University of Bahrain
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Programming a Slinky

PHYCS425: Computational Physics

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Abstract

The slinky is a flexible spring-like model that is consistent with oscillatory systems obeying Hooke's law. In our work, we considered a slinky model represented by a system of dimensionless particles attached together by springs. We developed a Fortran 90 code using the Euler-Cromer numerical method to adequately predict Three discrete models of the slinky motion. In section three, the vertical hanging slinky was implemented. We discussed its equilibrium length, and the framework for having the exact scenario but on the moon's surface. Section 4 exhibited a suspended slinky with its endpoints fixed, and compared it with the true catenary curve. In section 5, we demonstrated three scenarios in the context of free falling slinky. We analyzed the motion when the slinky is dropped from its equilibrium state, oscillating the first particle in the slinky, and charging the particles in the model to let them fall in a uniform electric field. At the end, we discussed the potential for further studies and modifications for the proposed model.

Keywords: Slinky, Elongation, Equilibrium length, Catenary curve

1. Introduction

The “slinky” was invented by the mechanical engineer Richard James in 1943. While James was working on his navigational device [2], he dropped a coiled wire on the floor and observed an unusual behavior. The coiled wires were tumbling head over heels on the floor!

This marvelous invention was introduced to the world as a playful tool and became a trendy toy named “the slinky” (comes from “sleek” referring to its sinuous motion [2]). A slinky is a wire that has a flexible property that allows it to stretch and compress simultaneously.

However, the slinky has much deeper physical insights consistent with oscillatory systems obeying Hook’s law [1]. Thereafter, many well-refereed research papers introduced several models to demonstrate the physics of a slinky [6],[5].

In this project, we used a numerical approach to write a Fortran 90 code, that is based on Euler-Cromer numerical method. This study will demonstrate a computational analysis adopting the model where a slinky is represented by dimensionless particles attached to each other by springs.

Section 3 will discuss vertical hanging slinky, and the factors involved in obtaining the corresponding equilibrium length. Section 4 will investigate a suspended slinky between two endpoints, and the deviations found when a comparison is done with the catenary curve function. In addition, we examined a unique motion for a popular slinky toy in which the center of mass is inspected. The last section (section 5) studies other dynamics of a free-falling vertical slinky. Subsections 5.2 & 5.3 contemplate two scenarios; adding energy to the system by letting the first particle oscillate, and dropping a charged slinky in a uniform electric field.

2. Theory

2.1 Setting the equations

The slinky is modeled as a system of massless springs containing N number of particles with masses m. Each particle is separated by a spring with spring constant k having length L_0. To damp the oscillations, an air friction term must be added to the equations to activate the damping force. There are 2 commonly used terms, either the force is proportional to v^2 in systems where the motion is fast, or as in our slinky it's proportional to v where the motion is considered to be slow. The forces on particle i can be summarized by **Equation.1** [1]:

$$F_i = k_{i-1}(L_{i-1} - L_0)\hat{r}_{i-1} + k_i(L_{i+1} - L_0)\hat{r}_{i+1} - m_i g\hat{j} - c(v)_i$$

Equation 1

Where the unit vectors $\hat{r}_{i-1}$ and $\hat{r}_{i+1}$ are:

$$\hat{r}_{i-1} = \frac{1}{L_{i-1}}(r_{i-1} - r_i)$$

Equation 2

$$\hat{r}_{i+1} = \frac{1}{L_{i+1}}(r_{i+1} - r_i)$$

Equation 3

For the first particle i = 0 the acting forces are set to:

$$(F_x)_0 = K_0\left(\frac{L_1 - L_0}{L_1}\right)(x_1 - x_0) \ - c(v_x)_0,$$

Equation 4

$$\left(F_y\right)_0 = k_0\left(\frac{L_1 - L_0}{L_1}\right)(y_1 - y_0) - m_0 g - c\left(v_y\right)_0$$

Equation 5

For any particle i:

$$(F_x)_i = K_{i-1}\left(\frac{L_{i-1} - L_0}{L_{i-1}}\right)(x_{i-1} - x_i) + k_i\left(\frac{L_{i+1} - L_0}{L_{i+1}}\right)(x_{i+1} - x_i) - c(v_x)_i,$$

Equation 6

$$\left(F_y\right)_i = k_{i-1}\left(\frac{L_{i-1} - L_0}{L_{i-1}}\right)(y_{i-1} - y_i) + k_i\left(\frac{L_{i+1} - L_0}{L_{i+1}}\right)(y_{i+1} - y_i) - m_i g - c\left(v_y\right)_i$$

Equation 7

The Forces acting on the last Particle N defined as:

$$(F_x)_N = K_{N-1}\left(\frac{L_{N-1} - L_0}{L_{N+1}}\right)(x_{N-1} - x_N) - c(v_x)_N,$$

Equation 8

$$\left(F_y\right)_N = k_{N-1}\left(\frac{L_{N-1} - L_0}{L_{N+1}}\right)(y_{N-1} - y_N) - m_N g - c\left(v_y\right)_N$$

Equation 9

Where x, y indicating the coordinates of each particle.

2.2 Numerical approach

The Euler-Cromer method was used to write the Fortran90 code instead of using Simple Euler Method, As Simple Euler assumes that the velocity and accelaration do not change signficantly with time step Δt. Thus, It dosen't take the current velocity into account so after many iterations results will be innacurate . On the other hand, Euler-Cromer is a modified Euler with a simple change, as it uses the current velocity to update the position at each time step Δt which will lead to more accurate results.The following equations shows the difference between Euler and Euler-Cromer: [8]

For Simple Euler:

$$v(t + \Delta t) = v(t) + a(t)\Delta t$$

Equation 10

$$y(t + \Delta t) = y(t) + v(t + \Delta t)\Delta t$$

Equation 11

For Euler Cromer:

$$v(t + \Delta t) = v(t) + a(t)\Delta t$$

Equation 12

$$y(t + \Delta t) = y(t) + v(t)\Delta t$$

Equation 13

For a conservative system, the total energy should be a constant of the motion. Beacuse Euler-Cromer method conserves the total energy, It's a natural choice for simulating conservative systems[7]

3. Vertical Hanging Slinky

Hereafter, if we consider an N + 1 number of particles for a slinky model and released it while hanging it vertically. We can simply say that the first particle is fixed and being at rest while the rest of the particles are oscillating under the effects of both Gravity and the elastic force (see **Equation 1)**. Accordingly, the program was designed to simulate the motion of the particles by pursuing their position i and setting the forces along the y-axis (since the acting forces are in one dimension and there is no force or motion in the x-axis).

The equations describing the motion of the first particle i = 0, and the last particle N, are given by **Equations (5)-(9)** respectively (Notice that they are only affected by one elastic force due to the particle underneath m_0 and the particle above the m_N). Meanwhile, particles in the middle were controlled by their own weight, and two elastic forces due to their neighbor particles as stated in **Equation.6**.

The damping force added to the acceleration will damp the motion of the slinky gradually, and the Drag coefficient was estimated to be roughly c = $5x10^{-3}$. For N = 10 particles with masses of m = 1g and spring constant k = 20. The results are simulated in **Figure 3.1**. We can see that the system exhibits a simple damped harmonic oscillation phenomenon, in which the slinky is compressed and stretched causing a change in the total length.

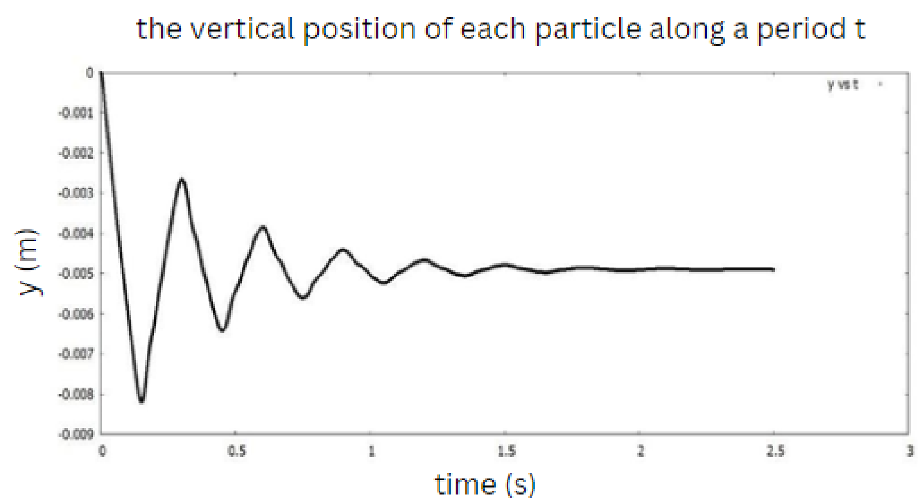


Figure 3.1 the plotted data of the vertical position of each particle along a period t = 2.5 s.

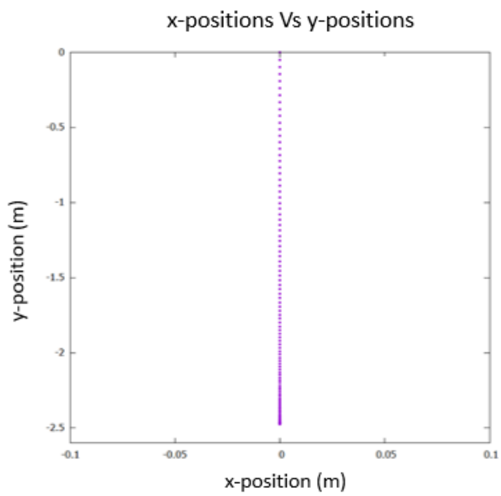
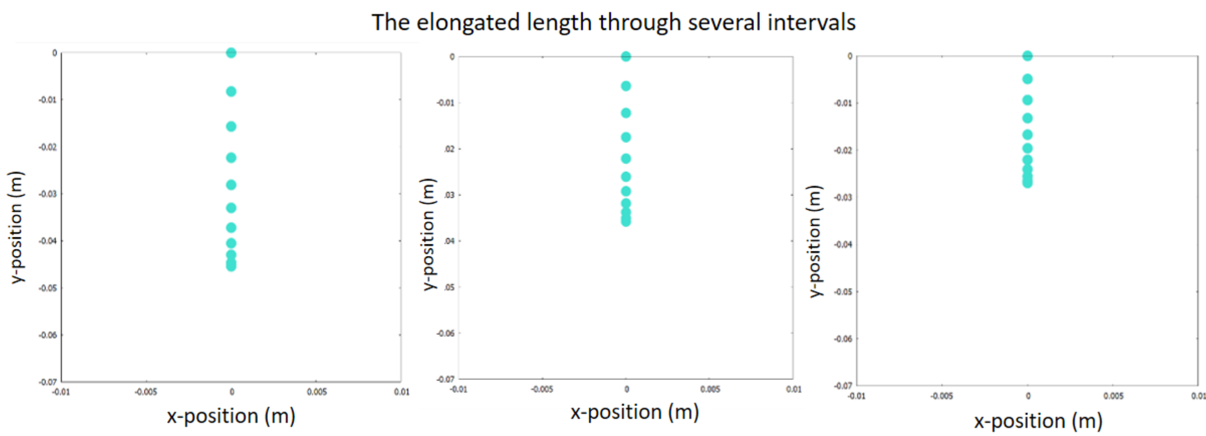


Figure 3.2 Vertical position of the particles corresponding to their horizontal positions.

Another point of visualization shown in **Figure 3.2.** It illustrates the coordinates of the particles when we simulate one hundred number of particles (N = 100). We can see that the slinky is elongated the most in the upper part since the center of mass is shifted towards the bottom.



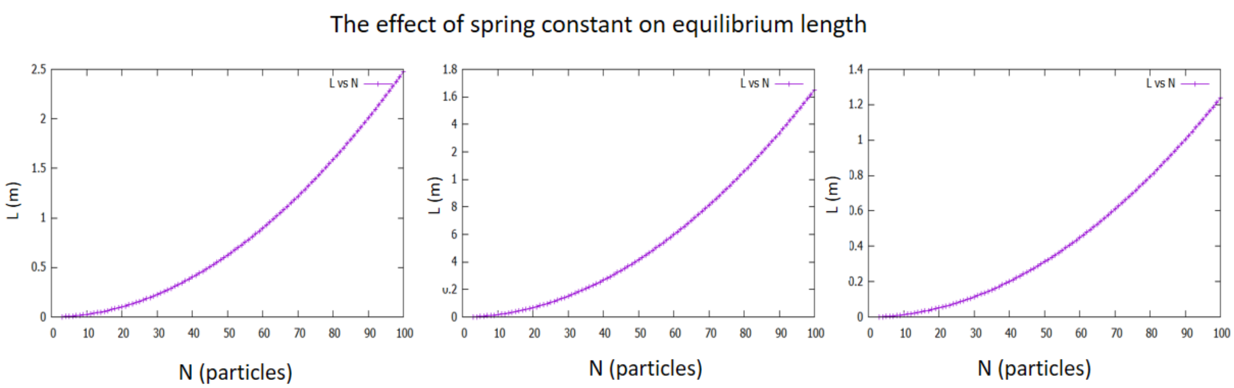
(a) (b) (c)

Figure3.3 The elongated length through three intervals for N = 10, (a) is the maximum elongated length, (b) is when the slinky is damping, (c) is the saturated elongated length (equilibrium length).

As we can see in **Figure3.3,** due to the conservation of energy the maximum elongation for the total slinky cannot exceed that of the first oscillation. In contrast, it is decreasing due to the damping force acting on the slinky model. In other words, no particle can reach a lower position than that of the first oscillation. In the absence of the damping force (virtually), we will have an infinite number of oscillations each of the same total elongated length (no energy loss). Furthermore, in the equilibrium length, the number of particles played an essential role in the slinky length. Increasing N means that the slinky will have a longer equilibrium length.

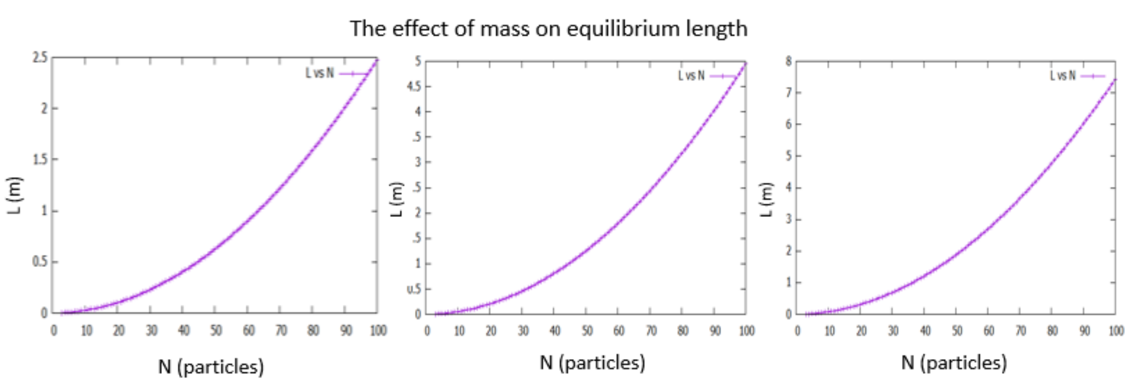
In addition, two factors can affect this relation: the masses of the particles and the spring constant K. The diagrams in **Figure 3.4** (Notice the scale difference on the y-axis) elucidate how the small value of the slinky constant K have a higher equilibrium length since the required force according to Hook's law to elongate it is relatively small, thus it will tend to have a longer elongated length.

On the other hand, the higher value of the spring constant k = 40 required much amount of the force to stretch the slinky, and therefore a shorter equilibrium length. Nevertheless, whenever we increase the masses of the particles the equilibrium length will be longer, according to the weight of the slinky as seen in **Figure 3.5** (Notice the scale difference on the y-axis) elucidate.



(a) K = 20 (b) k = 30 (c) k = 40

Figure 3.4 The effects of spring constant in equilibrium length.



(a) M = 1 (b) M = 2 (c) M = 3

Figure 3.5 The effects of mass in equilibrium length

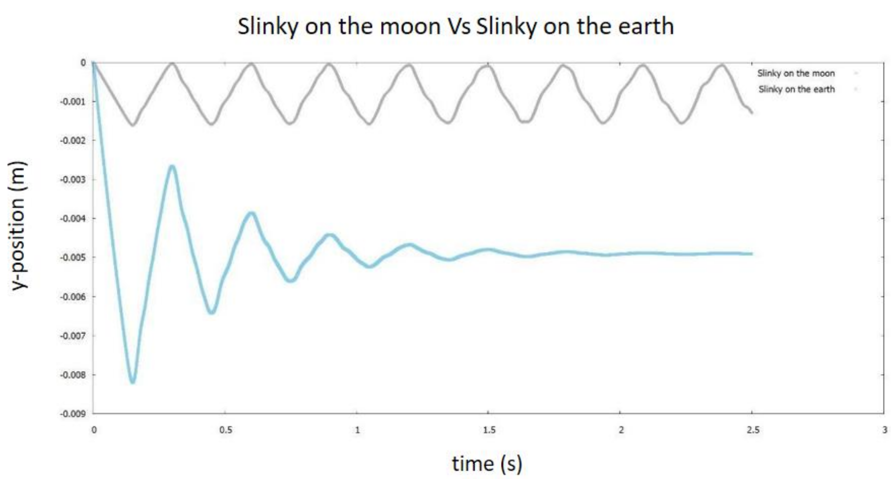
3.1 Slinky on the moon

Other circumstances have been adopted for the exact slinky model. What if a slinky was brought to the moon? We examined the Gedankenexperiment considering moon's lack of atmosphere, and the lunar gravity g = $1.6m\ /\ s^2$. The model displayed an oscillation behavior, but there are significant changes in its motion. Our finding can be deduced as follow:

1- the lack of atmosphere in the moon will prevent air from damping the slinky motion (where the most effective friction is), and thus it will oscillate forever. Figure 3.6 shows a clear damping for the oscillation of the slinky on Earth. While the oscillation of the slinky on the moon shows that it has a constant amplitude over time (no damping effect).

2- the frequency of oscillation is the same on both; Earth and the Moon, which is an expected result since the spring constant is the same in both cases. see **Figure 3.6**.

3- The total elongated length depends on the gravitational force, which is much less on the moon. Hence, the total elongation length (the amplitude of oscillation in the graph) would be shorter.



Figue3.6 The 'sky blue' shows the slinky model in earth gravity and the presence of air resistance, while the 'grey' trajectory shows the slinky model in moon gravity and the absence of air resistance.

4. Catenary -like Curve Slinky

4.1 The Suspended Slinky Shape

In a physical system, if a cable or a chain has been suspended by both endpoints and is hanging under the effect of its own weight. The obtained curve could be described mathematically as a catenary curve [3]. The catenary curve is a hyperbolic cosine function curve. In Cartesian coordinates, the catenary curve is in (**Equation.14**) [4].

$$y = a\cos h\left(\frac{x}{y}\right)$$

Equation 14

In the same context, a slinky suspended by its both ends would display a resemblance curve to the catenary curve but not exactly the same. In the code, the first particle of the slinky m_0 and the last particle m_N is being fixed at the x-axis, where m0 is located at the origin and m_N is separated from m_0 by a distance d = 1m. The masses are uniformly distributed on the x-axis. to achieve this, the separation distance was divided by the number of particles N in which del_x = d/N. By multiplying del_x with the sequence of the particle i, the particles will be distributed uniformly on the slinky. Initially, N = 10 and k is 20 N/m. Then, the slinky is let to move under the effect of its own weight only.

To compare the suspended slinky motion with the catenary curve, we must consider that the catenary curve is a hyperbolic cosine “cosh” function with a horizontal shift. So, to fix this issue the write condition was fixed to match the catenary characteristics by shifting the x positions of m_0 and m_N by 0.5m and the y position of the entire slinky by 1m.

In the graph below, we can see the fitted results of the slinky model with the catenary curve. Clearly, it shows that the slinky model with the previously mentioned parameters is way far from being like a catenary curve. This can be explained by the fact that the masses are non-continuously distributed along the x-axis. However, simulating a catenary-like curve needs continuous mass distribution. Also, the center of mass of any object tends to be as low as possible (getting closer to the ground) and the properties of the slinky “like stretching” will give the center of mass the freedom to be lower than the normal cable or chain that takes the shape of a catenary curve.

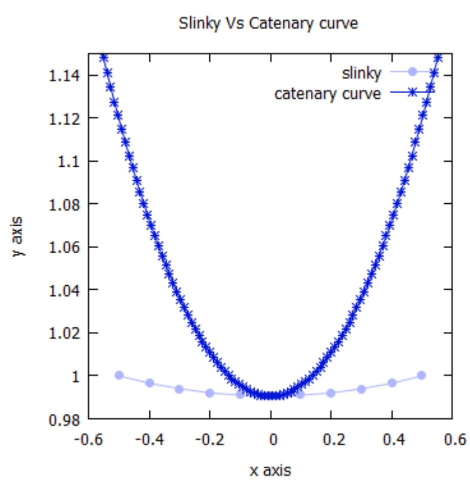


Figure 4.1 Slinky Vs Catenary Curve

4.2 Limit Approaching True Catenary Curve

To get a catenary-like curve, we tested the limits holding these questions in mind: what makes a slinky look closer to a true catenary curve? How can we balance the forces affecting the slinky for a more uniform shape?

After consideration, we noticed that by changing the number of particles to 1000 to get a saturated slinky, changing the spring constant using the following:

To balance the forces let $F_S = F_g$ assuming $\Delta l = l_0$ "no stretching involved".

For N number of particles,

$$kl_0 = mgN$$

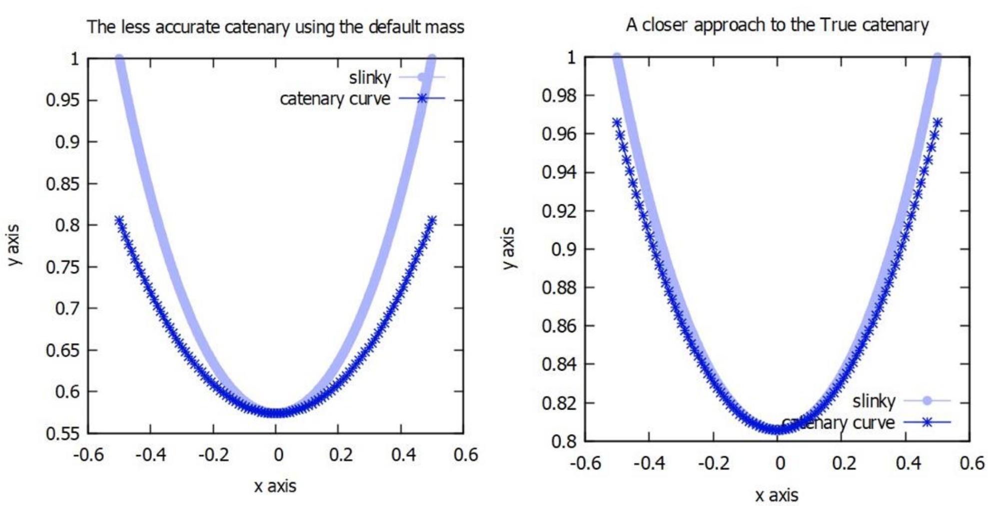
Equation 15

find an equation of k

$$k = \frac{mgN}{l_0}$$

Equation 16

when $m = l_0 = 1 \times 10^{-3}$, and since m = l_0; $k = g \times N = 9800\ N/m$. Adding some initial length l_0 = 1mm and changing the mass accordingly, we get closer and closer to the true catenary curve. The figures below show the attempts to get to the wanted results. Where **Figure 4.2a** uses the default mass m = 1×10^{-3} kg and **Figure 4.2.b** uses m=1.7×10^{-4} kg.



(a) (b)

Figure 4.2 The trials of deviation the slinky factors K=9800 and the particle masses

After several trials, the closest approach was found to be when m = 1.27×10^{-4} kg, k = 9800 N/m, and l_0 = 1 mm. The results are shown in **Figure 4.3** below.

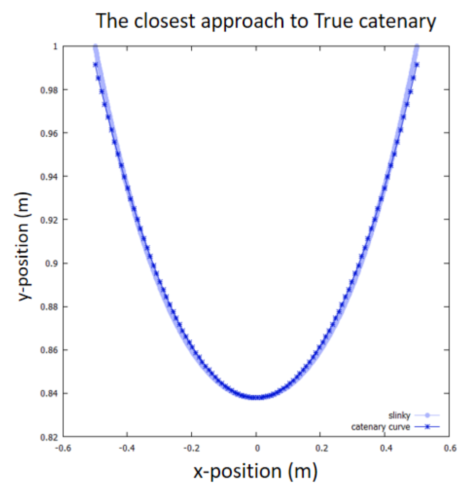


Figure 4.3 The closest approach to a true Catenary curve

4.3 Variable masses

This section will focus on the effect of varying the mass values of the particles on the slinky model. Hence, the masses have been alternating by giving even-numbered particles mass = 1g and odd-numbered particles mass = 2g. Subsequently, the slinky model will deviate from the true catenary curve due to the non-uniform mass distribution as illustrated in Figure 4.4 below. Conversely, to get a catenary curve the masses of the particles need to be uniformly distributed along the slinky.

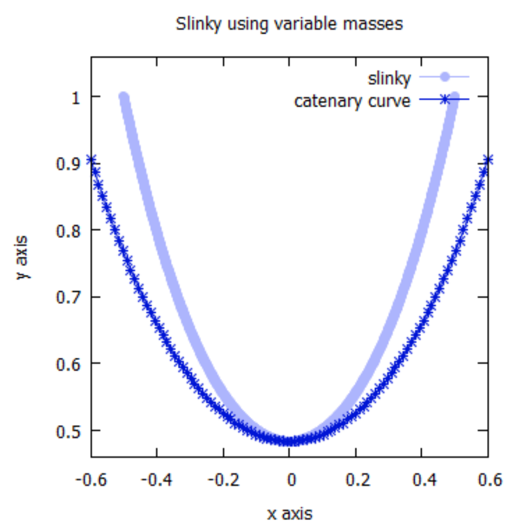


Figure 4.4 The Slinky model using variable masses.

4.4 Ask your questions: A Slinky Dog!

Figure 4.5 The Slinky dog "Slink" A toy dog with a spring.

Have you ever wondered what will happen if we raise mN above the rest of the masses in the slinky? What will happen to the position of the center of mass in this case? will it be at the center of the slinky? or will it be shifted towards one of the sides? These questions were inspired by the Slinky dog "Slink" from the movie "Toy Story". (**Figure 4.3**). As its neck rises above the rest of its body, it resembles a U-shaped suspended slinky. The center of mass of the assumed two-dimensional motion, was computed using the **Equation.17** for the center of mass along the x axis, and Equation.18 for the center of mass along the y axis.

$$x_{COM} = \frac{\sum_{i=0}^{N} m(i)x(i)}{M_{Total}}$$

Equation 17

Where m(i) is the mass of each particle I, x(i) is the corresponding position of each particle from the origin, and M_{Total} is the summed total of the particle's masses (the total mass of the slinky).

$$y_{COM} = \frac{\sum_{i=0}^{N} m(i)y(i)}{M_{Total}}$$

Equation 18

Where m(i), y(i), and M_{Total} are the mass of each particle i, y(i) is the corresponding position of each particle from the origin, and the total of the particle's masses respectively.

using N = 10, and fixing m0 at the origin, m_N at 1m on the x-axis, and 0.05m on the y-axis; the center of mass was found to be located at (0.0213,0.5000)m. This indicates that the center of mass was shifted towards m0 (the tail of Slink). That is because the center of mass would have been at 0.0250 on the y axis if the particles were uniformly distributed. **Figure 4.4** shows the instants of motion till equilibrium (**Figure 4.4.c**). This scenario was further studied in [6].

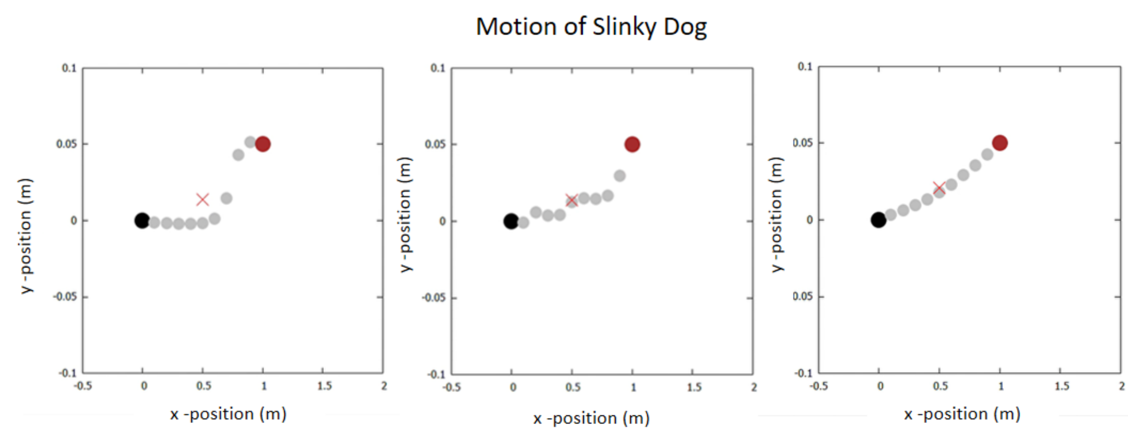


Figure 4.4 The motion of the slinky dog at several instants

5. Free Falling Slinky

5.1 Falling Slinky

In this section, the slinky is dropped and studied by considering different number of attached particles, and investigating the motion with respect to the center of mass of the slinky. The time required to move the last particle in the model was determined for different number of particles (N = 10, N = 20, N = 30, N = 40, and N = 50). The time it takes the last particle in the slinky to move was found after setting the condition where the velocity of the particle m_N is not equal to zero (this was achieved by choosing a suitable tolerance for the corresponding number of particles).

The following graph exhibits the relationship between the number of particles (N) and the time t(N) that it takes the bottom particle to move:

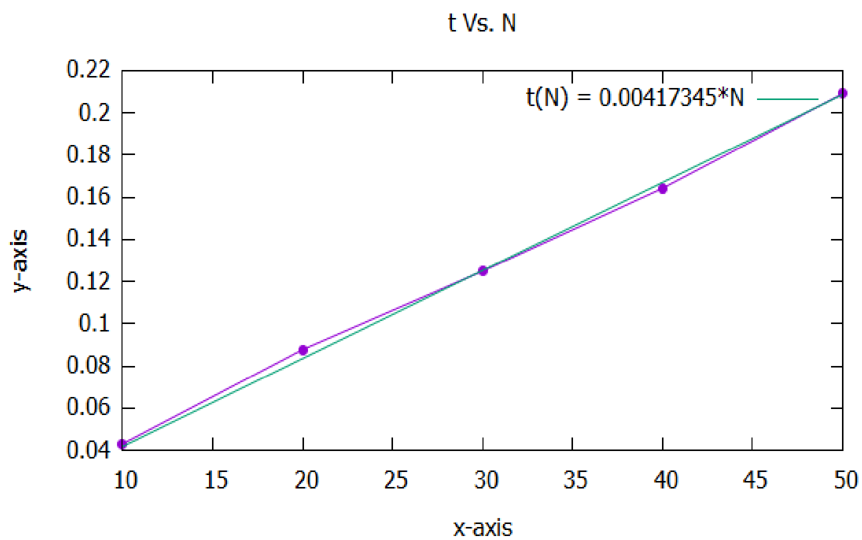


Figure 5.1 the relationship between t and N.

From the graph, it is clear that there is a direct relationship between the number of particles and the time needed for m_N to start moving. When the number of particles is increased, the corresponding m_N motion time increases as well. Using the linear equation fit, the equation describing this relationship is found to be:

$$t(N) \cong 0.0042N$$

Equation 19

From literature [5], The time t that it takes the last particle in the slinky to move was approximated to be the time it takes the center of mass to reach the bottom of the slinky:

$$t = \sqrt{\frac{2l}{3g}}$$

Equation 20

Where l is the length of the slinky at equilibrium (before letting it go), and g is the gravitational acceleration constant.

Comparing our findings with the values obtained from **Equation.20**, we found:

Number of particles	Computed time	Literature time	Percentage error
N = 10	0.0430	0.0428	0.47%
N = 20	0.0877	0.0837	4.78%
N = 30	0.1249	0.1245	0.32%
N = 40	0.1642	0.1652	0.61%
N = 50	0.2091	0.2062	1.41%

Table 1. The time it takes the bottom of the slinky to start moving in comparison with literature.

As **Table.1** shows, all the computed values had a percentage error < 5%, which makes them acceptable in our investigation. This error can be interpreted by considering the tolerance assigned to the last particle's velocity.

To study the motion of the center of mass of our model, the center of mass for the slinky was calculated using the following **Equation.21**.

$$y_{COM} = \frac{\sum_{i=0}^{N} m(i)y(i)}{M_{Total}}$$

Equation 21

Where m(i) is the mass of each particle i, y(i) is the corresponding position of each particle from the origin, and M_{Total} is the summed total of the particle's masses (the total mass of the slinky).

Figure 5.2 below shows the position of the center of mass of the slinky (with N = 10) before the motion (illustrated as a purple star). It is convenient that the center of mass of a slinky at equilibrium is not exactly at the center, but slightly shifted towards the bottom (where most of the particles are). This agrees with the discussion mentioned in **Section 4.1** that the center of mass tends to be at lower positions for any object to be stabilized.

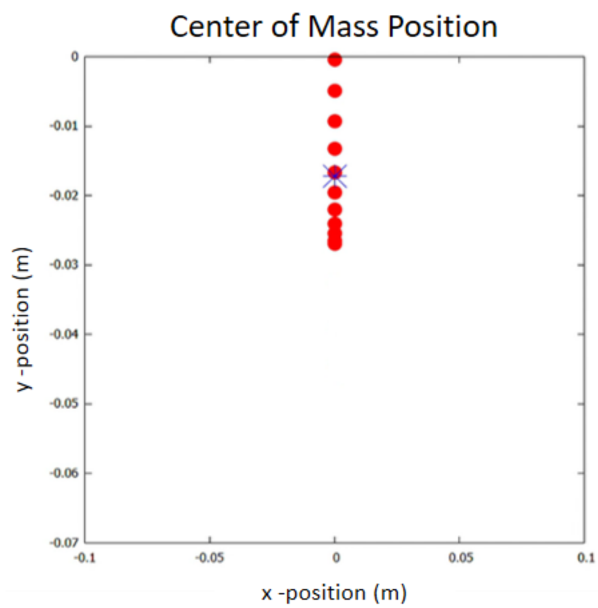


Figure 5.2 The location of the center of mass (represented by a purple star) relative to the particles in the slinky model.

The position of the center of mass during the motion of the slinky at equal time steps is illustrated in **Figure 5.3**.

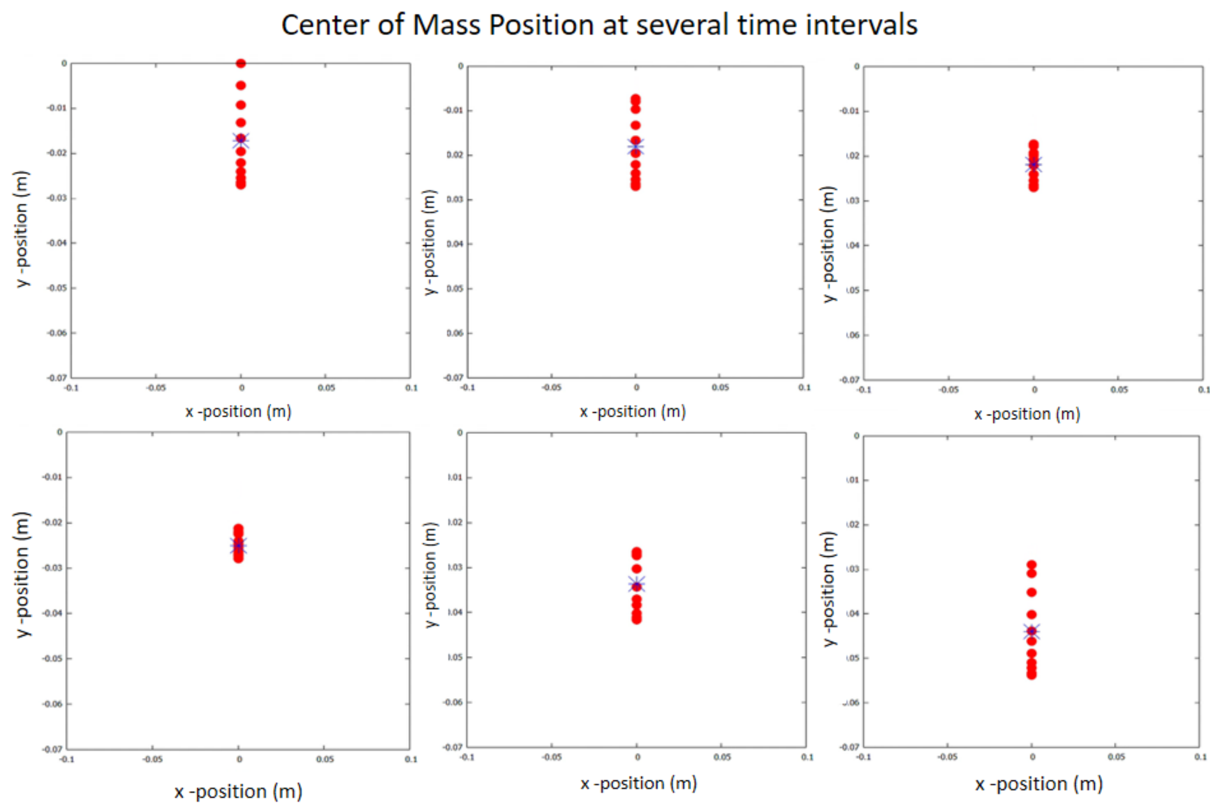


Figure 5.3 the motion of the falling slinky model with the position of its center of mass.

5.2 Adding Energy to the System

In this part, we considered two dynamics for the falling slinky model; the falling slinky model with mass m_0 oscillating with constant frequency (similar to the case of a child playing with it), and a charged falling slinky moving in a constant electric field directed in the positive y direction.

- **Jiggle-Jiggle!**

In this model, the first mass at the top of the slinky is moved with angular frequency = 30 rad s^{-1} and amplitude = $0.2\times \ell$ (0.2 of the total elongated length of a slinky at equilibrium). The oscillation of the mass m_0 took the form of **Equation.22**.

$$y_0(t) = Acos(wt)$$

Equation 22

Where A, w, and t are; amplitude, angular frequency, and time respectively. In this model, due to the external force (extra energy) applied by m_0 to the system, the particles can go to lower positions than that of the first oscillation (or go higher than their initial equilibrium positions). Since the particles in our adopted model are dimensionless, no collisions can occur between successive particles. Adding to that, the frequency of successive particles can match, which can make them resonate (have higher amplitudes than the initial one). As a result, some particles can jump and go higher than their former neighbor in sequence (or vice versa) as shown in **Figure 5.4**.

X vs. y for m_0 oscillating slinky

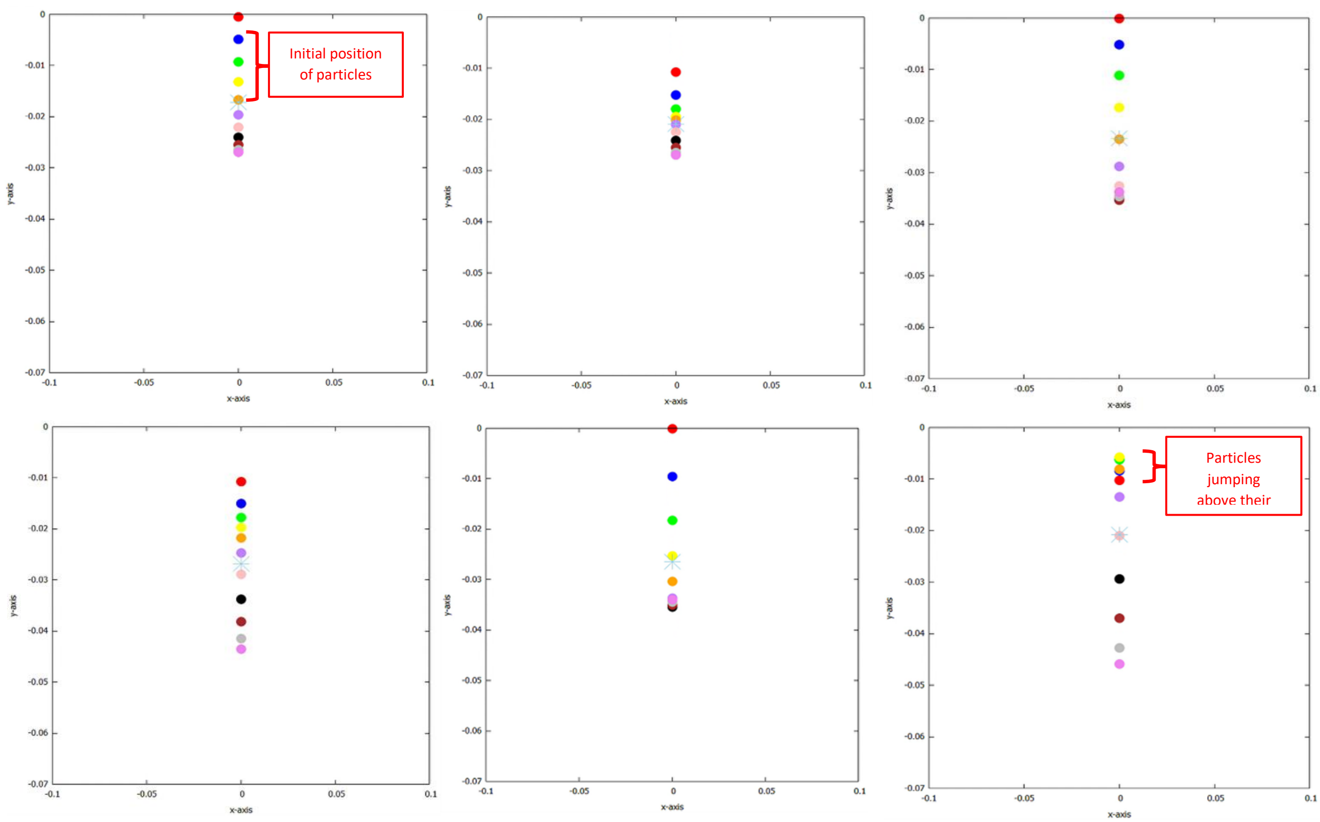


Figure 5.4 The motion of the slinky with m_0 oscillating at equal time steps. Notice the hop of some particles due to the absence of collisions between them.

5.3 Snail-like slinky

In this model, each particle in the slinky was assumed to be charged with q = 1 pC. The slinky was dropped from rest while it is being placed in a uniform electric field $\vec{E} = 9\hat{j}\ \frac{GN}{C}$. The behavior of the slinky following this scenario is shown in **Figure 5.6**.

Motion of Charged Slinky Through Uniform Electric Field

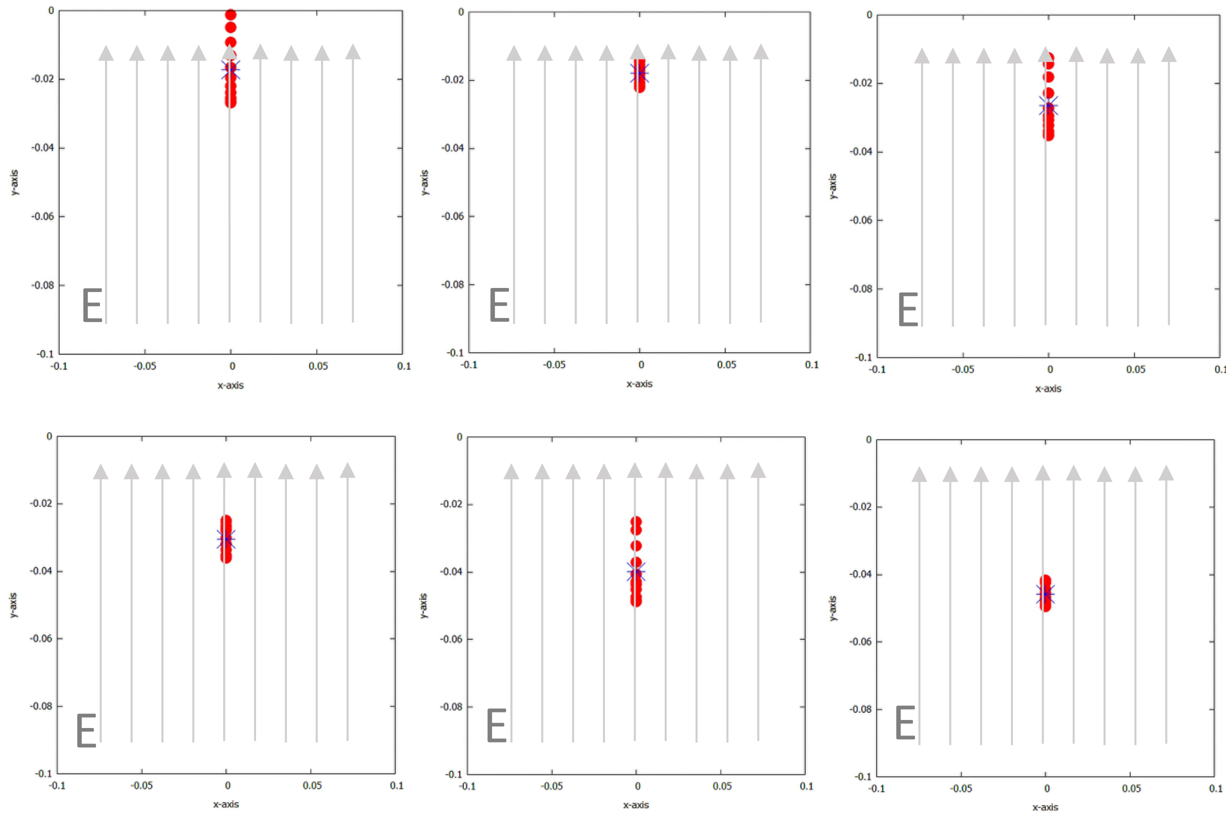


Figure 5.6 The motion of the charged slinky in a uniform electric field in the positive y direction.

As the figure shows, the particles start moving in the negative y direction slowly due to the presence of an upward force given by **Equation.23**.

$$\vec{F_e} = q\vec{E}$$

Equation 23

Where, F_e is the electrical force, q is the electrical charge, and E is the electrical field.

The charge of each particle is very small in a way that the interactions between particles due to their charges are minute, unless they were extremely close to each other. When the distance between the particles is sufficient for the repulsion to occur, we can see that the particles repel and separate from each other rapidly following the Coulomb force in **Equation.24**.

$$F_c = \sum_{i,j=0}^{N} \frac{kq^2}{r_{ij}^2} \ where\ i \neq j$$

Equation 24

Where k, q, and r_{ij} are the electric constant, electric charge, and the separation between each particle with its neighbors respectively. The equation describing the motion of such slinky is given by **Equation.25**.

$$\left(F_y\right)_i = k_{i-1}(y_{i-1} - y_i) + k_i(y_{i+1} - y_i) - m_i g + \sum_{i,j=0}^{N} \frac{kq^2}{r_{ij}^2} + qE - c(v_y)_i$$

Equation 25

Later on, the particles are pulled under their own weight, then collapsing over again. The motion of this charged slinky is very close to the motion of a slimy snail with its tail initially at the origin and heading its motion to something interesting located in the negative y direction!

Conclusion

In this project, the toy slinky is modeled as a set of dimensionless particles attached in series with massless springs. Due to the oscillatory nature of the slinky, the Euler-Cromer numerical method was adopted to conserve the total energy in simulation. The project considered three main aspects of study with further points of investigation within each aspect. The first investigation was regarding the vertical hanging slinky. While the other two were considered with the catenary comparison, and the dynamics of dropping the slinky from a certain initial state. For the vertical hanging slinky, we studied the relationship between the spring constant and the number of masses on the total elongated length of the slinky. For the optional task, we considered the case of having the exact scenario, but on the surface of the Moon. For the catenary like investigations, we observed the deviation of the suspended slinky from the catenary curve. Later on, we tried to find the best parameters that give us a catenary-like curve. The Slink toy ‘s center of mass was studied as an optional investigation. Finally, we analyzed the case where the vertically hanging slinky is dropped. We examined two additional dynamics; oscillating the first particle, and charging the slinky and dropping it in a constant electric field.

Future investigations

As a future work, we want to implement the code for freely-dropped suspended slinky with same the investigations done for the vertical one. We might also consider comparing our adopted model with the real slinky toy. Moreover, we want to examine the relationship between the stiffens of the springs and the period of oscillations for different number of masses. Another aspect of study is the relationship between the fluid that the slinky moves in (e.g., water, oil, honey, etc.) and the corresponding dynamics.

References

[1] D. J. Alsaei, "Slinky," *Project,* p. 4, 10 April 2023.

[2] R. Lallensack, "The Accidental Invention of the Slinky," 29 August 2019.

[3] CONVERSATIONOFMOMENTUM, "Catenary Curve," 10 AUGUST 2014.

[4] "Catenary Curves," *International Journal of Managment, Technology end Engineering,* vol. 8, no. XII, p. 644, 2018.

[5] R.C.Cross and M. S. Wheatland, "Modeling a falling slinky," *American Journal of Physics,* vol. 1051, p. 80, 2012.

[6] A. Eskandari-asl, "Some Static Properites of Slinky," *European Journal od Physics,* vol. 39, 2018.

[7] Nicholas J.Giordano - Hisao Nakanishi, in *Computational Physics*, 3 ed., Upper Saddler River, 2006, pp. 134-137.

[8] Harvey Gould- Jan Tobochnik- Wolfgang Christin, in *Computer Simulation Methods*, 3 ed., Pearson Addison Wesley, 2016, p. 45.