Approximation Algorithms

4.21.2017

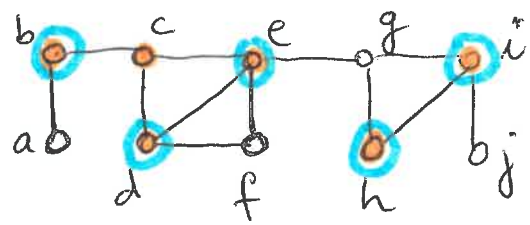
The Vertex-Cover (VC) problem

G(V,E) – undirected

A VC is a subset of vertices $V' \subseteq V$ such that for each edge $(u,v) \in E$, either $u \in V'$, or $v \in V'$, or both $u, v \in V'$

Objective: Given a graph G undirected, find a VC of minimum size.

- this is an NP-complete problem



$VC = \{b, e, i, c, h\}$ | $\{b, d, e, h, i\}$

$|VC| = 6$ | 5

Approximation alg.:

APPROX-VERTEX-COVER(G)

$C = \emptyset$

$E' = E$

while $E' \neq \emptyset$

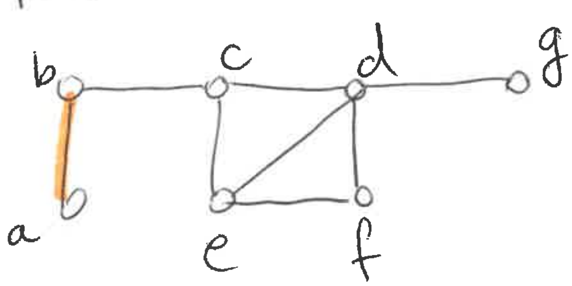
  let (u,v) be an arbitrary edge from E'

  $C = C \cup \{u, v\}$

  remove from E' every edge incident on u or v

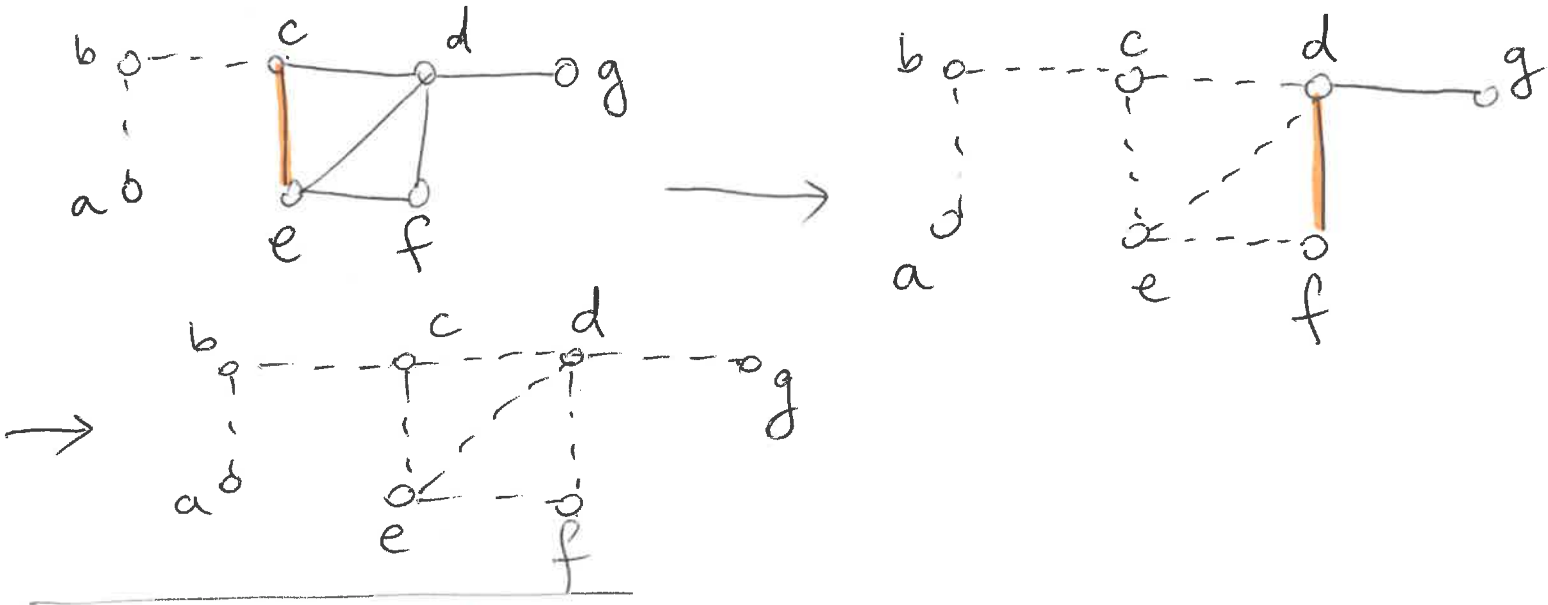
return C

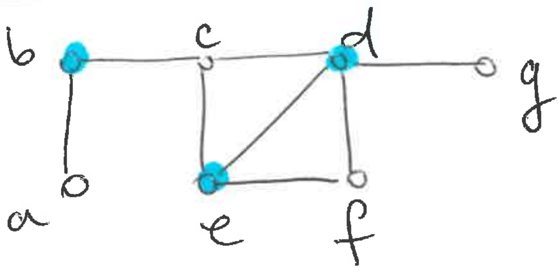
example



$C = \{a, b, c, e, d, f\}$

$|C| = 6$





optimal solution c^*

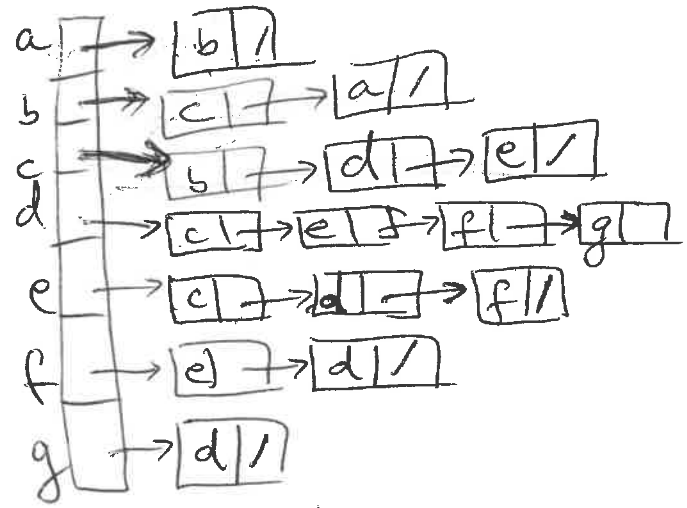
$c^* = \{b, e, d\}$

$|c^*| = 3$

RT analysis

- using adjacency-lists graph representation

$RT = O(V + E)$



graph G

$\boxed{\text{Theorem}}$ APPROX-VERTEX-COVER is a 2-approximation algorithm

proof

let C – VC returned by our alg. (it is a VC since every edge is covered)

C^* – optimal VC

– We need to show that $|C^*| \leq \underbrace{|C| \leq 2 \cdot |C^*|}$

$$\frac{|C|}{|C^*|} \leq 2$$

– let A be the set of edges selected by the alg.

$A = \{(a,b), (c,e), (d,f)\}$

$|C| = 2 \cdot |A|$

– all the edges in A are disjoint

$|C^*| \geq |A|$

$\Rightarrow |C| = 2 \cdot |A| \leq 2 \cdot |C^*| \quad \Rightarrow |C| \leq 2 \cdot |C^*| \Rightarrow$

$$\frac{|C|}{|C^*|} \leq 2$$

$\Rightarrow$ APPROX-VERTEX-COVER is a 2-approximation alg.

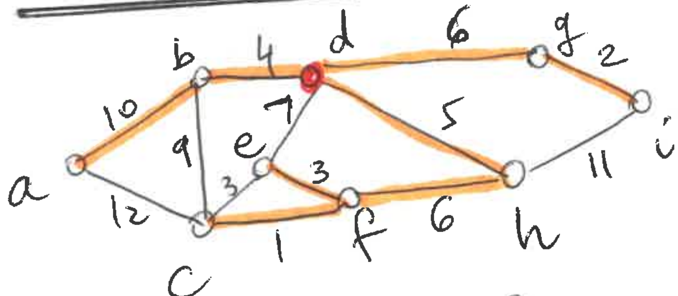
$(\rho = 2)$

Minimum Spanning Tree (MST) — spanning

- tree (which spans over all vertices in the graph) and which has a minimum weight.

- Prim's algorithm (ch 23.2 in CLRS)

d = root vertex

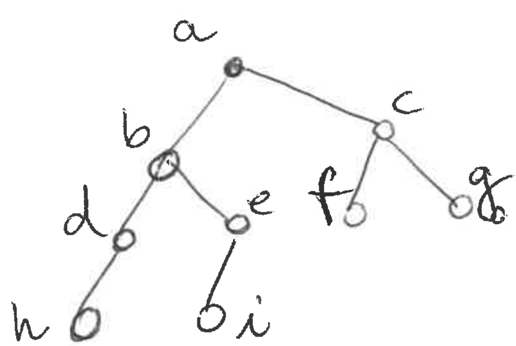


Prim's algorithm: $RT = O(E \cdot \lg V)$ using adjacency-lists graph representation

$RT = O(V^2)$ using adjacency-matrix graph representation

Preorder Tree Walk

- for any node x:
  - print x
  - print x's left subtree
  - print x's right subtree



a, b, d, h, e, i, c, f, g

TSP (Traveling Salesman Problem)

Given : graph $G(V, E)$ undirected, complete
$c(u, v)$ – cost function for each edge $(u, v) \in E$

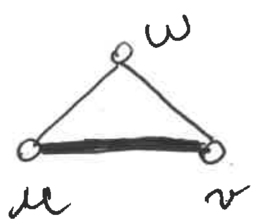
Find a tour that visits each vertex once and has a minimum cost.

Notation
- let A be a set of edges, $A \subseteq E$

$$\boxed{c(A) = \sum_{(u,v) \in A} c(u, v)}$$

- TSP with triangle inequality

$$c(u, v) \le c(u, w) + c(w, v)$$



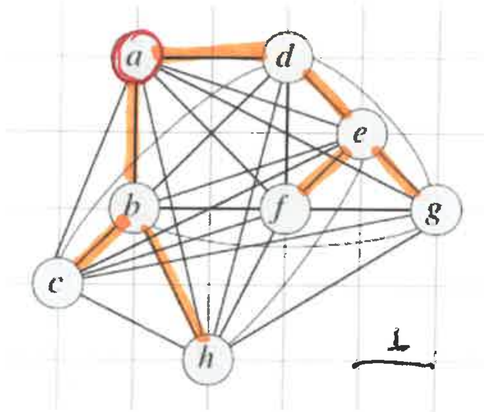
APPROX-TSP-TOUR(G, c)
1. select a vertex $r \in V$ as the "root" vertex
2. compute a MST T for G calling MST-Prim(G, c, r)
3. let H be the list of vertices, according to a preorder tree walk traversal of T
4. return H as the solution for the TSP

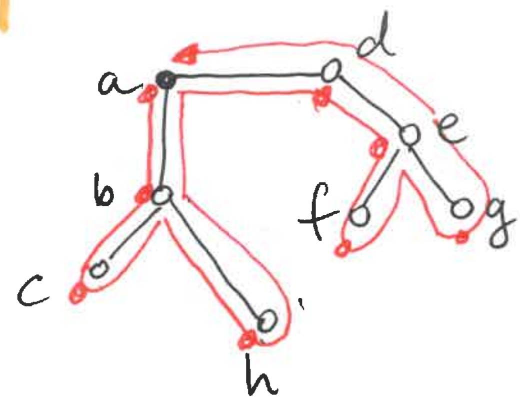
$\boxed{RT = O(V^2)}$ // assuming adj.-matrix graph representation

Example

root = a

MST T

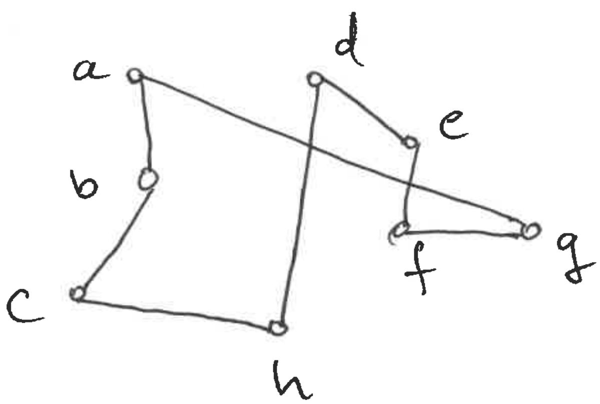


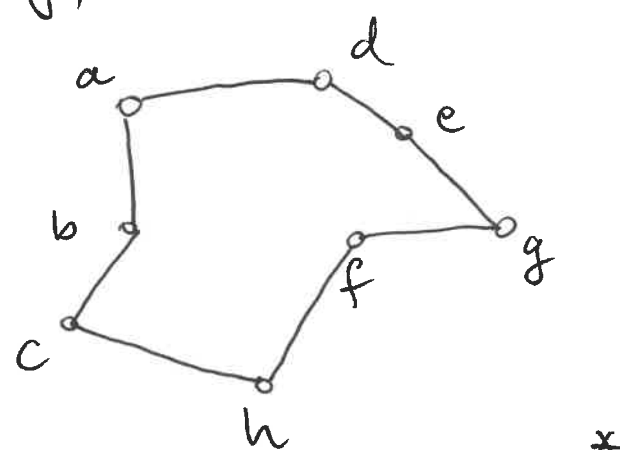


full walk of T, called W = a, b, c, b, h, b, a, d, e, f, e, g, e, d, a

preorder tree walk (each vertex is visited once):

H = a, b, c, h, d, e, f, g, a





approx. TSP **H**

cost = 19.074

optimal solution H^*

cost = 14.715

$$\text{ratio} = \frac{19.074}{14.715} = 1.29 \leq 2$$

Theorem APPROX-TSP-TOUR is a 2-approximation algorithm.

proof

let H^* - optimal solution

need to show that $c(H^*) \le c(H) \le 2 \cdot c(H^*)$

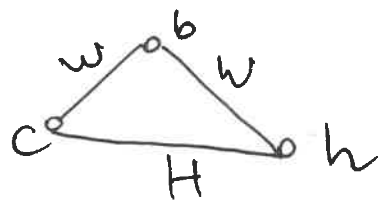
$c(W) = 2 \cdot c(T)$

let T_{H^*} - a spanning tree obtained by deleting an edge from H^*

$c(T) \le c(T_{H^*}) \le c(H^*) \Rightarrow c(T) \le c(H^*)$

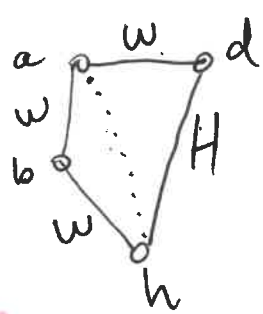
$c(W) = 2 \cdot c(T) \le 2 \cdot c(H^*) \Rightarrow \boxed{c(W) \le 2 \cdot c(H^*)}$

* Next, we'll show that $c(H) \le c(W)$



triangle inequality:

$\underbrace{c(c,h)}_{H} \le \underbrace{c(c,b) + c(b,h)}_{W}$



$\overbrace{c(h,d)}^{H} \le c(a,h) + c(a,d) \le$

$\le \underbrace{c(a,b) + c(b,h) + c(a,d)}_{W}$

- it follows $\boxed{c(H) \le c(W)}$

It results that $c(H) \le 2 \cdot c(H^*) \Rightarrow \boxed{\frac{c(H)}{c(H^*)} \le 2} \Rightarrow$ 2-approximation alg ($\rho = 2$)