3-coloring problem

4.19.2017

Decision problem: Given an undirected graph G, does G have a 3-coloring?

Theorem

3-coloring is NP-complete.

proof

- **3-coloring ∈ NP**

  certificate: the color of each vertex in G.

  verification algorithm (polynomial time)
  - for each edge (u,v) check if they are colored differently
  - check that each vertex is colored with one of the 3 colors

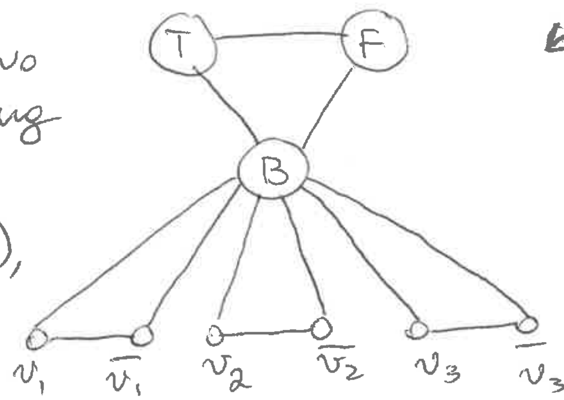
- **3-coloring is NP-hard**

  3-CNF-SAT $\leq_p$ 3-coloring

- start from an instance of 3-CNF-SAT and output an instance of 3-coloring

example $\phi(x_1, x_2, x_3) = (x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor x_2 \lor \overline{x_3})$

- start with a graph G:



graph G

- for each variable x_i, add two vertices v_i and $\overline{v_i}$ corresponding to x_i and $\overline{x_i}$
- add 3 special nodes: T (true), F (false), and B (base)
- join v_i to $\overline{v_i}$ by an edge, and join both to the base B
- join the vertices T, F, B into a triangle

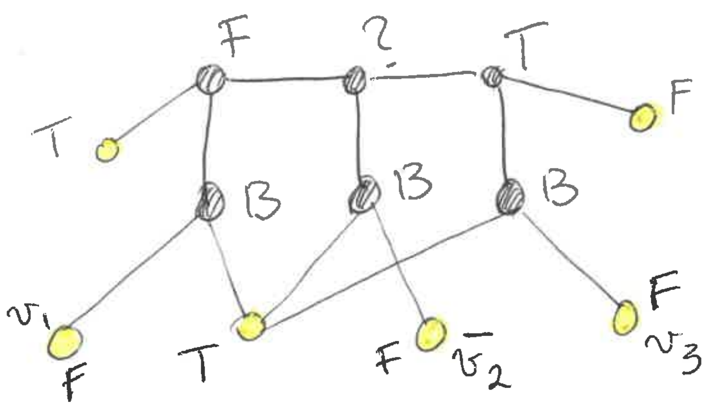
Observation: when coloring the graph G:

- v_i and $\bar{v}_i$ must get different colors, and the colors must be different than B's color
- the nodes T, F, B must get a permutation of the 3 colors. We can use "True" color, "False" color, and "Base" color
- one of $v_i, \bar{v}_i$ is colored with "True", and the other is colored with "False"

- idea: $x_i = 1$ iff v_i is colored True
- we need to design a reduction, such that 3-CNF ϕ is satisfiable iff G' is 3-colorable.
- we will add a graph component for each clause of ϕ

let us consider the clause: $(x_1 \vee \bar{x}_2 \vee x_3)$

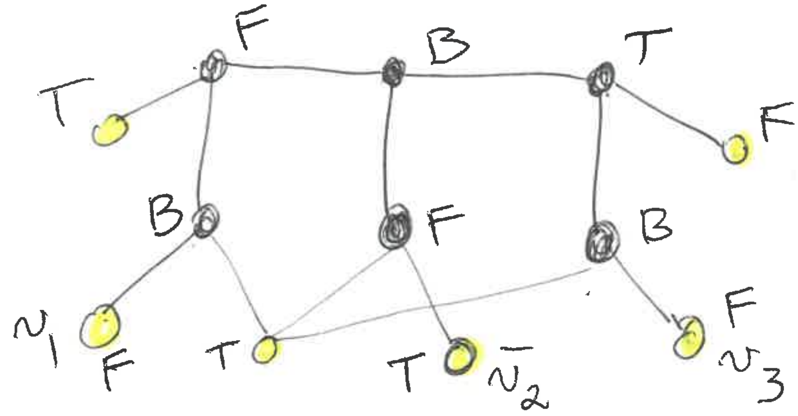
for each clause, add 6 new vertices as follows:



$\Rightarrow$ is NOT 3-colorable

this component is 3-colorable iff the clause evaluates to 1

- assume $x_1 = \bar{x}_2 = x_3 = 0 \Rightarrow$ clause evaluates to 0



(3-colorable)

- assume that the clause has a truth assignment:

  $x_1 = 0$, $\bar{x}_2 = 1$, $x_3 = 0$

- for each clause in ϕ, add a component with 6 vertices, as explained previously
- let us call the resulting graph G'

show that the alg. is a reduction:

ϕ is satisfiable iff G' has a 3-coloring

- <u>Suppose</u> ϕ is satisfiable, then it has a satisfying assignment.

  color the graph G as follows:

  if $x_i = 1$ $\Rightarrow$ color v_i true and $\bar{v}_i$ false

  if $x_i = 0$ $\Rightarrow$ color v_i false and $\bar{v}_i$ true

$\Rightarrow$ graph G' has a 3-coloring

- <u>Suppose</u> G' has a 3 coloring

  assign values for variables as follows:

  if v_i is colored true, then $x_i = 1$

  false, then $x_i = 0$

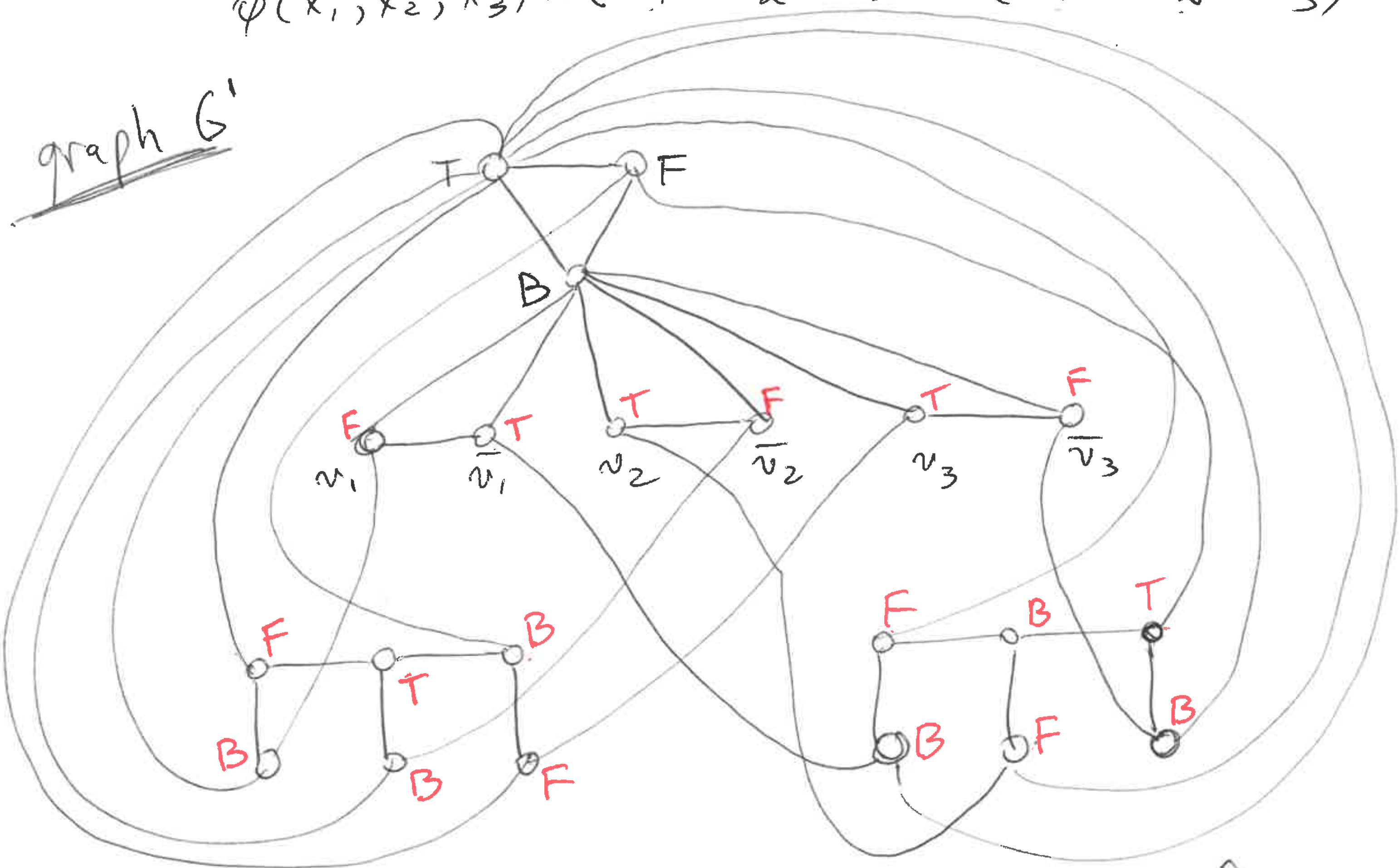
Using this assignment, ϕ evaluates to $\perp$.

- the reduction alg. takes polynomial time.

Example

$$\phi(x_1, x_2, x_3) = (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3})$$

graph G'



Satisfying assignment for ϕ: $\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 1 \end{cases}$

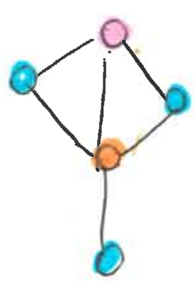
3-coloring

- K-coloring for K>3 is also NP-complete

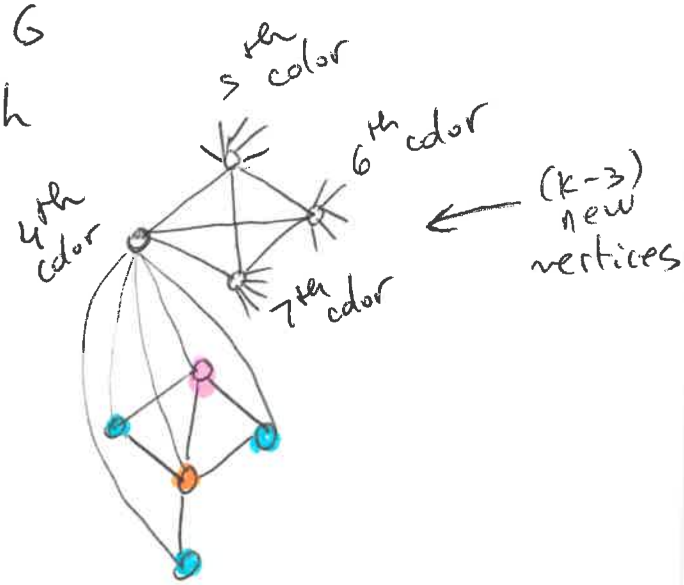
  3-coloring $\leq_p$ K-coloring

<u>reduction alg</u>

- takes an instance of 3-coloring and outputs an instance of K-coloring
- let G be the graph for the instance of 3-coloring
- build a graph G' - an instance of K-coloring - as follows:
  - add (k-3) new vertices to G
  - connect new vertices to each other and to all vertices in G



<u>graph G</u>
3-coloring



<u>graph G'</u>
7-coloring
(k=7)

G is 3 colorable iff G' is K-colorable

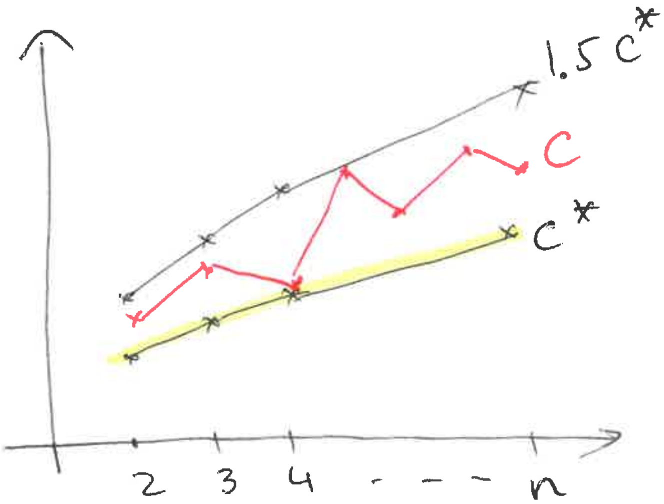
* reduction alg. takes polynomial time

Approximation Algorithms

- Minimization problem

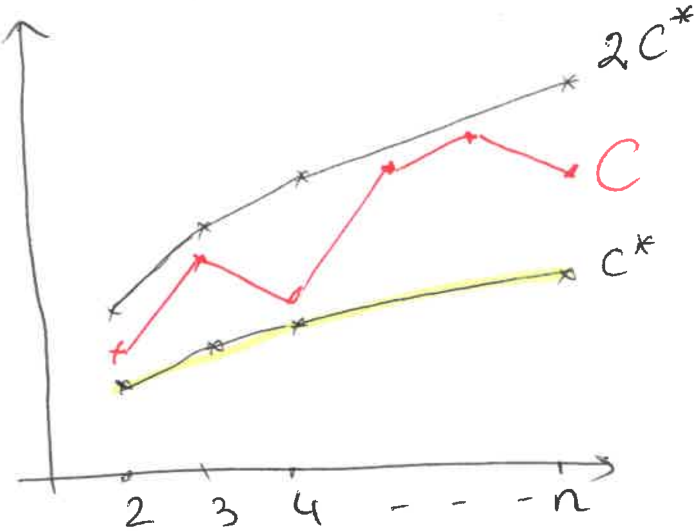
$$\boxed{\frac{C}{C^*} \leq \rho}$$

$$C^* \leq C \leq \rho \cdot C^*$$



1.5 - approximation alg
($\rho = 1.5$)

$C^* \leq C \leq 1.5\, C^*$



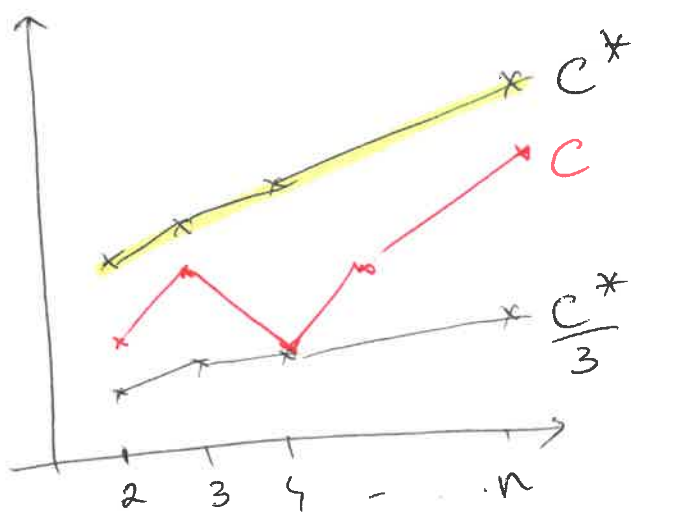
2- approximation alg.
($\rho = 2$)

$C^* \leq C \leq 2C^*$

- Maximization problem

$$\boxed{\frac{C^*}{C} \leq \rho}$$

$$\frac{C^*}{\rho} \leq C \leq C^*$$



3- approximation alg. ($\rho = 3$)

$\frac{C^*}{3} \leq C \leq C^*$