# Limits

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- Use limits to fill-out the table with YES/NO values

| $f(n)$ | $g(n)$ | $f=O(g)$ | $f=o(g)$ | $f=\Omega(g)$ | $f=\omega(g)$ | $f=\Theta(g)$ |
|---|---|---|---|---|---|---|
| $3^n+\lg n$ | $n^{100}$ | no | no | yes | yes | no |
| $3^{\lg n}+100$ | $n^{\log_2 5}$ | yes | yes | no | no | no |
| $n^4+100$ | $n^3\lg^2 n+n^4$ | yes | no | yes | no | yes |
| $n$ | $n^{1+\sin n}$ | — | — | — | — | — |
| $\lg^2 n$ | $n^7$ | yes | yes | no | no | no |

$$\lim_{n\to\infty}\frac{f(n)}{g(n)}=\lim_{n\to\infty}\frac{3^n+\lg n}{n^{100}}=\infty \Rightarrow \omega, \Omega$$

$$\lim_{n\to\infty}\frac{3^{\lg n}+100}{n^{\log_2 5}}=\lim_{n\to\infty}\frac{n^{1.58}+100}{n^{2.32}}=0 \Rightarrow o, O$$

$\log_2 5 =$ 2.32

$\lg 3 = 1.58$

$$a^{\log_b c}=c^{\log_b a}$$

$$3^{\lg n}=n^{\lg 3}=n^{1.58}$$

$$\lim_{n\to\infty}\frac{f(n)}{g(n)}=\lim_{n\to\infty}\frac{n^4+100}{n^3\lg^2 n+n^4}=1 \Rightarrow O, \Omega, \Theta$$

$$\lim_{n\to\infty}\frac{P(n)}{Q(n)} \qquad \frac{3n^4+2n}{5n^4-n^3+1}$$

$$\lim_{n\to\infty} \frac{n^1}{n^{1+\sin n}} = \text{undefined}$$

$-1 \le \sin n \le 1$

$0 \le 1+\sin n \le 2$

$$\lim_{n\to\infty} \frac{\lg^2 n}{n^7} = 0 \quad \Rightarrow o, O$$

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• Find $\Theta$-notation for the following expressions:

$$2+4+6+\dots+2n = 2(1+2+3+\dots+n) \overset{\text{arithmetic series}}{=} \cancel{2}\cdot\frac{n\cdot(n+1)}{\cancel{2}} = \Theta(n^2)$$

$$1+2+4+8+\dots+2^n = \frac{2^{n+1}-1}{2-1} = 2^{n+1}-1 = 2\cdot 2^n - 1 = \Theta(2^n)$$

$$1+\frac{1}{3}+\frac{1}{9}+\dots+\left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2} = \Theta(1)$$

geometric series

• Arrange the following functions in ascending order of growth rate:

$f_1(n) = n^{2.5} = \underline{n^2}\cdot n^{0.5}$

$f_2(n) = \sqrt{2n}$

$f_3(n) = n+10$

$f_4(n) = 2^n$

$f_5(n) = e^n$

$f_6(n) = \underline{n^2}\log_2^{100} n$

$f_7(n) = n!$

$e = 2.718$

Answer: $f_2, f_3, f_6, f_1, f_4, f_5, f_7$

# Recurrences

## Master Thm

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

case 1 $\quad f(n) = O\left(n^{\log_b a - \varepsilon}\right)$ for $\varepsilon > 0 \Rightarrow \boxed{T(n) = \Theta\left(n^{\log_b a}\right)}$

case 2 $\quad f(n) = \Theta\left(n^{\log_b a}\right) \Rightarrow \boxed{T(n) = \Theta\left(n^{\log_b a} \cdot \lg n\right)}$

case 3 $\quad f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right)$ for $\varepsilon > 0$

<u>regularity condition</u>

$$a \cdot f\left(\frac{n}{b}\right) \le c \cdot f(n) \qquad c < 1$$

$\Rightarrow \boxed{T(n) = \Theta(f(n))}$

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- Solve the recurrence

$$T(n) = 8 \cdot T\left(\frac{n}{2}\right) + n^3$$

$a = 8 \qquad f(n) = n^3$

$b = 2$

$f(n)$ vs $n^{\log_b a}$

$n^3$ vs $n^{\log_2 8} = n^3$

$n^3 = \Theta(n^3)$

case 2 Master Thm $\Rightarrow \boxed{T(n) = \Theta(n^3 \cdot \lg n)}$

- $T(n) = 3 \cdot T\left(\frac{n}{2}\right) + n^3 \lg n$

$f(n)$ vs $n^{\log_b a}$

$n^3 \lg n$ vs $n^{\log_2 3} = n^{1.58}$

$n^3 \lg n = \Omega\left(n^{1.58+\varepsilon}\right)$ $\varepsilon = 1$

<u>regularity condition</u>

$$a f\left(\frac{n}{b}\right) \le c \cdot f(n)$$

$$3\left(\frac{n}{2}\right)^3 \lg \frac{n}{2} \le c \cdot n^3 \lg n$$

$$\frac{3}{8}\cancel{n^3}(\lg n - \lg 2) \le c \cdot \cancel{n^3} \lg n$$

$$\frac{3}{8}(\lg n - 1) \le c \lg n$$

$$\frac{3}{8}\lg n - \frac{3}{8} \le c \cdot \lg n$$

$$\left(\frac{3}{8} - c\right)\lg n \le \frac{3}{8}$$

true for $c = \frac{3}{8}$

case 3 of the Master Thm $\Rightarrow \boxed{T(n) = \Theta(n^3 \lg n)}$

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$T(n) = 12 \cdot T\left(\frac{n}{3}\right) + \sqrt{n}$

$f(n)$ vs $n^{\log_b a} = n^{\log_3 12} = n^{2.26}$

$\sqrt{n}$ vs $n^{2.26}$

$\sqrt{n} = O(n^{2.26-\varepsilon})$ for $\varepsilon = 1$

case 1 of the Master Thm $\Rightarrow$ $\boxed{T(n) = \Theta(n^{2.26})}$

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$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n \cdot \lg n$

$f(n)$ vs $n^{\log_2 2} = n$

$n \lg n = \Omega(n^{1+\varepsilon})$ for $\varepsilon = ?$

$n \cdot n^{\varepsilon}$

cannot find an $\varepsilon > 0$ that works

$\Rightarrow$ cannot apply the Master Thm.

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change of variable:

$T(n) = T(\sqrt[3]{n}) + (\log_2 n)^2$

let $m = \lg n$ $\Rightarrow$ $n = 2^m$

$\sqrt[3]{n} = n^{1/3} = (2^m)^{1/3} = 2^{m/3}$

$T(2^m) = T(2^{m/3}) + m^2$

let $S(m) = T(2^m)$

$$S(m) = S(m/3) + m^2$$

$m^2$ vs $m^{\log_3 1} = m^0 = 1$

$m^2 = \Omega(m^{\varepsilon})$ for $\varepsilon = 1$

regularity condition

$$\left(\frac{m}{3}\right)^2 \le c \cdot m^2$$

$$\frac{m^2}{9} \le cm^2$$

$$\frac{1}{9} \le c \checkmark$$

case 3 Master Theorem $\Rightarrow S(m) = \Theta(m^2)$

$$T(2^m) = \Theta(m^2)$$

$m = \lg n \Rightarrow n = 2^m$

$$\boxed{T(n) = \Theta(\lg^2 n)}$$

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$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n \cdot \lg n$$

$$T(1) = \Theta(1)$$

Backward substitution

- assume $n$ is a power of 2

$n = 2^k \Rightarrow k = \lg n$

$$T\left(\frac{n}{2}\right) = 2\cdot T\left(\frac{n}{4}\right) + \frac{n}{2}\lg\frac{n}{2}$$

$$T\left(\frac{n}{4}\right) = 2\cdot T\left(\frac{n}{8}\right) + \frac{n}{4}\lg\frac{n}{4}$$

$n = 2^k$
$k = \lg n$

$$T(n) = 2\cdot T\left(\frac{n}{2}\right) + n\cdot\lg n = 2\left(2\cdot T\left(\frac{n}{4}\right) + \frac{n}{2}\lg\frac{n}{2}\right) + n\lg n =$$

$$= 4\cdot T\left(\frac{n}{4}\right) + n\lg\frac{n}{2} + n\lg n$$

$$= 4\left(2\cdot T\left(\frac{n}{8}\right) + \frac{n}{4}\lg\frac{n}{4}\right) + n\lg\frac{n}{2} + n\lg n$$

$$= 8\cdot T\left(\frac{n}{8}\right) + n\lg\frac{n}{4} + n\lg\frac{n}{2} + n\lg n$$

$$= 2^k\cdot T\left(\frac{n}{2^k}\right) + n\lg\frac{n}{2^{k-1}} + n\lg\frac{n}{2^{k-2}} + \dots + n\lg\frac{n}{2^0}$$

$$= n\cdot T(1) + n\lg\frac{2^k}{2^{k-1}} + n\lg\frac{2^k}{2^{k-2}} + \dots + n\lg\frac{2^k}{2^0}$$

$$= n\cdot\Theta(1) + n\lg 2 + n\lg 2^2 + n\lg 2^3 + \dots + n\lg 2^k =$$

$$= \Theta(n) + n\left(\lg 2 + \lg 2^2 + \lg 2^3 + \dots + \lg 2^k\right) =$$

$$= \Theta(n) + n\lg \underbrace{2\cdot 2^2\cdot 2^3\cdot\ldots\cdot 2^k} = \Theta(n) + n\lg 2^{1+2+3+\ldots+k} =$$

$$= \Theta(n) + n\lg 2^{\frac{k(k+1)}{2}} = \Theta(n) + n\cdot\frac{k(k+1)}{2} =$$

$$= \Theta(n) + n\,\frac{\lg n(\lg n+1)}{2} = \Theta(n\cdot\lg^2 n)$$

$$\boxed{T(n) = \Theta(n \cdot \lg^2 n)}$$

## Correctness

- show that $T(n) = O(n \cdot \lg^2 n)$ using induction

$T(n) \le c\, n \lg^2 n$ for some const $c > 0$

Inductive step

assume that $T(k) \le c k \lg^2 k$ for all $k < n$

show that $\underline{T(n) \le c n \lg^2 n}$

$$T(n) = 2 \cdot \underbrace{T\left(\frac{n}{2}\right)} + n \lg n \le \underbrace{2 \frac{n}{2} \lg^2 \frac{n}{2} + n \lg n \overset{?}{\le} c n \lg^2 n}$$