Statistics & Probability (IT): CSCI 1308

Chapter 04: Discrete Probability Distributions

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What We Will Look At

Chapter 04: Discrete Probability Distributions

Chapter Outline

Section 4.1

Probability Distributions

What We Will Look At

Section 4.1 Objectives

- **How to distinguish between discrete random variables and continuous random variables**
- **How to construct a discrete probability distribution and its graph and how to determine if a distribution is a probability distribution**
- **How to find the mean, variance, and standard deviation of a discrete probability distribution**
- **How to find the expected value of a discrete probability distribution**

Random Variables

Definition

- Represents a numerical value associated with each outcome of a probability distribution.
- Denoted by **X**.
- **Examples:**
  - Let **X** be the number of heads in four tosses of a fair coin(discrete).

    $$\mathbf{X} = \{0, 1, 2, 3, 4\}$$

  - Let **X** Number of sales calls a salesperson makes in one day (discrete).

    $$\mathbf{X} = \{0, 1, 2, 3, \cdots\}$$

  - Let **X** Hours spent on sales calls in one day(continuous).

    x can have any value between 0 and 24.

Definition: Discrete Random Variable

- Has a finite or countable number of possible outcomes that can be listed.

- **Examples:**
  - Let **X** be the number of heads in four tosses of a fair coin(discrete).

$$\mathbf{X} = \{0, 1, 2, 3, 4\}$$

  - Let **X** Number of sales calls a salesperson makes in one day (discrete).

$$\mathbf{X} = \{0, 1, 2, 3, \cdots\}$$

Number of Calls (Discrete)

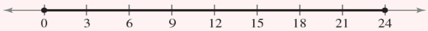
x can have only whole number values: 0, 1, 2, 3, . . .

Definition: Continuous Random Variable

- Has an uncountable number of possible outcomes, represented by an interval on the number line.

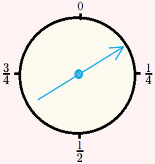
- **Examples:**
  - Let **X** Hours spent on sales calls in one day(continuous).

Hours Spent on Calls (Continuous)



x can have any value between 0 and 24.

  - Choose a number at random between 0 and 1, allowing **any** number between 0 and 1 as the outcome.



A spinner that generates a random number between 0 and 1.

Example 1:

Decide whether the random variable X is discrete or continuous.

(1) Let X represent the number of Fortune 500 companies that lost money in the previous year.

Solution:

Discrete random variable (The number of companies that lost money in the previous year can be counted, $\{0, 1, 2, 3, ..., 500\}$).

(2) Let X represent the volume of gasoline in a 21-gallon tank.

Solution:

The amount of gasoline in the tank can be any volume between 0 gallons and 21 gallons. So, X is a **continuous random variable**.

Discrete Probability Distributions

Definition: Discrete Probability Distributions

- Lists each possible value the random variable can assume, together with its probability.
- Must satisfy the following conditions:

IN WORDS	IN SYMBOLS
1. The probability of each value of the discrete random variable is between 0 and 1, inclusive.	$0 \leq P(x) \leq 1$
2. The sum of all the probabilities is 1.	$\Sigma P(x) = 1$

Because probabilities represent relative frequencies, a discrete probability distribution can be graphed with a relative frequency histogram.

Constructing a Discrete Probability Distribution

Guidelines: Constructing a Discrete Probability Distribution

Let X be a discrete random variable with possible outcomes $x_1, x_2, \cdots, x_n$.

(1) Make a frequency distribution for the possible outcomes.

(2) Find the sum of the frequencies.

(3) Find the probability of each possible outcome by dividing its frequency by the sum of the frequencies.

(4) Check that each probability is between 0 and 1, inclusive, and that the sum of all the probabilities is 1.

Example 2: Passive-Aggressive Traits

An industrial psychologist administered a personality inventory test for passive-aggressive traits to 150 employees. Each individual was given a score from 1 to 5, where 1 is extremely passive and 5 is extremely aggressive. A score of 3 indicated neither trait. The results are shown at the right. Construct a probability distribution for the random variable X. Then graph the distribution using a histogram.

Frequency Distribution

Score, x	Frequency, f
1	24
2	33
3	42
4	30
5	21

Solution:

Divide the frequency of each score by the total number of individuals in the study to find the probability for each value of the random variable.

$$P(1) = \frac{24}{150} = 0.16 \qquad P(2) = \frac{33}{150} = 0.22 \qquad P(3) = \frac{42}{150} = 0.28$$

$$P(4) = \frac{30}{150} = 0.20 \qquad P(5) = \frac{21}{150} = 0.14$$

Discrete Probability Distribution

Example 2 Cont..:

The discrete probability distribution is shown in the table below.

$X = x$	1	2	3	4	5
$P(x)$	0.16	0.22	0.28	0.20	0.14

Note that $0 \leq P(x) \leq 1$ and $\sum P(x) = 1$.

This is a valid discrete probability distribution since

(1) Each probability is between 0 and 1, inclusive, $0 \leq P(x) \leq 1$.

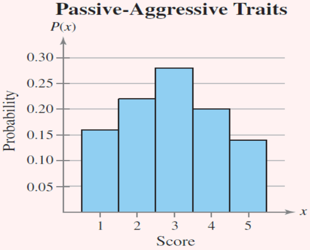
(2) The sum of the probabilities equals 1, $\sum P(x) = 0.16 + 0.22 + 0.28 + 0.20 + 0.14 = 1$.

(3) Histogram

Because the width of each bar is one, the area of each bar is equal to the probability of a particular outcome. Also, the probability of an event corresponds to the sum of the areas of the outcomes included in the event.

For instance, the probability of the event "having a score of 2 or 3" is equal to the sum of the areas of the second and third bars,
$(1)(0.22) + (1)(0.28) = 0.22 + 0.28 = 0.50$.

Interpretation You can see that the distribution is approximately symmetric.



Discrete Probability Distribution

Example 3: Grade Distributions

An (IT) college posts the grade distributions for its courses. In a recent semester, students in one section of Statistics & Probability 11308 received 32% A's, 42% B's, 19% C's, 3% D's, and 4% F's. Choose a Statistics & Probability 11308 student at random. To "choose at random" means to give every student the same chance to be chosen. The student's grade on a five-point scale (with A = 4) is a random variable X. Construct a probability distribution for the random variable X.

Solution:

The value of X changes when we repeatedly choose students at random, but it is always one of 0, 1, 2, 3, or 4. Here is the distribution of X:

Value of $X = x$	0	1	2	3	4
Probability $P(x)$	0.04	0.03	0.19	0.42	0.32

Again this is a valid discrete probability distribution since it satisfied the two conditions. (Show?)

Now, the probability that the student got a **B or better** is the sum of the probabilities of an A and a B. In the language of random variables,

$$P(X \geq 3) = P(X = 3) + P(X = 4)$$
$$= 0.42 + 0.32 = 0.74$$

Example 4: Number of heads in four tosses of a coin

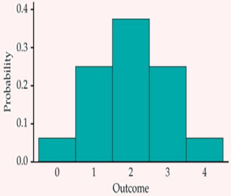
What is the probability distribution of the discrete random variable X that counts the number of heads in four tosses of a coin? Then graph the distribution using a histogram.

Solution:

The outcome of four tosses is a sequence of heads and tails such as $HTTH$. There are 16 possible outcomes in all. The Figure below lists these outcomes along with the value of X for each outcome. The multiplication rule for independent events tells us that, for example,

$$P(HTTH) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$$

		HHTT		
		HTTH		
	HTTT	HTHT	HHHT	
	THTT	THTH	HHTH	
	TTHT	THHT	HTHH	
TTTT	TTTH	TTHH	THHH	HHHH
$X = 0$	$X = 1$	$X = 2$	$X = 3$	$X = 4$



Each of the 16 possible outcomes similarly has probability 1/16. That is, these outcomes are equally likely. The discrete probability distribution is shown in the table below.

$X = x$	0	1	2	3	4
$P(x)$	0.06	0.25	0.38	0.25	0.06

Note that $0 \leq P(x) \leq 1$ and $\Sigma P(x) = 1$.

Mean, Variance and Standard Deviation

Mean Of A Discrete Random Variable

The **mean** of a discrete random variable is given by

$$\mu = \sum \mathbf{x}\mathbf{P}(\mathbf{x}).$$

Each value of x is multiplied by its corresponding probability and the products are added.

Mean Of A Discrete Random Variable

Example 5: Finding the Mean of a Probability Distribution

The probability distribution for the personality inventory test for passive-aggressive traits discussed in Example 2 is shown at the right. Find the mean score.

x	$P(x)$
1	0.16
2	0.22
3	0.28
4	0.20
5	0.14

Solution:

From the table, you can see that the mean score is $\mu = 2.94 \approx 2.9$. (Note that the mean is rounded to one more decimal place than the possible values of the random variable X.)

x	$P(x)$	$xP(x)$
1	0.16	$1(0.16) = 0.16$
2	0.22	$2(0.22) = 0.44$
3	0.28	$3(0.28) = 0.84$
4	0.20	$4(0.20) = 0.80$
5	0.14	$5(0.14) = 0.70$
	$\Sigma P(x) = 1$	$\Sigma xP(x) = 2.94 \approx 2.9$ ← Mean

Interpretation Recall that a **score of 3** represents an individual who exhibits neither passive nor aggressive traits and the mean is slightly less than 3. So, the mean personality trait is neither extremely passive nor extremely aggressive, but is slightly closer to passive.

Mean Of A Discrete Random Variable

Example 6: Finding the Mean of a Probability Distribution

The probability distribution for the grade of Statistics & Probability 11308 discussed in Example 3 is shown at the right. Find the mean score.

x	$P(x)$
0	0.04
1	0.03
2	0.19
3	0.42
4	0.32

Solution:

From the table, you can see that the mean grade is $\mu = 2.95 \approx 3.0$.

x	$P(x)$	$xP(x)$
0	0.04	$0(0.04) = 0.00$
1	0.03	$1(0.03) = 0.03$
2	0.19	$2(0.19) = 0.38$
3	0.42	$3(0.42) = 1.26$
4	0.32	$4(0.32) = 1.28$
	$\Sigma P(x) = 1$	$\Sigma xP(x) = 2.95 \approx 3.0$ ← Mean

Interpretation The mean is slightly less than 3. So, the level of a Statistics & Probability 11308 students is roughly high.

Mean, Variance and Standard Deviation

Variance and Standard Deviation Of A Discrete Random Variable

The **variance** of a discrete random variable is

$$\sigma^2 = \sum (x - \mu)^2 P(x).$$

The **standard deviation** is

$$\sigma = \sqrt{\sum (x - \mu)^2 P(x)}.$$

Variance and Standard Deviation Of A Discrete Random Variable

Example 7: Finding the Variance and Standard Deviation

The probability distribution for the personality inventory test for passive-aggressive traits discussed in Example 2 is shown at the right. Find the variance and standard deviation of the probability distribution.

x	$P(x)$
1	0.16
2	0.22
3	0.28
4	0.20
5	0.14

Solution:

From Example 5, you know that before rounding, the mean of the distribution is $\mu = 2.94$. Use a table to organize your work, as shown below.

x	$P(x)$	$x - \mu$	$(x - \mu)^2$	$(x - \mu)^2 P(x)$
1	0.16	−1.94	3.7636	0.602176
2	0.22	−0.94	0.8836	0.194392
3	0.28	0.06	0.0036	0.001008
4	0.20	1.06	1.1236	0.224720
5	0.14	2.06	4.2436	0.594104
	$\Sigma P(x) = 1$			$\Sigma(x - \mu)^2 P(x) = 1.6164$

Variance

So, the variance is $\sigma^2 = \mathbf{1.6164 \approx 1.6}$ and the standard deviation is $\sigma = \sqrt{\sigma^2} = \sqrt{\mathbf{1.6164}} \approx \mathbf{1.3}$. **Interpretation** Most of the data values differ from the mean by no more than 1.3.

Expected Value

Expected Value Of A Discrete Random Variable

The expected value of a discrete random variable is equal to the mean of the random variable.

$$\textbf{Expected Value} = \mathrm{E}(x) = \mu = \sum xP(x).$$

Although probabilities can never be negative, the expected value of a random variable can be negative.

Expected Value Of A Discrete Random Variable

Example 8: Finding an Expected Value

At a raffle, 1500 tickets are sold at \$2 each for four prizes of \$500, \$250, \$150, and \$75. You buy one ticket. What is the expected value of your gain?

Solution:

To find the gain for each prize, subtract the price of the ticket from the prize:

- Your gain for the \$500 prize is \$500 - \$2 = \$498
- Your gain for the \$250 prize is \$250 - \$2 = \$248
- Your gain for the \$150 prize is \$150 - \$2 = \$148
- Your gain for the \$75 prize is \$75 - \$2 = \$73
- If you do not win a prize, your gain is \$0 - \$2 = -\$2

Expected Value Of A Discrete Random Variable

Example 8: Finding an Expected Value (Cont.)

The probability distribution for the possible gains (or outcomes).

Gain, x	\$498	\$248	\$148	\$73	$-\$2$
Probability, $P(x)$	$\frac{1}{1500}$	$\frac{1}{1500}$	$\frac{1}{1500}$	$\frac{1}{1500}$	$\frac{1496}{1500}$

$-\$2$ represents a loss of \$2

Then, using the probability distribution, you can find the expected value.

$$\begin{aligned} E(x) &= \Sigma x P(x) \\ &= \$498 \cdot \frac{1}{1500} + \$248 \cdot \frac{1}{1500} + \$148 \cdot \frac{1}{1500} + \$73 \cdot \frac{1}{1500} + (-\$2) \cdot \frac{1496}{1500} \\ &= -\$1.35 \end{aligned}$$

Interpretation Because the expected value is negative, you can expect to lose an average of \$1.35 for each ticket you buy.

Exercise

(1) Determine whether each of the following variables X is discrete or continuous:

- (a) Let X represent the number of times you do laundry this month.
- (b) Let X represent your annual salary given to the nearest cent.
- (c) Let X represent your height at age 10.
- (d) Let X represent number of math classes that you have taken in your life.

(2) Determine if each of the following tables represents a probability distribution:

(a)

X	-5	6	9
$P(x)$	0.5	0.25	0.25

(b)

X	1	2	3	4
$P(x)$	0.4	0.4	0.4	0.2

(c)

X	1	2	3	4
$P(x)$	0.4	0.4	0.4	-0.2

(3) Make a probability distribution from the following frequency distribution represent the number of fish caught in a 6-hour period:

Number of fish caught	0	1	2	3	4
Frequency	88	72	30	8	2

(4) Calculate the expected value, variance, and standard deviation for each of the following probability distributions:

(a)

X	-5	6	9
$P(x)$	0.5	0.25	0.25

(b)

Number of fish caught	0	1	2	3	4
Frequency	0.44	0.36	0.15	0.04	0.01

What you have learned?

Section 4.1 Summary

- **Distinguished between discrete random variables and continuous random variables**
- **Constructed a discrete probability distribution and its graph and determined if a distribution is a probability distribution**
- **Found the mean, variance, and standard deviation of a discrete probability distribution**
- **Found the expected value of a discrete probability distribution**

Section 4.2

Binomial Distributions

What We Will Look At

Section 3.2 Objectives

- How to determine whether a probability experiment is a binomial experiment
- How to find binomial probabilities using the binomial probability formula
- How to find binomial probabilities using technology, formulas, and a binomial table
- How to construct and graph a binomial distribution
- How to find the mean, variance, and standard deviation of a binomial probability distribution

Binomial Experiment

Definition:

A **binomial experiment** is a probability experiment that satisfies these conditions.

- The experiment is repeated for a fixed number of trials, where each trial is independent of other trials.
- There are only two possible outcomes of interest for each trial. The outcomes can be classified as a success (S) or as a failure (F).
- The probability of a success, P(S), is the same for each trial.
- The random variable X counts the number of successful trials.

Notation for Binomial Experiments

Notation for Binomial Experiments:

S and F (**success** and **failure**) denote the two possible categories of all outcomes.

SYMBOL	DESCRIPTION
n	The number of trials
$P(S) = p$	The probability of success in a single trial
$P(F) = q$	The probability of failure in a single trial $(q = 1 - p)$
x	The random variable represents a count of the number of successes in n trials: $x = 0, 1, 2, 3, \ldots, n$.

CAUTION When using a binomial probability distribution, always be sure that x and p are *consistent* in the sense that they both refer to the *same* category being called a success.

Identifying and Understanding Binomial Experiments

Example 9: Twitter

When an adult is randomly selected (with replacement), there is a 0.85 probability that this person knows what Twitter is (based on results from a Pew Research Center survey). Suppose that we want to find the probability that exactly three of five randomly selected adults know what Twitter is.

(a) Does this procedure result in a binomial experiment?

(b) If this procedure does result in a binomial experiment, identify the values of $n, x, p,$ and q.

Solution:

(a) This procedure does satisfy the requirements for a **binomial experiment**, as shown below.

  (1) The number of trials (5) is fixed.

  (2) The 5 trials are independent because the probability of any adult knowing Twitter is not affected by results from other selected adults.

  (3) Each of the 5 trials has two categories of outcomes: The selected person knows what Twitter is or that person does not know what Twitter is.

  (4) For each randomly selected adult, there is a 0.85 probability that this person knows what Twitter is, and that probability remains the same for each of the five selected people.

(b) Having concluded that the given procedure does result in a **binomial experiment**, we now proceed to identify the values of $n, x, p,$ and q.

  (1) With five randomly selected adults, we have $n = 5$.

  (2) We want the probability of exactly three who know what Twitter is, so $x = 3$.

  (3) The probability of success (getting a person who knows what Twitter is) for one selection is 0.85, so $p = 0.85$.

  (4) The probability of failure (not getting someone who knows what Twitter is) is 0.15, so $q = 0.15$.

Identifying and Understanding Binomial Experiments

Example 10:

Determine whether the experiment is a binomial experiment. If it is, specify the values of $n, p,$ and q, and list the possible values of the random variable x. If it is not, explain why.

(1) A certain surgical procedure has an 85% chance of success. A doctor performs the procedure on eight patients. The random variable represents the number of successful surgeries.

(2) A jar contains five red marbles, nine blue marbles, and six green marbles. You randomly select three marbles from the jar, without replacement. The random variable represents the number of red marbles.

Solution:

(1) The experiment is a **binomial experiment** because it satisfies the four conditions of a binomial experiment.

- ➢ With eight surgeries, we have $n = 8$. Each surgery represents one trial, and each surgery is independent of the others.
- ➢ There are only two possible outcomes for each surgery—either the surgery is a success or it is a failure.
- ➢ Also, the probability of success for each surgery is $p = 0.85$.
- ➢ Finally, the random variable x represents the number of successful surgeries.

(2) The experiment is **not a binomial experiment** because it does not satisfy all four conditions of a binomial experiment. (Check !!)

- ➢ The probability of selecting a red marble on the first trial is $5/20$.
- ➢ Because the marble is not replaced, the probability of success (red) for subsequent trials is no longer $5/20$.
- ➢ The trials are not independent and the probability of a success is not the same for each trial.

Binomial Probability Formula

Binomial Probability Formula

In a binomial experiment, the probability of exactly x successes in n trials is

$$P(x) = {}_nC_x p^x q^{n-x} = \frac{n!}{(n-x)!\,x!} p^x q^{n-x}.$$

Note that the number of failures is $n - x$.

- $n =$ number of trials
- $p =$ probability of success
- $q = 1 - p$ probability of failure
- $x =$ number of successes in n trials

Binomial Probability Formula

Example 11: Finding a Binomial Probability

Rotator cuff surgery has a 90% chance of success. The surgery is performed on three patients. Find the probability of the surgery being successful on exactly two patients. (Source: The Orthopedic Center of St. Louis)

Solution:

Method 1: Draw a tree diagram and use the Multiplication Rule.

1st Surgery	2nd Surgery	3rd Surgery	Outcome	Number of Successes	Probability
S	S	S	SSS	3	$\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}=\frac{729}{1000}$
		F	SSF	2	$\frac{9}{10}\cdot\frac{9}{10}\cdot\frac{1}{10}=\frac{81}{1000}$
	F	S	SFS	2	$\frac{9}{10}\cdot\frac{1}{10}\cdot\frac{9}{10}=\frac{81}{1000}$
		F	SFF	1	$\frac{9}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}=\frac{9}{1000}$
F	S	S	FSS	2	$\frac{1}{10}\cdot\frac{9}{10}\cdot\frac{9}{10}=\frac{81}{1000}$
		F	FSF	1	$\frac{1}{10}\cdot\frac{9}{10}\cdot\frac{1}{10}=\frac{9}{1000}$
	F	S	FFS	1	$\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{9}{10}=\frac{9}{1000}$
		F	FFF	0	$\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}=\frac{1}{1000}$

There are three outcomes that have exactly two successes, and each has a probability of $\frac{81}{1000}$. So, the probability of a successful surgery on exactly two patients is $3\left(\frac{81}{1000}\right)=0.243$.

Binomial Probability Formula

Example 11: Finding a Binomial Probability (Cont.)

Rotator cuff surgery has a 90% chance of success. The surgery is performed on three patients. Find the probability of the surgery being successful on exactly two patients.

(Source: The Orthopedic Center of St. Louis)

Solution:

Method 2: Use the binomial probability formula.

In this binomial experiment, the values of n, p, q, and x are $n = 3$, $p = \frac{9}{10}$, $q = \frac{1}{10}$, and $x = 2$.
The probability of exactly two successful surgeries is

$$P(2) = \frac{3!}{(3-2)!2!}\left(\frac{9}{10}\right)^2\left(\frac{1}{10}\right)^1$$

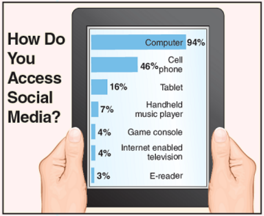
$$= 3\left(\frac{81}{100}\right)\left(\frac{1}{10}\right) = 3\left(\frac{81}{1000}\right)$$

$$= 0.243.$$

Binomial Probability Distribution

Example 12: Finding a Binomial Probability Distribution

In a survey, U.S. adults were asked to identify what devices they use to access social media. The results are shown in the figure. Seven adults who participated in the survey are randomly selected and asked whether they use a cell phone to access social media. Construct a binomial probability distribution for the number of adults who respond yes. (Source: Nielsen U.S. Social Media Survey)



Solution:

From the figure, you can see that 46% of adults use a cell phone to access social media. So, $p = 0.46$ and $q = 0.54$. Because $n = 7$, the possible values of x are $0, 1, 2, 3, 4, 5, 6,$ and 7.

$$
\begin{aligned}
P(0) &= {}_7C_0(0.46)^0(0.54)^7 = 1(0.46)^0(0.54)^7 \approx 0.0134 \\
P(1) &= {}_7C_1(0.46)^1(0.54)^6 = 7(0.46)^1(0.54)^6 \approx 0.0798 \\
P(2) &= {}_7C_2(0.46)^2(0.54)^5 = 21(0.46)^2(0.54)^5 \approx 0.2040 \\
P(3) &= {}_7C_3(0.46)^3(0.54)^4 = 35(0.46)^3(0.54)^4 \approx 0.2897 \\
P(4) &= {}_7C_4(0.46)^4(0.54)^3 = 35(0.46)^4(0.54)^3 \approx 0.2468 \\
P(5) &= {}_7C_5(0.46)^5(0.54)^2 = 21(0.46)^5(0.54)^2 \approx 0.1261 \\
P(6) &= {}_7C_6(0.46)^6(0.54)^1 = 7(0.46)^6(0.54)^1 \approx 0.0358 \\
P(7) &= {}_7C_7(0.46)^7(0.54)^0 = 1(0.46)^7(0.54)^0 \approx 0.0044
\end{aligned}
$$

x	$P(x)$
0	0.0134
1	0.0798
2	0.2040
3	0.2897
4	0.2468
5	0.1261
6	0.0358
7	0.0044
	$\Sigma P(x) = 1$

Notice in the table at the left that all the probabilities are between 0 and 1 and that the sum of the probabilities is 1.

Binomial Probability Distribution

Example 13: Twitter

Given that (Example 9) there is a 0.85 probability that a randomly selected adult knows what Twitter is, use the binomial probability formula to find the probability that when five adults are randomly selected, exactly three of them know what Twitter is. That is, apply Binomial Formula to find $P(3)$ given that $n = 5, x = 3,\ p = 0.85$, and $q = 0.15$.

Solution:

Using the given values of n, x, p, and q in the binomial probability formula, we get

$$\begin{aligned}
P(3) &= \frac{5!}{(5-3)!3!} \cdot 0.85^3 \cdot 0.15^{5-3} \\
&= \frac{5!}{2!3!} \cdot 0.614125 \cdot 0.0225 \\
&= (10)(0.614125)(0.0225) = 0.138178 \\
&= 0.138 \text{ (rounded to three significant digits)}
\end{aligned}$$

The probability of getting exactly three adults who know Twitter among five randomly selected adults is 0.138.

Example 14: Tossing a dice

A die is tossed 3 times. What is the probability of

(a) No fives turning up?

(b) 1 five?

(c) 3 fives?

Solution:

This is a binomial distribution because there are only 2 possible outcomes (we get a 5 or we don't). Now, $n = 3$ for each part. Let X = number of fives appearing.

(a) Here, $x = 0$.

$$P(X = 0) = {}_nC_x p^x q^{n-x} = {}_3C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^3 = \frac{125}{216} = 0.5787$$

(b) Here, $x = 1$.

$$P(X = 1) = {}_nC_x p^x q^{n-x} = {}_3C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^2 = \frac{75}{216} = 0.34722$$

(c) Here, $x = 3$.

$$P(X = 3) = {}_nC_x p^x q^{n-x} = {}_3C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^0 = \frac{1}{216} = 4.6296 \times 10^{-3}$$

Example 15: Telephone Call

In the old days, there was a probability of 0.8 of success in any attempt to make a telephone call. (This often depended on the importance of the person making the call, or the operator's curiosity!)
Calculate the probability of having 7 successes in 10 attempts.

Solution:

Probability of success $p = 0.8$, $q = 0.2$. Probability of 7 successes in 10 attempts:

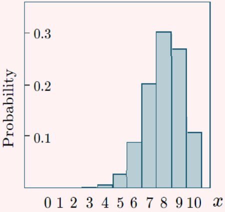
$$\begin{aligned}\text{Probability} &= P(X = 7)\\ &= {}_{10}C_7\,(0.8)^7(0.2)^{10-7}\\ &= 0.20133\end{aligned}$$

Histogram

We use the following function

$$P(X = x) = {}_{10}C_x\,(0.8)^x(0.2)^{10-x}$$

to obtain the probability histogram:



Histogram of the binomial distribution

Finding Binomial Probabilities Using Formulas

Example 16: Violent Video Games

A survey of U.S. adults found that 62% of women believe that there is a link between playing violent video games and teens exhibiting violent behavior. You randomly select four U.S. women and ask them whether they believe that there is a link between playing violent video games and teens exhibiting violent behavior. Find the probability that (1) exactly two of them respond yes, (2) at least two of them respond yes, and (3) fewer than two of them respond yes. (Source: Harris Interactive)

Solution:

1. Using $n = 4$, $p = 0.62$, $q = 0.38$, and $x = 2$, the probability that exactly two women will respond yes is

$$P(2) = {}_4C_2(0.62)^2(0.38)^2 = 6(0.62)^2(0.38)^2 \approx 0.333.$$

2. To find the probability that at least two women will respond yes, find the sum of $P(2)$, $P(3)$, and $P(4)$.

$$P(2) = {}_4C_2(0.62)^2(0.38)^2 = 6(0.62)^2(0.38)^2 \approx 0.333044$$
$$P(3) = {}_4C_3(0.62)^3(0.38)^1 = 4(0.62)^3(0.38)^1 \approx 0.362259$$
$$P(4) = {}_4C_4(0.62)^4(0.38)^0 = 1(0.62)^4(0.38)^0 \approx 0.147763$$

So, the probability that at least two will respond yes is

$$\begin{aligned} P(x \geq 2) &= P(2) + P(3) + P(4) \\ &\approx 0.333044 + 0.362259 + 0.147763 \\ &\approx 0.843. \end{aligned}$$

3. To find the probability that fewer than two women will respond yes, find the sum of $P(0)$ and $P(1)$.

$$\begin{aligned} P(0) &= {}_4C_0(0.62)^0(0.38)^4 \\ &= 1(0.62)^0(0.38)^4 \\ &\approx 0.020851 \end{aligned}$$

$$\begin{aligned} P(1) &= {}_4C_1(0.62)^1(0.38)^3 \\ &= 4(0.62)^1(0.38)^3 \\ &\approx 0.136083 \end{aligned}$$

So, the probability that fewer than two will respond yes is

$$\begin{aligned} P(x < 2) &= P(0) + P(1) \\ &\approx 0.020851 + 0.136083 \\ &\approx 0.157. \end{aligned}$$

Finding a Binomial Probability Using a Table

Example 17: Carpooling

About 10% percent of workers (ages 16 years and older) in the United States commute to their jobs by carpooling. You randomly select eight workers. What is the probability that exactly four of them carpool to work? Use a table to find the probability. (Source: American Community Survey)

Solution:

A portion of Table 2 in Appendix B is shown here. Using the distribution for n = 8 and p = 0.1, you can find the probability that x = 4, as shown by the highlighted areas in the table.

		p												
n	***x***	**.01**	**.05**	**.10**	**.15**	**.20**	**.25**	**.30**	**.35**	**.40**	**.45**	**.50**	**.55**	**.60**
2	0	.980	.902	.810	.723	.640	.563	.490	.423	.360	.303	.250	.203	.160
	1	.020	.095	.180	.255	.320	.375	.420	.455	.480	.495	.500	.495	.480
	2	.000	.002	.010	.023	.040	.063	.090	.123	.160	.203	.250	.303	.360
3	0	.970	.857	.729	.614	.512	.422	.343	.275	.216	.166	.125	.091	.064
	1	.029	.135	.243	.325	.384	.422	.441	.444	.432	.408	.375	.334	.288
	2	.000	.007	.027	.057	.096	.141	.189	.239	.288	.334	.375	.408	.432
	3	.000	.000	.001	.003	.008	.016	.027	.043	.064	.091	.125	.166	.216
8	0	.923	.663	.430	.272	.168	.100	.058	.032	.017	.008	.004	.002	.001
	1	.075	.279	.383	.385	.336	.267	.198	.137	.090	.055	.031	.016	.008
	2	.003	.051	.149	.238	.294	.311	.296	.259	.209	.157	.109	.070	.041
	3	.000	.005	.033	.084	.147	.208	.254	.279	.279	.257	.219	.172	.124
	4	.000	.000	.005	.018	.046	.087	.136	.188	.232	.263	.273	.263	.232
	5	.000	.000	.000	.003	.009	.023	.047	.081	.124	.172	.219	.257	.279
	6	.000	.000	.000	.000	.001	.004	.010	.022	.041	.070	.109	.157	.209
	7	.000	.000	.000	.000	.000	.000	.001	.003	.008	.016	.031	.055	.090
	8	.000	.000	.000	.000	.000	.000	.000	.000	.001	.002	.004	.008	.017

Interpretation So, the probability that exactly four of the eight workers carpool to work is 0.005. Because 0.005 is less than 0.05, this can be considered an unusual event.

Graphing a Binomial Distribution

Example 18: Cancer Survivors

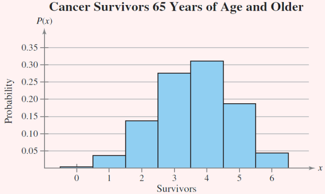
About 60% of cancer survivors are ages 65 years and older. You randomly select six cancer survivors and ask them whether they are 65 years of age and older. Construct a probability distribution for the random variable x. Then graph the distribution. (Adapted from National Cancer Institute)

Solution:

To construct the binomial distribution, find the probability for each value of x. Using $n = 6$, $p = 0.6$, and $q = 0.4$, you can obtain the following.

x	0	1	2	3	4	5	6
$P(x)$	0.004	0.037	0.138	0.276	0.311	0.187	0.047

The histogram of the probability distribution can be constructed easily as follows:



Mean, Variance, and Standard Deviation

Population Parameters of A Binomial Distribution

The **properties** of a **binomial distribution** enable you to use much simpler formulas.

Population Parameters of A Binomial Distribution
Mean: $\mu = np$
Variation: $\sigma^2 = npq$
Standard deviation: $\sigma = \sqrt{npq}$

Finding the Mean, Variance, and Standard Deviation

Example 19: Cloudy Days

In Pittsburgh, Pennsylvania, about 56% of the days in a year are cloudy. Find the mean, variance, and standard deviation for the number of cloudy days during the month of June. Interpret the results and determine any unusual values. (Source: National Climatic Data Center)

Solution:

There are 30 days in June. Using $n = 30$, $p = 0.56$, and $q = 0.44$, you can find the mean, variance, and standard deviation as shown below.

$$\begin{aligned} \mu &= np = 30 \cdot 0.56 \\ &= 16.8 && \text{Mean} \\ \sigma^2 &= npq = 30 \cdot 0.56 \cdot 0.44 \\ &\approx 7.4 && \text{Variance} \\ \sigma &= \sqrt{npq} = \sqrt{30 \cdot 0.56 \cdot 0.44} \\ &\approx 2.7 && \text{Standard deviation} \end{aligned}$$

Interpretation

- On average, there are 16.8 cloudy days during the month of June.
- The standard deviation is about 2.7 days.
- Values that are more than two standard deviations from the mean are considered unusual. Because:
  - $16.8 - 2(2.7) = 11.4$, a June with 11 cloudy days or less would be unusual.
  - $16.8 + 2(2.7) = 22.2$, a June with 23 cloudy days or more would also be unusual.

Exercise

(1) When flipping a weighted coin (with the probability of heads being 0.6), what is the probability that it will come up heads exactly 5 times when it is flipped 10 times?

(2) When randomly guessing on a multiple choice test with 8 questions, where each question has 4 options, what is the probability that you will get at least 7 questions correct? What is the expected number of questions a student will get correct without studying for the exam? What is the standard deviation?

What you have learned?

Section 4.2 Summary

- **Determined if a probability experiment is a binomial experiment**
- **Found binomial probabilities using the binomial probability formula**
- **Found binomial probabilities using technology, formulas, and a binomial table**
- **Constructed and graphed a binomial distribution**
- **Found the mean, variance, and standard deviation of a binomial probability distribution**

End of Chapter 04.

Thank you.!!!