Keeyon Ebrahimi
N14193968
Assignment 5

Exercise 9.1:

P(X = -1) = 0.2
P(X = 2) = 0.5
P(X = 6) = 0.3

Expected Value = $(-1 * 0.2) + (2 * 0.5) + (6 * 0.3) = 2.6$

Mean = (-1 + 2 + 6) / 3 = $2\frac{1}{3} = \frac{7}{3}$

Variance = $\frac{(-1-\frac{7}{3})^2+(2-\frac{7}{3})^2+(6-\frac{7}{3})^2}{2} = \frac{37}{3} = 12.\overline{3}$

This problem is about a Sample Variance and not a Population variance, which is why we divide by 2, $n-1$, instead of 3

Standard Deviation = $\sqrt{Variace} = \sqrt{\frac{37}{3}} = 3.5119$

Exercise 9.2:

(a) Marginal Distributions
$P(X) = [(0.12 + 0.08 + 0.10), (0.20 + 0.04 + 0.25), (0.08 + 0.10 + 0.03)]$
$P(Y) = [(0.12 + 0.20 + 0.08), (0.08 + 0.04 + 0.10), (0.10 + 0.25 + 0.03)]$

$$P(X) = [0.3, 0.49, 0.21]$$
$$P(Y) = [0.4, 0.22, 0.38]$$

	-1	1	2	$P(X)$
0	0.12	0.08	0.10	0.3
1	0.20	0.04	0.25	0.49
3	0.08	0.10	0.03	0.21
$P(Y)$	0.4	0.22	0.38	

(b) X and Y Independent? **Solution: No**

Lets label our original joint distribution as F. If we are dealing with something that is independent, we should get $F(X,Y) = P(X) * P(Y)$ for all (x, y) in range, or each cell in the table. This is because $P(A, B) = P(A) * P(B)$ with independent events.

$$F(X, Y)$$

	-1	1	2	$P(X)$
0	0.12	0.08	0.10	0.3
1	0.20	0.04	0.25	0.49
3	0.08	0.10	0.03	0.21
$P(Y)$	0.4	0.22	0.38	

$$P(X) * P(Y)$$

	-1	1	2
0	0.12	0.07	0.11
1	0.20	0.11	0.19
3	0.08	0.05	0.08

As we can see, when we multiply the margins, we do not get the same the same as $F(X, Y)$, so because $F(X, Y) \neq P(X) * P(Y)$, we know that they are not Independent.

(c) Exp(X) and Exp(Y)

$$\text{Exp(X)} = (0 * 0.3) + (1 * 0.49) + (3 * 0.21) = 1.12$$
$$\text{Exp(Y)} = (-1 * 0.4) + (1 * 0.22) + (2 * 0.38) = 0.58$$

(d) Distribution of $X + Y$.

We must first find all the possible values for $X + Y$. X can be $[0, 1, 3]$. Y can be $[-1, 1, 2]$ This means that the possible values for $X + Y$ are $[-1, 1, 2, 0, 3, 4, 5]$, thus

$P(-1) = 0.12$
$P(0) = 0.20$
$P(1) = 0.08$
$P(2) = 0.10 + 0.04 + 0.08 = 0.22$
$P(3) = 0.25$
$P(4) = 0.10$
$P(5) = 0.03$

(e) $P(X|Y = 2)$ and $P(Y|X = 1)$

i. $P(X|Y = 2)$

We know that
$$P(X|Y = 2) = \frac{P(X, Y = 2)}{P(Y = 2)}$$
Now lets compute
$$P(Y = 2) = 0.38$$

This means that
$$P(X = 0|Y = 2) = \frac{0.1}{0.38} = 0.26315 = 26.32\%$$
$$P(X = 1|Y = 2) = \frac{0.25}{0.38} = 0.65789 = 65.79\%$$
$$P(X = 3|Y = 2) = \frac{0.03}{0.38} = 0.07894 = 7.895\%$$

ii. $P(Y|X = 1)$

We know that
$$P(Y|X = 1) = \frac{P(X = 1, Y)}{P(X = 1)}$$

Now lets compute

$$P(X = 1) = 0.49$$

This means that

$$P(Y = -1|X = 1) = \frac{0.20}{0.49} = 0.40816 = 40.82\%$$

$$P(Y = 1|X = 1) = \frac{0.04}{0.49} = 0.08163 = 8.163\%$$

$$P(Y = 2|X = 1) = \frac{0.25}{0.49} = 0.5102 = 51.02\%$$

Exercise 9.3:

Let X be a random variable with values 0, 1, 3, and let Y be a random variable with values -1, 1, 2. Suppose that P(X=0) = 0.5, P(X=1) = 0.4, and P(X=3) = 0.1, with the following values of $P(Y|X)$

$P(Y|X)$

	-1	1	2
0	0.5	0.3	0.2
1	0.2	0.7	0.1
3	0.4	0.1	0.5

(a) Joint Distribution of X, Y

We are given all $P(X)$ and also all $P(Y|X)$. We must now solve for $P(X,Y)$ for the joint distribution. We know that $P(X) * P(Y|X) = P(X,Y)$, so we just need to multiply in the correct locations to have the correct values. This results in

$P(X,Y)$

	-1	1	2
0	0.25	0.15	0.1
1	0.08	0.28	0.04
3	0.04	0.01	0.05

(b) Distribution of Y

$$P(Y = -1) = 0.25 + 0.08 + 0.04 = 0.37 = 37\%$$
$$P(Y = 1) = 0.15 + 0.28 + 0.01 = 0.44 = 44\%$$
$$P(Y = 2) = 0.1 + 0.04 + 0.05 = 0.19 = 19\%$$

(c) Corresponding table for $P(X|Y)$

$$P(X,Y) = P(X|Y) * P(Y)$$
$$\frac{P(X,Y)}{P(Y)} = P(X|Y)$$

$P(X,Y)$

	-1	1	2
0	0.6757	0.341	0.5263
1	0.216	0.636	0.2105
3	0.108	0.0227	0.2623

(d) Distribution of $X + Y$

We must first find all the possible values for $X + Y$. X can be $[0, 1, 3]$. Y can be $[-1, 1, 2]$ This means that the possible values for $X + Y$ are $[-1, 1, 2, 0, 3, 4, 5]$, thus

$P(-1) = 0.25$
$P(0) = 0.08$
$P(1) = 0.15$
$P(2) = 0.10 + 0.28 + 0.04 = 0.42$
$P(3) = 0.04$
$P(4) = 0.01$
$P(5) = 0.05$

(e) Exp(X), Exp(Y), Exp($X + Y$).

(i) **Exp(X)** $= 0.7$

$$Exp(X) = (0 * 0.5) + (1 * 0.4) + (3 * 0.1) = 0.7$$

(ii) **Exp(Y)** $= 0.45$

$$Exp(Y) = (-1 * 0.37) + (1 * 0.44) + (2 * 0.19) = 0.45$$

(iii) **Exp($X + Y$)** $= 1.15$

By Theorem 9.8 from page 260, (Proof on page 261), we know that Exp($X + Y$) = Exp(X) + Exp(Y)

$$Exp(X + Y) = Exp(X) + Exp(Y) = 0.7 + 0.45 = 1.15$$

Problem 6

(a) What is the value of a (written as "a" in the drawing)

Solution: $a = 9$

With a probability density function, the area under the function has to go to 1. This is because $P(A) + P(\overline{A}) = 1$. So we must find the complete area under the function before a, and then find out how long a must be by finding the difference of 1 and the area under the function before a.

(i) Area under $x < 2.0 = 0$

(ii) Area under $2.0 \leq x < 4.0 = 0.4$

The equation of this section is $Y = .2$. To find the area under this part of the function, we take the integral of this equation and give it a range from 2.0 to 4.0.

$$\int_2^4 0.2\ dx = 4.0(0.2) - 2.0(0.2) = 0.4$$

(iii) Area under $4.0 \leq x < 6.0 = 0.3$

We must first find the equation of the line from $4.0 \leq x < 6.0$.

Slope $4.0 \leq x < 6.0$

$$m = \frac{0.1 - 0.2}{6.0 - 4.0} = -0.05$$

Equation Y

$$Y = -0.05(x - 4.0) + 0.2 = -0.05x + 0.4$$

Area Under Y from $4.0 \leq x < 6.0$

$$\int_4^6 -0.05x + 0.4 = \left(\frac{-0.05}{2} * 6^2 + (0.4 * 6)\right) - \left(\frac{-0.05}{2} * 4^2 + (0.4 * 4)\right) = 0.3$$

(iv) Area under $6.0 \leq a = 0.3$

So far our current area under the curve up until $x = 6.0$ is $0.4 + 0.3 = 0.7$ We need total area under the curve to be 1, so the remaining amount we need to have under the curve is $1 - 0.7 = 0.3$.

(v) Value of $a = 9$

We need 0.3 more under the curve, and we know that past 6, we keep the constant rate of 0.1. so

$$a = 6 + \frac{0.3}{0.1} = 9$$

(b) Using the given f, what is the probability that a randomly chosen real number will fall in the interval [5.0, 7.0]?

Solution $= 0.225 = 22.5\%$

From 5.0 to 6.0, we have a different probability than from 6.0 to 7.0. We need to find the probability through both these intervals and add them together.

(i) Probability from 5.0 to $6.0 = 0.125$

As discovered in $Parta.iii$ from this problem, we know that from 5.0 to 6.0 we do

$$\int_5^6 -0.05x+0.4 = (\frac{-0.05}{2}*6^2+(0.4*6))-(\frac{-0.05}{2}*5^2+(0.5*4)) = 0.125$$

(ii) Probability from 6.0 to $7.0 = 0.1$

$$Value = 0.1 * (7.0 - 6.0) = 0.1$$

(iii) Probability from 5.0 to $7.0 = 0.225$

$$Value = 0.125 + 0.1 = 0.225$$

(c) What is the cumulative distribution function? Please both define it as done in the bulleted list above above and produce a drawing.

(i) Definition
Area from 0 to $2 = 0$
Area from 2 to $4 = 0.4$
Area from 4 to $5 = .175$. We just apply equation from $PartBi$ but change the integral range to 4 and 5 to get this value
Area from 5 to $6 = .125$
Area From 6 to $9 = 0.3$

Know these values, we can easily represent the cumulative distribution function

Cumulative Distribution Function

x	$P[X \leq x]$
0	0
1	0
2	0
3	0.2
4	0.4
5	0.575
6	0.7
7	0.8
8	0.9
9	1.0

$P(X \leq 2) = 0$

$P(X \leq 3) = 0 + 0.2 = 0.2$

$P(X \leq 4) = 0 + 0.2 + 0.2 = 0.4$

$P(X \leq 5) = 0 + 0.2 + 0.2 + 0.175 = 0.575$

$P(X \leq 6) = 0 + 0.2 + 0.2 + 0.175 + 0.125 = 0.7$

$P(X \leq 7) = 0 + 0.2 + 0.2 + 0.175 + 0.125 + 0.1 = 0.8$

$P(X \leq 8) = 0 + 0.2 + 0.2 + 0.175 + 0.125 + 0.1 + 0.1 = 0.9$

$P(X \leq 9) = 0 + 0.2 + 0.2 + 0.175 + 0.125 + 0.1 + 0.1 + 0.1 = 1.0$

(ii) Drawing