $x_0=0, x_1, \dots, x_n$ be the given distinct numbers. The polynomial $y=y(x)$ of degree n assuming $n+1$ arbitrary values $y_i=y(x_i)$, $i=0, \dots, n$, can be written, by Lagrange's interpolation formula, as

$$(6) \quad y = \sum_{i=0}^n A_{xi} y_i.$$

If we write the k th derivative of $y(x)$ as

$$(7) \quad y^{(k)} = \sum_{i=0}^n A_{xi}^k y_i \quad (k = 1, \dots, n),$$

where

$$A_{xi}^k = \frac{d^k}{dx^k} A_{xi},$$

and set $x=x_0=0$, we have

$$(8) \quad y_0^{(k)} = \sum_{i=0}^n A_{0,i}^k y_i.$$

In the following, we obtain explicit expressions for the $A_{0,i}^k$, and then show that the $A_{0,i}^k$ satisfy systems of linear equations having the same coefficient matrix. Thus, we obtain the inverse of this matrix and, in turn, the inverse of $V(x_1, \dots, x_n)$ in terms of the $A_{0,i}^k$.

The functions A_{xi} , as given by the Lagrange interpolation formula, are

$$\begin{aligned} A_{xi} &= \frac{1}{p'_{n+1}(x_i)} \cdot \frac{p_{n+1}(x)}{(x - x_i)} \\ &= \frac{1}{p'_{n+1}(x_i)} (x - x_0) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n); \end{aligned}$$

and so

$$\begin{aligned} A_{xi} &= \frac{1}{p'_{n+1}(x_i)} [x^n - \sigma_{1,n-1}^i x^{n-1} + \sigma_{2,n-1}^i x^{n-2} - \cdots \\ &\quad + (-1)^{n-2} \sigma_{n-2,n-1}^i x^2 + (-1)^{n-1} \sigma_{n-1,n-1}^i x], \end{aligned}$$

where $\sigma_{j,n-1}^i$ is the sum of all products of j of the numbers $x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ without permutations or repetitions ($\sigma_{0,n-1}^i \equiv 1$). If we differentiate the above k times, set $x=0$, and notice that

$$p'_{n+1}(x_i) = \prod_{j=0}^n (x_i - x_j),$$

where the dash indicates that $i \neq j$, we obtain*

$$(9) \quad A_{0i}^k = \frac{(-1)^{n-k} \cdot k!}{\prod_{j=0}^n (x_i - x_j)} \cdot \sigma_{n-k,n-1}^i.$$

In order to exhibit the linear systems mentioned above, we expand $y(x)$ about $x_0=0$, and substitute $x=x_i$ to get

$$y_i = \sum_{\mu=0}^n \frac{1}{\mu!} y_0^{(\mu)} x_i^\mu.$$

* More generally, if we do not substitute $x=0$ after the above differentiation, and substitute the result in (7), we obtain

$$y^{(k)}(x) = \sum_{i=0}^n y_i \sum_{j=k}^n \frac{(-1)^{n-j} \sigma_{n-j,n-1}^i}{p_{n+1}(x_i)} j(j-1) \cdots (j-k+1) x^{j-k},$$

which is a formula for an arbitrary derivative of $y(x)$ at an arbitrary x .

Inserting this into (8) and rearranging, we get

$$y_0^{(k)} = \sum_{\mu=0}^n \frac{1}{\mu!} y_0^{(\mu)} \sum_{i=0}^n x_i^\mu A_{0i}^k \quad (k = 1, \dots, n).$$

Since these are identities in $y_0^{(k)}$, we must have

$$\sum_{i=0}^n x_i^\mu A_{0i}^k = \delta_{\mu k} k! \quad (\mu = 0, 1, \dots, n),$$

where $\delta_{\mu k}$ is the Kronecker delta. Since $x_0 = 0$, it follows that

$$\sum_{i=1}^n x_i^\mu A_{0i}^k = \delta_{\mu k} \cdot k! \quad (\mu = 1, \dots, n).$$

For each k ($k = 1, \dots, n$), this is a system of n linear equations in the unknowns $A_{01}^k, A_{02}^k, \dots, A_{0n}^k$. These n systems can be combined into the matrix equation

$$\{x_k^i\} \cdot \{A_{0i}^k\} = \begin{bmatrix} 1! & 0 & 0 & \dots & 0 \\ 0 & 2! & 0 & \dots & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & \dots & n! \end{bmatrix}$$

where A_{0i}^k is the element of $\{A_{0i}^k\}$ in the i th row and k th column, and similarly for the element x_k^i of $\{x_k^i\}$. (In x_k^i the i denotes an actual exponent.) Denoting the right member of the above equation by $D(n)$, we may write $\{A_{0i}^k\} = \{x_k^i\}^{-1} \cdot D(n)$. Thus, the element $b_{\lambda\mu}$ in the λ th row and μ th column of $\{x_k^i\}^{-1}$ ($\lambda, \mu = 1, \dots, n$) is given by

$$(10) \quad b_{\lambda\mu} = \frac{1}{\mu!} A_{0\lambda}^\mu,$$

where the $A_{0\lambda}^\mu$ may be obtained from (9).

Finally, if $V^{-1}(x_1, \dots, x_n) \equiv \{v_{\lambda\mu}\}$, a method similar to that used in Section 3 yields $v_{\lambda\mu} = x_\lambda b_{\lambda\mu}$, ($\lambda, \mu = 1, \dots, n$), which, together with (10), gives an explicit representation for the inverse of the general Vandermonde matrix.

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