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Social Networks
HW3

Problem 1

Q1

1.
   (a) Show by direct calculation that the expected degree is np.
   (b) Where is the mode of the binomial distribution?
   (c) Compute directly the variance of the distribution

Solution:

(a) Expected Degree:

$$E(K) = \sum_{k=1}^{n} k * \frac{n!}{k!(n-k)!} * p^k * (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n*(n-1)!}{(k-1)!(n-k)!} * p * p^{k-1} * (1-p)^{n-k}$$

$$= np * \sum_{k=1}^{n} k * \frac{(n-1)!}{(k-1)!(n-k)!} * p^{k-1} * (1-p)^{n-k}$$

Use substitution with a and b. $a = k-1$ and $b = n-1$

$$E(K) = np * \sum_{a=0}^{b} \frac{b!}{a!(b-a)!} * p^a * (1-p)^{b-a}$$

We now see that

$$\sum_{a=0}^{b} \frac{b!}{a!(b-a)!} * p^a * (1-p)^{b-a}$$

is just another binomial distribution that will sum to 1, because probability of random variables sum to one. Now we can deduce

$$E(K) = np * \sum_{a=0}^{b} \frac{b!}{a!(b-a)!} * p^a * (1-p)^{b-a}$$

$$= np * 1$$

Solution:

$$E(k) = np$$

(b) Mode of the distribution: We will find the mode of the Binomial distribution at

$$(n+1)p$$

(c) Variance of the distribution:

$Variance\ K = E(K^2) - (E(K))^2$

$(E(K))^2 = n^2p^2$ We know this from part (a).

$E(K^2) = n^2p^2 + np(1-p)$

When we subtract the two, we get

$Variance\ K = E(K^2) - (E(K))^2$

$= n^2p^2 + np(1-p) - n^2p^2$

$$= np(1-p)$$

Solution:

$$= np(1-p)$$

Problem 2

Q2

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 In G(n, 1/n) what is the probability that there is a vertex of
degree log n? Give an exact formula; also derive simple approximations.
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The exact formula is:

$$\binom{n}{\log(n)}\left(\frac{1}{n}\right)^{\log n}\left(1-\frac{1}{n}\right)^{n-\log n}$$

Here are a couple of examples. If $n = 3$, we get

$$\binom{3}{\log(3)}\left(\frac{1}{3}\right)^{\log 3}\left(1-\frac{1}{3}\right)^{3-\log 3} \approx 0.4338765$$

If $n = 5$, we get

$$\binom{5}{\log(5)}\left(\frac{1}{5}\right)^{\log 5}\left(1-\frac{1}{5}\right)^{5-\log 5} \approx 0.2931279$$

Problem 3

Q3

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(a)  What is the expected number of triangles in G(n,d/n)
(b)  What is the expected number of squares in G(n,d/n)
(c)  What is the expected number of 4-cliques in G(n,d/n)
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(a) Probability that there is a cycle of 3, means that three nodes have to have a certain edge, which is a probability of p^3. We know, $p = \frac{d}{n}$. The total amount of possible node triangle possibilities is $\binom{n}{3}$. When we multiply these two, we get

Solution:

$$\left(\frac{d}{n}\right)^3 * \binom{n}{3}$$

(b) With the same logic as part a, we can deduce the expected number of squares with

Solution:

$$\left(\frac{d}{n}\right)^4 * \binom{n}{4}$$

(c) Probability of 4 nodes being in a clique in this graph is

$$4 * \left(\frac{d}{n}\right)^3$$

Now the expected value of this means we have to see how many possible times this can happen, which we know is $\binom{n}{4}$. To find our solution, we multiply these two together and we get

Solution:

$$4 * \left(\frac{d}{n}\right)^3 * \binom{n}{4}$$