1.13.2017

(3) Find Θ-notation for the number of times the statement "$x = x+1$" is executed, where n- input size.

```
x = 100
i = n
while i >= 1
    x = x+1
    i = i/2
```

answer

$n=16$ i takes values: $\underbrace{16, 8, 4, 2, 1}$, 0.5

"$x=x+1$" executed for 5 times

i takes values: $\underbrace{\frac{n}{2^0}, \frac{n}{2}, \frac{n}{2^2}, \frac{n}{2^3}, \ldots, \frac{n}{2^k} \geq 1}, \frac{n}{2^{k+1}} < 1$

"$x=x+1$" executed for $(k+1)$ times

$2^k \leq n < 2^{k+1}$

$k \leq \lg n < k+1$

$k = \lfloor \lg n \rfloor$

"$x+1$" executed for $(\lfloor \lg n \rfloor + 1)$ times

$$\boxed{\Theta(\lg n)}$$

(4) same question for:

```
i=n
while i>=1
    for j=1 to n
        x=x+1
    i=i/3
```

answer $\boxed{\Theta(n \cdot \log_3 n)}$

---

(5) same question for:

```
for i=1 to n^2
    for j=1 to i
        x=x+1
```

answer

arithmetic series

$$\sum_{i=1}^{n^2} \sum_{j=1}^{i} 1 = \sum_{i=1}^{n^2} i = 1+2+3+\dots+n^2 = \frac{n^2(n^2+1)}{2} = \Theta(n^4)$$

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(6) RT for Bubble-Sort:

```
for i=1 to n-1
    for j=n downto i+1
        if A[j] < A[j-1]
            exchange A[j] with A[j-1]
```

arithmetic series

$$\sum_{i=1}^{n-1} \sum_{j=n}^{i+1 \text{ downto}} 1 = \sum_{i=1}^{n-1} (n-i) = (n-1)+(n-2)+\dots+1 = \frac{(n-1)\cdot n}{2} = \Theta(n^2)$$

Asymptotic Notations

• Show that $3n^2 - 4 = O(n^3)$ ($f(n)$ = $3n^2-4$, $g(n)$ = n^3)

$c, n_0 = ?$ $\quad 0 \le f(n) \le c \cdot g(n) \quad n \ge n_0$

$$0 \le 3n^2 - 4 \le c \cdot n^3$$

$0 \le 3n^2 - 4$

$3n^2 \ge 4$

$n^2 \ge \frac{4}{3}$

$n \ge \frac{2}{\sqrt{3}}$

$3n^2 - 4 \le cn^3$

$cn^3 - 3n^2 + 4 \ge 0$

let $c = 1$

$n^3 - 3n^2 + 4 \ge 0$

function calculator

$n \ge 0$

$$\boxed{\begin{array}{l} c = 1 \\ n_0 = \frac{2}{\sqrt{3}} \end{array}}$$

Another solution

$c = 1$

$n^3 - 3n^2 + 4 \ge 0$

$n^2(n-3) + 4 \ge 0$

$+$

$n \ge 3$

$$\boxed{\begin{array}{l} c = 1 \\ n_0 = 3 \end{array}}$$

• $2n^2 - 3 = \Omega(n)$

$c, n_0 = ?$ $\quad 0 \le c \cdot g(n) \le f(n) \quad n \ge n_0$

$$0 \le c \cdot n \le 2n^2 - 3$$

✓

$cn \le 2n^2 - 3$

$2n^2 - cn - 3 \ge 0$

let $c = 1$

$2n^2 - n - 3 \geq 0$

function calculater:

$n \geq 1.5$

$\boxed{\begin{aligned} c &= 1 \\ n_0 &= 1.5 \end{aligned}}$

Another solution:

$c = 1$

$2n^2 - n - 3 \geq 0$

$n^2 - n + n^2 - 3 \geq 0$

$n(n-1) + (n^2 - 3) \geq 0$

$n \geq 1 \qquad n \geq \sqrt{3}$

$\boxed{\begin{aligned} c &= 1 \\ n_0 &= \sqrt{3} \end{aligned}}$

- $3n^2 + 5 = \Theta(n^2)$

$c_1, c_2, n_0 = ?$

$0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2\, g(n) \qquad n \geq n_0$

$0 \leq c_1 n^2 \leq 3n^2 + 5 \leq c_2 n^2 \qquad n \geq n_0$

✓

$c_1 n^2 \leq 3n^2 + 5$

$(3 - c_1)n^2 + 5 \geq 0$

let $c_1 = 1$

$2n^2 + 5 \geq 0$

✓

$3n^2 + 5 \leq c_2 n^2$

$(c_2 - 3)n^2 - 5 \geq 0$

let $c_2 = 4$

$n^2 - 5 \geq 0$

$n \geq \sqrt{5}$

$\boxed{\begin{aligned} c_1 &= 1 \\ c_2 &= 4 \\ n_0 &= \sqrt{5} \end{aligned}}$

- $n^3 - 2 = o(n^4)$

$n_0 = ?$ $\quad 0 \le f(n) < c \cdot g(n) \quad n \ge n_0$

$0 \le n^3 - 2 < c \cdot n^4$

$0 \le n^3 - 2$

$n \ge \sqrt[3]{2}$

$n^3 - 2 < cn^4$

$cn^4 - n^3 + 2 > 0$

$cn^3 \left(n - \frac{1}{c}\right) + 2 > 0$

$+$

$n \ge \frac{1}{c}$

$$\boxed{n_0 = \max\left(\sqrt[3]{2}, \frac{1}{c}\right)}$$

- $n^2 - 50 = \omega(n)$

$n_0 = ?$ $\quad 0 \le c \cdot g(n) < f(n) \quad n \ge n_0$

$0 \le c \cdot n < n^2 - 50$

✓

$cn < n^2 - 50$

$n^2 - cn - 50 > 0$

$\frac{n^2}{2} - cn + \frac{n^2}{2} - 50 > 0$

$\frac{n}{2}(n - 2c) + \frac{1}{2}(n^2 - 100) > 0$

$+ \qquad +$

$n \ge 2c \qquad n > 10$

$n_0 = \max(2c, 11)$

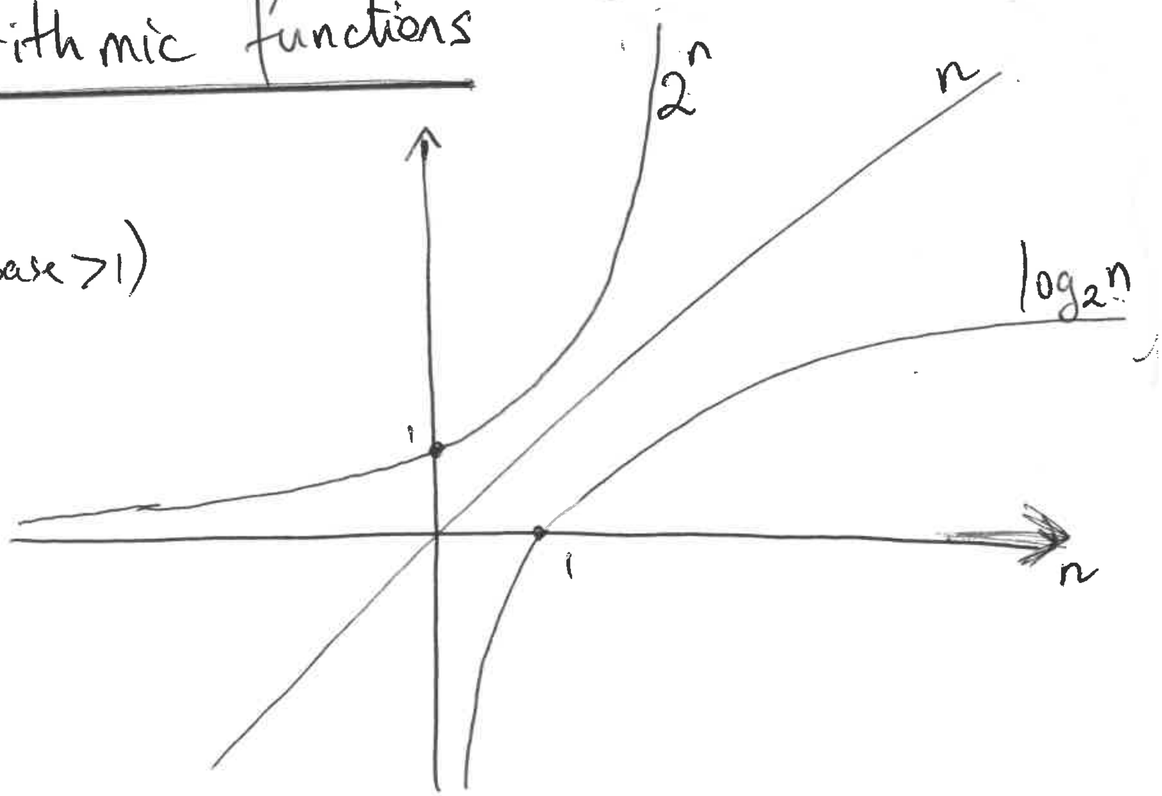
$$\boxed{n_0 = \max(2c, 10) + 1}$$

Rates of growth between polynomial, exponential, and polylogarithmic functions

n

2^n (base > 1)

$\log_2 n$



- Indicate which expressions are true / false:

$2^n + 5\log_2 n = O(n)$ false

$5n^3 + 2n - 100 = \Omega(\lg^7 n)$ true

$n^5 + 7n + n^2 \lg n = o(2^n)$ true

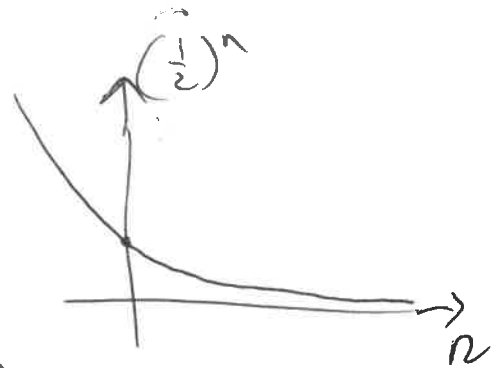
$2n^3 - 7n + 100 = \Theta(2^n)$ false

$n^5 + 7n + n^2 \lg n = \Omega(n)$ true

$\lg^{100} n + 3729 = o(n^2)$ true

- Find Θ-notation for each of the following expressions

$$n^{100} + 392n^2 + 5000 = \Theta(n^{100})$$



$$n^{1000} + 10^5 + 2^n = \Theta(2^n)$$

$$n + (\log_2 n)^{2596} = \Theta(n)$$

$$6n^3 + 12n\, lg^{525} n + (\tfrac{1}{2})^n = \Theta(n^3)$$

$n \cdot n^2$ $\quad$ $n \cdot lg^{525} n$

$$3n^2 + 1000\, n \cdot lg^{50} n = \Theta(n^2)$$

Limits

- Use limits to fill-out the table with yes/no values

$f(n)$	$g(n)$	$f=O(g)$	$f=o(g)$	$f=\Omega(g)$	$f=\omega(g)$	$f=\Theta(g)$
$3^n + lg n$	n^{100}	no	no	yes	yes	no
$3^{lg n} + 100$	$n^{\log_2 5}$	yes	yes	no	no	no
$n^4 + 100$	$n^3 lg^2 n + n^4$					
n	$n^{1+\sin n}$					
$lg^2 n$	n^7					

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{3^n + lg n}{n^{100}} = \infty$$

$$\lim_{n\to\infty} \frac{3^{lg n} + 100}{n^{\log_2 5}} = \lim_{n\to\infty} \frac{n^{1.58} + 100}{n^{2.32}} = 0.$$

$\log_2 5 = 2.32$

$lg\, 3 = 1.58$

$$a^{\log_b c} = c^{\log_b a}$$

$$3^{\lg n} = n^{\lg 3} = n^{1.58}$$