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Standard form of a Linear Program (LP)

maximize $c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

subject to

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \le b_1$$
$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \le b_2$$
$$\dots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \le b_m$$
$$x_1 \ge 0$$
$$x_2 \ge 0$$
$$\vdots$$
$$x_n \ge 0$$

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maximize $\sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j \le b_i$ for $i = 1, 2, \dots, m$

$x_j \ge 0$ for $j = 1, 2, \dots, n$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

maximize $c^T x$

subject to $A x \le \mathbf{b}$

$x \ge 0$

- Note: we can specify a LP by a tuple $(A, \mathbf{b}, c)$

slack form of a LP

maximize $x_1 + x_2$

subject to

$$\begin{aligned} 4x_1 - x_2 &\le 8 \\ 2x_1 + x_2 &\le 10 \\ -5x_1 + 2x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

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maximize $x_1 + x_2$

subject to

$$\begin{aligned} x_3 &= 8 - 4x_1 + x_2 \\ x_4 &= 10 - 2x_1 - x_2 \\ x_5 &= 2 + 5x_1 - 2x_2 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

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maximize $x_1 + x_2$

subject to

$$\begin{aligned} -4x_1 + x_2 - x_3 &= -8 \\ 2x_1 + x_2 + x_4 &= 10 \\ 5x_1 - 2x_2 - x_5 &= -2 \\ x_1, x_2, x_3, x_4, x_5 &\ge 0 \end{aligned}$$

slack form

Formulate the following problem as an LP:

A carpenter makes tables and chairs. Each table can be sold for a profit of \$30 and each chair for a profit of \$10. The carpenter can afford to spend up to 40 hours per week working and takes 6 hours to make a table and 3 hours to make a chair. Customer demand requires that he makes at least three times as many chairs as tables. Tables take up four times as much storage space as chairs and there is room for at most four tables each week. The objective is to maximize the profit per week.

Hint: use two variables, X_T – number of tables made per week, and X_C - number of chairs made per week

Integer Linear Program (ILP)

maximize $30X_T + 10X_C$

subject to

$$\begin{cases} 6X_T + 3 \cdot X_C \le 40 \\ X_C \ge 3 \cdot X_T \\ X_T + \frac{X_C}{4} \le 4 \end{cases}$$

$$X_C, X_T \in \mathbb{Z}$$

$$X_C, X_T \ge 0$$

The Knapsack Problem

Example: Knapsack capacity W = 16		
item	weight	value
1	2	\$20
2	5	\$30
3	10	\$50
4	5	\$10

Given:

- n objects and a knapsack
- object i (i = 1,..,n) has a positive weight w_i and a positive value v_i
- the knapsack can carry a weight $\leq W$

Objective: fill the knapsack s.t. to maximize the value of the included objects, while respecting the capacity constraints

Two variations:

- **0-1 knapsack problem**: you can only take the whole object
- **fractional knapsack problem**: you can take fractions of objects

Formulate this problem using Linear Programming.

0-1 Knapsack problem

n variables: $x_1, x_2, \ldots, x_n$

$x_i \in \{0,1\}$

$x_i = \begin{cases} 0 & \text{item } i \text{ is NOT selected} \\ 1 & \text{item } i \text{ is selected} \end{cases}$

Boolean Linear Programming (ILP)

maximize $v_1x_1 + v_2x_2 + \ldots + v_nx_n$

subject to $w_1x_1 + w_2x_2 + \ldots + w_nx_n \leq W$.

$x_i \in \{0,1\} \quad i = 1, 2, \ldots, n$

Fractional Knapsack problem

n variables $x_1, x_2, \ldots, x_n$

$0 \leq x_i \leq 1$

x_i – what fraction of item i is taken in the Knapsack

Linear Programming (LP)

maximize $\sum_{i=1}^{n} v_i x_i$

subject to $\sum_{i=1}^{n} w_i x_i \leq W$

$0 \leq x_i \leq 1 \qquad i = 1, 2, \ldots, n$

The Assignment Problem

There are n people who need to be assigned to n jobs, one person per job. The cost of assigning person i to job j is C[i,j]. Find an assignment that minimizes the total cost.

	Job 1	Job 2	Job 3	Job 4
Person 1	9 0	2 1	7 0	8 0
Person 2	6 1	4 0	3 0	7 0
Person 3	5 0	8 0	1 0	8 1
Person 4	7 0	6 0	9 1	4 0

x_{ij}

Formulate this problem using Linear Programming.

n^2 variables x_{ij} $\quad i = 1, 2, \dots, n$
$\quad\quad\quad\quad\quad\quad\quad\quad j = 1, 2, \dots, n$

$$x_{ij} = \begin{cases} 0 & \text{person } i \text{ is not assigned job } j \\ 1 & \text{person } i \text{ is assigned job } j \end{cases}$$

Boolean Linear Programming

$$\text{minimize} \sum_{\substack{i=1,\dots,n \\ j=1,\dots,n}} c_{ij} \cdot x_{ij}$$

$$\text{subject to} \begin{cases} \sum_{j=1}^{n} x_{ij} = 1 & \text{for all } i = 1, \dots, n \\ \sum_{i=1}^{n} x_{ij} = 1 & \text{for all } j = 1, 2, \dots, n \\ x_{ij} \in \{0, 1\} & \text{for all } \begin{cases} i = 1, \dots, n \\ j = 1, \dots, n \end{cases} \end{cases}$$