4.2 M/M/1/k Queueing Model

Characteristics

1. Interarrival time is exponential with rate λ
   - Arrival process is Poisson Process with rate λ

2. Interarrival time is exponential with rate μ
   - Number of services is Poisson Process with rate μ

3. Single Server

4. System size is finite = k

5. Queue Discipline : FCFS

Notation

M / M / 1 / k / FCFS

4.2 M/M/1/k Queueing Model

Steady-State Distribution

State of the system

system is in state **n** if there are n customers in the system (waiting or serviced)

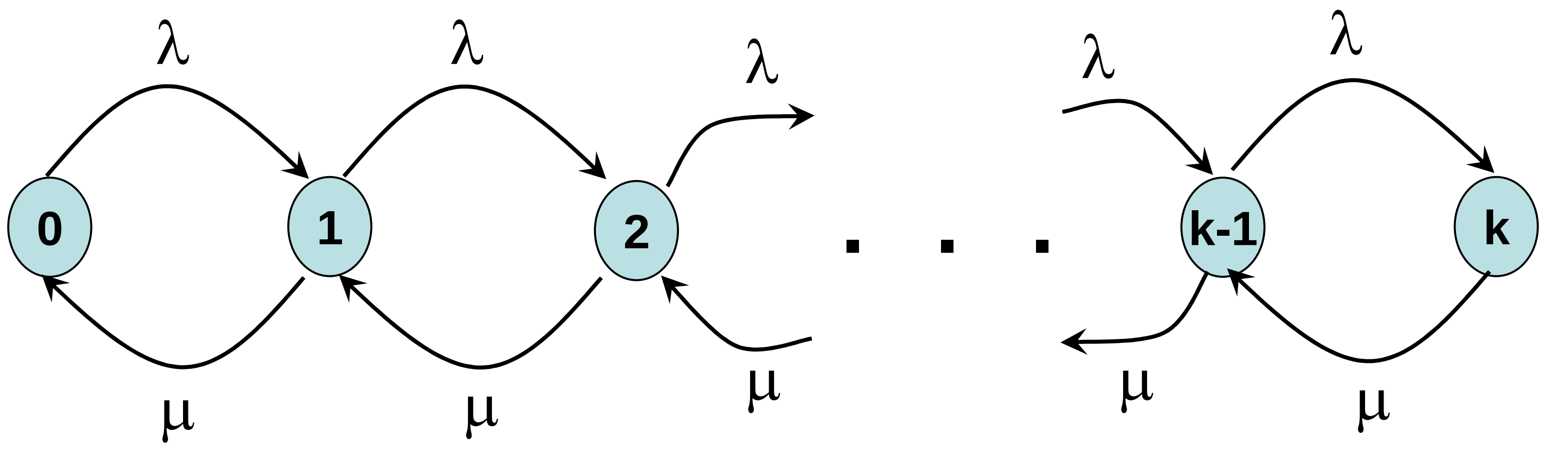
Let P_n be probability that there are n customers in the system in the steady-state. $n = 0 , 1 , 2 , 3 , \ldots, k$

4.2 M/M/1/k Queueing Model

Steady-State Distribution

<u>Rate Diagram:</u>

1. If system changes state, where to go?
2. How fast the system changes state?



4.2 M/M/1/k Queueing Model

Steady-State Distribution

<u>Balance Equations:</u>

For each state n:

$$\begin{bmatrix} \text{Average} \\ \text{Rate } \underline{\textbf{out of}} \\ \text{State } n \end{bmatrix} = \begin{bmatrix} \text{Average} \\ \text{Rate } \underline{\textbf{in to}} \\ \text{State } n \end{bmatrix}$$

$$\text{Average Rate out of state n} = \sum_{\forall k} (\text{rates n} \to \text{k}) \cdot \Pr\{\text{system in state n}\}$$

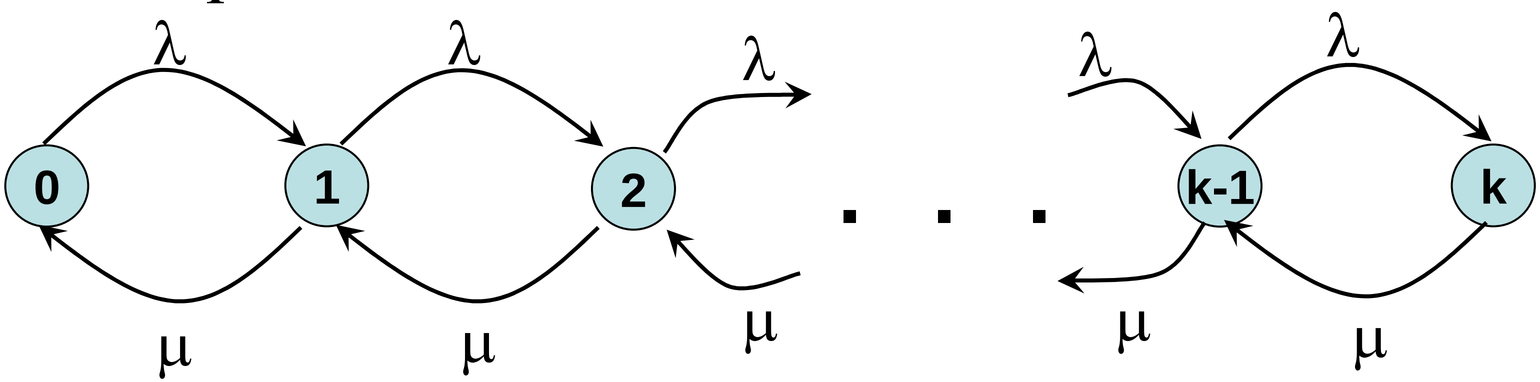
$$\text{Average Rate in to state n} = \sum_{\forall k} (\text{rates k} \to \text{n}) \cdot \Pr\{\text{system in state k}\}$$

4.2 M/M/1/k Queueing Model

Steady-State Distribution

[Average Rate **out of** State *n*] = [Average Rate **in to** State *n*]

Balance Equations:



$$n = 0 \Rightarrow \quad \lambda P_0 = \mu P_1$$

$$n = 1 \Rightarrow \quad \lambda P_1 + \mu P_1 = \lambda P_0 + \mu P_2 \quad \Leftrightarrow (\lambda+\mu)P_1 = \lambda P_0 + \mu P_1$$

$$n = 2 \Rightarrow \quad \lambda P_2 + \mu P_2 = \lambda P_1 + \mu P_3 \quad \Leftrightarrow (\lambda+\mu)P_2 = \lambda P_0 + \mu P_3$$

$$n = 3 \Rightarrow \quad \lambda P_3 + \mu P_3 = \lambda P_2 + \mu P_4 \quad \Leftrightarrow (\lambda+\mu)P_3 = \lambda P_2 + \mu P_4$$

. . . .

$$n = k \Rightarrow \quad \mu P_k = \lambda P_{k-1} \quad \Leftrightarrow \mu P_k = \lambda P_{k-1}$$

4.2 M/M/1/k Queueing Model

Steady-State Distribution

Solution of Balance Equations:

$$\lambda P_0 = \mu P_1$$

$$(\lambda+\mu)P_1 = \lambda P_0 + \mu P_2$$

$$(\lambda+\mu)P_2 = \lambda P_1 + \mu P_3$$

$$\cdots\cdots\cdots\cdots$$

$$\mu P_k = \lambda P_{k-1}$$

Eq-1 $\Leftrightarrow \quad \lambda P_0 = \mu P_1$

Eq-2 $\Leftrightarrow \quad (\lambda+\mu)P_1 - \lambda(\mu/\lambda)P_1 = \mu P_2 \quad \Leftrightarrow \quad \lambda P_1 = \mu P_2$

Eq-3 $\Leftrightarrow \quad (\lambda+\mu)P_2 - \lambda(\mu/\lambda)P_2 = \mu P_3 \quad \Leftrightarrow \quad \lambda P_2 = \mu P_3$

$\cdots\cdots \qquad \cdots\cdots$

Eq-k $\Leftrightarrow \quad \lambda P_{k-1} = \mu P_k$

4.2 M/M/1/k Queueing Model

Steady-State Distribution

Solution of Balance Equations:

Make all equations functions of P_0 only:

$$
\begin{array}{llll}
\text{Eq-1} \Leftrightarrow & \lambda P_0 = \mu P_1 & \Leftrightarrow & P_1 = (\lambda/\mu) P_0 \\
\text{Eq-2} \Leftrightarrow & \lambda P_1 = \mu P_2 & \Leftrightarrow \text{ from Eq-1} \Leftrightarrow & P_2 = (\lambda/\mu)^2 P_0 \\
\text{Eq-3} \Leftrightarrow & \lambda P_2 = \mu P_3 & \Leftrightarrow \text{ from Eq-2} \Leftrightarrow & P_3 = (\lambda/\mu)^3 P_0 \\
 & \cdots\cdots & & \cdots\cdots \\
\text{Eq-k} \Leftrightarrow & \lambda P_{k-1} = \mu P_k & \Leftrightarrow \text{ from Eq-(k-1)} \Leftrightarrow & P_k = (\lambda/\mu)^k P_0
\end{array}
$$

4.2 M/M/1/k Queueing Model

Steady-State Distribution

Solution of Balance Equations:

Computing P_0 : $$\sum_{\forall n} P_n = 1$$

$$P_0 + P_1 + P_2 + P_3 + \dots + P_k = 1$$

$$P_0 + (\lambda/\mu)P_0 + (\lambda/\mu)^2 P_0 + (\lambda/\mu)^3 P_0 + \dots + (\lambda/\mu)^k P_0 = 1$$

$$P_0 [\, 1 + (\lambda/\mu) + (\lambda/\mu)^2 + (\lambda/\mu)^3 + \dots + (\lambda/\mu)^k \,] = 1$$

$$P_0 = [1 + (\lambda/\mu) + (\lambda/\mu)^2 + (\lambda/\mu)^3 + \dots + (\lambda/\mu)^k \,]^{-1}$$

4.2 M/M/1/k Queueing Model

Steady-State Distribution

<u>Solution of Balance Equations:</u>

Computing P_0 :

$$\boxed{P_0 = \frac{1}{\sum_{n=0}^{k} \left(\frac{\lambda}{\mu}\right)^n}}$$

All P_n are functions of P_0 . Then $P_n > 0$ <u>if and only if</u> $P_0 > 0$

Then $P_0 > 0$ for any value of λ and μ since $\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n$ is finite sum

4.2 M/M/1/k Queueing Model

Steady-State Distribution

Solution of Balance Equations:

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$\boxed{P_n = \frac{\left(\frac{\lambda}{\mu}\right)^n}{\sum_{n=0}^{k}\left(\frac{\lambda}{\mu}\right)^n}}$$

$n = 1,2,3, \ldots,k$

For any value of λ and μ

4.2 M/M/1/k Queueing Model

Steady-State Distribution

Solution of Balance Equations:

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$\rho = \frac{\lambda}{\mu} \qquad\qquad P_0 = \frac{1-\rho}{1-\rho^{k+1}}$$

$$\boxed{P_n = \rho^n \frac{1-\rho}{1-\rho^{k+1}}} \qquad n = 1,2,3,\ldots,k$$

for any value of ρ (ρ can be > 1)

4.2 M/M/1/k Queueing Model

Steady-State Distribution

<u>Solution of Balance Equations:</u>

why (λ/μ) can be >1 ??

If system is full arrival rate $\lambda = 0$

Number of customers in system does not go to ∞

4.2 M/M/1/k Queueing Model

Performance Measures

In steady state

$$\lambda_e \, , \, \mu \, , \, P_0$$

$$L_B = E[\text{busy servers}] = E[\#\text{Cust. in service}]$$

$$L_s = L_q + L_B$$

$$W_s = W_q + (1/\mu)$$

$$L_s = \lambda W_s$$

$$L_q = \lambda W_q$$

$$L_B = \lambda W_B$$

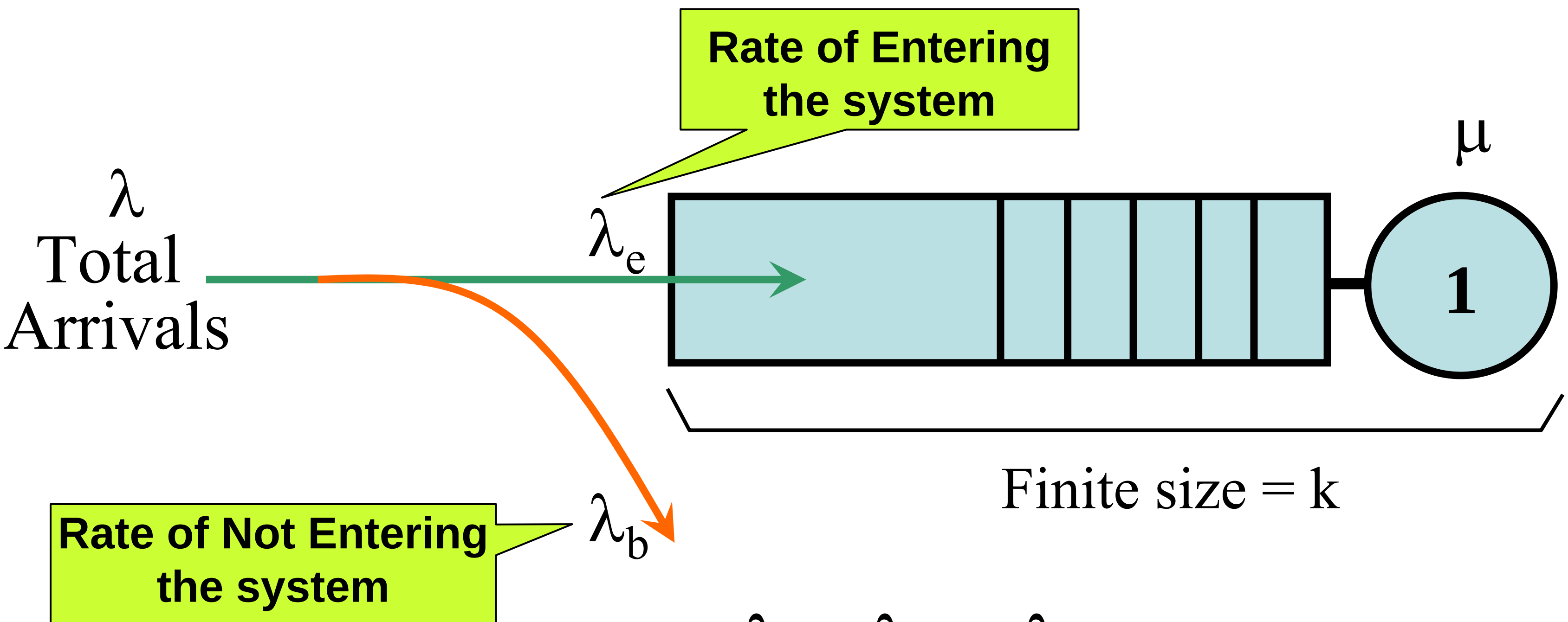
System is in Steady Stead

Know 4 measures $\Rightarrow$ all measures are known

4.2 M/M/1/k Queueing Model

Performance Measures

1. Effective Arrival Rate λ_e:



$$\lambda = \lambda_e + \lambda_b$$

4.2 M/M/1/k Queueing Model

Performance Measures

1. Effective Arrival Rate λ_e:

$$\begin{aligned}\lambda_e &= \lambda \,.\, \Pr\{\text{an arrival } \underline{\textbf{enters}} \text{ the system}\}\\ &= \lambda \,.\, \Pr\{\text{system is } \underline{\textbf{not}} \text{ full}\}\\ &= \lambda \,.\, [P_0 + P_1 + P_2 + \ldots + P_{k-1}]\\ &= \lambda \,.\, [1 - P_k] \quad = \textbf{\textit{Through-put Rate}}\end{aligned}$$

$$\begin{aligned}\lambda_b &= \lambda \,.\, \Pr\{\text{an arrival } \underline{\textbf{can't enter}} \text{ the system}\}\\ &= \lambda \,.\, \Pr\{\text{system is full}\}\\ &= \lambda \,.\, P_k\end{aligned}$$

4.2 M/M/1/k Queueing Model

Performance Measures

<u>2. Average Customers in System L_s:</u>

$$L_s = \sum_{n=0}^{k} n \cdot P_n$$

Finite sum

<u>3. Average Busy servers L_B:</u>

$$L_B = E[\text{busy servers}] = E[\#\text{Cust. in service}]$$

$$L_B = 0\ .\ P_0 + 1\ (P_1 + P_2 + P_3 + \ldots) = 1 - P_0$$

$$= 1 - \frac{1-\rho}{1-\rho^{k+1}} \qquad \boxed{\rho = \frac{\lambda}{\mu}}$$

4.2 M/M/1/k Queueing Model

Performance Measures

4. Utilization of the System U:

$$U = \Pr\{ n > 0 \} = P_1 + P_2 + P_3 + \ldots + P_k = 1 - P_0$$

5. Average Customers in Queue L_q:

$$\rho = \frac{\lambda}{\mu}$$

$$L_q = L_s - L_B$$

or

$$L_q = 0.(P_0 + P_1) + 1.P_2 + 2.P_3 + \ldots + (k-1)P_k$$

4.2 M/M/1/k Queueing Model

Performance Measures

6. Average Waiting time in System W_s:

$$L_s = \lambda_e . W_s \qquad \Leftrightarrow \qquad \boxed{W_s = \frac{L_s}{\lambda_e}}$$

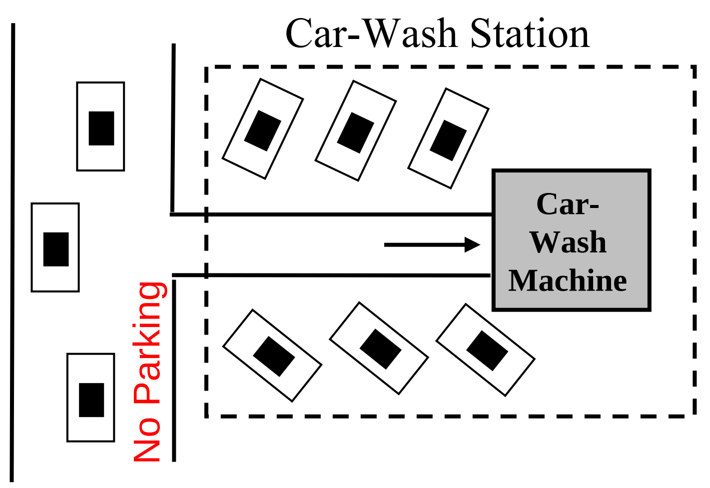
7. Average Time Spent in Queue W_q:

$$L_q = \lambda_e . W_q \qquad \Leftrightarrow \qquad \boxed{W_q = \frac{L_q}{\lambda_e}}$$

4.2 M/M/1/k Queueing Model

Example

Consider the car-wash station in Example-2. Assume now that the it is not allowed for care to wait on the side of the road. So, station has made some modifications so that 6 cars can wait inside the station (See diagram). Also, a driver is hired to move cars from parking to the machine. The driver takes an average of 2 minutes to move the car to the machine.



4.2 M/M/1/k Queueing Model

Example

Assuming that the arrival rate is 9 cars per hour and the washing time is 6 minutes. Also, assume Poisson arrivals and exponential service. Answer the following questions in steady-state:

1. What is the average number of cares waiting in station?
2. If the car wash costs 15 SR and the station works from 8:00am to 8:00 pm how much money the collects per day on average? How much the station losses?
3. On average How much it takes for a customer until he leaves with his car washed?
4. The management decided to buy another machine if the old machine works more than 85% of the time. Will the management buy a new machine?

4.2 M/M/1/k Queueing Model

Example

λ = 9 cars/hour

E[S]= E[driving]+E[washing] = 2 min + 6 min = 8 min

μ = 1/8 cars/hr = 7.5 cars/hr   single machine

ρ = 9/7.5 = 1.2

k =(max. # waiting) + (max. # in service) = 6+1= 7 (max. system size)

M/M/1/k queueing system

$$\boxed{P_n = \frac{\rho^n}{\sum_{n=0}^{k} \rho^n}} \qquad n = 0,1,2, \ldots ,7 \qquad \boxed{\rho = \frac{\lambda}{\mu}}$$

4.2 M/M/1/k Queueing Model

Example

$\lambda = 9$ cars/hour $\mu = 7.5$ cars/hr M/M/1/k=7 system

1. Average number of cares waiting in station = $L_q = L_s - (1 - P_0)$

n	0	1	2	3	4	5	6	7	Σ
ρ^n	1	1.2	1.44	1.73	2.07	2.49	2.99	3.58	16.50
P_n	0.061	0.073	0.087	0.105	0.126	0.151	0.181	0.217	1.00
nP_n	0.000	0.073	0.175	0.314	0.503	0.754	1.086	1.520	4.424

$L_q = L_s - (1 - P_0) = 4.424 - (1 - 0.061) = 3.485$ cars

4.2 M/M/1/k Queueing Model

Example

$\lambda = 9$ cars/hour $\mu = 7.5$ cars/hr M/M/1/k=7 system

2. car wash costs = 15 SR
   works hours = 12 hours
   E[money collected per day] =(15SR) E[cars washed per day] (12hr)
   E[cars washed per day] = $\lambda_e = \lambda\,(1 - P_7) = 9\,(1-0.217) = 7.047$ car

$\Rightarrow$ E[money collected per day] =(15SR)(7.047)(12hr) = **1268.46 SR**

   E[money lost per day] =(15SR) E[cars not washed per day] (12hr)
   E[cars not washed per day] = $\lambda_b = \lambda\,.\,P_7 = 9\,(0.217) = 1.953$ car

$\Rightarrow$ E[money lost per day] =(15SR)(1.953)(12hr) = **351.54 SR**

4.2 M/M/1/k Queueing Model

Example

$\lambda = 9$ cars/hour $\qquad \mu = 7.5$ cars/hr $\qquad$ M/M/1/k=7 system

3. E[time until customer leaves with his car washed] = W_s

$$W_s = L_s / \lambda_e = 4.424/7.047 = 0.6278 \text{ hrs}$$

4. The management decided to buy another machine if the old machine works more than 85% of the time. Will the management buy a new machine?

   Percentage of working time for the old machine = $\Pr\{ n > 0 \} = U$

   $U = 1 - P_0 = 1 - 0.061 = 0.939 > 0.85$

   $\Rightarrow$ Buy a new machine