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PROOF, EXPLANATION AND EXPLORATION: AN OVERVIEW

ABSTRACT. This paper explores the role of proof in mathematics education and provides justification for its importance in the curriculum. It also discusses three applications of dynamic geometry software – heuristics, exploration and visualization – as valuable tools in the teaching of proof and as potential challenges to the importance of proof. Finally, it introduces the four papers in this issue that present empirical research on the use of dynamic geometry software.

KEY WORDS: dynamic geometry, proof, technology, visualization

INTRODUCTION

There has been a recent upsurge in papers on the teaching and learning of proof: Between 1990 and 1999 the leading journals of mathematics education published over one hundred research papers on this topic. In addition, the *International Newsletter on the Teaching and Learning of Mathematical Proof*, which Nicolas Balacheff has maintained as a Web site since 1997, has been visited over five thousand times to date. This newsletter publishes information on theoretical and empirical research on proof, primarily on papers and books, and is updated six times a year. These are certainly indications that proof is a prominent issue in mathematics education.

Some of this activity has to do with the very 'raison d'être' of proof. This is not surprising, since certain developments in both mathematics and mathematics education have called into question the role of proof. Let me start, then, by stating categorically that proof is alive and well in mathematical practice, and that it continues to deserve a prominent place in the mathematics curriculum. One of our key tasks as mathematics educators, however, is to understand the role of proof in teaching, so that we can enhance its use in the classroom.

Proof is an important part of mathematics itself, of course, and so we must discuss with our students the function of proof in mathematics, pointing out both its importance and its limitations. But in the classroom the key role of proof is the promotion of mathematical understanding, and thus our

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most important challenge is to find more effective ways of using proof for this purpose.

One of these potentially more effective ways is to use dynamic geometry software, which opens up entirely new approaches to the teaching of proof. The four research papers in this special issue address this opportunity, discussing empirical research into the degree to which such software can be used effectively with students to help them develop sound mathematical reasoning, produce valid proofs of geometric propositions, and enhance their understanding of mathematics in the broadest sense. To place this issue in context, these articles also review the research literature on the range of difficulties encountered when teaching proof in the secondary classroom.

My paper, on the other hand, is not empirical in focus. It deals, rather, with some theoretical aspects of questions considered in the four research papers. It first explores the role of proof in mathematics education, and provides some justification for the importance of proof in the curriculum and in particular for its usefulness in promoting understanding. It then goes on to look at heuristics, exploration and visualization, three important potential applications of dynamic software (though of course not exclusive to it), discussing each of them both as a valuable tool in the teaching of proof and as a potential challenge to the importance of proof in the mathematics curriculum. The paper continues with some observations on the need for a clear view, on the part of educators, of the nature of mathematics in general and of the relationship between deduction and experimentation in particular, and concludes with a brief review of the empirical research presented in detail in the four research papers that follow.

THE ROLE OF PROOF

Over a number of years I had been asking myself what role proof ought to play in mathematics education. Of course this raised the underlying question of the role of proof in mathematics itself. An examination of the philosophy and history of mathematics made it clear to me, first of all, that there long have been and still are conflicting opinions on the role of proof in mathematics and in particular on what makes a proof acceptable. This topic is discussed at some length in *Rigorous proof in mathematics education* (Hanna, 1983), which was in essence a critique of the view of proof adopted by the 'new math' movement of the 1950s and 1960s, and in particular of its emphasis on rigour. Then, through a closer examination of mathematical practice, I came to the further conclusion that even in the eyes of practising mathematicians rigorous proof, however it is defined, is secondary in importance to understanding. It became clear to me that a proof, valid as it might be in terms of formal derivation, actually becomes both convincing and legitimate to a mathematician only when it leads to real mathematical understanding.

Mathematical understanding

All four research papers in this issue explore the effect on mathematical understanding of the use of dynamic geometry software in the classroom. Mathematics teachers are well aware that the term 'mathematical understanding' is somewhat elusive. This is not surprising, since the nature of understanding is a topic of discussion among practising mathematicians as well. Mathematicians, however, know that there is such a thing, and in fact most share the view that a proof is most valuable when it leads to understanding, helping them think more clearly and effectively about mathematics (Rav, 1999; Manin, 1992, 1998; Thurston, 1994).

Mathematicians, then, see proofs not only as syntactic derivations (sequences of sentences, each of which is either an axiom or the immediate consequence of preceding sentences by application of rules of inference). Rather, they see proofs as primarily conceptual, with the specific technical approach being secondary. Proofs are the "mathematician's way to *display the mathematical machinery* for solving problems and to *justify* that a proposed solution to a problem is indeed a solution" (Rav, 1999, p. 13).

Rav suggests we think of proofs as "a network of roads in a public transportation system, and regard statements of theorems as bus stops". A similar metaphor is used by Manin (1992) when he says that "Axioms, definitions and theorems are spots in a mathscape, local attractions and crossroads. Proofs are the roads themselves, the paths and highways. Every itinerary has its own sightseeing qualities, which may be more important than the fact that it leads from A to B."

These metaphors speak directly to mathematics education, where a proof is important precisely for its 'sightseeing' qualities. Clearly students ought to be taught the nature and standards of deductive reasoning, so that they can tell when a result has or has not been established. But proof can make its greatest contribution in the classroom only when the teacher is able to use proofs that convey understanding.

The functions of proof

It is useful, when attempting to set out the role of proof in the classroom in a systematic fashion, to consider the whole range of functions which proof performs in mathematical practice. Proof in the classroom would be expected to reflect all of them in some way. But these functions are not all relevant to learning mathematics in the same degree, so of course they should not be given the same weight in instruction (de Villiers, 1990; Hersh, 1993).

As mentioned above, mathematicians clearly expect more of a proof than justification. As Manin (1977) pointed out, they would also like it to make them wiser. This means that the best proof is one that also helps understand the meaning of the theorem being proved: to see not only *that* it is true, but also *why* it is true. Of course such a proof is also more convincing and more likely to lead to further discoveries. A proof may have other valuable benefits as well. It may demonstrate the need for better definitions, or yield a useful algorithm. It may even make a contribution to the systematization or communication of results, or to the formalization of a body of mathematical knowledge.

The following is a useful list of the functions of proof and proving (Bell, 1976; de Villiers, 1990, 1999; Hanna and Jahnke, 1996):

- *verification* (concerned with the truth of a statement)
- *explanation* (providing insight into why it is true)
- *systematisation* (the organisation of various results into a deductive system ofaxioms, major concepts and theorems)
- *discovery* (the discovery or invention of new results)
- *communication* (the transmission of mathematical knowledge)
- *construction* of an empirical theory
- *exploration* of the meaning of a definition or the consequences of an assumption
- *incorporation* of a well-known fact into a new framework and thus viewing it from a fresh perspective

But just as such a richly differentiated view of proof and proving could arise only as the product of a long historical development, so must every student just entering the world of mathematics start with the fundamental functions: verification and explanation. (Scholars pointed out the importance of clarification, as distinct from justification, as early as the 17th century.) But in the classroom, the fundamental question that proof must address is surely 'why?'. In the educational domain, then, it is only natural to view proof first and foremost as explanation, and in consequence to value most highly those proofs which best help to explain.

Some proofs are by their nature more explanatory than others. An insight into what distinguishes an explanatory proof is provided by Steiner (1978), who says that such a proof will make "... reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the results depend on the property" (p. 143). Closely related to Steiner's definition is the concept of *'inhaltlich-anschaulicher Beweis'*. Wittmann and Müller (1990) use this term to characterize a proof in which "[the] method of demonstration calls upon the meaning of the term employed, as distinct from abstract methods, which escape from the interpretation of the terms and employ only the abstract relations between them".

An example is the usual proof that the three angle bisectors of a triangle meet at a single point, the incentre of the triangle. This proof, which makes use of the characterizing property that an angle bisector is the locus of all points equidistant from the edges of the angle, has been found to be both convincing and illuminating, helping students see why the theorem must be true.

For teachers there are great advantages to be gained by choosing explanatory proofs, as Hanna (1990) has discussed in some detail. Of course one cannot always find an explanatory proof for every theorem one wishes to present. In many mathematical subjects some theorems need to be proved using contradiction, mathematical induction or other non-explanatory methods. It so happens that geometry enjoys a special position in this regard, however, in that most of its proofs are explanatory.

HEURISTICS VS. PROOF

Some educators, concerned for the effective allocation of valuable classroom time, maintain that in the competition for classroom focus the topic of proof should take second place to heuristics. They believe that they have to make a choice between developing investigative and problem-solving skills (which in their opinion make mathematics look 'useful', 'enjoyable', and more of 'a human activity'), and instilling the ostensibly less useful and enjoyable skills needed to construct proofs (Simon and Blume, 1996; Simpson, 1995). They would seem to see proof as a chore, and as an impediment to understanding rather than as a route to it.

Simpson (1995) differentiates between 'proof through logic', which emphasizes the formal, and 'proof through reasoning', which involves investigations. The former is 'alien' to students, in his view, since it has no connection with their existing mental structure, and so can be mastered only by a minority. He believes that the latter appeals to the 'natural' learner, however, because it embodies heuristic argument, and so is accessible to a greater proportion of students.

Other educators, in expressing the view that deductive proof need no longer be taught, have focussed not only on reasoning, but also on justification. Their belief is that heuristic techniques are more useful than proof in developing skills even in justification, where proof might have been expected to have enjoyed the advantage. Their argument is that much of what parades in the classroom as the teaching of proof is actually the rote learning of mathematical proofs, devoid of any educational value. They see a more significant educational role for investigation, exploration and informal justification, all of which make use of intuition and yet are more likely than proof, in their view, to engender mathematical insight and even technical fluency. Accordingly they would support cultivating a perception of mathematics as a science that stresses heuristics and the inductive approach. This view has found expression in the NCTM *Standards* (1989) and the British National Curriculum (Noss, 1994).

The NCTM Standards

By the time the *Standards* (1989) was published by the National Council of Teachers of Mathematics (NCTM) in the United States, the concept of proof had all but disappeared from the curriculum (Greeno, 1994) or shrunk to a meaningless ritual (Wu, 1996). The NCTM did not seek to reverse this situation in the curriculum as a whole. It even proposed a shift in the teaching of geometry, the traditional stronghold of proof in the United States, recommending that less emphasis be given to two-column proofs and to Euclidean geometry as an axiomatic system.

On the other hand, the *Standards* did propose greater emphasis on the testing of conjectures, the formulation of counterexamples and the construction and examination of valid arguments, as well as on the ability to use these techniques in the context of non-routine problem solving. There are even two topics, among the seven recommended for greater attention, which have a distinct flavour of proof: (1) short sequences of theorems, and (2) deductive arguments expressed orally and in sentence form (pp. 126–127).

But the NCTM's strategy for the reform of the mathematics curriculum stopped short of mathematical proof as such. Its approach was to stress motivation and 'heuristic argument'. As a result the *Standards* (1989) failed to exploit the potential of proof as a teaching tool. Nor did this document reflect mathematical practice, where heuristic arguments, important as they are in discovery and understanding, are no substitute for proof.

The new version of the NCTM *Principles and Standards* (2000) has remedied this situation by recommending that reasoning and proof be a part of the mathematics curriculum at all levels from prekindergarten through grade 12. The section of this document called 'Reasoning and Proof' states that students should be able to:

- recognize reasoning and proof as fundamental aspects of mathematics;

- make and investigate mathematical conjectures;
- develop and evaluate mathematical arguments and proofs;
- select and use various types of reasoning and methods of proof.

British National Curriculum

Proof has lost ground to heuristics in the United Kingdom as well. Noss (1994) observed that "some educators are convinced that proof in the curriculum is a barrier to investigative and creative activity, and that it is inherently opposed to the spirit of exploration and investigation which has permeated the UK mathematics curriculum for much of the recent past" (p. 6).

The London Mathematical Society, the Institute of Mathematics and its Applications and the Royal Statistical Society reacted to this tendency in a joint paper published in October, 1995. Addressing the British mathematics curriculum from primary school through university, the paper expressed concern for what it termed "a changed perception of what mathematics is – in particular of the essential place within it of precision and proof" (quoted in Barnard et al., 1996, p. 6), and also criticised schools for failing to prepare students adequately for university mathematics.

This statement from the three professional bodies prompted the editors of *Mathematics Teaching* to invite a number of people to discuss the issue of proof in mathematics and to publish their remarks under the title *Teaching Proof* (Barnard et al., 1996). The extreme viewpoint was perhaps that of MacKernan, who deplored "proofs of the ghastliness required by today's academic journals", and went so far as to ask, "So, do we really need proof at all? Especially in schools? ... Why on earth can't we – the overwhelming majority – simply be allowed to accept that something is intuitive, or very probably true, or just simply obvious?" (p. 16).

The views of mathematicians: A recent debate

It is instructive to examine what some mathematicians have had to say about the relative importance of heuristics and proof in mathematics. Jaffe and Quinn (1993), while acknowledging that the use of rigorous proof had been a blessing to mathematics, bringing "a clarity and reliability unmatched by other sciences", identified a trend "toward basing mathematics on intuitive reasoning without proof" (p. 1). They attributed this trend to the influence of the less rigorous standards of reasoning commonly employed in the physical sciences. In reaction to it, they suggested that two distinct types of mathematical endeavour, employing different types of justification, be accorded legitimacy, but that they be clearly distinguished. In their proposal, mathematical results based upon speculative and intuitive reasoning or upon the examination of test cases would be referred to as 'theoretical mathematics', while the label 'rigorous mathematics' would be applied to the results of what has traditionally been regarded as proper mathematics, in which theorems are proven rigorously.

The editors of the *Bulletin of the American Mathematics Society* (1994) invited a number of prominent mathematicians to respond to Jaffe and Quinn's paper, receiving a long contribution from Thurston (1994) and fifteen shorter ones (Atiyah et al., 1994). Most of the respondents rejected the idea of recognizing a separate, speculative branch of mathematical activity. Glimm expressed the general feeling best, perhaps, when he wrote that if mathematics is to cope with the "serious expansion in the amount of speculation" it will have to adhere to the "absolute standard of logically correct reasoning [which] was developed and tested in the crucible of history. This standard is a unique contribution of mathematics to the culture of science. We should be careful to preserve it, even (or especially) while expanding our horizons" (p. 184).

There was agreement that intuition, speculation and heuristics are very useful in the preliminary stages of obtaining mathematical results. Indeed, Schwartz (Atiyah et al., 1994) proposed that mathematicians abandon the old bias that dictates that only rigorous results are accepted for publication, preferring to see scholars "sometimes acting as rigorous mathematicians (if possible), sometimes writing heuristic papers (if rigorous methods do not work)" (p. 199). But all agreed that it is imperative to make a clear distinction between a correct proof and a heuristic argument, and that the validity of mathematical results ultimately rests on proof.

EXPLORATION VS. PROOF

The availability in the classroom of software with dynamic graphing capabilities has given a new impetus to mathematical exploration, and in particular has brought a welcome new interest in the teaching of geometry. Geometer's Sketchpad (Jackiw, 1991) and Cabri Geometry (1996), for example, help students understand propositions by allowing them to perform geometric constructions with a high degree of accuracy. This makes it easier for them to see the significance of propositions. Students can also easily test conjectures by exploring given properties of the constructions they have produced, or even 'discover' new properties. The Sketchpad workbook discusses exploration under seven headings, most of them not part of the traditional geometry curriculum: Investigation, Exploration, Demonstration, Construction, Problem, Art and Puzzle.

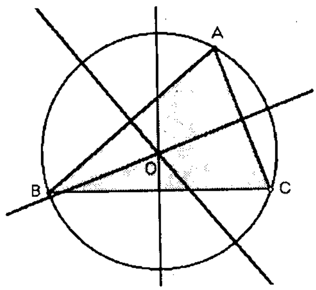

Figure 1.

Dynamic software has the potential to encourage both exploration and proof, because it makes it so easy to pose and test conjectures. But unfortunately the successful use of this software in exploration has lent support to a view among educators that deductive proof in geometry should be downplayed or abandoned in favour of an entirely experimental approach to mathematical justification. Mason (1991), for example, maintains that with dynamic software one can check a large number of cases or even, "by appeal to continuity, an infinite number of cases", and concludes that "truth will be ascribed to observations made in a huge range of cases explored rapidly on a computer" (p. 87).

An example will show how the capabilities of dynamic software could move some to question the need for analytical proof. Suppose a student wants to 'prove' the theorem that in any triangle the perpendicular edge bisectors intersect at a single point. The student could, on paper, construct a triangle and its three perpendicular bisectors and show that this is true. But carrying out this construction with Cabri Geometry or Geometer's Sketchpad has an important advantage. It allows the student to grab a point on the triangle and pull the triangle over the screen in such a manner that it changes its shape. As this is done, the perpendicular bisectors are continuously redrawn correctly. This shows the student that the three perpendicular bisectors still intersect at a single point, called the circumcentre of triangle, no matter what the shape of the triangle (Figure 1). The procedure is at least equivalent to drawing a large number of triangles on paper, or imagining that one had drawn them.

Such a powerful feature provides the student with strong evidence that the theorem is true (and reinforces the value of exploration in general in giving students confidence in a theorem). As Mason (1991) put it, it helps the student form a mental image. It would only be natural if the student were to jump to the conclusion that this exploration is entirely sufficient to establish that the perpendicular bisectors always intersect in a single point. Perhaps less understandable, however, is that some educators have leapt to the same conclusion, misinterpreting the power of computers in demonstration as an indication that proof is no longer a central aspect of mathematical theory and practice.

This is not a new issue. Exploration was an important facet of mathematical practice long before computers were invented, and was not seen as inconsistent with the view of mathematics as an analytic science or with the central role of proof. What we really need to do, of course, is not to replace proof by exploration, but to make use of both.

This is clear, first of all, when one considers that mathematical exploration itself, with or without the aid of a computer, makes much use of deductive reasoning, the very foundation of proof. Polya (1957) has discussed in some detail the role of deductive reasoning in exploration and problem solving. He points out that solving a problem amounts to finding the connection between the data and the unknown, and for this one must use what he calls a 'heuristic syllogism', a kind of reasoning that uses deduction, in addition to circumstantial, inductive and statistical evidence.

In the second place, it is a simple fact that, while exploring and proving are separate activities, they are complementary and reinforce each other. Not only are they both part of problem solving in general, they are both needed for success in mathematics in particular. Exploration leads to discovery, while proof is confirmation. Exploration of a problem can lead one to grasp its structure and its ramifications, but cannot yield an explicit understanding of every link. Thus exploration can lead to conclusions which, though precisely formulated, must remain tentative. Though the truth of a proposition may seem apparent from exploration, it still needs, as Giaquinto (1994) points out, 'demonstrable justification'. Only a proof, by providing a derivation from accepted premises, can provide this.

The teacher's classroom challenge is to exploit the excitement and enjoyment of exploration to motivate students to supply a proof, or at least to make an effort to follow a proof supplied. One reason to go this extra step is that exploration does not reflect the totality of mathematics itself, because mathematicians aspire to a degree of certainty that can only be achieved by proof. A second reason is that students should come to understand the first reason: As most mathematics educators would still agree, students need to be taught that exploration, useful as it may be in formulating and testing conjectures, does not constitute proof.

VISUALIZATION AND VISUAL PROOFS

A number of mathematicians and logicians are now investigating the use of visual representations, and in particular their potential contribution to mathematical proofs. In the last decade or so such investigations have gained in scope and status, in part because computers have increased the possibilities of visualization so greatly. Such studies are being pursued at many places, such as the Visual Inference Laboratory at Indiana University and the Centre for Experimental and Constructive Mathematics (CECM) at Simon Fraser University in British Columbia. At most of these institutions, the departments of philosophy, mathematics, computer science and cognitive science cooperate in research projects devoted to developing computational and visual tools to facilitate reasoning.

Researchers who recommend the use of visual representations in mathematics and mathematics teaching realize, of course, that misleading diagrams abound. Brown (1999) has presented some of the well-known examples of diagrams that might lead to error. This fact alone, however, does not give reason to believe that visualization does not have promise for investigation and teaching.

A key question raised by the intensified study of visualization is whether, or to what extent, visual representations can be used, not only as evidence for a mathematical statement, but also in its justification. Diagrams and other visual aids have long been used to facilitate understanding, of course. They have been welcomed as heuristic accompaniments to proof, where they can inspire both the theorem to be proved and approaches to the proof itself. In this sense it is well accepted that a diagram is a legitimate component of a mathematical argument. Every mathematics educator knows that diagrams and other visual representations are also an essential component of the mathematics curriculum, where they can convey insight as well as knowledge. They have not been considered substitutes for traditional proof, however, at least until recently. Today there is much controversy on this topic, and the question is now being explored by several researchers.

According to Francis (1996), for example, the fact that more and more mathematicians turn to computer graphics in mathematical research does not obviate the need for rigour in verifying the knowledge acquired through visualization. He does recognize that "the computer-dominated information revolution will ultimately move mathematics away from the sterile formalism characteristic of the Bourbaki decades, and which still dominates academic mathematics". But he adds that it would be absurd to expect computer experimentation to "replace the rigour that mathematics has achieved for its methodology over the past two centuries". For Francis, visual reasoning is clearly not on a par with sentential reasoning.

Other researchers have come to similar conclusions. Palais (1999), for example, is a mathematician at Brandeis University who has been working on a mathematical visualization program called 3D-Filmstrip for more than five years. He reports on his use of computers to model mathematical objects and processes, where he defines a process as "an animation that shows a related family of mathematical objects or else an object that arises by some procedure naturally associated to another object" (p. 650). He observes that visualization through computer graphics makes it possible not only to transform data, alter images and manipulate objects, but also to examine features of objects that were inaccessible without computers. Palais concludes that visualization can directly show the way to rigorous proofs, but he stops short of saying that visual representations could be accepted as legitimate proofs in themselves.

Borwein and Jörgenson of CECM, as well, have examined the role of visualization in reasoning in general and in mathematics in particular. The two questions they posed to themselves were: "Can it contribute directly to the body of mathematical knowledge?" and "Can an image act as a form of 'visual proof'?" They answer both these questions in the positive, though they would insist that a visual representation meet certain qualifications if it is to be accepted as a proof.

Borwein and Jörgenson cite the many differences between the visual and the logical modes of presentation. Whereas a mathematical proof, as a sequence of valid inferences, has traditionally been presented in sentential mode, a visual representation purporting to constitute a 'visual proof' would be presented as a static picture. They point out that such a picture may well contain the same information as the traditional sentential presentation, but would not display an explicit path through that information and thus, in their opinion, would leave "the viewer to establish what is important (and what is not) and in what order the dependencies should be assessed".

For this reason these researchers believe that successful visual proofs are few and far between, and tend to be limited in their scope and generalizability. They nevertheless concede that a number of compelling visual proofs do exist, such as those published in the book *Proofs without words* (Nelsen, 1993). As one example, they present the following heuristic diagram, which proves that the sum of the infinite series $1/4 + 1/16 + 1/64 + \ldots = 1/3$ (See Figure 2).

Borwein and Jörgenson suggest three necessary (but not sufficient) conditions for an acceptable visual proof:

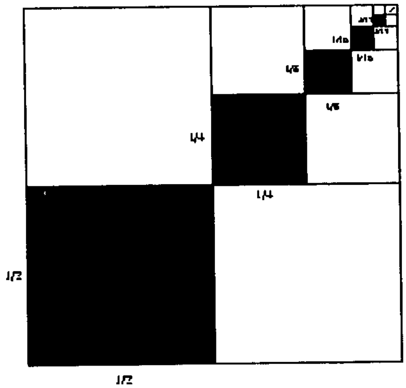

Figure 2.

- Reliability: That the underlying means of arriving at the proof are reliable and that the result is unvarying with each inspection
- Consistency: That the means and end of the proof are consistent with other known facts, beliefs and proofs
- Repeatability: That the proof may be confirmed by or demonstrated to others

One might wonder whether these criteria would not apply to proofs in general, not only to visual ones. One might also object that the first criterion in particular, lacking as it does a definition of 'reliable means', does not provide sufficient specific guidance in separating acceptable from unacceptable visual proofs. Indeed, Borwein and Jörgenson make no claim to have answered this question definitively. Nevertheless, they would assign to visual reasoning a greater role in general in mathematics, and believe that some visual representations can constitute proofs.

For other researchers, too, the idea that visual representations are no more than heuristic tools is a dogma that needs to be challenged. Barwise and Etchemendy (1991, 1996) sought ways to formalize diagrammatic reasoning and make it no less precise than traditional deductive reasoning. They acknowledge that the age-old notion of proof as a derivation, consisting of a sequence of steps leading from premises to conclusion by way of valid reasoning, and in particular the elaboration of this notion in modern mathematical logic, have contributed enormously to progress in mathematics. They claim, however, that the focus on logical structures and sentential reasoning has led to the neglect of many other forms of mathematical thinking, such as diagrams, charts, nets, maps, and pictures, that do not fit the traditional inferential model. They also argue that it is possible to build logically sound and even rigorous arguments upon such visual representations.

These two researchers proceeded from what they call an informational perspective, building upon the insight that inference is "the task of extracting information implicit in some explicitly presented information" (Barwise and Etchemendy, 1996, p. 180). This view leads to a criterion for the validity of a proof in the most general sense: "As long as the purported proof really does clearly demonstrate that the information represented by the conclusion is implicit in the information represented by the premises, the purported proof is valid" (p. 180). The authors go on to point out that whenever "there is structure, there is information", and that a visual representation, which may be highly structured, can carry a wealth of information very efficiently. Because information may be presented in both linguistic and non-linguistic ways, they conclude that strict adherence to inference through sentential logic is too restrictive, inasmuch as sentential logic applies solely to linguistic representations.

The question is how to extract the information implicit in a visual representation in such a manner as to yield a valid proof. Barwise and Etchemendy show examples of informal derivations, such as the use of Venn diagrams, and suggest that perfectly valid visual proofs can be built in a similar fashion upon the direct manipulation of visual objects. Unfortunately, as they point out, the focus on sentential derivation in modern mathematics has meant that little work has been done on the development of protocols for derivation using visual objects, so that there is much catching up to do if visual proof is to realize its considerable potential.

Though the view of these researchers is that proof does not depend on sentential representation alone, they do not believe that visual and sentential reasoning are mutually exclusive. On the contrary, much of their work has been aimed at elaborating the concept of 'heterogeneous proof'. Building upon this position, Barwise and Etchemendy (1991) have developed *Hyperproof*, an interactive program which facilitates reasoning with visual objects. It is designed to direct the attention of students to the content of a proof, rather than to the syntactic structure of sentences, and teaches logical reasoning and proof construction by manipulating both visual and sentential information in an integrated manner. With this program, proof goes well beyond simple inspection of a diagram. A proof proceeds on the basis of explicit rules of derivation that, taken as a whole, apply to both sentential and visual information.

EPISTEMOLOGICAL PITFALLS

As the four research papers in this issue clearly show, dynamic geometry software can be used to enhance the role of heuristics, exploration and visualization in the classroom. The use of such approaches has always posed two questions that are really a single issue: How can these approaches best be integrated into the mathematics curriculum as a whole, and how can they best be used to promote understanding? The advent of the attractive and engaging techniques offered by computer software only raises the profile of this issue and the urgency with which it needs to be addressed.

In attempting to think about the issue, it is revealing to examine the epistemology of mathematics implied by much of present classroom practice and to compare it with the epistemology, as unformed and uncertain as it may be, that students bring to the classroom. As Hanna and Jahnke (1999) have discussed, the discrepancies can have significant educational implications.

Students are often taught, for example, that the angle-sum theorem for triangles is true in general only because it has been proven mathematically. There is no reference to the measurement of real triangles. This practice implies a very specific and limited view of the nature of mathematics and in particular of its relationship to the outside world. Students do not share this view, however; they typically come to class with the belief that geometry has something to say about the triangles they find around them. For this reason it should come as no surprise to educators when students misinterpret the teacher's assertion that mathematical proof is sufficient in geometry to mean that empirical truth can be arrived at by pure deduction.

It has been observed, to cite another example of the implications of epistemological confusion, that students, having been shown the proof of a theorem, will quite often ask for empirical testing, even though they say they understand the proof (Fischbein, 1982). From a purely mathematical viewpoint such a request seems unreasonable, and teachers usually take it as an indication that the students do not really understand what a mathematical proof is. From the viewpoint of an experimental scientist, however, it seems quite natural. No physicist, for example, would accept a fact as true on the basis of a theoretical deduction alone. Thus a consideration of the role of mathematical proof in the experimental sciences may well shed light on how students view proof.

Computer-supported heuristics, exploration and visualization can be valuable tools for fostering understanding. But with the increased use of such techniques in the classroom, there is also increased potential for misunderstanding, with epistemology at its root. The challenge for educators is to convey very clearly to students the interplay of deduction and experimentation and the relationship between mathematics and the real world.

To do so, educators will need to gain more insight into the assumptions of their students. They will need to examine, for example, how students approach proof while using dynamic geometry software such as Cabri Geometry or the Sketchpad, which allow explorative work similar to that of experimental physics. But they may need to reassess their own assumptions as well, asking in particular whether these assumptions truly reflect the accounts of the nature of mathematics that are implicit in the practice of mathematics itself and espoused by mathematicians and philosophers of mathematics. Only through such a reassessment will they be in a position to cope with the essentially epistemological questions that are bound to be created in students' minds when they construct mathematical proofs using concepts and arguments derived from their own preliminary explorations.

The four papers that follow present the results of empirical research which demonstrates that the judicious use of dynamic geometry software in heuristics, exploration and visualization can foster an understanding of proof. Mariotti looks at how the students' view of geometry moves from an 'intuitive' one, in which it is seen as a collection of evident properties, to a 'theoretical' one, in which it is seen as a system of related statements that are validated by proof. According to her, this transition is greatly facilitated by the use of dynamic software that affords visualization (a 'by eye' strategy, as she calls it), exploration and in particular the use of heuristics. The latter starts with revisiting and manipulating drawn objects and leads to conjectures, discussion and finally to a mathematical proof. In Mariotti's view, the dynamic software contributes to the understanding of 'theoretical' geometry by providing a 'semiotic mediation' that helps students make sense of the process of exploring, conjecturing, and arguing as a way of arriving at a valid proof.

Jones' paper focuses on the evolution of students' ability to make use of precise language and to classify correctly a family of quadrilaterals. Starting with a visual exploration of the similarities and differences between quadrilaterals, the classroom experiment aims to provide a chance for the students to engage in deductive reasoning to arrive at an understanding of the relationships between the various properties of quadrilaterals. The students then display their new understanding by creating a hierarchical classification for quadrilaterals. Jones observes that the dynamic software clearly helped students to formulate reasonably precise statements about properties and relationships and to carry out correct deductions – both important steps in constructing proofs.

Marrades and Gutiérrez examine ways in which dynamic geometry software can be used to improve students' understanding of the nature of mathematical proof and to improve proof skills. They report on the progress made by two pairs of students in justifying mathematical statements, moving from informal reasons to proper mathematical proofs by adopting increasingly sophisticated modes of reasoning. The authors also show how dynamic geometry software, in affording students greater access to exploration, heuristics and visualization, actually increased their understanding of the limitations of such informal approaches and thus of the need for deductive proof.

Hadas, Hershkowitz and Schwartz investigate the ability of students to reflect and offer valid arguments when confronted with surprise and uncertainty. The students taking part in the study found a contradiction between their own conjectures on the sum of the exterior angles of a polygon and the results they obtained through measurement. It was observed that these students were able to arrive at a resolution of the contradiction by using dynamic geometry software to help them work through heuristic strategies, explorations and visual arguments.

All four papers describe the successful use of dynamic software. It should be noted that in all four cases this use was accompanied by carefully designed tasks, by professional teacher input, and by opportunities for students to notice details, to conjecture, to make mistakes, to reflect, to interpret relationships among objects, and to offer tentative mathematical explanations. It seems reasonable to assume that the use of dynamic software in the classroom might not be as effective in the absence of these supporting factors.

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