Finite Element Method (AE675)

Why do we compute?

- To "solve" real-life problems – given in terms of their mathematical definitions
  - PHYSICS of things in the language of MATHEMATICS
- Solve real life problems to understand whats happening and to DESIGN.
- ANALYZE effect of changes – WHAT IF MOMENTS!

How do we compute?

- MODEL reality using mathematical laws in terms of FIELD variables (e.g. displacement, temperature, velocity, electric displacement, etc).
  - HOW GOOD IS THE MODEL?
    Counter question ~ ⟨WITH RESPECT TO WHAT?⟩
  - IDEALIZATION – removal of "unnecessary" complexity to get simplified mathematical equations (linear, static, small perturbation, ...)
  - Global conservation laws (of mass, momentum, energy, charge, ...)
- Convert mathematical conservation laws into IBVP's (initial-boundary value problems) ~ PARTIAL DIFFERENTIAL EQUATIONS.
- Convert IBVP's into forms that a COMPUTER can handle ~ MATRIX PROBLEM

$$[K]\{x\} = \{F\}$$

WHERE DO WE COMPUTE ?

- All disciplines — engineering, biology, chemistry, physics, geo-tech, etc.

WHAT do we compute ?

- Compute to obtain field variables that define the state of the system (as we understand it!)

- Will it break ? (stress, strain, fracture parameters) Will it melt ? (Temperature, heat flux) Will it choke ? (Flow rate, pressure)

* $\langle I \rangle$ IDEALIZED MODEL OF REALITY $\langle R \rangle$

MODELING ERROR

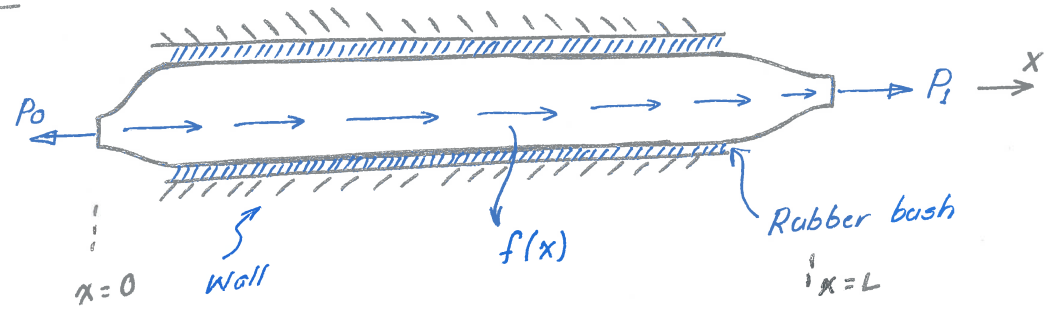
* $\langle I_h \rangle$ APPROXIMATE solutions of the mathematical problems

DISCRETISATION / APPROXIMATION ERROR

Questions:

- Does it converge to EXACT solution (of idealized model)? How close is I_h to I ?
- How fast does it converge ?
- What is good enough for me ?

EXAMPLE:



Reality: A metal rod of cross-sectional area $A(x)$, with a snugly fit rubber bush, is enclosed in a container and is subjected to a distributed axial load $f(x)$ and end-loads P_0, P_1 as shown.

also idealizations

Idealized model:
* Rod ~ linear elasticity, only axial stress
  i.e. $\sigma_{xx} = E\,\varepsilon_{xx} = E\dfrac{du}{dx}$
* Rubber bush ~ linear spring (distributed) with constant $k(x)$. $\langle N/m \rangle$
* Small deformation, slender member

$\Downarrow$

1D equation instead of 3D elasticity

$$\boxed{\frac{d}{dx}\left(EA\frac{du}{dx}\right) + \underset{-k(x)\,u(x)}{f(x)} = 0 \quad , \quad 0 < x < L} \quad \text{— (A)}$$

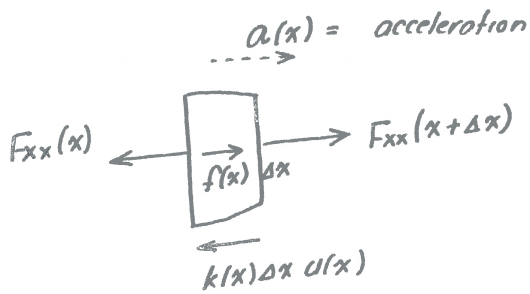
← governing eqn.

with $F_{xx}(x) = \sigma_{xx}(x)\,A(x) = EA\dfrac{du}{dx}$ ← AXIAL FORCE

Also, $\boxed{F_{xx}\big|_L = P_1 \; ; \; F_{xx}\big|_0 = P_0}$ ⇐ Boundary Condition — (B)

Concept of residual:

Let $r(x) = \dfrac{d}{dx}\left(EA\dfrac{du}{dx}\right) - ku + f$ be the residual or the RESULTANT FORCE on an infinitesimal piece of size Δx.



Since it is a STATIC problem, $a(x) = 0 \Rightarrow$

$$\cancel{\Delta x}\, r(x) = F_{xx}(x+\Delta x) - F_{xx}(x) + f(x)\Delta x - k(x)\,u(x)\,\Delta x = 0$$

$$= \left(\frac{d}{dx}(F_{xx}) + f(x) - k(x)\,u(x) \right) \cancel{\Delta x}$$

$$\int_0^L r(x)\, v(x)\, dx = W = \text{work done by the residual}$$

against a "suitable" test function $v(x)$.

Clearly, $\underline{W = \int_0^L r(x)\, v(x)\, dx = 0}$

$\Uparrow$

WEIGHTED RESIDUAL FORMULATION.

* For W to be finite,
  * $r(x)$ has to exist, i.e. $\frac{d^2u}{dx^2}$ has to exist or can have jumps (worst case)

  * $v(x)$ has to exist or $v(x)$ can be discontinuous (worst case).

Thus $v(x)$ and $u(x)$ need to satisfy different conditions.

What we will ~~due~~ do now is create SERIES representation of $u(x)$, $v(x)$ as:

$$u(x) = \sum_{i=0}^{\infty} \alpha_i \varphi_i(x) \quad \text{— (a)}$$

$$v(x) = \sum_{i=0}^{\infty} \beta_i \psi_i(x) \quad \text{— (b)}$$

such that they satisfy the existence property.

- We are assuming that the series (a), (b) are "convergent" and complete.
- Here $\{\varphi_i(x)\}_{i=0}^{\infty}$, $\{\psi_i(x)\}_{i=0}^{\infty}$ are the "basis" functions for the 2 series (a), (b).

WILL BE CHOSEN BY US $\Rightarrow$ <u>Basis functions</u>: Infinite such functions, <u>linearly independent</u> and capable of representing any function via (a), (b) which satisfy the desired property.

Linear independence: $\sum_{i=0}^{\infty} \gamma_i \varphi_i(x) = 0 \Rightarrow \gamma_i = 0$, $i = 0, 1, 2, \ldots \infty$

$\sum_{i=0}^{\infty} \delta_i \psi_i(x) = 0 \Rightarrow \delta_i = 0$

From residual, $u(x)$ has to satisfy $\int_0^L (u_{xx})^2 dx < \infty$

$u \in H_2(\Omega) \rightsquigarrow$ or $u(x)$ has square-integrable second derivative

$v \in L_2(\Omega) \rightsquigarrow$ $v(x)$ has to satisfy $\int_0^L v^2 dx < \infty$ or $v(x)$ is square-integrable

$\uparrow$ FUNCTION SPACES

$\Omega = [0, L] \leftarrow$ domain of interest.

Since $\varphi_i(x) \neq \psi_i(x) \rightsquigarrow$ weighted residual form is called Petrov-Galerkin formulation

* $v(x)$ is under "our" control, or the test function is CHOSEN by me; $u(x)$ has to be "solved" for, or we need to get coefficients $\{\alpha_i\}_{i=0}^{\infty}$ in order to construct $u(x)$.

$\rightarrow$ Note that the fact that $F_{xx}|_0 = P_0$, $F_{xx}|_L = P_L$ has not figured in W means that we have to <u>enforce</u> these explicitly. Often tough to do (in 2, 3D). Can we do better?

WEAK FORMULATION

$$W = \int_0^L \left(\frac{d}{dx}(F_{xx}) - ku + f \right) v \, dx = 0 \quad \text{— (c)}$$

Now let us integrate the first term by parts, i.e.

$$\frac{d}{dx}(F_{xx}) \cdot v = \frac{d}{dx}(F_{xx} \cdot v) - F_{xx} \cdot \frac{dv}{dx}$$

$$\Rightarrow \int_0^L \frac{d}{dx}(F_{xx}) \cdot v \, dx = \int_0^L \frac{d}{dx}(F_{xx} \cdot v) \, dx - \int_0^L F_{xx} \cdot \frac{dv}{dx} dx$$

$$= F_{xx}\big|_L \cdot v\big|_L - F_{xx}\big|_0 \cdot v\big|_0 - \int_0^L F_{xx} \frac{dv}{dx} dx \quad \text{— (c1)}$$

Putting (c1) in (c) we get:

$$W = -\int_0^L F_{xx} \cdot \frac{dv}{dx} dx + \left(-\int_0^L kuv \, dx \right) + \left[\int_0^L fv \, dx + F_{xx}\big|_L v\big|_L - F_{xx}\big|_0 v\big|_0 \right] = 0$$

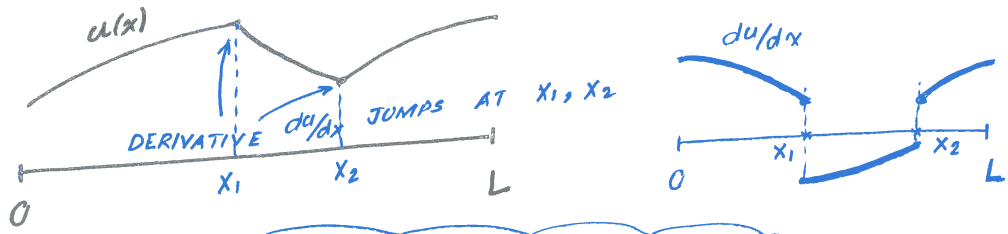
$$\Rightarrow \underbrace{\int_0^L \left(\overbrace{EA \frac{du}{dx}}^{F_{xx}} \frac{dv}{dx} + kuv \right) dx}_{\substack{B(u,v) \\ \text{bilinear form}}} = \underbrace{\int_0^L fv \, dx + F_{xx}\big|_L v\big|_L - F_{xx}\big|_0 v\big|_0}_{\substack{F(v) \\ \text{linear functional}}} \quad \text{— (c2)}$$

(c2) has a few salient features :

(i) The LHS now has $\frac{du}{dx}$, $\frac{dv}{dx}$, i.e. 1st derivatives of both u, v. Any more integration by parts will not help. (WHY?). Thus (c2) demands EXISTENCE of $\frac{du}{dx}$, $\frac{dv}{dx}$ for $B(u,v)$ to be FINITE.

This is a RELAXATION with respect to the demand for existence of $\frac{d^2u}{dx^2}$ earlier. Hence the name

WEAK FORMULATION for (C2). This says that the worst-case scenario that we can have is: $u(x)$, $v(x)$ are barely continuous in $(0, L)$, i.e. they can have KINKS at some points which means that derivative can jump at SOME POINTS.



WHEN DO WE ENCOUNTER THIS SITUATION ?

(ii) $B(u, v) = B(v, u)$, i.e. is symmetric

Also $B(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1 B(u_1, v) + \alpha_2 B(u_2, v)$

$B(u, \gamma_1 v_1 + \gamma_2 v_2) = \gamma_1 B(u, v_1) + \gamma_2 B(u, v_2)$

$F(\alpha v) = \alpha F(v)$

(iii) LHS of (C2) is work done by internal force F_{xx} against strain due to v; RHS is work done by external forces $f(x)$, P_L, P_0 + work done by spring

$$\Downarrow$$

$$RHS = \int_0^L f v \, dx + P_L v|_L - P_0 v|_0$$

So (C2) is a generalized version of principle of virtual work.

(iv) P_0, P_L come in naturally in the formulation.

(v) If $v(x) = u(x)$ is chosen (if allowed), then

$$B(u,u) = \int_0^L \left(EA \left(\frac{du}{dx}\right)^2 + k u^2 \right) dx = 2 \underbrace{U(u)}_{\text{strain energy}}$$

$B(u,u) \geq 0$ with $\underbrace{u = 0}_{\text{TRIVIAL SOLUTION}}$ giving $B(u,u) = 0$

Thus, $B(u,u)$ can be seen as a measure of "length" of the function $u(x)$, i.e.

$$\|u\|_{E(\Omega)} = \sqrt{B(u,u)}$$

signify length or NORM

specific measure of length here is strain energy or energy E over domain $\Omega = (0,L)$.

* Measure of length is needed to define "CLOSENESS" of functions, i.e. if I claim the $u_1(x)$ is "close" to $u(x)$ in Ω, then how do I show it?

We will MEASURE

$$\|u-u_1\|_{E(\Omega)} = \sqrt{B(u-u_1, u-u_1)} = \eta$$

If η is "SMALL" (compared to $\|u\|_{E(\Omega)}$) then we can say that $u_1(x)$ is close to $u(x)$ in Ω with respect to the norm $\|u-u_1\|_{E(\Omega)}$.

(vi) Note that (C2) gives us the TYPE of possible boundary-conditions at $x=0$ and $x=L$, i.e. RHS has

$$F_{xx}|_L \, v|_L - F_{xx}|_0 \, v|_0$$

either $F_{xx}|_L$ or $v|_L$ is given and is ZERO

(in example, $F_{xx}|_L = P_L$)

either $F_{xx}|_0$ is given (in this case $F_{xx}|_0 = P_0$) or $v|_0$ is given and is ZERO

* $v(0) = 0$, i.e. $v(0)$ is given is an outcome of $u(0) = u_0$ being given as KNOWN displacement at $x=0$. So,

EITHER

a) $F_{xx}|_0 = P_0$ is given

* No constraint on $v(0)$

OR

$u(0) = u_0$ is given which implies we choose $v(0) = 0$

b) $F_{xx}|_L = P_1$ is given

* No constraint on $v(L)$

$\Uparrow$

FORCE / NEUMANN / NATURAL boundary condition

$u(L) = u_L$ is given which implies we choose $v(L) = 0$

$\Uparrow$

DISPLACEMENT / DIRICHLET / ESSENTIAL boundary condition

* Nothing more is possible as boundary - conditions.
* The example is ONE case. We want to handle ALL possible cases.

■ HOW TO GET $u(x)$?

- We will again assume $u(x) = \sum_{i=0}^{\infty} \alpha_i \varphi_i(x)$

But now (GALERKIN APPROACH) $v(x) = \sum_{i=0}^{\infty} \beta_i \varphi_i(x)$

will be CHOSEN, i.e. $v(x)$ is in terms of the same basis functions as for $u(x)$.

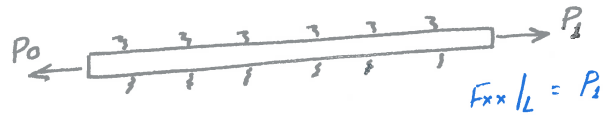
APPROXIMATION : Assuming that indeed a convergent (uniformly) series representation of $u(x)$ is possible, we can take TRUNCATED series $u_N(x) = \sum_{i=0}^{N} \alpha_i \varphi_i(x)$.

Then we choose $v(x) = v_N(x) = \sum_{i=0}^{N} \beta_i \varphi_i(x)$ such that $v(x)$ (or $v_N(x)$) is ZERO wherever $u(x)$ (or $u_N(x)$) is given (as B.C.).

i.e. $\boxed{u_N(\bar{x}) = \bar{u} \text{ (given value) has to be enforced too}}$

EXAMPLES :

1)



$F_{xx}|_0 = P$

$u(0)$ unconstrained

$v(0)$ "

$F_{xx}|_L = P_1$

$u(L)$ unconstrained

$v(L)$ "

2)

$u(0) = 5$

$\downarrow$

$v(0) = 0$ to be enforced

$u_N(0) = 5$ to be enforced

$u(L) = 0$

$\downarrow$

$v(L) = 0$ to be enforced

$u_N(L) = 0$ to be enforced

3)

$u(0) = 0$

$\downarrow$

enforce

$u_N(0) = 0$

$\updownarrow$

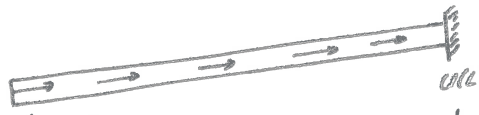
$v_N(0) = 0$

$F_{xx}|_L = P_1$

$\downarrow$

no constraint on $u(L)$, $v(L)$.

4)



$F_{xx}|_0 = 0$

no constraint on $u(0)$, $v(0)$

$u(L) = 0$

$\downarrow$

enforce

$u_N(L) = 0$

$\updownarrow$

$v_N(L) = 0$

CHOICE OF $\varphi_i(x)$

A simple choice is $\boxed{\varphi_i(x) = x^i}$; $i = 0, 1, 2, \cdots N$

Monomials

Linearly independent, complete

i.e. $\sum_{i=0}^{N} \gamma_i \varphi_i(x) = 0 \Rightarrow \gamma_i = 0, \quad i = 0, 1, 2, \cdots, N$

$\sum_{i=0}^{N} \delta_i \varphi_i(x) = f(x) \leftarrow = f_0 + f_1 x + \cdots + f_N x^N$

a generic polynomial of degree upto N can be exactly represented by LHS.

* It is easy to see that indeed

$\delta_i = f_i$ will give an EXACT match!

Now let $u_N(x)$ be $\sum_{i=0}^{N} \beta_i \varphi_i(x)$.

From (C2), using u_N instead of u and u_N as v (since we can choose $v(x)$) we get:

$$B(u_N, u_N) = F(u_N) \qquad \text{—— (C3)}$$

Now we choose β_i judiciously, i.e. let $\beta_i = 1$, $\beta_j\big|_{j \neq i} = 0$ to get a CHOICE of a valid $u_N(x)$ as $u_N(x) = \varphi_i(x)$, $i = 0, 1, \ldots, N$.

$$\therefore \quad B(u_N, \varphi_i) = F(\varphi_i), \qquad i = 0, 1, 2, \ldots, N \qquad \text{—— (C4)}$$

$$\Rightarrow B\left(\sum_{j=0}^{N} \alpha_j \varphi_j, \varphi_i\right) = \sum_{j=0}^{N} \alpha_j B(\varphi_j, \varphi_i) = F(\varphi_i)$$

$$\Rightarrow \sum_{j=0}^{N} \alpha_j \underbrace{B(\varphi_i, \varphi_j)}_{K_{ij}} = \underbrace{F(\varphi_i)}_{F_i}$$

or $[K]\{\alpha\} = \{F\}$ displacement vector

with $K_{ij} = B(\varphi_i, \varphi_j) = \int_0^L \left(EA \frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} + k\,\varphi_i \varphi_j\right) dx$

$$F_i = F(\varphi_i) = \int_0^L f \cdot \varphi_i \, dx + P_L \cdot \varphi_i\big|_L - P_0 \varphi_i\big|_0$$

For $\varphi_i(x) = x^i$, we get $[i = 0, 1, 2, \ldots, N]$

$$K_{ij} = \int_0^L \Big(\underbrace{EA\,(i)\,x^{i-1}\,(j)\,x^{j-1}}_{=0 \text{ for either } i=0 \text{ or } j=0} + k \cdot x^{i+j}\Big)\, dx$$

EXAMPLE: $EA = 10$, $k = 5$, $f(x) = f_0 = 1$, $P_0 = 5$, $P_1 = 7$

Let $N = 2$ or $u_2(x) = \alpha_0 \cdot 1 + \alpha_1 x + \alpha_2 x^2$

$v_2(x) = \beta_0 \cdot 1 + \beta_1 \cdot x + \beta_2 x^2$

* No constraints on $u(x)$, hence on $v(x)$.

$\therefore$ we have:

$$\left[\begin{array}{lll}
\int_0^L (0 + 5 \cdot x \cdot 1 \cdot 1)\,dx & \int_0^L (0 + 5 \cdot 1 \cdot x \cdot x)\,dx & \int_0^L (0 + 5 \cdot 1 \cdot x \cdot x^2)\,dx \\
 & & \int_0^L (2 \cdot EA \cdot x + 5 x^3)\,dx \\
K_{21} = K_{12} & \int_0^L (EA + 5 \cdot x^2)\,dx & \\
 & & \int_0^L (4 \cdot EA \cdot x^2 + 5 \cdot x^4)\,dx \\
K_{31} = K_{13} & K_{32} = K_{23} &
\end{array}\right] = [K]_{3\times 3}$$

$$\left\{\begin{array}{l}
\int_0^L 1 \cdot 1\,dx + P_1 \cdot 1 - P_0 \cdot 1 \\
\int_0^L 1 \cdot x\,dx + 7 \cdot L - 5 \times 0 \\
\int_0^L 1 \cdot x^2\,dx + 7 \cdot L^2 - 5 \times 0
\end{array}\right\} = \left\{\begin{array}{c} F_0 \\ F_1 \\ F_2 \end{array}\right\}_{3\times 1}$$

LOAD VECTOR

$$\boxed{\{\alpha\} = [K]^{-1}\{F\}}$$

stiffness matrix

* PROPERTIES OF $[K]$: Symmetric, positive definite (How?)

Positive definiteness: $\{\alpha\}^T [K] \{\alpha\} = B(u_N, u_N) \geq 0$ for non-trivial $u(x)$ $\Rightarrow$ $[K]$ is positive definite.

- $[K]$ is invertible or $[K]^{-1}$ exists.

<u>Discussion</u>: If $k = 0$ was chosen in the example then you can see that $K_{00}, K_{01}, K_{02}, K_{10}, K_{20} = 0$ or $[K]$ has ZEROES in 1st row and column.

$\Rightarrow$ $[K]$ is singular OR $[K]^{-1}$ does not exist!

Then the problem has a solution if and only if

$$F_0 = 0 \quad (0 = 0) \quad \text{or} \quad \underbrace{\int_0^L 1\,dx}_{L} + \underbrace{P_1 \cdot 1}_{7} - \underbrace{P_0 \cdot 1}_{5} = 0$$

For $L = 1$ (say), $1 + 7 - 5 = 3 \neq 0 \Rightarrow F_0 \neq 0 \Rightarrow$

NO SOLUTION EXISTS.

Hence P_1, P_0, f <u>should</u> be such that

$$(\star) \quad \int_0^L f \cdot 1\,dx + P_1 \cdot 1 - P_0 \cdot 1 = 0 \Leftarrow$$

CONSISTENCY CONDITION OR STATIC EQUILIBRIUM REQUIREMENT FROM MECHANICS!

If $(\star)$ is satisfied then solution exists (infact infinite solutions exist), i.e. if $u_2(x)$ is a solution then $u_2(x) + C$ is also a solution! i.e. add a rigid displacement and nothing changes.

* Then we will choose a specific C such that $\underline{u_2(0) = 0}$

or we CHOOSE $\underline{u(0) = 0}$

$\Downarrow$

$\alpha_0 = 0$

$\Rightarrow \beta_0 = 0$ has to be enforced

Thus $u_2(x) = \alpha_1 x + \alpha_2 x^2$

$v_2(x) = \beta_1 x + \beta_2 x^2$

So we need to ONLY knock off the 1st row & column from the $[K]$ above

Recipe :

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \int_0^L EA\,dx & \int_0^L 2EA\,x\,dx \\ 0 & & \int_0^4 4\,EA\,x^2\,dx \end{bmatrix}}_{[K]} \underbrace{\begin{Bmatrix} d_0 \\ d_1 \\ d_2 \end{Bmatrix}}_{\{d\}} = \underbrace{\begin{Bmatrix} \int_0^L f\,dx + \overset{0}{P_1} - P_0 \\ \int_0^L f \cdot x\,dx + P_2 L - 0 \\ \int_0^L f \cdot x^2\,dx + P_3 L^2 - 0 \end{Bmatrix}}_{\{F\}}$$

Make $K_{00} = 1$, $F_0 = 0$

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & K_{11} & K_{12} \\ 0 & K_{12} & K_{22} \end{bmatrix}}_{[\bar{K}]} \begin{Bmatrix} d_0 \\ d_1 \\ d_2 \end{Bmatrix} = \underbrace{\begin{Bmatrix} 0 \\ F_1 \\ F_2 \end{Bmatrix}}_{\{\bar{F}\}}$$

Now $[\bar{K}]$ is invertible with the 1st row giving

$1 \cdot d_0 = 0 \Rightarrow \boxed{d_0 = 0} \Longleftarrow$ REQUIRED VALUE OF $d_0 = u_2(0) = 0$!

* This is a way of smartly using a tweak to $[K]$, $\{F\}$ to get a certain d_k to have a GIVEN value

This is called PENALTY METHOD ~ used to ENFORCE Dirichlet B.C.

END OF DISCUSSION

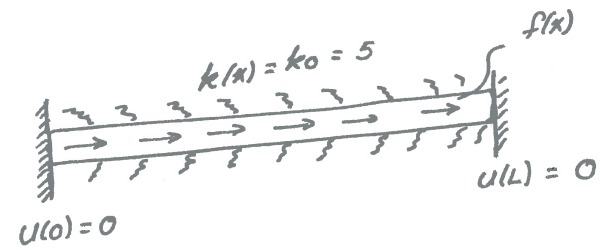
LAB. WORK 1

My teachers used to say that "You will not learn till you get your hands dirty".

(1) — So get your hands dirty & solve the example problem for $N = 2, 4; 6$. Plot the displacement $u(x)$ in $(0,1)$ (take $L = 1$). Report on convergence.

(2) — Solve the next (discussion) problem too, but with $P_0 = 8$ $(L = 1)$ $\langle \Rightarrow$ such that $P_1 - P_0 + \int_0^L f dx = 0 \rangle$ fixing $u(0) = 0$.

EXAMPLE 2



$EA(x) = 10$

$L = 1$

$f(x) = x$

Again $u_N(x) = \sum_{i=0}^{N} \alpha_i x^i$ with $u_N(0) = 0$, $u_N(L) = 0$

$\Rightarrow \boxed{\alpha_0 = 0}$ and $\alpha_1 L + \alpha_2 L^2 + \dots + \alpha_N L^N = 0$

$$\Downarrow$$

$$\boxed{\alpha_1 = -\left(\alpha_2 L + \alpha_3 L^2 + \dots + \alpha_N L^{N-1}\right)}$$

$\Rightarrow$ $u_N(x)$ can now be written as:

$$u_N(x) = \sum_{i=2}^{N} \alpha_i \left(x^i - L^{i-1} x\right) = \sum_{i=2}^{N} \alpha_i \underbrace{x\left(x^{i-1} - L^{i-1}\right)}_{\tilde{\varphi}_i(x)}$$

$\therefore$ We choose $v_N(x)$ to also be $v_N(0) = v_N(L) = 0$

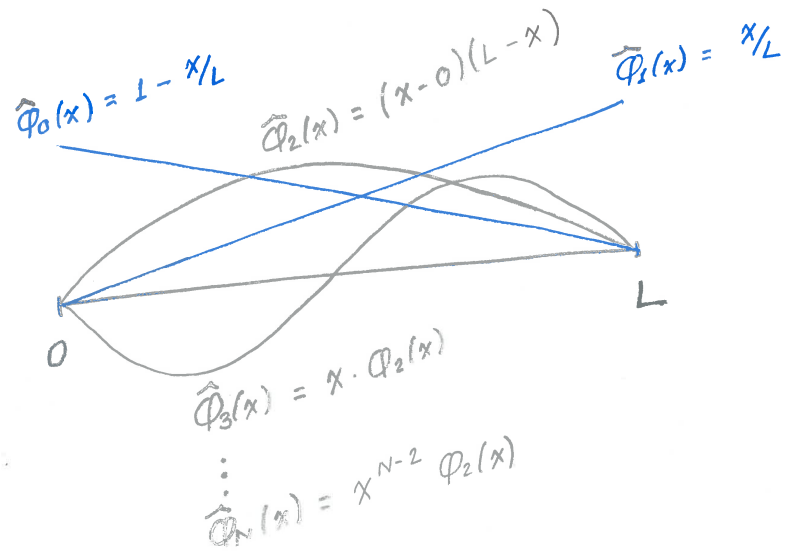
$$\Rightarrow v_N(x) = \sum_{i=2}^{N} \beta_i \tilde{\varphi}_i(x)$$

Only $(N-1)$ unknowns to be SOLVED for

(3) SOLVE THIS FOR $N = 2, 4$

The example 2 shows that in satisfying the displacement constraint at $x = L$, the basis functions had to be <u>redefined</u> as $\tilde{\varphi}_i(x) = x(x^{i-1} - L^{i-1})$ so that ALL $\tilde{\varphi}_i(x)$ are zero at $x = 0$, $x = L$! PLOT THESE FUNCTIONS!

- We want to do better than this, so that we do not have to redefine the basis functions and the conditions of $x = 0, L$ can be easily applied. HOW?



Then $$u_N(x) = \sum_{i=0}^{N} \alpha_i \hat{\varphi}_i(x)$$

Note that as $\hat{\varphi}_1(x), \hat{\varphi}_2(x), \ldots, \hat{\varphi}_N(x) = 0$ at $x = 0$ and $\hat{\varphi}_0(0) = 1$,

$$u_N(0) = \bar{u}_0 \Rightarrow \alpha_0 = \bar{u}_0$$

Similarly, $u_N(L) = \bar{u}_L \Rightarrow \alpha_1 = \bar{u}_L$ can be easily applied

* ARE THESE LINEARLY INDEPENDENT? IS THIS SET OF $\hat{\varphi}_i(x)$ COMPLETE?

CONVERGENCE

We have a solution / method to solve the problem. How do we assure that the solution (truncated series) is good? Lets establish some properties.

PROPERTIES OF U_N

Since $B(u, v) = F(v) \Rightarrow B(u, v_N) = F(v_n)$

(choosing $v(x) = v_N(x)$)

$$B(u_N, v_N) = F(v_N) \quad \text{—— (1)}$$

$$\Rightarrow \quad B(u, v_N) = F(v_N) \quad \text{—— (2)}$$

$$\Rightarrow \boxed{B(\underbrace{u-u_N}_{e_N}, v_N) = 0} \sim (3)$$

$\Leftarrow$ e_N is "perpendicular" to v_N OR error is ORTHOGONAL to all functions $v_N(x)$.

ORTHOGONALITY OF ERROR

Thus $B(u-u_N, u-u_N) = B(u,u) + B(u_N, u_N) - 2B(u, u_N)$

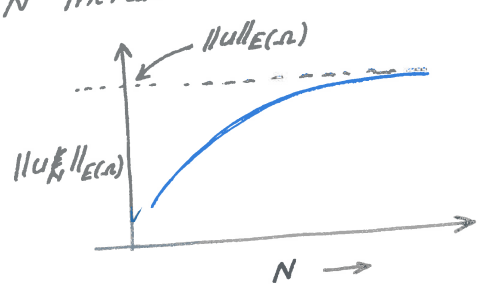
But from (3) above, $B(u, u_N) = B(u_N, u_N)$ <if $v_N = u_N$ can be chosen>

Thus,

$$\boxed{B(e_N, e_N) = \|e_N\|^2_{E(\Omega)} = \|u\|^2_{E(\Omega)} - \underbrace{\|u_N\|^2_{E(\Omega)}}_{\geq 0}} \sim (4)$$

From (4) it is clear that $\|u\|^2_{E(\Omega)} \geq \|u_N\|^2_{E(\Omega)}$ ~ (5)

OR, as N increases $\|u_N\|_{E(\Omega)}$ converges to $\|u\|_{E(\Omega)}$ from below



* Can you show that the convergence is monotonic, or $\|u_{N+1}\|_{E(\Omega)} \geq \|u_N\|_{E(\Omega)}$??

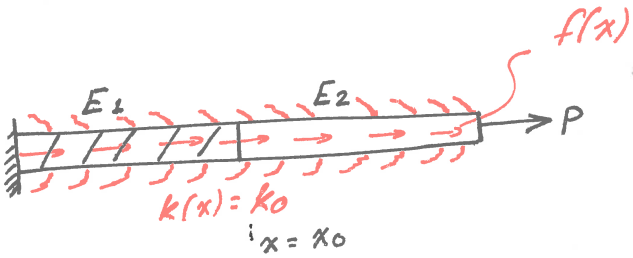
HINT: USE (3) & (4) above

TRANSITION TO FEM

If the polynomial series does the job, why talk of anything else?

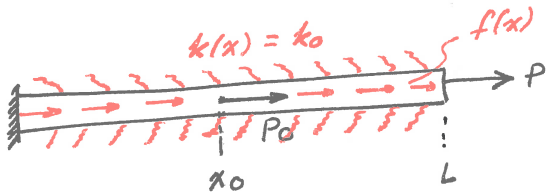
Let us see the reason for moving on:

(a)



Different material

(b)

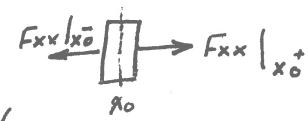


Point Load

Case (a): What happens at x_0?

You know that the differential equation (of equilibrium) is valid for $(0, x_0)$ and (x_0, L), i.e. at $x = 0, x_0, L$ the differential equation is NOT valid. Note that at $x = 0, L$ boundary conditions are PRESCRIBED; at $x = x_0$ we have to enforce continuity conditions OR

(1) $\boxed{u|_{x_0^+} = u|_{x_0^-}}$ CONTINUITY OF VALUE AT x_0 &



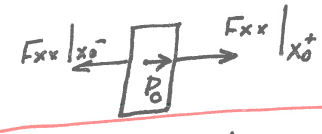
(2) $F_{xx}|_{x_0^+} = F_{xx}|_{x_0^-} \Rightarrow E_1 A \left.\frac{du}{dx}\right|_{x_0^-} = E_2 A \left.\frac{du}{dx}\right|_{x_0^+}$

or $\boxed{\left.\frac{du}{dx}\right|_{x_0^-} = \left(\frac{E_2}{E_1}\right) \left.\frac{du}{dx}\right|_{x_0^+}}$ JUMP IN DERIVATIVE AT x_0

Case (b): Again at $x = x_0$ we have:

(1) $u|_{x_0^+} = u|_{x_0^-}$

(2)

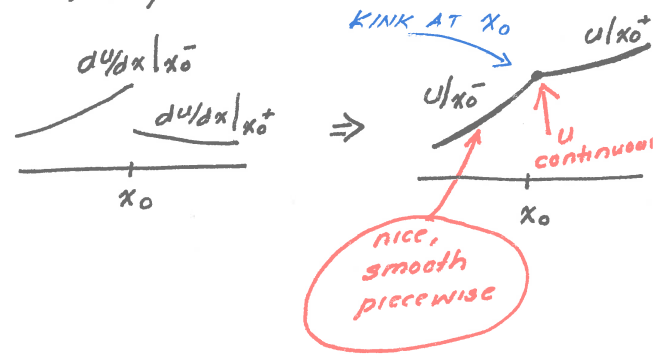


AXIAL FORCE JUMPS at x_0

$\Rightarrow \left.du/dx\right|_{x_0^-} - \left.\frac{du}{dx}\right|_{x_0^+} = \frac{P_0}{EA(x_0)} \Longrightarrow \boxed{F_{xx}|_{x_0^-} - F_{xx}|_{x_0^+} = P_0}$

Both cases are practically feasible, important and relevant.

If $\frac{du}{dx}$ jumps at a point

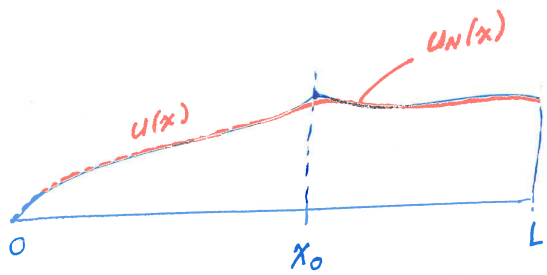


So, we CANNOT use the basis functions $\varphi_i(x) = x^i$, as

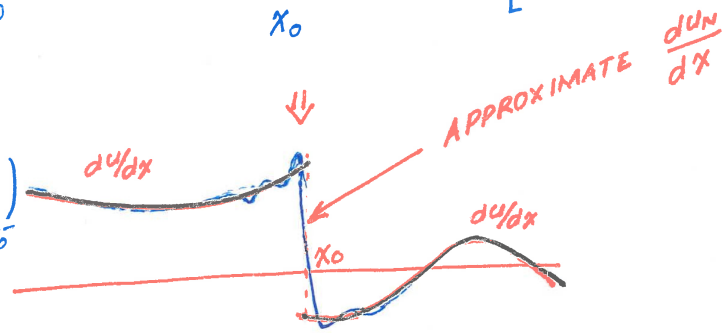
$$\frac{d\varphi_i}{dx} = i\,x^{i-1} \quad \text{which gives} \quad \left.\frac{d\varphi_i}{dx}\right|_{x_0^+} = \left.\frac{d\varphi_i}{dx}\right|_{x_0^-} = i\,x_0^{i-1}$$

and hence does not give the required jump in derivatives at $x = x_0$

WHAT IF I SOLVE WITH $\varphi_i(x) = x^i$ ANYWAY ?

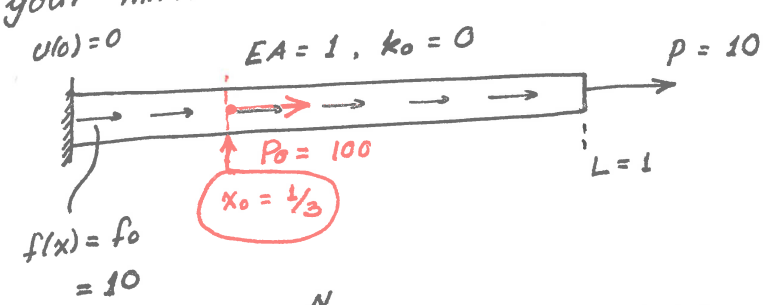


$$\lim_{N\to\infty} \frac{du_N}{dx} \to \frac{1}{2}\left(\left.\frac{du}{dx}\right|_{x_0^+} + \left.\frac{du}{dx}\right|_{x_0^-}\right)$$



HOME - WORK :

Use your MATLAB code to solve for



$$F_i = \int_0^1 10 \cdot x^i\,dx + p \cdot (L^i) + P_0 \cdot (1/3)^i$$

ADDITIONAL PART IN LOAD VECTOR

With $u_N(x) = \sum_{i=0}^{N} d_i\,\varphi_i(x)$; $\varphi_i(x) = x^i$

USE $N = 2, 4, 8$

LESSON:

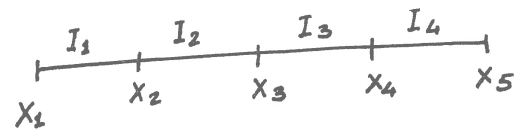
At $x = x_0$, the solution gives a wrong picture of $F_{xx}\big|_{x_0}$ or $\frac{du}{dx}\big|_{x_0}$. This is a problem as point load application points or MATERIAL transition points are CRITICAL stress points & the ANSWER we got does not help!

So, we need to create a better approximation. How?

$\boxed{\text{Piecewise polynomials?}}$

BASIC IDEA: 1) Divide domain $\Omega = (0, L)$ into some pieces. (i.e. NE pieces), as

Piece $I_k = (x_k, x_{k+1})$



$NE = 4$

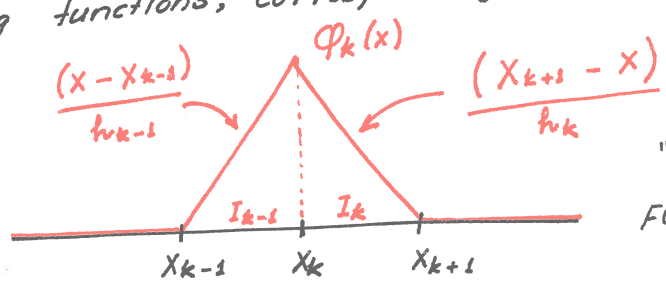
* Here No. of pieces, or ELEMENTS or SUBDOMAINS

$NE = 4$

x_ℓ, $\ell = 1, 2, \ldots, (NE+1)$, are the NODES or VERTICES

Finite (NE) no. of Elements $\quad I_k \rightarrow$ elements

2) Define the following functions, corresponding to each vertex x_k:



"HAT" FUNCTION

$h_{k-1} = x_k - x_{k-1}$

$h_k = x_{k+1} - x_k$

h_ℓ = length of $I_\ell = (x_{\ell+1} - x_\ell)$ is also called mesh-size

The function $\varphi_k(x) = \begin{cases} \dfrac{(x - x_{k-1})}{h_{k-1}} & \text{in } I_{k-1} \\ -\dfrac{(x_{k+1} - x)}{h_k} & \text{in } I_k \\ 0 & \text{everywhere else} \end{cases}$

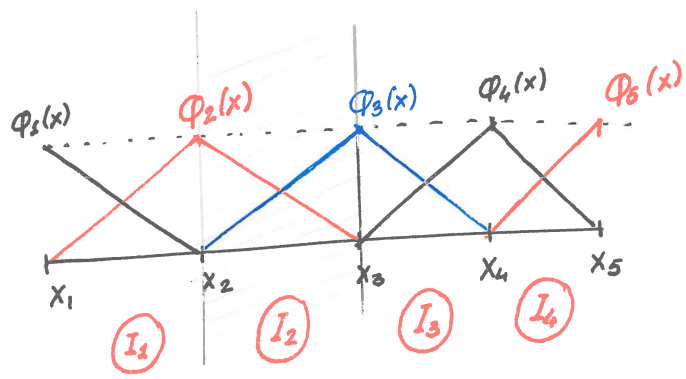
Note that (a) $\varphi_k(x)$ is defined in $\Omega = (0, L)$

(b) $\varphi_k(x)$ is non-zero in $I_{k-1} \cup I_k$ and is zero in rest of Ω ; $\varphi_k(x)$ is said to have <u>LOCAL SUPPORT</u>

(c) $\varphi_k(x)$ is LINEAR in I_{k-1} and I_k, i.e. it is piecewise linear.

(d) $\varphi_k(x_k) = 1$; $\varphi_k(x_{k-1}) = \varphi_k(x_{k+1}) = 0$

So, we have :



<u>Claim</u>: (1) $\sum_{i=1}^{NE+1} \varphi_i(x) = 1$ everywhere

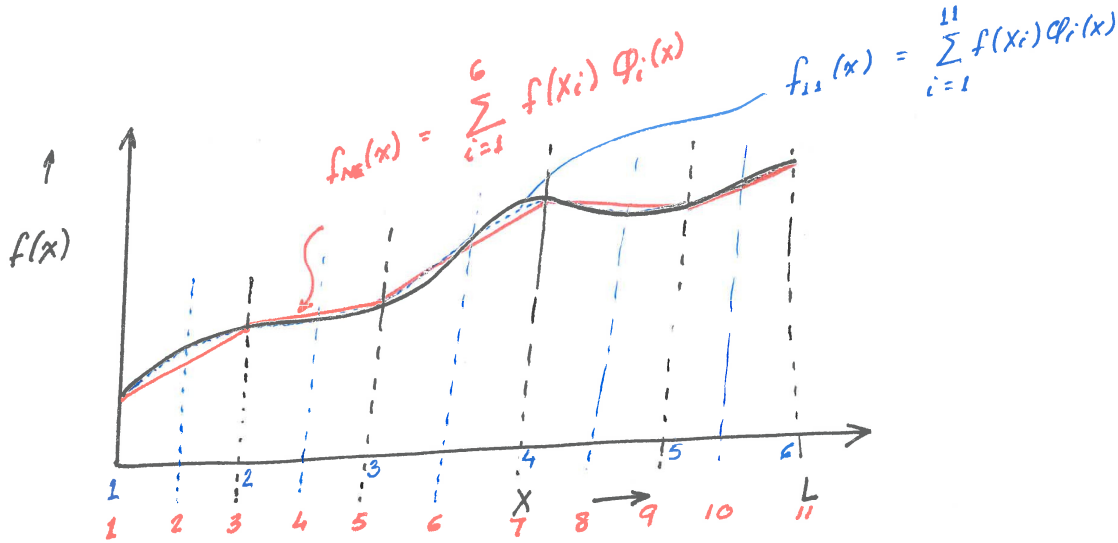
(2) In each I_ℓ ; $a_0 + a_1 x = \alpha_1^\ell \varphi_\ell(x) + \alpha_2^\ell \varphi_{\ell+1}(x)$

can be represented, i.e. piecewise linear functions can be EXACTLY represented in terms of $\varphi_k(x)$

In fact : $\alpha_1^\ell = a_0 + a_1 x_\ell$; $\alpha_2^\ell = a_0 + a_1 x_{\ell+1}$

(3) The $\varphi_k(x)$ are C^0 or continuous with jump in derivative at x_{k-1}, x_k, x_{k+1}

(4) Any ~~polynom~~ function can be approximated well by a combination of the $\{\varphi_k(x)\}_{k=1}^{NE+1}$



* - $f_6(x)$ is a six-term interpolation of $f(x)$

- $f_{11}(x)$ is a refined eleven-term interpolation of $f(x)$

$f_{11}(x)$ is "closer" to $f(x)$ than $f_6(x)$.

☆ Thus, by taking smaller elements or more no. of elements NE, we can get a better approximation of $f(x)$. In the limit, i.e.

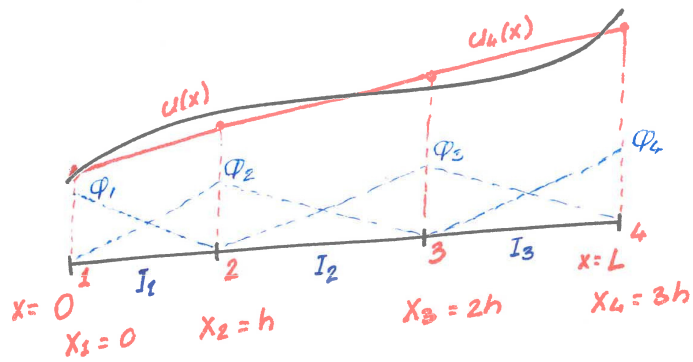
$$f(x) = \lim_{NE \to \infty} \sum_{i=1}^{NE+1} f(x_i)\varphi_i(x)$$

With $x_i = (i-1)\cdot L/(NE) = (i-1)h$

$(h = L/NE)$. $h \rightarrow$ UNIFORM MESH-SIZE

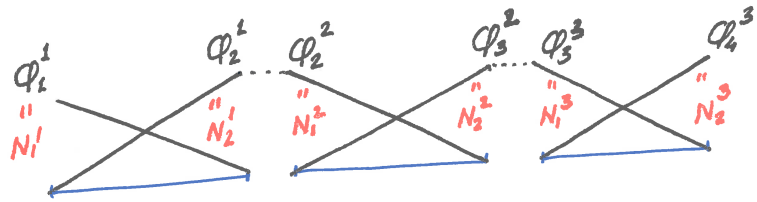
$\therefore$ We can approximate $u(x)$ by $u_h(x) = \sum_{i=1}^{NE+1} \alpha_i \varphi_i(x)$

. We will also choose $v(x) = v_h(x) = \sum_{i=1}^{NE+1} \beta_i \varphi_i(x)$ with β_i chosen by us.



$h = \frac{L}{3}$

$$u_4(x) = \sum_{i=1}^{4} d_i \, \varphi_i(x)$$



Element No.	Local	Global
k	1	k
k	2	$k+1$

$$\varphi_k^k\Big|_{I_k} = N_1^k$$

$$\varphi_{k+1}\Big|_{I_k} = N_2^k$$

element shape function

Following our earlier derivation, we will create equation

$$B(u_h, \varphi_i) = F(\varphi_i) \qquad \text{for } i = 1, 2, \dots, (NE+1)$$

$$\Rightarrow \sum_{j=1}^{NE+1} d_j \, B(\varphi_j, \varphi_i) = F(\varphi_i)$$

Note that from the figure it is easy to see that:

$$B(\varphi_{i-1}, \varphi_i); \quad B(\varphi_i, \varphi_i); \quad B(\varphi_{i+1}, \varphi_i) \neq 0$$

all other $\quad B(\varphi_j, \varphi_i) = 0$

Also

$$B(\varphi_{i-1}, \varphi_i) = B(\overbrace{\varphi_{i-1}|_{I_{i-1}}}^{N_1^{i-1}}, \overbrace{\varphi_i|_{I_{i-1}}}^{N_2^{i-1}})$$

$$B(\varphi_{i+1}, \varphi_i) = B(\overbrace{\varphi_{i+1}|_{I_i}}^{N_2^{i}}, \overbrace{\varphi_i|_{I_i}}^{N_1^{i}})$$

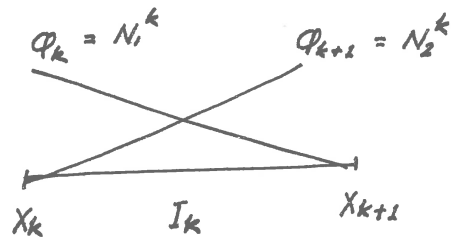
$$B(\varphi_i, \varphi_i) = \underbrace{B(\overbrace{\varphi_i|_{I_{i-1}}}^{N_2^{i-1}}, \overbrace{\varphi_i|_{I_{i-1}}}^{N_2^{i-1}})} + \underbrace{B(\overbrace{\varphi_i|_{I_i}}^{N_1^{i}}, \overbrace{\varphi_i|_{I_i}}^{N_1^{i}})}$$

Element wise integrals

$$F(\varphi_i) = F(\varphi_i|_{I_{i-1}}) + F(\varphi_i|_{I_i})$$

ELEMENT - LEVEL INTEGRALS

$$\left.\begin{array}{l} B(u_n, \varphi_k) \\ B(u_n, \varphi_{k+1}) \\ F(\varphi_k) \\ F(\varphi_{k+1}) \end{array}\right\} \text{will have NON-ZERO contribution from } I_k$$



$$u_n|_{I_k} = \alpha_k \varphi_k + \alpha_{k+1} \varphi_{k+1} \equiv \alpha_k N_1^k + \alpha_{k+1} N_2^k \equiv \alpha_1^k N_1^k + \alpha_2^k N_2^k$$

$$\alpha_1^k \rightarrow \alpha_k \quad ; \quad N_1^k \rightarrow \varphi_k$$

$$\alpha_2^k \rightarrow \alpha_{k+1} \quad ; \quad N_2^k \rightarrow \varphi_{k+1}$$

$$\therefore \quad B(\alpha_k \varphi_k + \alpha_{k+1} \varphi_{k+1}, \varphi_k) = \alpha_k \overbrace{B(\varphi_k, \varphi_k)}^{B_k(1,1)} + \alpha_{k+1} \overbrace{B(\varphi_{k+1}, \varphi_k)}^{B_k(2,1) = B_k(1,2)}$$

$$B(\alpha_k \varphi_k + \alpha_{k+1} \varphi_{k+1}, \varphi_{k+1}) = \alpha_k \underbrace{B(\varphi_k, \varphi_{k+1})}_{B_k(N_i^k, N_j^k)} + \alpha_{k+1} \underbrace{B(\varphi_{k+1}, \varphi_{k+1})}_{B_k(2,2)}$$

$i, j = 1, 2$

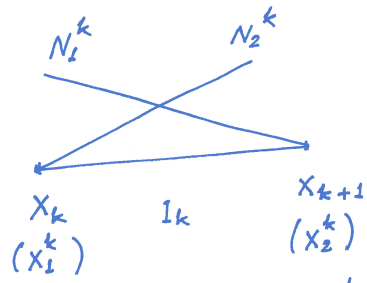
Thus, we need to obtain $B_k(N_i^k, N_j^k)$ ← Element stiffness matrix

and $F_k(N_i^k)$ ← Element Load Vector

Element Calculation (Element I_k)

$$N_1^k(x) = \frac{(x_2^k - x)}{(x_2^k - x_1^k)} = \frac{(x_2^k - x)}{h_k}$$

$$N_2^k(x) = \frac{(x - x_1^k)}{(x_2^k - x_1^k)} = \frac{(x - x_1^k)}{h_k}$$



$$\Rightarrow \frac{dN_1^k}{dx} = -\frac{1}{h_k} \; ; \; \frac{dN_2^k}{dx} = \frac{1}{h_k}$$

$$K_{ij}^k = B_k(N_i^k, N_j^k) = \int_{x_1^k}^{x_2^k} \left(EA \frac{dN_i^k}{dx}\frac{dN_j^k}{dx} + k_0 N_i^k N_j^k \right) dx$$

$$\Rightarrow K_{11}^k = \int_{x_1^k}^{x_2^k} \left(EA\left(-\frac{1}{h_k}\right)^2 + k_0 \frac{(x_2^k - x)^2}{h_k^2} \right) dx = \frac{EA}{h_k} + \frac{k_0}{3h_k^2}(h_k)^3$$

$$= \frac{EA}{h_k} + \frac{k_0 h_k}{3}$$

$$K_{22}^k = \int_{x_1^k}^{x_2^k} \left[EA\left(1/h_k\right)^2 + k_0 \frac{(x - x_1^k)^2}{h_k^2} \right] dx = \frac{EA}{h_k} + \frac{k_0}{3h_k^2} h_k^3$$

$$= \frac{EA}{h_k} + \frac{k_0 h_k}{3}$$

$$K_{12}^k = K_{21}^k = \int_{x_1^k}^{x_2^k} \left[EA\left(-1/h_k\right)^2 + \frac{k_0}{h_k^2}(x - x_1^k)(x_2^k - x) \right] dx$$

$$= -\frac{EA}{h_k} + k_0 \frac{h_k}{6}$$

$$\Rightarrow [K^k] = \begin{bmatrix} \left(\frac{EA}{h_k} + \frac{k_0 h_k}{3}\right) & \left(-\frac{EA}{h_k} + \frac{k_0 h_k}{6}\right) \\ \left(-\frac{EA}{h_k} + \frac{k_0 h_k}{6}\right) & \left(\frac{EA}{h_k} + \frac{k_0 h_k}{3}\right) \end{bmatrix}$$

$$F_1^k = \int_{x_1^k}^{x_2^k} f_0 N_1^k \, dx = \frac{f_0 \cdot h_k}{2} \; ; \; F_2^k = \int_{x_1^k}^{x_2^k} f_0 N_2^k \, dx = \frac{f_0 h_k}{2}$$

$$\{F^k\} = \begin{Bmatrix} f_0 h_k/2 \\ f_0 h_k/2 \end{Bmatrix}$$

For $h_k = h = \frac{L}{NE}$ (uniform mesh), we can now write the following assembled global stiffness matrix and load vector :

$$K_{ii} = K_{22}^{i-1} + K_{11}^{i} = \frac{2EA}{h} + \frac{2k_o h}{3}$$

$$K_{i,(i-1)} = K_{12}^{i-1} = -\frac{EA}{h} + \frac{k_o h}{6}$$

$$K_{i,(i+1)} = K_{12}^{i} = -\frac{EA}{h} + \frac{k_o h}{6}$$

$$F_i = F_2^{i-1} + F_1^{i} = f_o h = f_o h$$

For $i = 1$

$K_{11} = K_{11}^{1}$

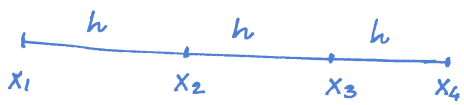
$K_{1,0}$ means nothing & hence ignore

For $i = NE+1$

$K_{(NE+1),(NE+1)} = K_{22}^{NE}$

$K_{(NE+1),(NE+2)}$ means nothing & hence ignore

EXAMPLE:



$h = L/3$

$$\Rightarrow [K] = \begin{bmatrix} \left(\frac{\cancel{6}^{3}EA}{L} + \frac{\cancel{2}^{1}k_oL}{9}\right) & \left(-\frac{3EA}{L} + \frac{k_oL}{18}\right) & 0 & 0 \\ \left(\frac{6EA}{L} + \frac{2k_oL}{9}\right) & \left(-\frac{3EA}{L} + \frac{k_oL}{18}\right) & 0 \\ & \left(\frac{6EA}{L} + \frac{2k_oL}{9}\right) & \left(-\frac{3EA}{L} + \frac{k_oL}{18}\right) \\ & & \left(\frac{\cancel{6}^{3}EA}{L} + \frac{\cancel{2}^{1}k_oL}{9}\right) \end{bmatrix}$$

$$\{F\} = \begin{Bmatrix} f_o h/2 \\ f_o h \\ f_o h \\ f_o h/2 \end{Bmatrix} = \begin{Bmatrix} f_o L/6 \\ f_o L/3 \\ f_o L/3 \\ f_o L/6 \end{Bmatrix}$$

APPLICATION OF B-C.

If $u(0) = 0$ and $F_{xx}|_L = P$, then we need to

(a) Take care of $P \cdot v|_L$, i.e. $\phi_i|_L = 0$ except $\phi_4|_L = 1$

$\Rightarrow P \cdot \phi_4|_L = P$ has to be added to 4th row of $\{F\}$

OR

$$F_4 = \frac{f_0 h}{2} + \underline{\underline{P}}$$

HAS TO BE ADDED TO $F_{(NE+1)}$

Now the load vector (global) is complete

(b) In order to set $u(0) = 0$, we need to set $\alpha_1 = 0$

or we modify $[K]$ and $\{F\}$ as:

$$\begin{array}{c} \\ \rightarrow 0 \\ \rightarrow 0 \\ \rightarrow 0 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ & \left(\frac{6EA}{L} + \frac{2k_0L}{9}\right) & \left(-\frac{3EA}{L} + \frac{k_0L}{18}\right) & 0 \\ & & \left(\frac{6EA}{L} + \frac{2k_0L}{9}\right) & \left(-\frac{3EA}{L} + \frac{k_0L}{18}\right) \\ & & & \left(\frac{6EA}{L} + \frac{2k_0L}{9}\right) \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_0 h \\ f_0 h \\ \frac{f_0 h}{2} + P \end{Bmatrix}$$

Now the matrix problem is properly posed & ready to be solved

- Solve this to get $\{\alpha\}$
- Now we have $u_h(x) = \sum_{i=1}^{4} \alpha_i \phi_i(x)$

* Note that in element I_k, $u_h(x) = \alpha_k \varphi_k + \alpha_{k+1} \varphi_{k+1}$

$$= \alpha_k N_1^k + \alpha_{k+1} N_2^k$$

$$= \alpha_1^k N_1^k + \alpha_2^k N_2^k$$

[Locally, we RENAME

$\varphi_k\big|_{I_k} = N_1^k$, $\alpha_k = \alpha_1^k$;

$\varphi_{k+1}\big|_{I_k} = N_2^k$, $\alpha_{k+1} = \alpha_2^k$]

E.g. in element I_2, $u_h(x) = \alpha_2 \varphi_2(x) + \alpha_3 \varphi_3(x)$

$$= \alpha_2 N_1^2(x) + \alpha_3 N_2^2(x)$$

$$= \alpha_1^2 N_1^2(x) + \alpha_2^2 N_2^2(x)$$

with $\alpha_1^2 = \alpha_2$; $\alpha_2^2 = \alpha_3$

Lab. Work 2 : * For the problems ① and ② on page L1 - 15 solve using NE = 2, 4, 8, 16.

- Plot the results & compare with the solution for N = 6 used in the earlier assignment

Can we formolize the assembly of $[K]$, $\{F\}$?

Lets do it for this case:

$K[i,j] = 0$; $F(i) = 0$ $\quad i,j = 1,2,\dots,(NE+1)$

For $i = 1, NE$ ← loop over elements

get $[K^i]$, $\{F^i\}$ ← element stiffness matrix and load vector

For $j = 1, 2$ ← Loop over the local d.o.fs

$J = (i-1) + j$ ← Local to Global connectivity

$F(J) = F(J) + F^i(j)$ ← Add the element Load vector term

For $k = 1, 2$

$K = (i-1) + k$

$K(J,K) = K(J,K) + K^i(j,k)$ ← Add the element stiffness term

End

End

End

Now we will apply the Neumann B.C. first, i.e.

Assign a BCTYPE to each boundary vertex

at node 1

i.e. IBCTYPE(1) = 1 [Choices are: TYPE = 1 ← DIRICHLET
= 2 ← NEUMANN]

IBCTYPE(2) = 2

at node (NE+1)

Since IBCTYPE(2) = 2 ⇒ Neumann B.C. at node (NE+1)

⇒ $F(NE+1) = F(NE+1) + P$ ← Add P to the last row of load vector

Since IBCTYPE(1) = 1 ⇒ Dirichlet B.C. at node (1).

$F(1) = U_0$ ← Set F(1) to given displacement

Set

$$F_i = F_i - K_{i1} \cdot u_0 \quad \longleftarrow \quad \text{move the effect of known } \alpha_1 \text{ or } u_0 \text{ to RHS}$$

$$i = 2, 3, \ldots \; (NE+1)$$

Set $\quad K_{i1} = K_{1i} = 0 \quad \longleftarrow$ make all entries in 1st row and 1st column ZERO

Set $\quad K_{11} = 1 \;, \quad F_1 = u_0$

$$\Downarrow$$

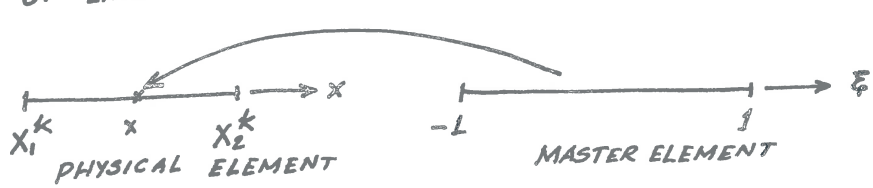
$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & K_{22} & K_{23} & \cdots & \\ 0 & K_{23} & K_{33} & \cdots & \\ 0 & & & \ddots & \\ \vdots & & & & \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{NE+1} \end{Bmatrix} = \begin{Bmatrix} u_0 \\ F_2 - K_{21} u_0 \\ F_3 - K_{31} u_0 \\ \vdots \\ F_{(NE+1)} - K_{(NE+1),1} u_0 \end{Bmatrix}$$

$$\boxed{\text{We now have "a" functional code using linear FEM}}$$

NUMERICAL INTEGRATION + MAPPING TO MASTER ELEMENT

We will now focus on a computer implementation that is general, versatile and correct. How do we do it ?

AFFINE or LINEAR MAPPING :



$$x(\xi) = a_0 + a_1 \xi \quad \text{with} \quad x(-1) = x_1^k, \; x(+1) = x_2^k$$

$$\Rightarrow \boxed{x(\xi) = x_1^k \left(\frac{1-\xi}{2}\right) + x_2^k \left(\frac{1+\xi}{2}\right)} \quad \Rightarrow \frac{dx}{d\xi} = \frac{x_2^k - x_1^k}{2} = \frac{h_k}{2}$$

$$\therefore \quad N_1^k(x) \longrightarrow \hat{N}_1(\xi) = \frac{1-\xi}{2}$$

$$N_2^k(x) \longrightarrow \hat{N}_2(\xi) = \frac{1+\xi}{2}$$

and
$$\int_{x_1^k}^{x_2^k} g(x)\,dx = \int_{\xi=-1}^{1} \hat{g}(\xi) \frac{dx}{d\xi}\, d\xi = \int_{-1}^{1} \hat{g}(\xi)\, |J|\, d\xi$$

$$= \frac{h_k}{2} \int_{\xi=-1}^{1} \hat{g}(\xi)\, d\xi \qquad \hat{g}(\xi) = g(x(\xi))$$

NO. OF INTEGRATION POINTS

Now we want to obtain $\displaystyle\int_{\xi=-1}^{1} \hat{g}(\xi)\, d\xi \approx \sum_{k=1}^{NINT} \hat{g}(\xi_k)\, w_k$

Where $\{\xi_k\}_{k=1}^{NINT}$ are the INTEGRATION points and $\{w_k\}_{k=1}^{NINT}$ are the weight functions corresponding to the integration points.

Gauss Legendre integration rule:

1) If ξ_k is an integration point then so is $-\xi_k$

2) Both ξ_k and $-\xi_k$ have same weight w_k.

Ex: $NINT = 1 \Rightarrow \xi_1^1$ is pt. but $-\xi_1^1$ should also be possible

$\therefore \; \xi_1^1 = -\xi_1^1 \Rightarrow 2\xi_1^1 = 0 \Rightarrow \xi_1^1 = 0$ is the point

It should integrate polynomials of "certain" orders exactly

Least order polynomial $\hat{g}(\xi) = C$

$$\Rightarrow \int_{-1}^{1} C \cdot d\xi = C \cdot w_1 = C \cdot 2 \Rightarrow \boxed{w_1 = 2}$$

What if $\hat{g}(\xi) = C_1 + C_2 \xi$ (linear)?

$$\int_{-1}^{1} (C_1 + C_2 \xi)\, d\xi = C_1 \cdot 2 = (C_1 + C_2 \times 0) \cdot w_1 = \underline{\underline{2C_1}}$$

EXACT!

Quadratic ?

$$\hat{g}(\xi) = C_1 + C_2\xi + C_3\xi^2$$

$$\Rightarrow \int_{-1}^{1}(C_1 + C_2\xi + C_3\xi^2)d\xi = 2C_1 + \frac{2C_3}{3}$$

1-pt. rule gives $2C_1 \neq \int_{-1}^{1}\hat{g}(\xi)\,d\xi$

2-pt. rule ? $\xi_1^2 = -\xi_2^2 \Rightarrow \xi_1^2, -\xi_1^2$ are points

$$\omega_1^2 = \omega_2^2 = \omega_1^2$$

$$\int_{-1}^{1}(C_1 + C_2\xi + C_3\xi^2)d\xi = 2C_1 + \frac{2C_3}{3} = \Big[\big(C_1 + C_2\xi_1^2 + C_3(\xi_1^2)^2\big) + \big(C_1 - C_2\xi_1^2 + C_3(\xi_1^2)^2\big)\Big]\omega_1$$

Term by term match ?

$$2C_1\omega_1 = 2C_1 \Rightarrow \boxed{\omega_1 = 1}$$

$$2C_3(\xi_1^2)^2 = \frac{2}{3}C_3 \Rightarrow (\xi_1^2)^2 = \frac{1}{3} \Rightarrow \boxed{\xi_1^2 = \pm\frac{1}{\sqrt{3}}}$$ EXACT!

Cubic ?

$$\hat{g}(\xi) = C_1 + C_2\xi + C_3\xi^2 + C_4\xi^3$$

odd term

$$\Rightarrow \int_{-1}^{1}\hat{g}(\xi)\,d\xi = 2C_1 + \frac{2}{3}C_2 \Rightarrow$$ NUMERICAL INTEGRATION WITH 2-POINT ROLE IS ALSO EXACT.

For $\hat{g}(\xi) = C_1 + C_2\xi + C_3\xi^2 + C_4\xi^3 + C_5\xi^4$ 2-point will not work

Then 3-point rule!

$$\xi_1,\ \xi_2 = 0,\ \xi_3 = -\xi_1$$

$$\omega_1,\ \omega_2,\ \omega_3 = \omega_1$$

We can solve for $\boxed{\omega_1, \omega_2, \xi_1}$ easily as:

$$\int_{-1}^{1}(C_1 + C_2\xi + C_3\xi^2 + C_4\xi^3 + C_5\xi^4)d\xi = 2C_1 + \frac{2}{3}C_3 + \frac{2}{5}C_5$$

$$= \omega_1(2C_1 + 2C_3\xi_1^2 + 2C_5\xi_1^4) + C_1\cdot\omega_2$$

$$\Rightarrow 2C_1\omega_1 + C_1\omega_2 = 2 \Rightarrow \boxed{2\omega_1 + \omega_2 = 2}$$

$$\frac{2}{3}\cancel{C_3} = 2\cancel{C_3}\, w_1 \xi_1^2 \qquad \Rightarrow \quad \frac{3}{5} = \xi_1^2 \Rightarrow \boxed{\xi_1 = \pm\sqrt{\frac{3}{5}}}$$

$$\frac{2}{5}\cancel{C_5} = 2\cancel{C_5}\, w_1 \xi_1^4$$

$$\frac{2}{3} = 2 \times w_1 \times \frac{3}{5} \Rightarrow \boxed{w_1 = \frac{5}{9}} \qquad \Rightarrow \boxed{w_2} = 2 - \frac{10}{9} = \boxed{\frac{8}{9}}$$

Thus we can create the set of integration points and corresponding weight functions.

* Note that if $\hat{g}(\xi)$ is a polynomial of degree p then $\boxed{NINT \geq \frac{p+1}{2}}$ for EXACT integral is <u>REQUIRED</u>!

LAB. WORK

1) Follow the process above to create integration points for

NINT = 1, 2, 3, 4, 5, 6, 7, 8

2) Show that integration ~~order~~ rule of NINT points corresponds to the roots of a Legendre polynomial of order NINT.

× × × × × × **Diversion on Legendre polynomials**

Legendre polynomials: $\mathcal{P}_k(x) \rightarrow$ of polynomial order k

such that (1) $\int_{-1}^{1} \mathcal{P}_k \mathcal{P}_\ell \, dx = 0$ if $k \neq \ell$ ⟨"orthogonal" in $(-1, 1)$⟩

(2) Normalize the size of $\mathcal{P}_k(x)$ such that $\int_{-1}^{1} \mathcal{P}_k^2 \, dx = 1$

How to construct?

$$\boxed{\mathcal{P}_0(x) = C} \Rightarrow \int_{-1}^{1} \mathcal{P}_k^2 (dx) = 1 \Rightarrow C^2 2 = 1 \Rightarrow C = 1/\sqrt{2}$$

$$\mathcal{P}_1(x) = C_0 + C_1 x \quad \text{s.t.} \quad \int_{-1}^{1} \mathcal{P}_0 \cdot \mathcal{P}_1 \, dx = 0 \Rightarrow C_0 \sqrt{2} = 0 \Rightarrow C_0 = 0$$

$$\int_{-1}^{1} \mathcal{P}_1^2 \, dx = C_1^2 \int_{-1}^{1} x^2 dx = C_1^2 \cdot \frac{2}{3} = 1 \Rightarrow C_1 = \sqrt{3/2} \Rightarrow \boxed{\mathcal{P}_1(x) = \sqrt{3/2}\, x}$$

$$P_2(x) = C_0 + C_1 x + C_2 x^2$$

$$\int_{-1}^{1} P_0 P_2 \, dx = C_0 \cdot 2 + C_2 \frac{2}{3} = 0 \quad \text{— (1)} \quad \Rightarrow \quad C_0 = -\frac{1}{3} C_2$$

$$\int_{-1}^{1} P_1 P_2 \, dx = \int_{-1}^{1} (C_0 x + C_1 x^2 + C_2 x^3) \, dx = \frac{C_1}{3} \cdot 2 = 0 \quad \Rightarrow \quad C_1 = 0$$

$$\int_{-1}^{1} P_2^2 \, dx = \int_{-1}^{1} C_2^2 \left(-\frac{1}{3} + x^2\right)^2 dx = C_2^2 \int_{-1}^{1} \left(x^4 - \tfrac{2}{3} x^2 + \tfrac{1}{9}\right) dx = C_2^2 \cdot \left[\tfrac{2}{5} - \tfrac{4}{9} + \tfrac{2}{9}\right]$$

$$= C_2^2 \left[\tfrac{2}{5} - \tfrac{2}{9}\right] = C_2^2 \left[\tfrac{8}{45}\right] = 1 \quad \Rightarrow \quad C_2 = \sqrt{\tfrac{45}{8}}$$

$$\Rightarrow \quad \boxed{P_2(x) = \sqrt{\tfrac{45}{8}} \left(x^2 - \tfrac{1}{3}\right)}$$

* Put the approach above to the test and get $P_3(x), \ldots, P_8(x)$.
* Plot $P_k(x)$, $k = 2, 3, \ldots, 8$ in the interval $(-1, 1)$

x x x x x x x

SHOW OFF TIME FOR GALERKIN APPROACH!

So, we have a weak formulation, i.e. $B(u,v) = F(v)$ with $u|_{\Gamma_D} = u_0 = 0$ (Γ_D is the Dirichlet boundary)

Thus, $B(u-u_h, u-u_h) = B(u,u) + B(u_h, u_h) - 2B(u, u_h)$

Since orthogonality of error gives $B(u-u_h, u_h) = 0$ (with u_h as a SPECIAL choice of v_h).

Thus $\boxed{B(u, u_h) = B(u_h, u_h)}$ or $B(\overbrace{u-u_h}^{e_h}, \overbrace{u-u_h}^{e_h}) = \|e_h\|_\Omega^2$

$B(u,u) - B(u_h, u_h) \geq 0$

$\Rightarrow B(u,u) \geq B(u_h, u_h)$.

Now let $\bar{u}_h$ be "a" solution obtained from "some" method such that $\bar{u}_h(x) = \sum_{i=0}^{N} \bar{\alpha}_i \cdot \phi_i(x)$. Then $\tilde{e}_h = u - \bar{u}_h = u - u_h + u_h - \bar{u}_h$

$\Rightarrow \tilde{e}_h = e_h + e_h^*$ $\quad [e_h = u - u_h \,;\; e_h^* = u_h - \bar{u}_h]$

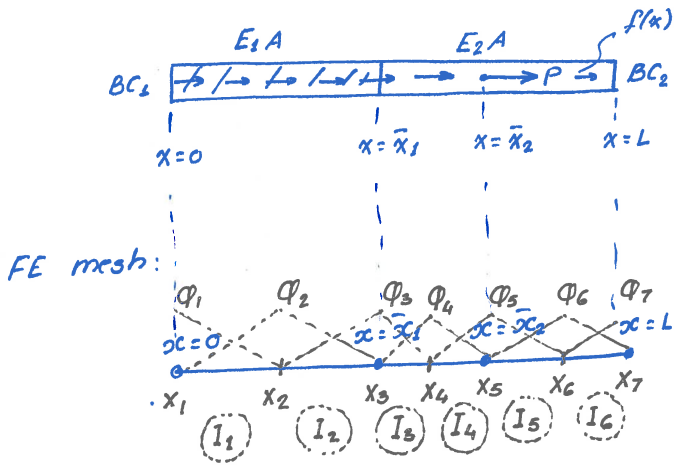
Thus $B(\tilde{e}_h, \tilde{e}_h) = B(e_h + e_h^*, e_h + e_h^*) = B(e_h, e_h) + B(e_h^*, e_h^*)$
$+ 2B(e_h, e_h^*) = 0$ (by orthogonality and $v_h = e_h^*$)

$\Rightarrow \|\tilde{e}_h\|_\Omega^2 = \|e_h\|_\Omega^2 + \|e_h^*\|_\Omega^2 \Rightarrow \boxed{\|\tilde{e}_h\|_\Omega \geq \|e_h\|_\Omega}$ — (A)

---

* Result (A) means that the Finite element solution is the <u>CLOSEST</u> to $u(x)$, of all the possible solutions of the form $\bar{u}_h(x)$, in ~~the~~ terms of the strain energy. The FE/Galerkin solution is said to be the BEST APPROXIMATION of $u(x)$

Let us now look at the FE solution of a specific bar problem:



BC_1 : $u(0) = 0$ (say)

BC_2 : $E_2 A \frac{du}{dx}\Big|_L = P_L$

$\{A$ is constant$\}$

* NOTE that points of MATERIAL transition and application of point loads HAVE to be nodes.

Let $u_h(x) = \sum_{i=1}^{7} \alpha_i \varphi_i(x)$ with $\alpha_1 = 0$ as $u(0) = 0$

Let $B(u_h, v_h) = F(v_h)$ with $B(u_h, v_h) = \int_0^{\bar{x}_1} E_1 A \frac{du_h}{dx}\frac{dv_h}{dx}\,dx$

$+ \int_{\bar{x}_1}^{\bar{x}_2} E_2 A \frac{du_h}{dx}\frac{dv_h}{dx}\,dx + \int_{\bar{x}_2}^{L} E_2 A \frac{du_h}{dx}\frac{dv_h}{dx}\,dx$; $F(v_h) = \int_0^L f\, v_h\,dx + P_L v_L + P v\big|_{\bar{x}_2}$ —(1)

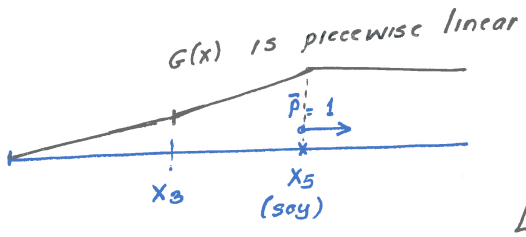
Let us also solve the problem for a different loading, given by $f(x) = 0$; $P_L = 0$; $P = 0$ and point load $\bar{P} = 1$ at $x = x_k$ (where k is a node of the mesh). Let the exact solution of this problem be $G(x)$ and its finite element solution $G_h(x)$ with:

$G(x)\big|_0 = G_h(x)\big|_0 = 0$; $G_h(x) = \sum_{i=1}^{7} \beta_i \varphi_i(x)$ [$\beta_1 = 0$ from $G(0) = 0$]

$\therefore$ G and G_h satisfy $B(G, v_h) = 1 \cdot v_h\big|_{x=x_k}$; $B(G_h, v_h) = 1 \cdot v_h\big|_{x_k}$ — (2)

$\Rightarrow B(G - G_h, v_h) = 0$. — (2a)

What is G?



$G(x)$ is piecewise linear

$\Rightarrow$ as $\bar{P}$ is at a node,

$G_h(x) = G(x)$

[from completeness, best approximation

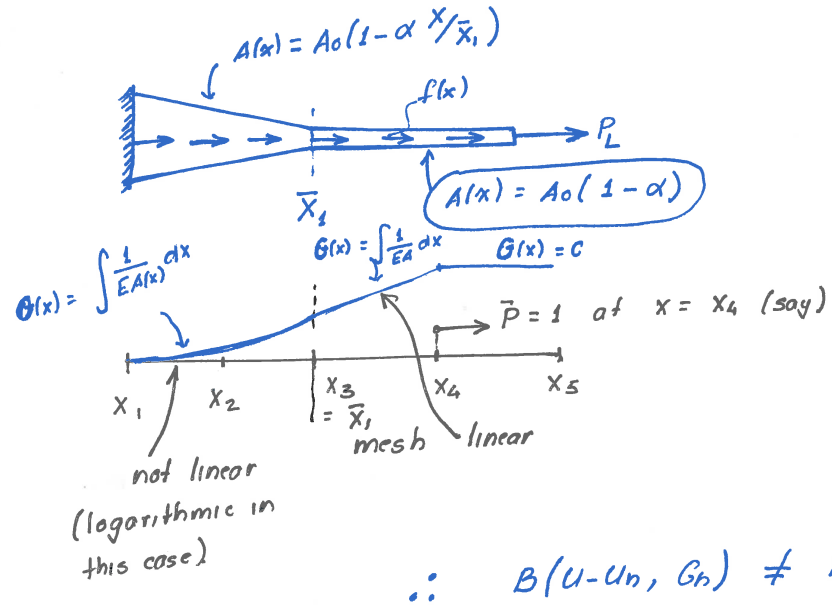
$\Rightarrow B(u - u_h, v_h) = 0$ gives $B(u - u_h, G_h) = 0 = B(u - u_h, G)$.

$\Rightarrow B(G, u - u_h) = 1 \cdot (u - u_h)\big|_{x_k} = 0 \Rightarrow \boxed{u\big|_{x_k} = u_h\big|_{x_k}}$ NODAL EXACTNESS!

* Note that $G(x)$ is the solution of the UNIT LOAD problem (Castigliano's theorems in basic strength of materials course!)

Is $G(x) = G_h(x)$ always? NO!

Ex



$G(x)$ IS NOT piecewise linear

Thus $G_h(x) = \sum_{i=1}^{5} \beta_i \varphi_i(x) \neq G(x)$

$$\therefore \quad B(u-u_h, G_h) \neq B(u-u_h, G)$$

But $B(G, u-u_h) = 1 \cdot (u-u_h)\big|_{x_4} = B(G-G_h, u-u_h)$

[As $B(G_h, u-u_h) = B(u-u_h, G_h) = 0$]

$$\Rightarrow B(G-G_h, u-u_h) = (u-u_h)\big|_{x_4} \Rightarrow |u-u_h|_{x_4} = |B(u-u_h, G-G_h)|$$

$$\leq \underbrace{\|u-u_h\|_\Omega \, \|G-G_h\|_\Omega}_{\text{SCHWARZ INEQUALITY}}$$

* Now we need to know the SIZE of $\|u-u_h\|_\Omega$, $\|G-G_h\|_\Omega$!

$\Downarrow$

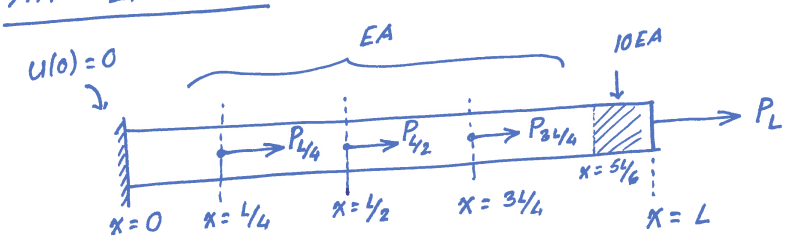
Thus we need to create estimates of error with respect to a parameter – which is mesh-size "h" in this case; OR convergence is achieved by decreasing h (or increasing NE) $= L/NE$.

It turns out that $\|u-u_h\|_\Omega \leq C_1 \cdot h$; $\|G-G_h\|_\Omega \leq C_2 h$

$$\Rightarrow \boxed{|u-u_h|\big|_{x_4} \leq \bar{C} h^2}$$

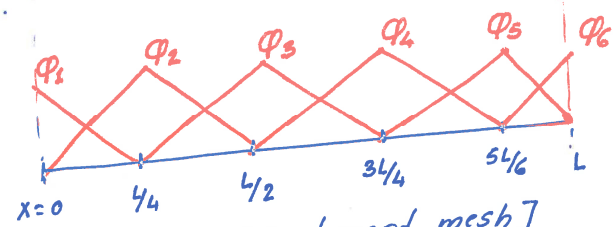
* Nodal value of $u_h(x) \to u(x)$ as $h \to 0$

AN EXAMPLE



* Bar with several point loads and a material interface

- Its easy to see that the EXACT solution, u_{ex}, is piecewise linear.

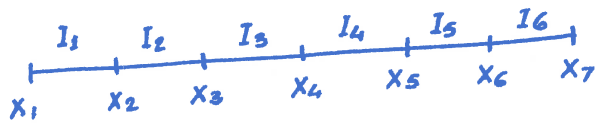


[5 element mesh]

$$u_h(x) = \sum_{i=1}^{6} \alpha_i \varphi_i(x)$$

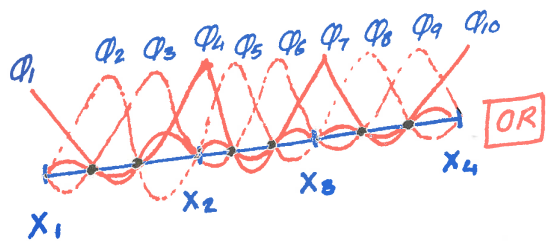
- We have put nodes at points where material transitions or point loads are applied.
- We can show that $\boxed{u_h(x) = u_{ex}(x)}$ {WHY ?}
  - FEM gives us a convenient method to resolve the point loads and material transitions.
  - Using $\varphi_i(x) = x^i$, $i = 1, 2, \ldots N$ would not have helped!

MORE GENERAL F.E. ANALYSIS IN 1-D

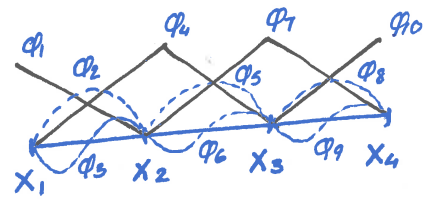


- Till now we have chosen piecewise linear approximation ($p=1$) and convergence is achieved via reduction of mesh size h_k.
- Can we do better? YES!
- What about piecewise p-order approximation? For this we can do the following.

C^0-functions with "kink" at certain points $\Rightarrow$



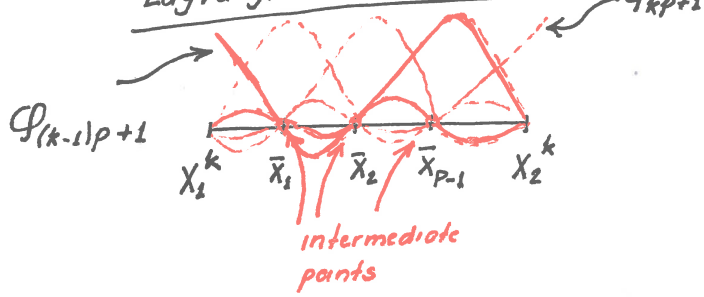
OR



Lagrangian basis functions

Hierarchic basis functions

Lagrangian of order p

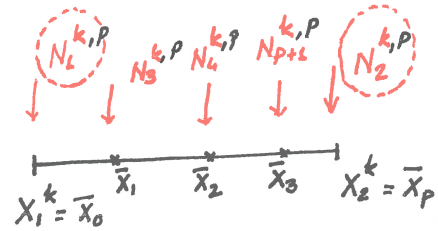


* $(p-1)$ intermediate points $\bar{x}$ located at $\bar{x}_\ell = x_1^k + \ell \bar{h}$

$$\boxed{\text{with } \bar{h} = \frac{h_k}{p}}$$

For $\ell = 1, 2, \ldots, (p+1) \Rightarrow$

Element I_k:



Corresponding to each of the $\{\bar{x}_m\}_{m=0}^{p}$ define the basis functi[illegible] or element shape function as:

$$N_2^{k,p}(x) = \prod_{\substack{j=0 \\ j \neq p}}^{p} \frac{(x - \bar{x}_j)}{(\bar{x}_p - \bar{x}_j)} \; ; \quad N_\ell^{k,p} = \prod_{\substack{j=0 \\ j \neq (\ell-1)}}^{p} \frac{(x - \bar{x}_j)}{(\bar{x}_{\ell-1} - \bar{x}_j)}$$

- The given choice of $N_2^{k,p}(x)$ is my preference. You can number these shape-functions differently.

- Note that

$$N_\ell^{k,p} = \varphi^{(p)}_{p(k-1)+(\ell-1)} \quad \text{for } \ell = 3, 4, \cdots, p+1$$

$$N_1^{k,p} = \varphi^{(p)}_{p(k-1)+1}$$

$$N_2^{k,p} = \varphi^{(p)}_{pk+1}$$

$\Leftarrow$ local to global connectivity

$$\therefore \quad u_h(x) = \sum_{i=1}^{NE*p+1} \alpha_i \, \varphi_i^{(p)}(x) \quad \text{globally}$$

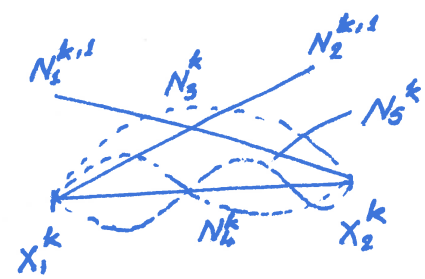
In I_k :

$$u_h(x)\Big|_{I_k} = \sum_{i=1}^{(p+1)} \alpha_i^k \cdot N_i^{k,p}(x)$$

with $\alpha_1^k = \alpha_{p(k-1)+1}$; $\alpha_2^k = \alpha_{pk+1}$;

$$\alpha_\ell^k = \alpha_{p(k-1)+(\ell-1)} \quad \text{for } \ell = 3, 4, \cdots, (p+1)$$

Hierarchic Basis of degree p



$$N_1^k(x) = \frac{(x - x_2^k)}{(x_1^k - x_2^k)} \; ;$$

$$N_2^k(x) = \frac{(x - x_1^k)}{(x_2^k - x_1^k)} \; ;$$

$$N_\ell^k(x) = \frac{\int_{x_1^k}^{x} \mathcal{P}_{\ell-2}(x)\, dx}{\int_{x_1^k}^{x_2^k} \left(\mathcal{P}_{\ell-2}(x)\right)^2 dx}$$

where $\int_{x_1^k}^{x_2^k} \mathcal{P}_i \, \mathcal{P}_j \, dx = 0$ if $i \neq j$

[Actually these are "Legendre" polynomials in (x_1^k, x_2^k)]

MASTER ELEMENT CALCULATIONS:

- Instead of the physical element, we will work in the master element – it is more convenient...
- Linear mapping defined on L2-13 can be used to get

$$x(\xi) = x_1^k \frac{(1-\xi)}{2} + x_2^k \frac{(1+\xi)}{2} \Rightarrow dx = \frac{h_k}{2} d\xi$$

$\downarrow$

The mapping ensures that a polynomial of order p in x, will map to a corresponding polynomial of order p in ξ.

- So $N_\ell^{k,p}(x) \longrightarrow N_\ell^{k,p}(x(\xi)) = \hat{N}_\ell^{(p)}(\xi)$

$\Uparrow$

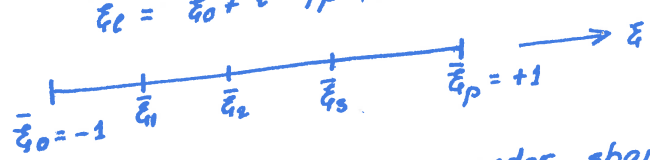
– shape function in the master element.

– We need to thus define the shape functions in the MASTER ELEMENT only!

Define $\bar{\xi}_\ell$ such that

$\bar{\xi}_0 = -1, \quad \bar{\xi}_p = +1$;

$\bar{\xi}_\ell = \bar{\xi}_0 + \ell \cdot 2/p, \quad \ell = 1, 2, \dots, p-1$



Thus, we can define the p-order shape functions in the master element $\hat{I}$ as:

LAGRANGIAN

$$\hat{N}_2^{(p)}(\xi) = \prod_{\substack{j=0 \\ j \neq p}}^{p} \frac{(\xi - \bar{\xi}_j)}{(\bar{\xi}_p - \bar{\xi}_j)} \quad ; \quad \hat{N}_\ell^{(p)}(\xi) = \prod_{\substack{j=0 \\ j \neq (\ell-1)}}^{p} \frac{(\xi - \bar{\xi}_j)}{(\bar{\xi}_{(\ell-1)} - \bar{\xi}_j)}$$

$\ell = 1, 3, 4 \dots, p$

HIERARCHIC:

$$\hat{N}_1^{(p)}(\xi) = \frac{(1-\xi)}{2} \; ; \; \hat{N}_2^{(p)}(\xi) = \frac{(1+\xi)}{2} \; ; \; \hat{N}_\ell^{(p)}(\xi) = \frac{\int_{-1}^{\xi} \mathcal{P}_{\ell-2}(\xi)\, d\xi}{\int_{-1}^{1} \mathcal{P}_{\ell-2}^2\, d\xi}$$

$\langle \ell = 3, 4, \dots \rangle$

where $\mathcal{P}_j(\xi)$ is the jth Legendre polynomial

* Note that $\hat{N}_\ell^{(p)}(\xi = -1) = 0$; $\hat{N}_\ell^{(p)}(\xi = +1) = 0$

i.e. $\hat{N}_\ell^{(p)}\big|_{\ell \geq 3}$ vanishes at $\xi = \pm 1$. ~ BUBBLE FUNCTIONS

PROVE THE FOLLOWING :

(1) The shape functions $\hat{N}_i^{(p)}(\xi)$ are LINEARLY INDEPENDENT and COMPLETE.

(2) The diagonal entries of element stiffness matrix for high p become small as p increases. Check for Lagrangian family.

Now, $$u_h\big|_{I_k} = \sum_{i=1}^{p+1} d_i^k \underbrace{N_i^{k,p}(x)}_{\text{PHYSICAL}} = \sum_{i=1}^{p+1} d_i^k \underbrace{\hat{N}_i^{(p)}(\xi)}_{\text{MASTER}}$$

$$\therefore \quad \frac{du_h}{dx}\bigg|_{I_k} = \sum_{i=1}^{p+1} d_i^k \frac{d\hat{N}_i^{(p)}}{d\xi} \cdot \frac{d\xi}{dx} = \boxed{\frac{2}{h_k} \sum_{i=1}^{(p+1)} d_i^k \frac{d\hat{N}_i^{(p)}}{d\xi}}$$

Element Calculation :

$$K_{ij}^k = \int_{x_1^k}^{x_2^k} \left(EA \frac{dN_i^{k,p}}{dx} \frac{dN_j^{k,p}}{dx} + k\, N_i^{k,p} N_j^{k,p} \right) dx$$

$$= \int_{\xi=-1}^{1} \left(EA(x(\xi)) \frac{d\hat{N}_i^{(p)}}{d\xi} \frac{d\hat{N}_j^{(p)}}{d\xi} \times \left(\frac{h_k}{2}\right)^2 + k(x(\xi))\, \hat{N}_i^{(p)} \hat{N}_j^{(p)} \right) \frac{2}{h_k} d\xi$$

INTEGRAL OVER MASTER ELEMENT

Now $EA(x(\xi)) = \widehat{EA}(\xi)$ ← just a name!

$k(x(\xi)) = \hat{k}(\xi)$

$$F_i^k = \int_{x_1^k}^{x_2^k} f(x)\, N_i^{k,p}\, dx = \int_{\xi=-1}^{1} f(x(\xi))\, \hat{N}_i^{(p)} \frac{h_k}{2} d\xi$$

If $EA(x), k$ is piecewise linear, $f(x)$ is piecewise quadratic, then order of the 2 integrands is :

$K_{ij}^k \rightarrow \max(1 + 2(p-1),\ 1 + 2p) = 2p+1$

$F_i^k \rightarrow 2 + p = p+2$

Highest order is $q = 2p+1$

Thus, we use $NINT \geq \frac{q+1}{2} \geq \frac{2p+2}{2} \geq \underline{(p+1)}$

atleast a $\underline{(p+1)}$ pt. rule needed.

$NINT = (p+1)$ has to be taken

* Note : If $k(x) \equiv 0$, then for K_{ij}^k $NINT = \frac{2p}{2} = p$ point rule will also work!
<Not for F_i^k though...>

NUMERICAL INTEGRATION:

We can now do the element level integrations as:

$$K_{ij}^k = \sum_{\ell=1}^{NINT} \left[EA(x(\xi_\ell)) \frac{d\hat{N}_i^{(p)}}{d\xi}\bigg|_{\xi_\ell} \frac{d\hat{N}_j^{(p)}}{d\xi}\bigg|_{\xi_\ell} \left(\frac{2}{h_k}\right)^2 + k(x(\xi_\ell))\, \hat{N}_i^{(p)}\big|_{\xi_\ell}\, \hat{N}_j^{(p)}\big|_{\xi_\ell} \right] w_\ell \, \frac{h_k}{2}$$

$$F_i^k = \sum_{\ell=1}^{NINT} \left(f(x(\xi_\ell))\, \hat{N}_i^{(p)}\big|_{\xi_\ell} \cdot \frac{h_k}{2} \cdot w_\ell \right)$$

→ Thus, we can replace the symbollic integration by a summation.

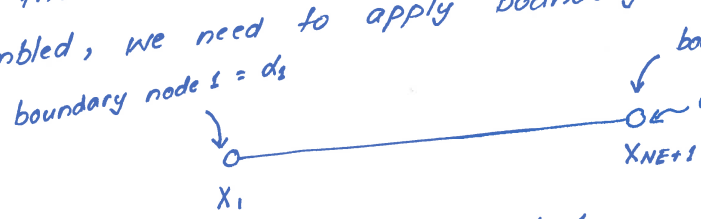
→ If $\underline{p \leq 8 \text{ used}}$, then $\underline{NINT \leq 9}$ will be required.

→ Now use the mapping of L4-2 to update the Global stiffness matrix and load vector.

```
For k = 1, NE          ← Loop over elements
    For i = 1,(p+1)    ← Loop over rows
        If i = 1 , then I = p(k-1)+1
    else If i = 2, then I = pk + 1
    else
            I = p(k-1) + (i-1)
    endif
            F(I) = F(I) + F^k(i)   ← Assemble load vector
```

```
For j = 1, p+1                    ← Loop over columns
    if j = 1, then J = p(k-1) + 1
  elseif j = 2, then J = pk + 1
  else
            J = p(k-1) + (j-1)
  endif
            K(I,J) = K(I,J) + K^k(i,j).
End. loop                         ← Over columns
End Loop                          ← Over rows
End Loop                          ← Over elements
```

Now that the GLOBAL stiffness matrix [K] and load vector {F} are assembled, we need to apply boundary conditions.



IBCTYPE(1) = 1 or 2 or 3

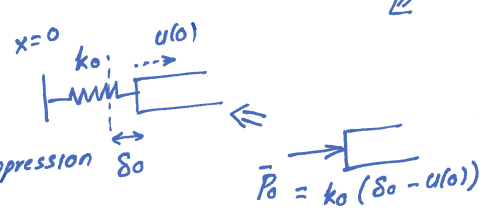
IBCTYPE(2) = 1 or 2 or 3

IBCTYPE(1) ← B.C. type at boundary node 1

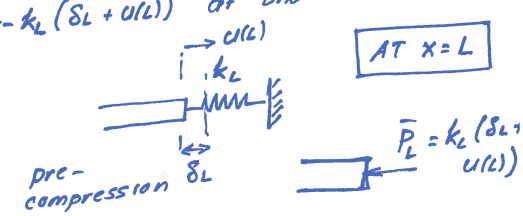
IBCTYPE(2) ← " " " " " 2

IBCTYPE =
- → 1 ⇒ Dirichlet B.C. ⇒ u given ($u(0)$ at bnd. node 1; $u(L)$ at " " $p(NE)+1$.
- → 2 ⇒ Neumann B.C. ⇒ F_x given (P_0 at bnd. node 1; P_L at bnd. node $p(NE)+1$
- → 3 ⇒ Mixed B.C. ⇒ $F_x = -k_0(\delta_0 - u(0))$ at bnd. node 1; $F_x = -k_L(\delta_L + u(L))$ at bnd. node $p(NE)+1$

$\boxed{\text{AT } X=0}$



$\boxed{\text{AT } X=L}$



How to apply the B.C. ?

If ibctype(1) = 1 then ⇐ APPLY DIRICHLET B.C. AT x = 0

For i = 1, p*NE + 1

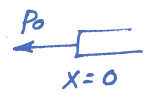
F(i) = F(i) − K(i,1) * u(0) ← change the load vector

K(1,i) = 0 } Set 1st row & column to zero.
K(i,1) = 0

End

F(1) = u(0)] Reset the stiffness and load terms to get
K(1,1) = 1.0 $\alpha_1 = u(0)$

elseif ibctype(1) = 2 then ⇐ APPLY NEUMANN B.C. AT x = 0



F(1) = F(1) − Po ← Add/subtract the end load at x = 0

elseif ibctype(1) = 3 then ⇐ APPLY ROBIN B.C. AT x = 0

K(1,1) = K(1,1) + ko.

F(1) = F(1) + ko δo

endif

Total no. of degrees of freedom

If ibctype(2) = 1 then ⇐ APPLY DIRICHLET B.C. AT x = L

[NDOF = p*NE + 1]

For i = 1, p*NE + 1

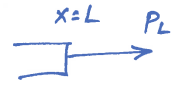
F(i) = F(i) − K(i, NDOF) * u(L) ← Change Load vector

K(i,NDOF) = 0 } Set last row and column to ZERO
K(NDOF, i) = 0

End

F(NDOF) = u(L) } Reset load and stiffness to get
K(NDOF, NDOF) = 1.0 $\alpha_{NDOF} = u(L)$

elseif ibctype(2) = 2 then



F(NDOF) = F(NDOF) + P_L

elseif ibytype(2) = 3 then

K(NDOF, NDOF) = K(NDOF, NDOF) + k_L

F(NDOF) = F(NDOF) − $k_L \cdot \delta_L$

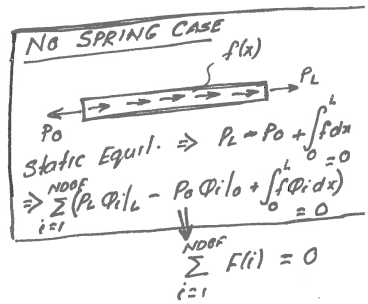
endif

So, we have taken care of all B.C. types at both ends.

We have to make the following check:

If $k(x) = 0$ and (ibctype(1) and ibctype(2) $= 2$) then

If $\sum_{i=1}^{NDOF} F(i) = 0$ then problem is consistent, i.e. infinite solutions exist.



Now set $u(0) = 0$ or

For $i = 1, NDOF$

$K(1,i) = 0$

$K(i,1) = 0$

End

$F(1) = 0$
$K(1,1) = 1.0$ } Forces $\alpha(1) = 0$

else if $\sum_{i=1}^{NDOF} F(i) \neq 0$ then $\boxed{\text{EXIT}}$ giving error message

WARN THAT LOADS ARE INCONSISTENT; STOP CODE!

endif.

$\Rightarrow$ Now the system of equations is ready, or.

$$[K]\{\alpha\} = \{F\}$$

Find $\{\alpha\}$ by $\{\alpha\} = [K]^{-1}\{F\}$ using either Fully pivoted Gauss elimination; $[L][U]$ decomposition of $[K]$ or by Conjugate Gradient method.

Next we will discuss POST-PROCESSING of the solution, i.e. what to do with $u_h(x) = \sum_{i=1}^{NDOF} d_i \phi_i(x)$

Post – Processing

There are several ways of post-processing the FE solution.

- What is post-processing?

  Extracting useful information that the analyst wants. E.g. stresses, strains, max. deflection, support loads, etc.

- Why post-process?

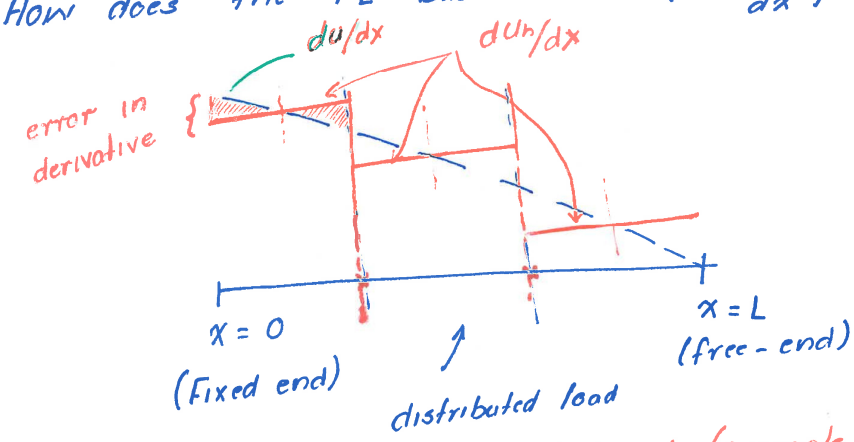
  To make design/certification decisions.

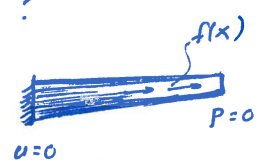
- Can we use the FE results directly?

  Yes – for displacement

  No – for strains and stresses, as these jump at inter-element boundaries and hence can portray the wrong picture!

* How does the FE-based strain (or $\frac{du_h}{dx}$) do?





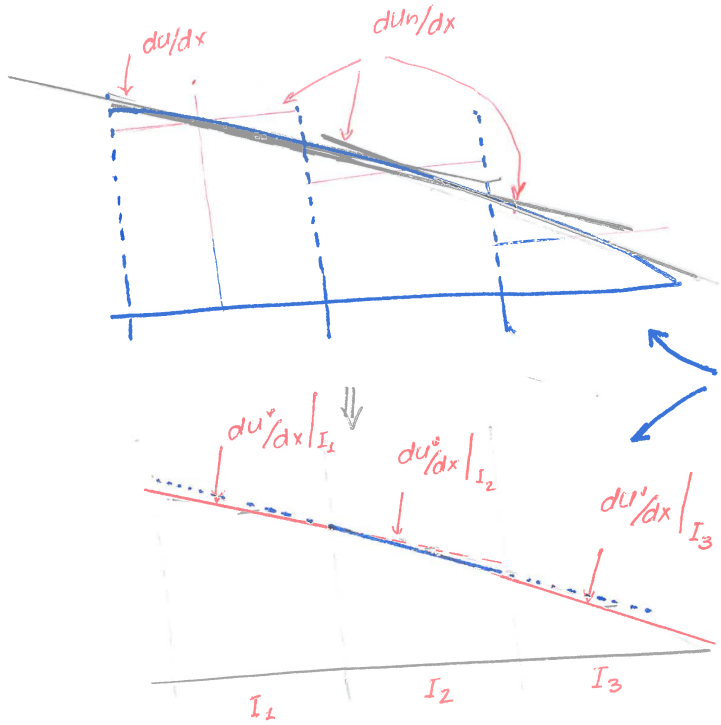
Linear approx. ($p = 1$)

3-element mesh (example)

* Constant strain in each element

- Strains from FE solution are "good" at the centre of the element (i.e. at $\xi = 0$ in master element) ~ GAUSS INTEGRATION POINT (1-PT. RULE)!
- strains from FE solution are "bad" at the nodes
- FE strain at $x=0$ (i.e. boundary node) is bad – leading to bad prediction of reaction force there.

Can we do better ?



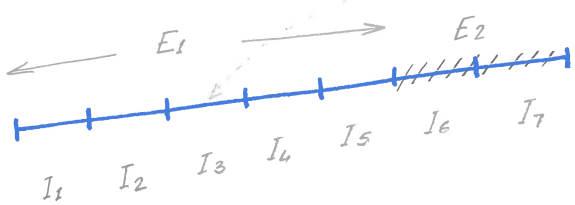
How to get better fits of $\frac{du^*}{dx}$ and/or F_{xx}^* ?

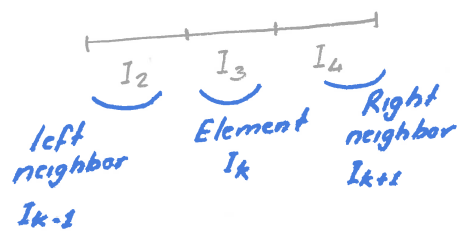
$\Downarrow$

Zienkiewicz - Zhu recipe or patch - recovery scheme (Babuska, Strouboulis, CSU).

<u>Basic Idea</u>: Sliding averages or Least - square fit.

Let us take a generic mesh :





We want to find $F_x^*\big|_{I_k}$ <i.e. recovered axial force>

$$F_x = EA\frac{du}{dx} \Rightarrow F_x^*\Big|_{I_k} = a_0^k + a_1^k x$$ $\Leftarrow$ i.e. choose it to be linear

If elements of order p, $F_x^*\big|_{I_k} = \sum_{i=0}^{p} a_i^k x^i$

where a_i^k are coefficients corresponding to the "fit" for element I_k.

<u>Define</u> (over $I_{k-1} \cup I_k \cup I_{k+1}$):

$$J_k = \frac{1}{2} \sum_{l=k-1}^{k+1} \int_{I_l} (F_{x,h} - F_x^*)^2 \, dx \qquad \text{—— (1)}$$

with $F_{x,h} = EA \dfrac{du_h}{dx}$ ← axial force due to FE solution

* Find $\{a_i^k\}_{i=0}^{p}$ such that they MINIMIZE J_k

⇒ Note that we can alternatively define

$$\tilde{J}_k = \frac{1}{2} \sum_{l=k-1}^{k+1} \int_{I_l} \frac{(F_{x,h} - F_x^*)^2}{EA} \, dx \qquad \text{—— (2)}$$

⇑

STRAIN ENERGY IN THE PATCH $\Omega_k = I_{k-1} \cup I_k \cup I_{k+1}$

Minimizing (1) we get:

$$\frac{\partial J_k}{\partial a_i^k} = 0 \Rightarrow \sum_{l=k-1}^{k+1} \int_{I_l} F_x^* \cdot x^i \, dx = \sum_{l=k-1}^{k+1} \int_{I_l} F_{x,h} \cdot x^i \, dx$$

$i = 0, 1, 2, \dots, p$

⇙

$$[\tilde{\tilde{K}}^k]\{a^k\} = \{\tilde{\tilde{F}}^k\} \qquad \text{—— (3)}$$

with $\tilde{\tilde{K}}_{ij}^k = \sum_{l=k-1}^{k+1} \int_{I_l} x^i \cdot x^j \, dx$; $\tilde{\tilde{F}}_i^k = \sum_{l=k-1}^{k+1} \int_{I_l} F_{x,h} \cdot x^i \, dx$

* If $k = 1$ (i.e. 1st element), then $\Omega_1 = I_1 \cup I_2$
  or l varies from 1 to 2
* If $k = NE$ (i.e. last element), then $\Omega_{NE} = I_{NE-1} \cup I_{NE}$
  or l varies from $(NE-1)$ to NE.
* In this case, for I_5 also use only $I_4 \cup I_5$ and for I_6 use $I_6 \cup I_7$ AS these are sitting at material interface.

* Now one can repeat the process for each element I_k $k = 1, 2, \cdots, NE$.

Solving (3) gives us $\{a^k\}_{k=0}^{p}$. Now recreate the recovered or post-processed $F_x^*\big|_{I_k} = \sum_{i=0}^{p} a_i^k x^i$.

* At the nodes $\{X_j\}_{j=1}^{NE+1}$, use $F_x^*\big|_{X_j} \approx \frac{1}{2}\left(F_x^*\big|_{I_{j-1}} + F_x^*\big|_{I_j}\right)\Big|_{X_j}$ or average from both sides.

> This is said to be a SUPERCONVERGENT recovery as F_x^* is "closer" to $F_x|_{exact}$ than $F_{x,h}$.

Lets get greedy! If F_x^* is "better" than $F_{x,h}$ then we can use:

$$\eta_k = \sqrt{\int_{I_k} \frac{\left(F_x^*\big|_{I_k} - F_{x,h}\right)^2 dx}{EA}} \quad \text{— (4)}$$

as the ERROR INDICATOR FOR ELEMENT I_k.

— Note that η_k is strain energy norm of the difference of F_x^* and $F_{x,h}$.

Now we can define $\bar{\eta}_k = \dfrac{\eta_k}{\eta_{max}}$ with $\eta_{max} = \max\limits_{k=1,\cdots,NE}\{\eta_k\}$

Also $\xi = \sqrt{\sum_{k=1}^{NE} \eta_k^2}$ — (5)

GLOBAL ERROR INDICATOR

→ CHECK IF $\dfrac{\xi}{\|u_h\|_\Omega} \times 100 < \text{tolerance } t$ — (6)

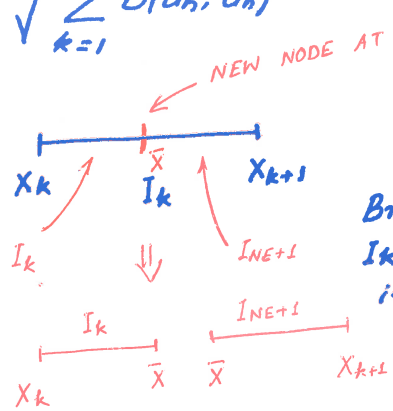
USER SPECIFIED

IF NOT, then REFINE all elements with $\bar{\eta}_k \geq 0.8$ (say).

(4) - (6) gives us a recipe for ADAPTIVE F.E.M., i.e. we can "ESTIMATE" the error in FEM solution and then improve (by refining elements) the FE solution to achieve the desired tolerance τ given in (6).

<u>Note:</u> (1) $\|u_n\|_\Omega = \sqrt{\sum_{k=1}^{NE} B(u_n, u_n)}$

(2) Refining an element $\Rightarrow$



Break element I_k into 2, i.e. I_k (new) and I_{NE+1}

NEW ELEMENT, COUNT OF TOTAL ELEMENTS INCREASE

(3) τ is normally taken as 5%

<u>QUESTIONS</u>: (1) Look up what stress recovery is done in the commercial FE code ABAQUS.

(2) Show (using your code) that FE stress is "good" at the points corresponding to the roots of the p-degree Legendre polynomial (where p is the order of approximation) ~ or the points of the p-point Gauss-Legendre integration rule.

A look at another error estimator (for h-version)

We have seen that

$$\|u - u_h\|_\Omega \le C h^p \qquad (p \text{ - is order of approx.})$$

Let us assume that $\|u - u_h\|_\Omega \approx C h^\beta$ <ACTUALLY THIS IS FINE>

Let us solve the problem with meshes of sizes $2h, h, h/2$

$$\therefore \|u - u_{2h}\| \approx C(2h)^\beta \Rightarrow \|u - u_{2h}\|^2 = \|u\|^2 - \|u_{2h}\|^2 = \bar{C}(2h)^{2\beta}$$

$$\|u - u_h\| \approx C(h)^\beta \Rightarrow \|u - u_h\|^2 = \|u\|^2 - \|u_h\|^2 = \bar{C}(h)^{2\beta}$$

$$\|u - u_{h/2}\| \approx C(h/2)^\beta \Rightarrow \|u - u_{h/2}\|^2 = \|u\|^2 - \|u_{h/2}\|^2 = \bar{C}\left(\frac{h}{2}\right)^{2\beta}$$

$$(\tilde{\beta} = 2\beta)$$

$$\therefore \frac{\|u\|^2 - \|u_{2h}\|^2}{\|u\|^2 - \|u_h\|^2} = \frac{\bar{C}(2h)^{\tilde{\beta}}}{\bar{C}(h)^{\tilde{\beta}}} = 2^{\tilde{\beta}}$$

and

$$\frac{\|u\|^2 - \|u_h\|^2}{\|u\|^2 - \|u_{h/2}\|^2} = 2^{\tilde{\beta}} \Rightarrow (\overset{A}{\|u\|^2} - \overset{B}{\|u_{2h}\|^2})(\overset{A}{\|u\|^2} - \overset{C}{\|u_{h/2}\|^2}) = (\overset{A}{\|u\|^2} - \overset{D}{\|u_h\|^2})^2$$

$$\Rightarrow \cancel{A^2} - A(B+C) + BC = \cancel{A^2} - 2AD + D^2 \qquad \sim (1)$$

$$\Rightarrow \boxed{A = \frac{D^2 - BC}{(2D - B - C)}} \Rightarrow \text{We get } \|u\|^2 \text{ !!}$$

* $u_{2h}, u_h, u_{h/2}$ will be obtained using FEM.

Also,

$$\|u - u_h\|^2 / \|u - u_{h/2}\|^2 = 2^{\tilde{\beta}} \Rightarrow \ln(\|u - u_h\|^2) - \ln(\|u - u_{h/2}\|^2) = \tilde{\beta} \ln 2 \qquad (3)$$

$$\Rightarrow \tilde{\beta} = \frac{\ln(A - D) - \ln(A - C)}{\ln 2} \Rightarrow \bar{C} = \frac{\|u - u_{h/2}\|^2}{(h/2)^{2\beta}}$$

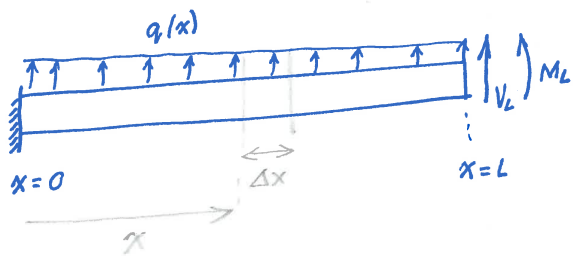
(2)

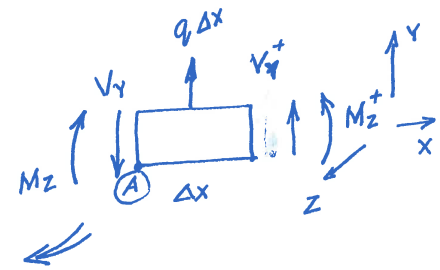
$$\boxed{\tilde{\beta} = 2\beta} \longrightarrow \beta \approx p \text{ should come!}$$

- Now we have an <u>error estimator</u> and a way to IMPROVE the FE solution.
- Can all this be generalized ?
  - YES. We will show how, as we go along.

* Lets move on to something new — <u>Fourth Order Diff. eqns.</u>

Where do they arise ?





$$\sum F_Y = 0$$

$$\Rightarrow V_Y^+ - V_Y + q\Delta x = 0$$

$$\Rightarrow \boxed{\frac{dV_y}{dx} + q = 0} \quad \text{— (A)}$$

$$\sum M_Z\Big|_A = 0 \Rightarrow V_Y^+ \cdot \Delta x + M_Z^+ - M_Z = 0 \Rightarrow \frac{dM_Z}{dx} + V_Y = 0$$

$$\Rightarrow \boxed{V_Y = -\frac{dM_Z}{dx}} \text{ — (B)} \Rightarrow \text{Using (B) in (A) we get:}$$

$$\boxed{\frac{d^2 M_Z}{dx^2} = q} \quad \text{— (C)}$$

Constitutive relationship : $EI_{zz}\dfrac{d^2v}{dx^2} = M_Z$ — EULER - BERNOULLI THEORY FOR BEAMS

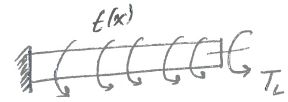
$$\Rightarrow \boxed{\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = q} \quad \text{— (D)}$$

Similarly, torsion of a non-circular shaft with constraint to warping leads to :

$$\boxed{\underbrace{E\Gamma \frac{d^4\theta}{dx^4}}_{-T_{\Gamma,x}} - \underbrace{GJ \frac{d^2\theta}{dx^2}}_{T_{J,x}} = t(x)} \quad \text{— (E)}$$

bending - torsion coupling

St. Venant torsion



[Acro guys see section 27.3 of Megson]

$$T_\Gamma = -E\Gamma \frac{d^3\theta}{dx^3}$$
$$T_J = GJ \frac{d\theta}{dx}$$

Define the residual :

$$r(x) = q(x) - \frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) \quad \sim (1) \quad \text{— EB - beam bending}$$

$$r(x) = t(x) - \frac{d^2}{dx^2}\left(E\Gamma\frac{d^2\theta}{dx^2}\right) + \frac{d}{dx}\left(GJ\frac{d\theta}{dx}\right) \quad \sim (2) \quad \text{— Bending - torsion coupling}$$

<u>Weighted residual form:</u>

$$\int_0^L r(x)\, w(x)\, dx = 0 = \begin{cases} \displaystyle\int_0^L \left(q - \frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right)\right) w\, dx & \text{— (3a)} \\ \displaystyle\int_0^L w\left(t - \frac{d^2}{dx^2}\left(E\Gamma\frac{d^2\theta}{dx^2}\right) + \frac{d}{dx}\left(GJ\frac{d\theta}{dx}\right)\right) dx & \text{— (3b)} \end{cases}$$

* Note that (1) is a force balance equation, so w is a virtual displacement in y-direction in (3a)
  (2) is a torque balance equation, so w is a virtual "twist" about x-axis in (3b)

* This is the power of the WEIGHTED RESIDUAL formulation!

<u>Integration by parts</u> $\xrightarrow{\langle IBP \rangle}$ <u>Weak formulation</u> ⟨Twice IBP to be done⟩

$$\int_0^L q\, w\, dx - \int_0^L EI_{zz}\, v''\, w''\, dx + M_z|_L\, w'|_L - M_0\, w'|_0 + V_y|_L\, w|_L - V_y|_0\, w|_0 = 0$$

or

$$\int_0^L EI_{zz}\, v''\, w''\, dx = \int_0^L q\, w\, dx + M_z|_L\, w'|_L - M_0\, w'|_0 + V_y|_L\, w|_L - V_y|_0\, w|_0 \quad \text{— (4a)}$$

(3b) gives :

$$\int_0^L t\,w\,dx - \int_0^L GJ\theta' w'\,dx - \int_0^L E\Gamma\theta'' w''\,dx + \left[\left(GJ\theta' - \frac{d}{dx}(E\Gamma\theta'')\right)w\right]_0^L$$

$$+ \left[E\Gamma\theta'' w'\right]_0^L = 0$$

or

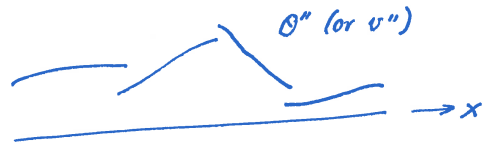
$$\int_0^L (E\Gamma\theta'' w'' + GJ\theta' w')\,dx = \int_0^L t\,w\,dx + \underbrace{[T\,w]_0^L}_{\text{total torque at end}} + \underbrace{\left[(E\Gamma\theta'')w'\right]_0^L}_{\text{end axial force}} \qquad \text{(4b)}$$

warping

$$\left\{ v' \to \frac{dv}{dx} \; ; \; v'' \to \frac{d^2v}{dx^2} \right\}$$

Eqns. (4a) and (4b) are similar and show the following :

(1) For LHS to be <u>finite</u> ; θ'' or v'' has to <u>exist</u> everywhere i.e. at most v'' (or θ'') can have JUMPS at some points.



(2) There are 2 boundary conditions that need to be specified at EACH boundary point, i.e.

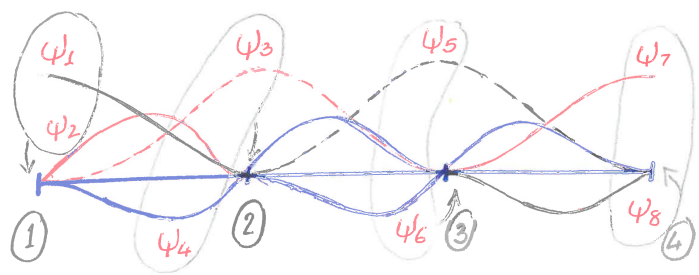
<u>EITHER</u>		<u>OR</u>	
v	(θ)	V_y	(T)
	AND		
v'	(θ')	M_z	(F_x)

From (1) above $v(x)$ or $\theta(x)$ have to have CONTINUOUS first derivative (as 2nd derivative jumps) $\Rightarrow$ $v(x)$ (or $\theta(x)$) have to be C^1 - <u>atleast</u>.

$\rightarrow$ We will build the corresponding basis functions next . . .

HERMITE (C^1) BASIS FUNCTIONS

We will now need to enforce continuity of displacement $U_h(x)$ and $\frac{dU_h}{dx}(x)$ at the nodes. The basis functions are constructed as:



Each node I has $\underline{\underline{2}}$ basis functions attached as: ψ_{2I-1} and ψ_{2I} such that: (In element I_I)

$$\psi_{2I-1}(x_I) = 1 \quad ; \quad \frac{d\psi_{2I-1}}{dx}\Big|_{x_I} = 0 \quad ; \quad \psi_{2I-1}\Big|_{x_{I+1}} = 0 \quad ;$$

$$\frac{d\psi_{2I-1}}{dx}\Big|_{x_{I+1}} = 0$$

And $\psi_{2I}(x_I) = 0$; $\frac{d\psi_{2I}}{dx}\Big|_{x_I} = 1$; $\psi_{2I}(x_{I+1}) = 0$; $\frac{d\psi_{2I}}{dx}\Big|_{x_{I+1}} = 0$

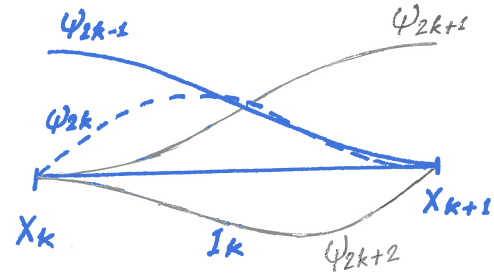
In element I_{I-1}:

$$\psi_{2I-1}(x_{I-1}) = 0 \; ; \; \frac{d}{dx}\psi_{2I-1}(x_{I-1}) = 0$$

$$\psi_{2I}(x_{I-1}) = 0 \; ; \; \frac{d}{dx}\psi_{2I}\Big|_{x_{I-1}} = 0$$

* Note that ψ_{2I-1}, ψ_{2I} are non-zero in element I_{I-1}, I_I only.

As before, note that in EACH element, the non-zero ψ_i's are



Local numbering:

$$N_1^{H,k}(x) = \psi_{2k-1}\Big|_{I_k} \quad ; \quad N_2^{H,k}(x) = \psi_{2k}\Big|_{I_k} \quad ;$$

$$N_3^{H,k}(x) = \psi_{2k+1}\Big|_{I_k} \quad ; \quad N_4^{H,k} = \psi_{2k+2}\Big|_{I_k}$$

* Note that such a definition of basis functions allows for continuity of value and derivative, at each node.
* Each ψ_i (or $N_i^{H,k}$) satisfies 4 conditions in each element.

$N_i^{H,k}$ is thus a polynomial of degree 3 (4 constants define a cubic) – hence called HERMITE cubic polynomials or basis functions. Example:

Let $N_1^{H,k}(x) = a_0 + a_1(x - x_k) + a_2(x - x_k)^2 + a_3(x - x_k)^3$

$N_1^{H,k}(x_k) = 1 = a_0$

$\frac{dN_1^{H,k}}{dx}(x_k) = 0 \Rightarrow a_1 = 0$

$N_1^{H,k}(x_{k+1}) = 0 \Rightarrow 1 + a_2 h_k^2 + a_3 h_k^3 = 0 \quad (*)$

$\frac{dN_1^{H,k}}{dx}(x_{k+1}) = 0 \Rightarrow 2a_2 h_k + 3a_3 h_k^2 = 0$

$\Rightarrow a_2 = -\frac{3}{2} a_3 \cdot h_k$

From $(*)$: $1 - \frac{3}{2} a_3 h_k^3 + a_3 h_k^3 = 0 \Rightarrow a_3 = \frac{2}{h_k^3}$

$$\Rightarrow \boxed{N_1^{H,k}(x) = 1 - 3\frac{(x - x_k)^2}{h_k^2} + 2\frac{(x - x_k)^3}{h_k^3}}$$

Its easy to show that $\boxed{N_3^{H,k}(x) = 3\dfrac{(x-x_k)^2}{h_k^2} - 2\dfrac{(x-x_k)^3}{h_k^3}}$

Further, $N_2^{H,k}(x) = (x-x_k)\left(1 - \dfrac{x-x_k}{h_k}\right)^2$

$$N_4^{H,k}(x) = (x-x_k)\left[\frac{(x-x_k)^2}{h_k^2} - \frac{(x-x_k)}{h_k}\right]$$

Now we can create $v_h(x) = \sum_{i=1}^{2(NE+1)} \alpha_i \varphi_i(x)$

In an element I_k,

Show that :

$\alpha_{2i-1} = v(x_i)$

$\alpha_{2i} = \left.\dfrac{dv}{dx}\right|_{x_i}$

$$v_h\Big|_{I_k} = \alpha_{2k-1}\varphi_{2k-1}(x) + \alpha_{2k}\varphi_{2k}(x) + \alpha_{2k+1}\varphi_{2k+1}(x) + \alpha_{2k+2}\varphi_{2k+2}(x)$$

$$= \alpha_1^k N_1^{H,k} + \alpha_2^k N_2^{H,k} + \alpha_3^k N_3^{H,k} + \alpha_4^k N_4^{H,k}$$

The weight function $w_h(x) = \sum_{i=1}^{2(NE+1)} \beta_i \varphi_i(x)$ with β_i chosen by us

Defining $B(v,w) = \sum_{k=1}^{NE} \int_{I_k} EI_{zz} \dfrac{d^2v}{dx^2}\dfrac{d^2w}{dx^2}\,dx$; $F(w) = \sum_{k=1}^{NE}\int_{I_k} q\,w\,dx + M_L \cdot w'|_L - M_0 w'|_0 + V_L w|_L - V_0 w|_0$

we get $B(v,w) = F(w)$

Putting v_h instead of v and choosing $w = \varphi_i$ $(i = 1, 2, \ldots, 2(NE+1))$

we get $B(v_h, \varphi_i) = F(\varphi_i)$

$$\Rightarrow \sum_{j=1}^{2(NE+1)} \alpha_j \left(\sum_{k=1}^{NE}\int_{I_k} EI_{zz}\frac{d^2\varphi_j}{dx^2}\frac{d^2\varphi_i}{dx^2}\,dx\right) = \sum_{k=1}^{NE}\int_{I_k} q\,\varphi_i\,dx + M_L\varphi_i'|_L - M_0\varphi_i'|_0 + V_L\varphi_i|_L - V_0\varphi_i|_0$$

This is equivalent to computing for element I_k :

$$[K^k]_{4\times 4} \quad \text{with} \quad K_{ij}^k = \int_{I_k} EI_{zz} \frac{d^2 N_i^{H,k}}{dx^2} \cdot \frac{d^2 N_j^{H,k}}{dx^2} dx$$

and $$\{F^k\}_{4\times 1} \quad \text{with} \quad F_i^k = \int_{I_k} q\, N_i^{H,k}\, dx$$

<u>ASSEMBLY</u> $\rightarrow$ $[K]_{2(NE+1)\times 2(NE+1)}$, $\{F\}_{2(NE+1)}$ $\leftarrow$ GLOBAL

For element I_k :

$i = 1, 2$

$i_1 = 2i - 1$

$i_2 = 2i$

$I_1 = 2(k-1) + i_1$

$I_2 = 2(k-1) + i_2$

$$\boxed{\begin{aligned} F(I_1) &= F(I_1) + F^k(i_1) \\ F(I_2) &= F(I_2) + F^k(i_2) \end{aligned}}$$

$j = 1, 2$

$j_1 = 2j - 1$

$j_2 = 2j$

$J_1 = 2(k-1) + j_1$

$J_2 = 2(k-1) + j_2$

$K(I_1, J_1) = K(I_1, J_1) + K^k(i_1, j_1)$

$K(I_1, J_2) = K(I_1, J_2) + K^k(i_1, j_2)$

$K(I_2, J_1) = K(I_2, J_1) + K^k(i_2, j_1)$

$K(I_2, J_2) = K(I_2, J_2) + K^k(i_2, j_2)$.

enddo

enddo

<u>Boundary Condition</u> : Corresponding to rows 1, 2 and $2(NE+1)-1$, $2(NE+1)$

Use ibgtype (1,1) ← Node 1, Force / Displacement B.C.
ibctype (1,2) ← Node 1, Moment / Rotation B.C.

ibctype (2,1) ← Node (NE+1), Force / Displacement B.C.
ibctype (2,2) ← Node (NE+1), Moment / Rotation B.C.

If ibctype(ℓ,1) = 1 ⇒ Displacement given, i.e. v is given ($v(0)$ or $v(L)$ is given depending on ℓ)
2 ⇒ Shear force given, i.e. V_y is given

If ibctype(ℓ,2) = 1 ⇒ Slope given, i.e. $\frac{dv}{dx}$ is given ($dv/dx|_0$ or $dv/dx|_L$ is given depending on ℓ)
2 ⇒ Bending moment M_z given

* Apply B.C. in the usual way!

Also note that :

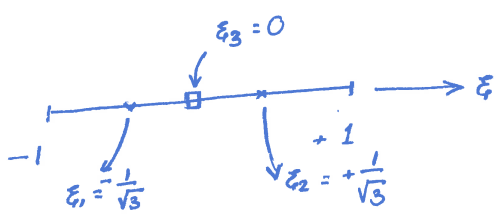
1) $B(v - v_h, w_h) = 0$ ← ORTHOGONALITY OF ERROR

2) $B(v - v_h, v - v_h) = B(v,v) - B(v_h, v_h) \geq 0$

$\Rightarrow \underbrace{B(v,v)}_{2 * \text{ Strain energy for bending problem}} = \|v\|^2 \geq B(v_h, v_h) = \|v_h\|^2$

3) $\|v - v_h\|_\Omega \leq Ch^2$

4)



* $\frac{d^2v}{dx^2}$ or curvature (implies also $M_z = EI_{zz}\frac{d^2v}{dx^2}$) is "EXACT" at ξ_1, ξ_2

* $\frac{d^3v}{dx^3}$ or $V_y = -\frac{d}{dx}\left(EI_{zz}\frac{d^2v}{dx^2}\right)$ is "EXACT" at $\xi_3 = 0$

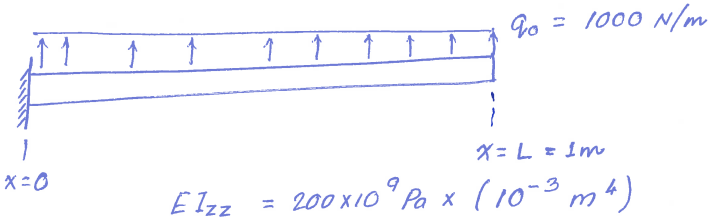
Note that ξ_1, ξ_2 are roots of the second order Legendre polynomial
ξ_3 is root of linear Legendre polynomial.

Q Can you follow the approach of L5-2 to L5-4 to construct a post-processed moment ($M_{zz}(x)$) field for the mesh given?

Hint: Let $M_{zz}\big|_{I_k}(x) = a_0^k + a_1^k x + a_2^k x^2$

HW (Lab Work 4)

Q Write a matlab code to solve the following problem:



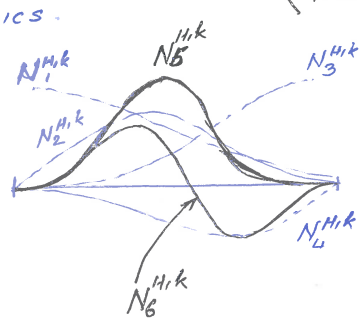
$$EI_{zz} = \underbrace{200\times10^9 \text{ Pa}}_{\text{steel}} \times (10^{-3}\text{ m}^4)$$

Use Hermite cubic elements with NE = 2, 4, 8. Solve the problem. Plot $u_h(x)$ vs x and compare with exact $u(x)$.

END OF ASSIGNMENT

Higher Order elements?

Can go higher than the cubics. ⟨HIERARCHIC CONSTRUCTION⟩



$N_5^{H,k}(x)$ is quartic with $N_5^{H,k}(x_1^k) = 0$, $N_5^{H,k}(x_2^k) = 0$

$\frac{dN_5^{H,k}}{dx}(x_1^k) = 0$, $\frac{dN_5^{H,k}}{dx}(x_2^k) = 0$

and $\int_{x_1^k}^{x_2^k} (N_5^{H,k})^2 dx = 1$

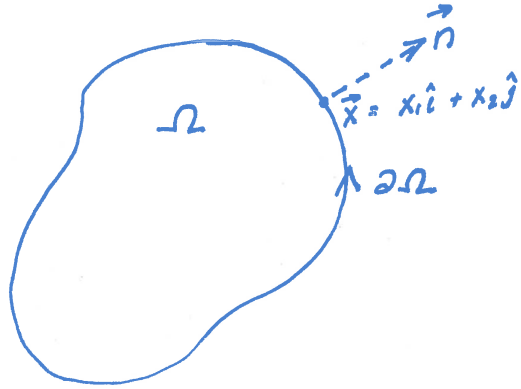
$N_6^{H,k}$ is quintic with value and derivative zero at x_1^k, x_2^k. Further,

$\int_{x_1^k}^{x_2^k} N_6^{H,k} N_5^{H,k} dx = 0$

$\int_{x_1^k}^{x_2^k} (N_6^{H,k})^2 dx = 1$

FE for 2D problems

Now we have a change of domain:



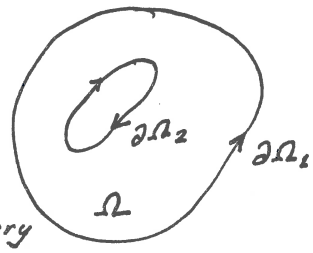
Ω : 2D domain

$\partial\Omega$: "directed" boundary of Ω

~ directed means we traverse along the boundary in the counter-clockwise direction such that the area Ω lies to the <u>LEFT</u> of the boundary.

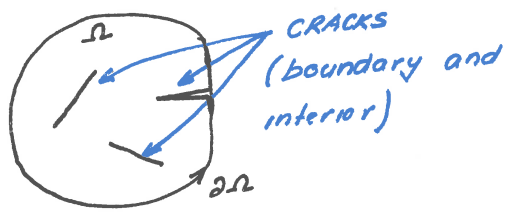
$\vec{n}$: Outer normal to $\partial\Omega$ at the point $\vec{x} = (x_1, x_2)$.

<u>Domain with cut-outs</u>:



The boundary of the cut-out is $\partial\Omega_2$ with $\partial\Omega_2$ directed CLOCKWISE - again keeping Ω to the LEFT of the boundary

* Now the domain plays a more strategic role - it decides the smoothness of the solution. For example: CRACKED DOMAIN



Interior cracks are cut-outs, and hence have to be treated as such.

- Crack-tips,

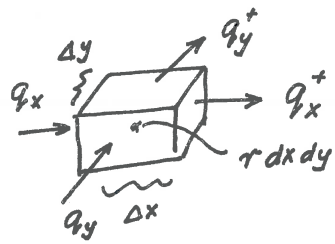
e.g. A, B are points of singularity (of what?) and the solution (of elasticity, heat conduction, torsion, etc) "misbehaves" in the vicinity of the crack-tips.

- Cut-outs in general cause some problems, i.e. concentration of some field variables.

<u>Lets get started!</u>

Let us take a <u>MODEL PROBLEM</u>:

(Steady-state heat conduction)



$$q_x = -\left(k_{11}\frac{\partial T}{\partial x} + k_{12}\frac{\partial T}{\partial y}\right)$$

$$q_y = -\left(k_{12}\frac{\partial T}{\partial x} + k_{22}\frac{\partial T}{\partial y}\right)$$

$\vec{q} = q_x\,\hat{i} + q_y\,\hat{j}$ ← HEAT FLUX VECTOR

(heat fluxes <u>from</u> hotter <u>to</u> colder part)

<u>Heat balance</u>:

$$q_x^+ - q_x + q_y^+ - q_y - r = 0$$

$$\Rightarrow \boxed{\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = r} \quad \text{in } \Omega \quad \text{or} \quad \frac{\partial}{\partial x}\left(k_{11}\frac{\partial T}{\partial x} + k_{12}\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(k_{12}\frac{\partial T}{\partial x} + k_{22}\frac{\partial T}{\partial y}\right) + r = 0 \quad \text{—(1)}$$

* $T(x,y)$ is absolute temperature at a point in Ω.

<u>Residual</u>: $R(x,y) = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}\right) + r = 0$ for exact T.

<u>Weighted residual form</u>: $\int_\Omega R \cdot v\, dA = 0 \Rightarrow \int_\Omega (r - q_{x,x} - q_{y,y})\, v\, dA = 0$ —(2)

What is $v(x,y)$? A "test" "temperature".

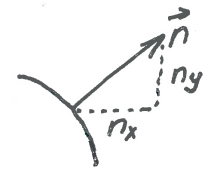
<u>Weak formulation</u>: Same idea as before, i.e. integration by parts till T, v have same order derivatives. Hence integration by parts ONCE on (2), to get:

$$\frac{\partial}{\partial x}(q_x \cdot v) = q_x \cdot \frac{\partial v}{\partial x} + \frac{\partial q_x}{\partial x} \cdot v \quad ; \quad \frac{\partial}{\partial y}(q_y\, v) = q_y\, v_{,y} + q_{y,y} \cdot v$$

$$\Rightarrow \int_\Omega r\, v\, dA + \int_\Omega (q_x\, v_{,x} + q_y\, v_{,y})\, dA - \oint_{\partial\Omega} \{(q_x\, v)\, n_x + (q_y\, v)\, n_y\}\, ds = 0$$

* $\int_\Omega (q_x\, v)_{,x}\, dA = \int_{\partial\Omega} q_x\, v\, n_x\, ds \quad ; \quad \int_\Omega (q_y\, v)_{,y}\, dA = \int_{\partial\Omega} q_y\, v\, n_y\, ds$

Where outer normal $\vec{n} = n_x \hat{i} + n_y \hat{j}$



$\therefore$ Weak formulation is :

$$-\int_{\Omega} (q_x v_{,x} + q_y v_{,y})\, dA = \int_{\Omega} (r\, v)\, dA - \int_{\partial\Omega} q_n v\, ds \qquad \text{— (3)}$$

($q_n = q_x n_x + q_y n_y \rightarrow$ normal outgoing flux)

OR

$$\underbrace{\int_{\Omega} \Big((k_{11} T_{,x} + k_{12} T_{,y}) v_{,x} + (k_{12} T_{,x} + k_{22} T_{,y}) v_{,y} \Big) dA}_{B(T,v)} = \int_{\Omega} r\, v\, dA + \underbrace{\int_{\partial\Omega} \Big((k_{11} T_{,x} + k_{12} T_{,y}) n_x + (k_{12} T_{,x} + k_{22} T_{,y}) n_y \Big) v\, ds}_{F(v)} \qquad \text{— (3A)}$$

Note that:

- Weak form here is NOT a virtual work form
- In this case it may be thought of as a flux balance form.
- In general it is not obvious to fit a "WORK" type definition to the weak form.

In short : $B(T, v) = F(v)$ — (3B)

Boundary conditions : From RHS of (3), we get the following possibilities :

Either q_n is given on a point on $\partial\Omega$

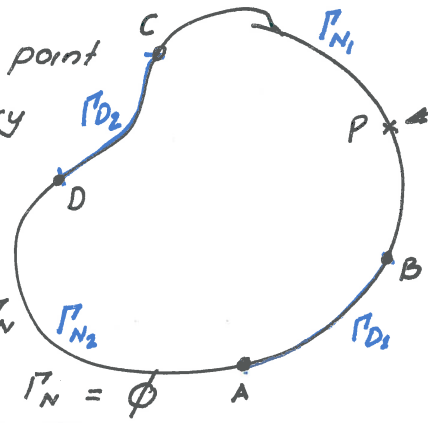
Or v (i.e. T) is given (=0) at that point on $\partial\Omega$

If $q_n(x,y)\big|_{\partial\Omega}$ given $\rightarrow$ NEUMANN/NATURAL B.C.

If $T(x,y)\big|_{\partial\Omega}$ given $\rightarrow$ DIRICHLET/ESSENTIAL B.C. ($v(x,y)\big|_{\partial\Omega} = 0$)

More on B.C.

- B.C. is not specific to a point but is specified for a boundary segment Γ. Thus, part of $\partial\Omega$ may be Dirichlet or Γ_D and part may be Neumann or Γ_N with $\partial\Omega = \Gamma_D \cup \Gamma_N$; $\underbrace{\Gamma_D \cap \Gamma_N = \phi}$



Either q_n or T given

$\Gamma_D = \Gamma_{D_1} \cup \Gamma_{D_2}$

$\Gamma_N = \Gamma_{N_1} \cup \Gamma_{N_2}$

this means the Γ_D and Γ_N cannot intersect i.e. a segment cannot have both q_n and T specified

* At points A, B, C, D in figure there is a transition of B.C. For example, point A. Here the ESSENTIAL condition will be enforced at POINT A, or $T|_A = \bar{T}$ will be imposed.

What about k_{11}, k_{12}, k_{22} ?

Note that
$$\begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = - \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} T_{,x} \\ T_{,y} \end{Bmatrix}$$

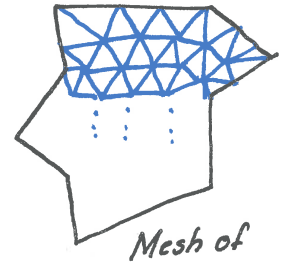
$[k]$ has to be symmetric, positive definite.

Conditions on T, v

* From (3A) it is easy to see that for the LHS to be finite, $T_{,x}$, $T_{,y}$, $v_{,x}$, $v_{,y}$ have to be DEFINED – but not necessarily continuous. So $T_{,x}$; $T_{,y}$; $v_{,x}$; $v_{,y}$ can have jumps across lines (WHY??); i.e. they have to be ATLEAST $\underline{\underline{C^0}}$.

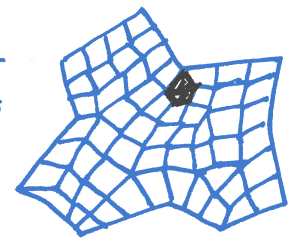
How to create the approximation?

- Consider a polygonal domain.
- Mesh it with triangles or quadrilaterals (should exactly cover full domain)



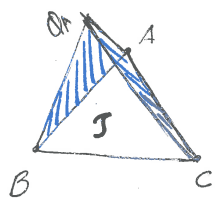
Mesh of triangles

~ Can we create piecewise polynomial basis functions over this mesh? Ensure C^0 continuity

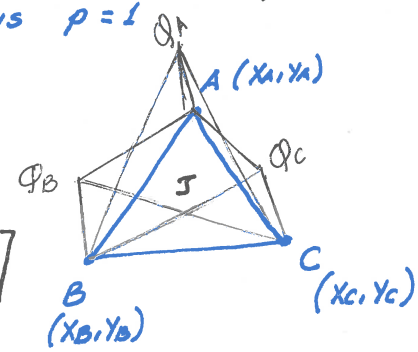


Mesh of quadrilaterals.

~ Smallest order approximation is $p=1$ or piecewise linear. How?



[$\mathcal{T}$ – generic element]



$$\varphi_A\big|_{\mathcal{T}} = a_0 + a_1 x + a_2 y$$

with $\varphi_A(x_A, y_A) = 1$; $\varphi_A(x_B, y_B) = 0$; $\varphi_A(x_C, y_C) = 0$

$$\Rightarrow \quad \begin{aligned} a_0 + a_1 x_A + a_2 y_A &= 1 \\ a_0 + a_1 x_B + a_2 y_B &= 0 \\ a_0 + a_1 x_C + a_2 y_C &= 0 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} 1 & x_A & y_A \\ 1 & x_B & y_B \\ 1 & x_C & y_C \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

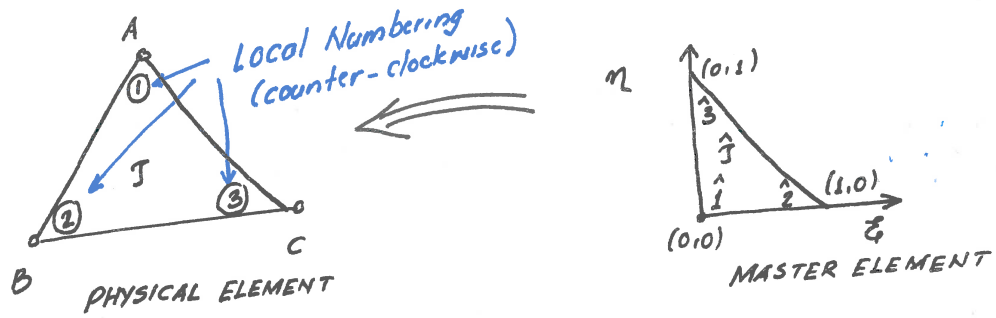
From this condition a_0, a_1, a_2 can be found.

Similarly, $\varphi_B\big|_{\mathcal{T}} = b_0 + b_1 x + b_2 y$ with $\varphi_B(x_A, y_A) = \varphi_B(x_C, y_C) = 0$

$\varphi_B(x_B, y_B) = 1$ AND $\varphi_C\big|_{\mathcal{T}} = c_0 + c_1 x + c_2 y$ with

$\varphi_C(x_A, y_A) = \varphi_C(x_B, y_B) = 0$; $\varphi_C(x_C, y_C) = 1$

This is messy! So let us seek a CLEANER form, i.e. go to a MASTER element $\hat{\mathcal{T}}$



Let us take a linear map from $\hat{J}$ to J as:

$$x(\xi,\eta) = \alpha_0 + \alpha_1 \xi + \alpha_2 \eta$$

with $x(0,0) = x_1$; $x(1,0) = x_2$; $x(0,1) = x_3$

It is easy to show that then

$$x(\xi,\eta) = x_1(1-\xi-\eta) + x_2 \xi + x_3 \eta$$

Similarly

$$y(\xi,\eta) = y_1 \underbrace{(1-\xi-\eta)}_{\hat{N}_1(\xi,\eta)} + y_2 \underbrace{\xi}_{\hat{N}_2(\xi,\eta)} + y_3 \underbrace{\eta}_{\hat{N}_3(\xi,\eta)}$$

Note that : $\hat{N}_1(\xi,\eta) = 1-\xi-\eta$; $\hat{N}_2(\xi,\eta) = \xi$; $\hat{N}_3(\xi,\eta) = \eta$

are the <u>linear SHAPE FUNCTIONS</u> defined in $\hat{J}$.

Also $N_1^J = \varphi_A|_J = \hat{N}_1(\xi(x,y), \eta(x,y))$ or

$$\boxed{\hat{N}_1(\xi,\eta) = \varphi_A(x(\xi,\eta), y(\xi,\eta))} = N_1^J(x,y)$$

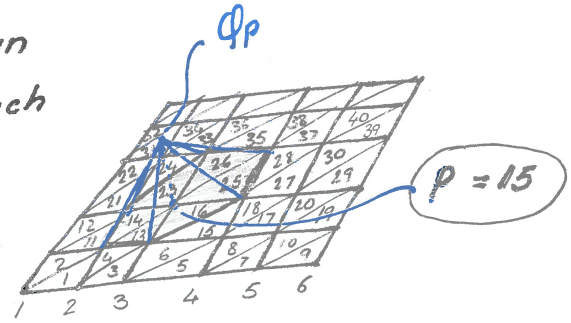
Similarly $\hat{N}_2(\xi,\eta) = \underbrace{\varphi_B(x(\xi,\eta), y(\xi,\eta))}_{N_2^J(x,y)}$; $\hat{N}_3(\xi,\eta) = \underbrace{\varphi_C(x(\xi,\eta), y(\xi,\eta))}_{N_3^J(x,y)}$

so

$$\left.\begin{aligned} \varphi_A|_J &= N_1^J \longrightarrow \hat{N}_1 \\ \varphi_B|_J &= N_2^J \longrightarrow \hat{N}_2 \\ \varphi_C|_J &= N_3^J \longrightarrow \hat{N}_3 \end{aligned}\right\} \text{ for any generic element } J.$$

What are the GLOBAL basis functions?

$\varphi_p(x,y)$ is NON-ZERO in only the elements for which it is a vertex.



$\Rightarrow \varphi_{p=15}$ is non-zero in elements $T_{13}, T_{14}, T_{16}, T_{23}, T_{25}, T_{26}$.

$$T_h(x,y) \approx \sum_{i=1}^{NDOF} \alpha_i \varphi_i(x,y) \quad ; \quad v_h(x,y) = \sum_{i=1}^{NDOF} \beta_i \varphi_i(x,y)$$

Now, we put T_h, v_h in the weak formulation to get

$$B(T_h, v_h) = F(v_h) \quad \text{——} \quad (4)$$

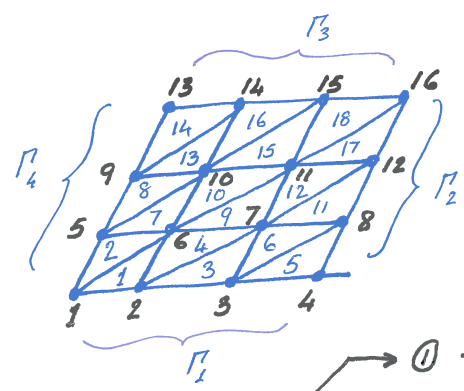
Choosing $v_h = \varphi_i(x,y)$ for $i = 1, 2, \ldots, NDOF$ we get

$$B(T_h, \varphi_i) = F(\varphi_i) \Rightarrow \sum_{j=1}^{NDOF} \alpha_j B(\varphi_j, \varphi_i) = F(\varphi_i), \quad i = 1, 2, \ldots NDOF.$$

Thus, with $K_{ij} = B(\varphi_i, \varphi_j)$, $F_i = F(\varphi_i)$ we get

$$[K]\{\alpha\} = \{F\}_{NDOF \times 1} \quad \text{——} \quad (5)$$

EXAMPLE:



Boundary segments: $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$.

On each segment we will specify a B.C. (our choice!)

Algorithm: $J_{6(i-1)+2j-1}$

① $\rightarrow 4(i-1)+j$ ← NODE 1

② $\rightarrow 4(i-1)+j+1$ ← NODE 2

③ $\rightarrow 4i+j+1$ ← NODE 3

$$J_6(i-1)+2j \begin{cases} \rightarrow (1) \rightarrow 4i+j+1 \\ \rightarrow (2) \rightarrow 4i+j \\ \rightarrow (3) \rightarrow 4(i-1)+j \end{cases}$$

$i = 1, 2, \ldots NDIVX$

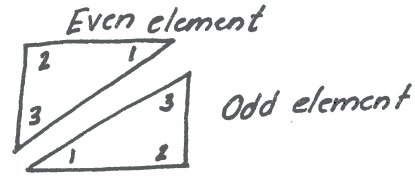
$j = 1, 2, \cdots NDIVY$

NDIVX → NO. OF SUBDIVISIONS IN X-DIRECTION

NDIVX = 3 HERE

NDIVY → NO. OF SUBDIVISIONS IN Y-DIRECTION (NDIVY = 3)

NELEM = 18 HERE ; NDOF = 16



<u>Case 1</u> : Let Γ_1 → DIRICHLET with $T = 300K$

$\Gamma_2, \Gamma_3, \Gamma_4$ → NEUMANN with $q_n = 0$

* Note that as $T = 300K$ on Γ_1, $T_1 = T_2 = T_3 = T_4 = 300$ has to be fixed.
* Further, $\beta_1, \beta_2, \beta_3, \beta_4 = 0$ has to be enforced.

→ How do we enforce $T_1 = T_2 = T_3 = T_4 = 300$?

By making $\alpha_1, \alpha_2, \alpha_3, \alpha_4 = 300$

Thus in Eqn. 5 of L6-7, we set

$$F_\ell = \alpha_\ell \text{ (or } T_\ell) \; ; \quad F_k\Big|_{k \neq \ell} = F_k - K_{k\ell}\,\alpha_\ell \; ; \quad \text{for } \ell = 1, 2, 3, 4$$

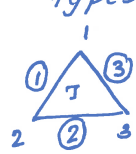
$K_{k\ell} = K_{\ell k} = 0$; $K_{\ell\ell} = 1$

In fact for any generic fixed d.o.f.

Element Calculations

— There are 3 basic "types" of elements

1) Interior



— all edges are interior to the domain, i.e. there is no edge on the boundary ($\partial T \cap \partial\Omega = \phi$)

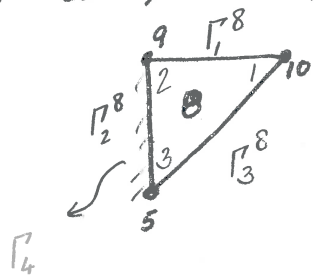
2) Dirichlet boundary — one or more edge on Dirichlet boundary Γ_D ($\partial T \cap \Gamma_D \neq \phi$)

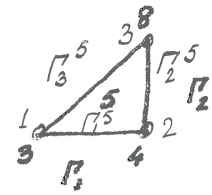
3) Neumann boundary — one or more edge on Neumann boundary Γ_N ($\partial T \cap \Gamma_N \neq \phi$)

- Thus, we have to keep track of edges of elements (Γ_i^T, $i=1,2,3$ for triangles; $i=1,2,3,4$ for quadrilaterals)

Edge Integrals :

I will take 2 examples: (a) Element 5 (b) Element 8



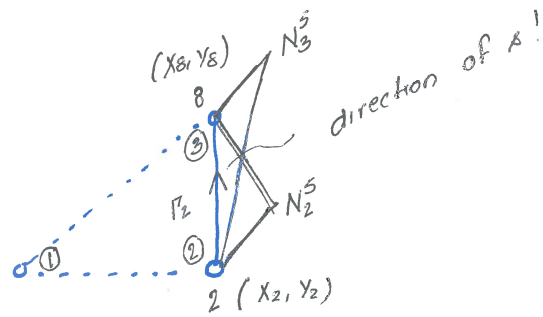


→ Let Γ_2^5 be on Γ_2 which is a Neumann boundary with $q_n = q_2$ (constant) given

→ Let Γ_2^8 be on Γ_4 with Γ_4 a Neumann boundary with $q_n = q_4$ (constant) given

Boundary Integrals :

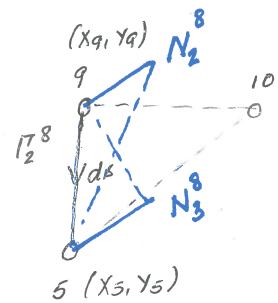
On Γ_2^5 :



$ds = +dy$, y goes from y_2 to y_8

$$\therefore \int_{\Gamma_2^5} q_n N_2^5 \, ds = \int_{y_2}^{y_8} q_2 \cdot N_2^5 \cdot dy = q_2 \int_{y_2}^{y_8} N_2^5 \, dy = q_2 \cdot \frac{(y_8 - y_2)}{2}$$

On Γ_2^8 :



$ds = -dy$, ds goes from y_9 to y_5

$$\int_{\Gamma_2^8} q_n N_2^8 \, ds = \int_{y_9}^{y_5} q_4 N_2^8 (-dy) = q_4 \frac{(y_9 - y_5)}{2}$$

~ IN CLASS I HAD WRONGLY PUT A MINUS SIGN

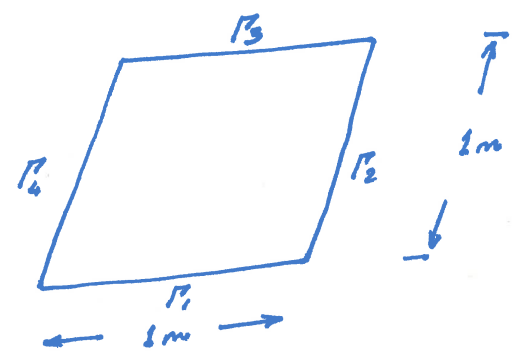
Similarly

$$\int_{\Gamma_2^5} q_n N_3^5 \, ds = \frac{q_2 (y_8 - y_2)}{2}$$

$$\int_{\Gamma_2^8} q_n N_3^8 \, ds = \frac{q_4 (y_9 - y_6)}{2}$$

* So, atleast for constant flux case, it is easy to compute the addition to the load vectors F_9, F_5, F_2, F_8 due to the boundary flux.

HOME WORK :



(a) $K_{11} = K_{22} = 1$

$K_{12} = 0$

$f = 5$

$u|_{\partial\Omega} = 0$.

Solve using $p = 1$ and a mesh of 6×6, 12×12 elements.

(b) $T|_{\Gamma_1} = 0$; $q_n|_{\Gamma_2, \Gamma_4} = 0$; $q_4|_{\Gamma_3} = 2$; $f = 5$;

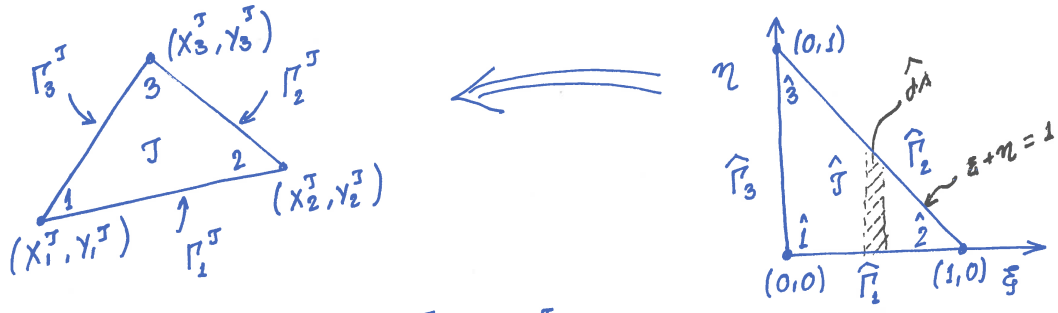
$K_{11} = K_{22} = 1$; $K_{12} = 0$

Use $p = 1$ and a mesh of 6×6 elements.

$P = 1$, ELEMENT CALCULATION

- We have the structure of what has to be done. Let us do the element calculation. For brevity, I will do for $k_{11} = k_2 = 1$, $k_{12} = 0$; which you should be able to generalize.

Step 1 : Go from physical to master element.



$$x(\xi,\eta) = x_1^J(1-\xi-\eta) + x_2^J\xi + x_3^J\eta \quad \Rightarrow \quad dx = \frac{\partial x}{\partial \xi}d\xi + \frac{\partial x}{\partial \eta}d\eta$$

$$y(\xi,\eta) = y_1^J(1-\xi-\eta) + y_2^J\xi + y_3^J\eta \qquad\qquad dy = \frac{\partial y}{\partial \xi}d\xi + \frac{\partial y}{\partial \eta}d\eta$$

$$\Rightarrow \underbrace{\begin{Bmatrix} dx \\ dy \end{Bmatrix}}_{\{dx\}} = \begin{bmatrix} (x_2^J - x_1^J) & (x_3^J - x_1^J) \\ (y_2^J - y_1^J) & (y_3^J - y_1^J) \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix}$$

$$\{dx\} = \underbrace{[J]}_{\text{Jacobian matrix}} \{d\xi\}$$

constant for an element

$$\Rightarrow \frac{\partial x}{\partial \xi} = x_2^J - x_1^J \qquad \frac{\partial x}{\partial \eta} = x_3^J - x_1^J \qquad \frac{\partial y}{\partial \xi} = y_2^J - y_1^J \qquad \frac{\partial y}{\partial \eta} = y_3^J - y_1^J$$

$$|J| = (x_2^J - x_1^J)(y_3^J - y_1^J) - (x_3^J - x_1^J)(y_2^J - y_1^J)$$

Constant for J

$$\Rightarrow \{d\xi\} = [J]^{-1}\{dx\} \quad \Rightarrow \quad \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} (y_3^J - y_1^J) & -(x_3^J - x_1^J) \\ -(y_2^J - y_1^J) & (x_2^J - x_1^J) \end{bmatrix} \begin{Bmatrix} dx \\ dy \end{Bmatrix}$$

$$\left.\begin{aligned} \Rightarrow d\xi &= \frac{\partial \xi}{\partial x}dx + \frac{\partial \xi}{\partial y}dy \\ d\eta &= \frac{\partial \eta}{\partial x}dx + \frac{\partial \eta}{\partial y}dy \end{aligned}\right\}$$

METRICS OF INVERSE TRANSFORMATION

with: $\dfrac{\partial \xi}{\partial x} = (y_3^J - y_1^J)/|J|$; $\dfrac{\partial \xi}{\partial y} = -\dfrac{(x_3^J - x_1^J)}{|J|}$

$$\frac{\partial \eta}{\partial x} = -\frac{(y_2^J - y_1^J)}{|J|} \; ; \; \frac{\partial \eta}{\partial y} = \frac{(x_2^J - x_1^J)}{|J|}$$

Derivatives of shape functions:

$$\frac{\partial N_i^T}{\partial x} = \frac{\partial \hat{N}_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \hat{N}_i}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial N_i^T}{\partial y} = \frac{\partial \hat{N}_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \hat{N}_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\therefore \quad \frac{\partial N_1^T}{\partial x} = \frac{\partial \hat{N}_1}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \hat{N}_1}{\partial \eta}\frac{\partial \eta}{\partial x} = -1 \times \frac{\partial \xi}{\partial x} - 1 \times \frac{\partial \eta}{\partial x}$$

$$\frac{\partial N_1^T}{\partial y} = \frac{\partial \hat{N}_1}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \hat{N}_1}{\partial \eta}\frac{\partial \eta}{\partial y} = -1 \times \frac{\partial \xi}{\partial y} - 1 \times \frac{\partial \eta}{\partial y}$$

Also: $\frac{\partial N_2^T}{\partial x} = 1 \times \frac{\partial \xi}{\partial x} + 0$; $\frac{\partial N_2^T}{\partial y} = 1 \times \frac{\partial \xi}{\partial y} + 0$

$$\frac{\partial N_3^T}{\partial x} = 0 + 1 \times \frac{\partial \eta}{\partial x} \; ; \; \frac{\partial N_3^T}{\partial y} = 0 + 1 \times \frac{\partial \eta}{\partial y}$$

* Note that $N_1^T + N_2^T + N_3^T = 1$ <u>everywhere</u>

and hence $\frac{\partial N_1^T}{\partial x} + \frac{\partial N_2^T}{\partial x} + \frac{\partial N_3^T}{\partial x} = 0$ & $\frac{\partial N_1^T}{\partial y} + \frac{\partial N_2^T}{\partial y} + \frac{\partial N_3^T}{\partial y} = 0$ <u>everywhere</u>

Stiffness matrix for element T, $[K^T]_{3\times 3}$ with:

$$K_{ij}^T = \int_T \left\{ \left(k_{11}\frac{\partial N_i^T}{\partial x} + k_{12}\frac{\partial N_i^T}{\partial y} \right)\frac{\partial N_j^T}{\partial x} + \left(k_{12}\frac{\partial N_i^T}{\partial x} + k_{22}\frac{\partial N_i^T}{\partial y} \right)\frac{\partial N_j^T}{\partial y} \right\} dA$$

But $dA = |d\vec{x} \times d\vec{y}| = |(x_{,\xi}\, d\xi\, \hat{\imath} + x_{,\eta}\, d\eta\, \hat{\jmath}) \times (y_{,\xi}\, d\xi\, \hat{\imath} + y_{,\eta}\, d\eta\, \hat{\jmath}$

$$= |J|\, d\xi\, d\eta = |J|\, d\hat{A}$$

Thus, we can perform this integral over the master element $\hat{T}$:

$$K_{ij}^T = \int_{\hat{T}} \left\{ \left(k_{11} N_{i,x}^T + k_{12} N_{i,y}^T \right) N_{j,x}^T + \left(k_{12} N_{i,x}^T + k_{22} N_{i,y}^T \right) N_{j,y}^T \right\} \cdot |J|\, d\hat{A}$$

This gives:

$$K_{ij}^{J} = \int_{\xi=0}^{1} \int_{\eta=0}^{1-\xi} \underbrace{\left[(k_{11} N_{i,x}^{J} + k_{12} N_{i,y}^{J}) N_{j,x}^{J} + (k_{12} N_{i,x}^{J} + k_{22} N_{i,y}^{J}) N_{j,y}^{J} \right] |J|}_{\substack{I_{ij} \\ = I_{ij} |J| * \frac{1}{2}}} d$$

* Note that here (i.e. for $p=1$) the integrand is constant (i.e. I_{ij} is a constant)
* This can be obtained explicitly!

Element Load Vector $\{F^J\}_{3\times1}$ is given as:

$$F_i^J = \int_J r\, N_i^J \, dA = \int_{\hat{J}} r(x(\xi,\eta), y(\xi,\eta))\, \hat{N}_i \, |J| \, d\xi \, d\eta$$

If $r(x,y)$ is a constant r_0, then we have:

$$F_i^J = r_0 \int_{\hat{J}} \hat{N}_i \, |J| \, d\xi \, d\eta \Rightarrow F_1^J = r_0 |J| \int_{\xi=0}^{1} \int_{\eta=0}^{1-\xi} (1-\xi-\eta) \, d\eta \, d\xi$$

$$= r_0 |J| \int_{\xi=0}^{1} \left(\eta - \xi\eta - \eta^2/2 \right) \Big|_0^{(1-\xi)} d\xi = r_0 |J| \int_{\xi=0}^{1} \left((1-\xi) - \xi(1-\xi) - \frac{(1-\xi)^2}{2} \right) d\xi$$

$$= r_0 |J| \int_{\xi=0}^{1} \frac{(1-\xi)^2}{2} \, d\xi = r_0 |J| \left[-\frac{(1-\xi)^3}{6} \right]_0^1 = \frac{r_0 |J|}{6} (1)$$

Similarly, $F_2^J = F_3^J = \dfrac{r_0 |J|}{6}$

Now it is a matter of <u>assembling</u> $[K^J]$, $\{F^J\}$ in global $[K]$, $\{F\}$.

<u>EXAMPLE</u>: For element $J = 7$ in mesh of L6-7, we get

Local i	Global No.
1	6
2	7
3	11

Let ieldofs be a simple array IELDOFS $\left(NELEM, \frac{(p+1)(p+2)}{2}\right)$

Then IELDOFS(7,1) = 6 ; IELDOFS(7,2) = 7 ; IELDOFS(7,3) = 11

```
i = 1, (p+1)(p+2)/2
    I = IELDOFS(7, i)

    F(I) = F(I) + F^J(i)
j = 1, (p+1)(p+2)/2
    J = IELDOFS(7, j)
    K(I,J) = K(I,J) + K^J(i,j)
enddo
enddo
```

generalize it for all elements!

* Take care of Neumann B.C. right here, i.e. check

for iside = 1, 3

if id-side(7, iside) = >0 means side has an element as a neighbor

<0 means side has $\Gamma_{-\text{id-side}(7,\text{iside})}$ as neighbor

Example:

id-side(11, 2) = −2 ⇒ $\underbrace{\Gamma_2'' \text{ is on } \Gamma_2}$

if id-side(7, iside) < 0 ⇒ edge on bndary.

Check identity of boundary segment Γ_ℓ → iden-bnd(ℓ)

1 (zero Dirichlet) | 2 (non-zero Dirichlet) | 3 (zero Neumann) | 4 (non-zero Neumann)

if iden-bnd(−id-side(7, iside)) = 4

– then find global identity of the two nodes on the side, i.e. GNODE-1 ; GNODE-2.

Then $F(GNODE\text{-}1) = F(GNODE\text{-}1) - \frac{q_0 \ell_s}{2}$

$F(GNODE\text{-}2) = F(GNODE\text{-}2) - \frac{q_0 \ell_s}{2}$

ℓ_s = length of side

- Dirichlet B.C. will be applied last, the way discussed in L6 - 8 (after assembly)

* You can see that the 2D problem is more complicated with respect to data-handling, assembly, B.C., etc.

* Next, we will recreate the elemental solution as:

$$T_h(x,y)\Big|_J = \sum_{i=1}^{3} d_i^J N_i^J(x,y) \equiv \sum_{i=1}^{3} d_i^J \hat{N}_i(\xi,\eta)$$

with $\boxed{d_i^J = d\,(\text{IELDOFS}(J,i))}$

$$\left.\begin{aligned} \frac{\partial T_h}{\partial x}\Big|_J &= \sum_{i=1}^{3} d_i^J \left(\frac{\partial \hat{N}_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \hat{N}_i}{\partial \eta}\frac{\partial \eta}{\partial x} \right) \\ \frac{\partial T_h}{\partial y}\Big|_J &= \sum_{i=1}^{3} d_i^J \left(\frac{\partial \hat{N}_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \hat{N}_i}{\partial \eta}\frac{\partial \eta}{\partial y} \right) \end{aligned}\right\}$$

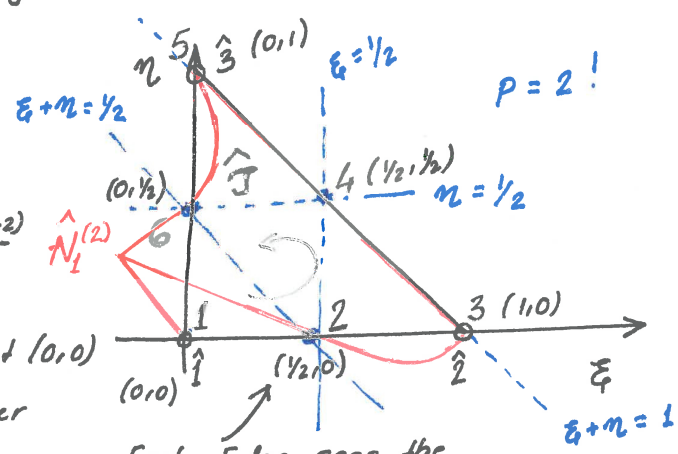
at any pt. (ξ,η) which will have a corresponding point (x,y)

Higher Order Approximation

In 2D we follow the Pascal's triangle for completeness :

$$
\begin{array}{ccccccc}
 & & & 1 & & & \text{Linear} \\
 & & x & & y & & + \\
 & x^2 & & xy & & y^2 & \text{quadratic} \\
 & & & & & & + \\
x^3 & & x^2y & & xy^2 & & y^3 \quad \text{Cubic} \\
 & & & & & &
\end{array}
$$

For completeness, a polynomial of order p has $ndof = \dfrac{(p+1)(p+2)}{2}$ terms. So we need $\underline{ndof}$ basis functions. We will carry this exercise in the master triangle $\hat{T}$.



* We will call shape functions by $\hat{N}_i^{(p)}$, $i = 1, 2, \ldots, \dfrac{(p+1)(p+2)}{2}$

<u>EXAMPLE</u> : • $\hat{N}_1^{(2)}(\xi,\eta) = 1$ at $(0,0)$

• $\hat{N}_1^{(2)}(\xi,\eta) = 0$ at all other master "nodes", i.e. at 2, 3, 4, 5, 6.

• $\hat{N}_1^{(2)}(\xi,\eta)$ is quadratic in (ξ,η)

• Each Edge sees the 1D shape functions, as the projection of the 2D shape functions on that edge

Note that in order to satisfy all these conditions $\hat{N}_1^{(2)}$ can be made as $\hat{N}_1^{(2)}(\xi,\eta) = \alpha_1 \left(\xi + \eta - \tfrac{1}{2}\right)(\xi + \eta - 1)$

product of equations of $\underline{\underline{2}}$ lines on which all the OTHER points lie.

- This is quadratic
- Vanishes at "nodes" 2, 3, 4, 5, 6
- For $\hat{N}_1^{(2)}(0,0) = 1$, $\alpha_1 = \underline{\underline{2}}$

Similarly, we can create:

$$\hat{N}_2^{(2)}(\xi,\eta) = \alpha_2(\xi-0)(\xi+\eta-1) \qquad \hat{N}_4^{(2)} = \alpha_4(\xi-0)(\eta-0)$$

$$\hat{N}_3^{(2)}(\xi,\eta) = \alpha_3(\xi-0)(\xi-1/2) \qquad \hat{N}_5^{(2)} = \alpha_5(\eta-0)(\eta-1/2)$$

$$\hat{N}_6^{(2)} = \alpha_6(\eta-0)(\xi+\eta-1)$$

With $\hat{N}_2^{(2)}(1/2,0) = 1 \Rightarrow \alpha_2 = -4$ $\qquad \hat{N}_5^{(2)}(0,1) = 1 \Rightarrow \alpha_5 = 2$

$\hat{N}_3^{(2)}(1,0) = 1 \Rightarrow \alpha_3 = 2$ $\qquad \hat{N}_6^{(2)}(0,1/2) = 1 \Rightarrow \alpha_6 = -4$

$\hat{N}_4^{(2)}(1/2,1/2) = 1 \Rightarrow \alpha_4 = 4$

* These are the "Lagrangian" shape functions for triangles.

I will partially repeat the process for $p=3$. You can complete the rest and generalize for any p.

<u>Step 1</u>: On each edge create p equal partitions

<u>Step 2</u>: Create a grid of $3(p+1)$ lines, passing through some of the points and parallel to one of the 3 edges of $\hat{\mathcal{T}}$. Points of intersection are the NODES

<u>Step 3</u>: Appropriately choose product of equations of p such lines to create a shape function that is 1 at its node and 0 at all other nodes.

$$\hat{N}_1^{(3)}(\xi,\eta) = \alpha_1(\xi+\eta-1/3)(\xi+\eta-2/3)(\xi+\eta-1)$$

$$\hat{N}_2^{(3)} = \alpha_2(\xi-0)(\xi+\eta-2/3)(\xi+\eta-1)$$

$$\hat{N}_3^{(3)} = \alpha_3(\xi-0)(\xi-1/3)(\xi+\eta-1)$$

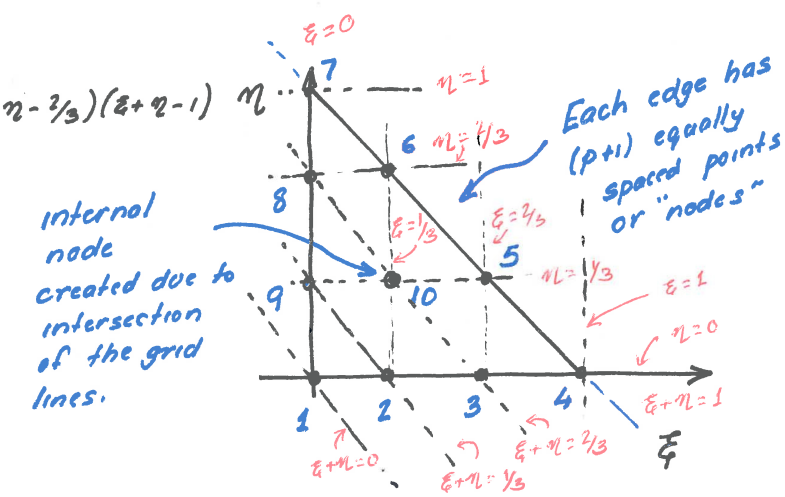
$$\hat{N}_4^{(3)} = \alpha_4(\xi-0)(\xi-1/3)(\xi-2/3)$$

$$\hat{N}_5^{(3)} = \alpha_5(\xi-0)(\xi-1/3)(\eta-0)$$

$$\hat{N}_6^{(3)} = \alpha_6(\xi-0)(\eta-1/3)(\eta-0)$$

$$\hat{N}_7^{(3)} = \alpha_7(\eta-0)(\eta-1/3)(\eta-2/3) \; ; \; \hat{N}_8^{(3)} = \alpha_8(\eta-0)(\eta-1/3)(\xi+\eta-1);$$

$$\hat{N}_9^{(3)} = \alpha_9(\xi+\eta-2/3)(\xi+\eta-1)(\eta-0) \; ; \; \hat{N}_{10}^{(3)} = \alpha_{10}(\xi-0)(\eta-0)(\xi+\eta-1)$$



→ So, we have the finite element solution in element J as:

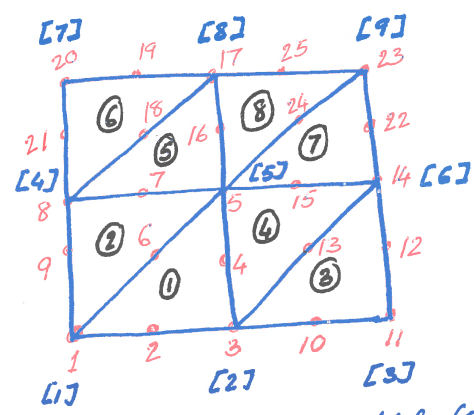
$$u_h\big|_J = \sum_{i=1}^{ndof} \alpha_i^J \hat{N}_i^{(p)}(\xi,\eta) \qquad \text{— (A)}$$

→ Mapping <u>remains affine</u> i.e. $X(\xi,\eta) = X_1^J(1-\xi-\eta) + X_2^J\xi + X_3^J\eta$

$$Y(\xi,\eta) = Y_1^J(1-\xi-\eta) + Y_2^J\xi + Y_3^J\eta \qquad \text{— (B)}$$

* We can also define "higher" order mapping. But note that mapping of domain IS NOT NECESSARILY of same order (or form as the finite element solution u_h^J given by (A).

GLOBAL PICTURE ?



- There are 8 elements; 9 vertices and with $p=2$ there are 25 solution nodes of nndof = 25
- Define eldofs(element, dof) (NELEM, NDOF)

  E.g. eldofs(5,1) = 8; eldofs(5,2) = 7; eldofs(5,3) = 5; eldofs(5,4) = 16; eldofs(5,5) = 17
  eldofs(5,6) = 18.

- $$u_h = \sum_{i=1}^{NNDOF} \alpha_i \underline{\varphi_i^{(p)}(x,y)} \Rightarrow u_h^J = u_h\big|_J = \sum_{i=1}^{ndof} \alpha_i^J \hat{N}_i^{(p)}(\xi,\eta)$$

  we do not explicitly create these but implicitly create them using element shape functions

  eldofs(*,*) will give the global names of corresponding $\alpha_k, \varphi_k^{(p)}$

A note on element calculations

$$[K^{\mathcal{T}}]_{ndof \times ndof} \Rightarrow \underbrace{B^{\mathcal{T}}(N_i^{\mathcal{T},(p)}, N_j^{\mathcal{T},(p)})}_{= K_{ij}^{\mathcal{T}}} \; ; \quad \{F^{\mathcal{T}}\}_{ndof \times 1} \Rightarrow F(N_i^{\mathcal{T}(p)}) = F_i^{\mathcal{T}}$$

With

$$K_{ij}^{\mathcal{T}} = \iint_{\hat{\mathcal{T}}} \Big[\Big(k_{11}\Big(\frac{\partial \hat{N}_i}{\partial \xi}\xi_{,x} + \hat{N}_{i,\eta}\eta_{,x}\Big) + k_{12}\Big(\hat{N}_{i,\xi}\xi_{,y} + \hat{N}_{i,\eta}\eta_{,y}\Big)\Big)\Big(\hat{N}_{j,\xi}\xi_{,x} + \hat{N}_{j,\eta}\eta_{,x}\Big) + \Big(k_{12}\big(\hat{N}_{i,\xi}\xi_{,x} + \hat{N}_{i,\eta}\eta_{,x}\big) + k_{22}\big(\hat{N}_{i,\xi}\xi_{,y} + \hat{N}_{i,\eta}\eta_{,y}\big)\Big)\Big(\hat{N}_{j,\xi}\xi_{,y} + \hat{N}_{j,\eta}\eta_{,y}\Big)\Big] |J| \, d\xi \, d\eta = \int_{\hat{\mathcal{T}}} I_{ij} \, |J| \, d\xi \, d\eta$$

$$= \sum_{\ell=1}^{NINT} I_{ij}(\hat{\xi}_\ell, \hat{\eta}_\ell) \, |J| \cdot w_\ell$$

NUMERICAL INTEGRATION

NINT - pt. integration rule with points $(\hat{\xi}_\ell, \hat{\eta}_\ell)$ and weight w_ℓ, $\ell = 1, 2, \cdots, NINT$

→ Integration points for an element can be obtained from any FEM book or through online resources.

* NINT depends on the order of integrand I_{ij}, which in this case is $2(p-1)$

$$F_i^{\mathcal{T}} = \int_{\hat{\mathcal{T}}} r(x(\xi,\eta), y(\xi,\eta)) \, \hat{N}_i \, |J| \, d\xi \, d\eta$$

+ edge contribution (if edge on Neumann boundary)

$$\boxed{\begin{array}{l} \ell_{\Gamma_i^{\mathcal{T}}} \rightarrow \text{length of actual edge} \\ \ell_{\hat{\Gamma}_i} \rightarrow \text{length of master triangle edge} \end{array}}$$

integral on corresponding master element edge

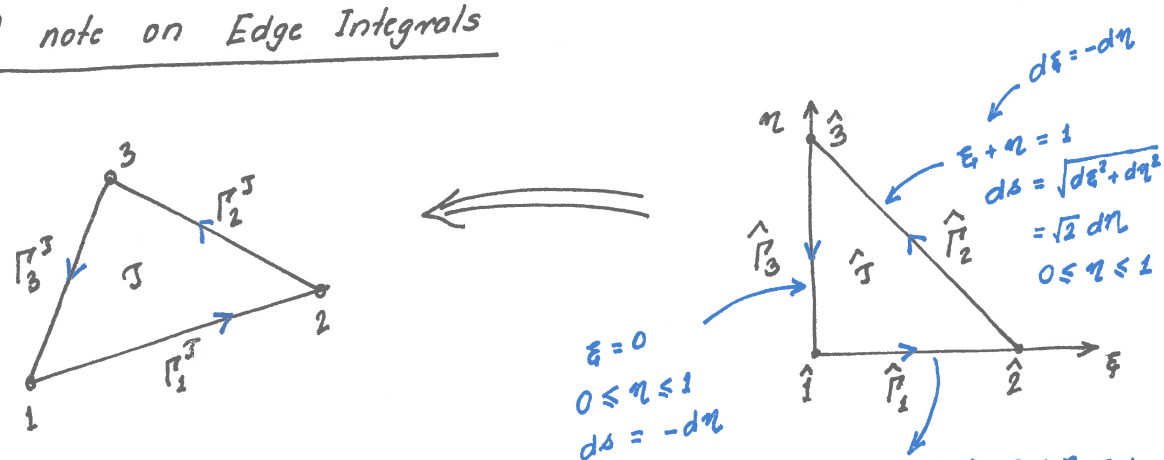
$$-\int_{\hat{\Gamma}_i} q_n(x(\xi,\eta), y(\xi,\eta))^* \, \hat{N}_i(\xi,\eta) \, \ell_{\Gamma_i^{\mathcal{T}}} / \ell_{\hat{\Gamma}_i} \, d\hat{s}$$

$$-\int_{\Gamma_i^{\mathcal{T}}} q_n\Big|_{\Gamma_i^{\mathcal{T}}} \cdot N_i^{\mathcal{T}} \, ds$$

integral on actual edge

WORK IT OUT! HOW WILL YOU DO EDGE NUMERICAL INTEGRATION?

A note on Edge Integrals



Edge 1

$$\int_{\Gamma_1^J} q_n N_i^{\Gamma_1^J} d\Delta \Rightarrow \int_{\Gamma_1^J} q_n N_1^J d\Delta \,, \quad \int_{\Gamma_1^J} q_n N_2^J d\Delta$$

2 non-zero shape functions on this edge (for higher p, it will be $(p+1)$ such shape functions)

$$\int_{\Gamma_1^J} q_n N_1^J d\Delta = \int_{\hat{\Gamma}_1} q_n(x(\xi,0), y(\xi,0))\, \hat{N}_1(\xi,0) \cdot \frac{l_{\Gamma_1^J}}{1}\, d\xi$$

$$= \int_{\xi=0}^{1} q_n(x(\xi,0), y(\xi,0))\, \hat{N}_1(\xi,0)\, l_{\Gamma_1^J}\, d\xi$$

Similarly $\displaystyle\int_{\Gamma_1^J} q_n N_2^J d\Delta = \int_{\hat{\Gamma}_1} q_n(x(\xi,0), y(\xi,0))\, \hat{N}_2(\xi,0) \cdot l_{\Gamma_1^J}\, d\xi$

Edge 2 : We evaluate

$$\int_{\Gamma_2^J} q_n N_2^J d\Delta \,, \quad \int_{\Gamma_2^J} q_n N_3^J d\Delta \Rightarrow \int_{\hat{\Gamma}_2} q_n(x(1-\eta,\eta), y(1-\eta,\eta)\, \hat{N}_2(1-\eta,\eta)\, \frac{l_{\Gamma_2^J}}{\sqrt{2}}\, d$$

$$= \int_{\eta=0}^{1} q_n(x(1-\eta,\eta), y(1-\eta,\eta))\, \hat{N}_2(1-\eta,\eta)\, \frac{l_{\Gamma_2^J}}{\cancel{\sqrt{2}}} \cdot \cancel{\sqrt{2}}\, d\eta$$

and $\displaystyle\int_{\eta=0}^{1} q_n(x(1-\eta,\eta), y(1-\eta,\eta))\, \hat{N}_3(1-\eta,\eta)\, l_{\Gamma_2^J}\, d\eta$

Edge 3 : We evaluate $\displaystyle\int_{\Gamma_3^J} q_n N_3^J d\Delta \,, \quad \int_{\Gamma_3^J} q_n N_1^J d\Delta$

$$= \int_{\eta=1}^{\eta=0} q_n(x(0,\eta), y(0,\eta))\, \hat{N}_3(0,\eta)\, \ell_{\Gamma_3^I}(-d\eta) = \int_{\eta=0}^{1} q_n(x(0,\eta), y(0,\eta))\, \hat{N}_3(0,\eta) \cdot \ell_{\Gamma_3^I} \cdot d\eta$$

and $$\int_{\eta=0}^{1} q_n(x(0,\eta), y(0,\eta))\, \hat{N}_1(0,\eta)\, \ell_{\Gamma_3^I}\, d\eta$$

* For p-order approximation, identify the shape functions that are non-zero on the given edge Γ_ℓ^I, and proceed with integration as shown.

* Can this be done numerically? Yes, by mapping the interval $\underbrace{(0,1)}_{\xi \text{ or } \eta}$ to the interval $\underbrace{(-1,1)}_{\zeta}$ as $\xi(\zeta) = 0 \times \frac{(1-\zeta)}{2} + 1\,\frac{(1+\zeta)}{2}$

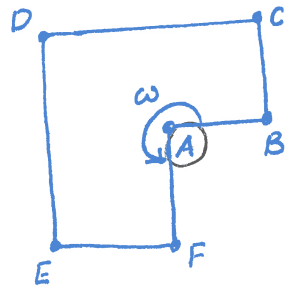
and doing the NUMERICAL integration over $-1 \le \zeta \le 1$.

Note that $\zeta = 2\xi - 1$ $\Rightarrow$ If an integration point is ζ_ℓ^{int}

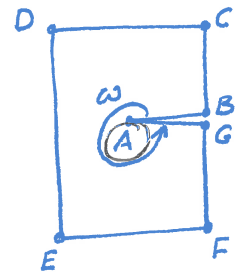
$\Rightarrow$ $\xi_\ell^{int} = \frac{(1+\zeta_\ell^{int})}{2}$; if weight is w_ℓ^{int} $\Rightarrow$ MAPPED wt. is $\tilde{w}_\ell^{int} = \frac{w_\ell^{int}}{2}$

as $$\int_{\xi=0}^{1} g(\xi)\, d\xi = \int_{\zeta=-1}^{1} g(\xi(\zeta)) \cdot \frac{1}{2}\, d\zeta = \sum_{k=1}^{NINT} g(\xi(\zeta_k^{int})) \cdot \frac{w_k^{int}}{2}$$
$$= \sum_{k=1}^{NINT} g(\xi_k^{int}) \cdot \tilde{w}_k^{int}$$

DOMAINS WITH RE-ENTRANT CORNERS, CRACKS



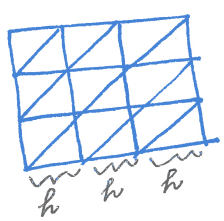
L-shaped domain



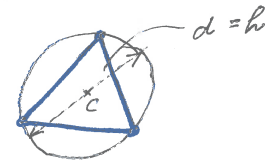
Domain with edge crack.

- Vertex A corresponds to a re-entrant corner, as corner angle $\omega > \pi$
- What is special about these vertices, or domains?

Let us imagine that we use a UNIFORM mesh (i.e. mesh with elements of same size) with fixed order of approximation p.



For a triangle h could be length of a chosen edge or the diameter of circumscribed circle



* We choose h as shown.

- As mesh is refined, i.e. h reduces, we expect to get (from the 1D study)

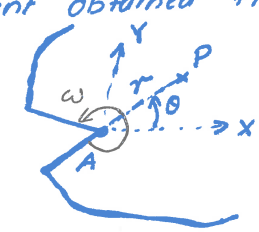
$$B(T-T_h, T-T_h) = \|e\|_\Omega^2 \leq C h^{2p} \quad \text{or} \quad \|e\|_\Omega \leq C h^p$$

This is true almost always!

- For re-entrant corners, however, for uniform meshes, we get

$$\|e\|_\Omega \leq C h^{\min(p,\beta)}, \quad \text{with } \beta < 1, \text{ i.e.}$$

$$\|e\|_\Omega \leq C h^\beta$$ ~ β is the exponent obtained from the representation of the exact solution $T(x,y)$

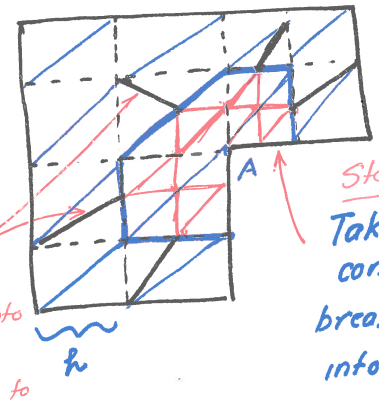


$$T(r,\theta) \approx A r^\beta \sin(\beta\theta)$$

How to improve the rate of convergence?

This will give

$$\|e\| \approx c h^p$$



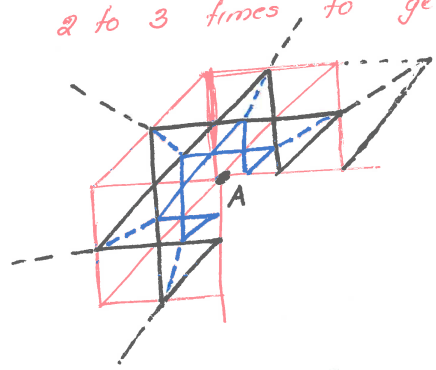
Step 2

division of these elements into 2 pieces is required to have continuity of displacement

Step 1

Take all elements connected to A; break each element into 4 elements as

Step 3: Repeat steps 1 & 2 about 2 to 3 times, to get



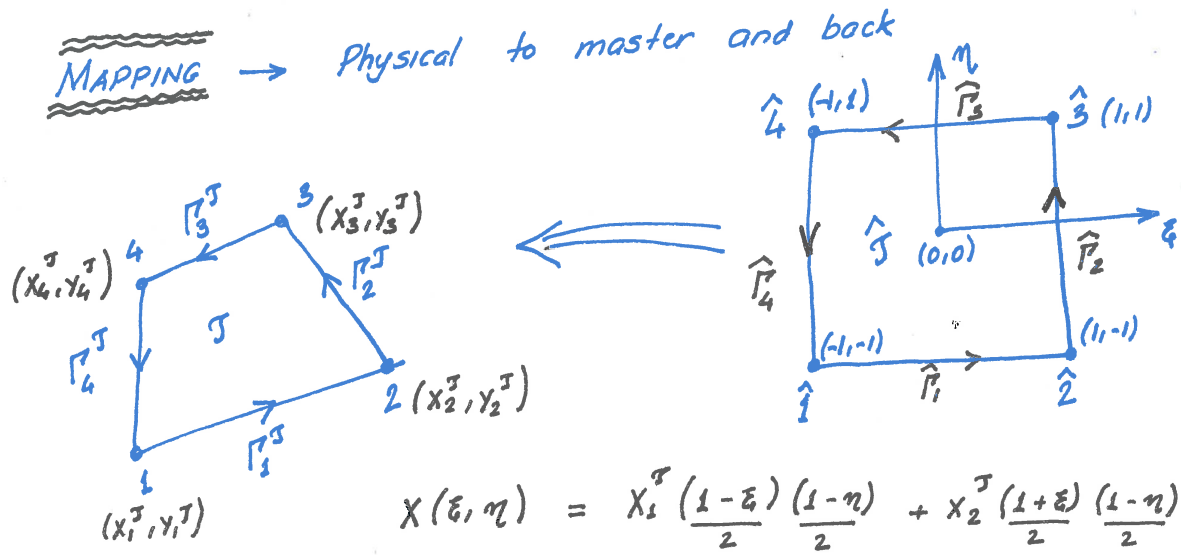
– This is called <u>geometric refinement</u> as the mesh at A has sizes h, $\frac{1}{2}h$, $(\frac{1}{2})^2 h$, ... as we refine as shown.

QUADRILATERAL ELEMENTS

— Its a matter of preference and need, as to whether I solve a problem with a mesh of triangles or quadrilaterals.

Let us look at a simple construction — bi-p elements or tensor product elements

MAPPING → Physical to master and back



$$X(\xi,\eta) = X_1^J \frac{(1-\xi)}{2}\frac{(1-\eta)}{2} + X_2^J \frac{(1+\xi)}{2}\frac{(1-\eta)}{2} + X_3^J \frac{(1+\xi)}{2}\frac{(1+\eta)}{2} + X_4^J \frac{(1-\xi)}{2}\frac{(1+\eta)}{2}$$

linear in ξ — linear in η

[BILINEAR MAP]

will have terms with $1, \xi, \eta, \underline{\underline{\xi\eta}}$

↑ bilinear (more than linear!)

METRICS OF TRANSFORMATION:

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \underbrace{\begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}}_{[J]} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \begin{bmatrix} \left[(x_2^J - x_1^J)\frac{(1-\eta)}{4} + (x_3^J - x_4^J)\frac{(1+\eta)}{4}\right] & \left[(x_4^J - x_1^J)\frac{(1-\xi)}{4} + (x_3^J - x_2^J)\frac{(1+\xi)}{4}\right] \\ \downarrow \text{ same thing for } y & \downarrow \end{bmatrix}$$

* Note that $[J]$ is <u>not</u> a constant matrix

$$\begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{,\eta} & -x_{,\eta} \\ -y_{,\xi} & x_{,\xi} \end{bmatrix} \begin{Bmatrix} dx \\ dy \end{Bmatrix}$$

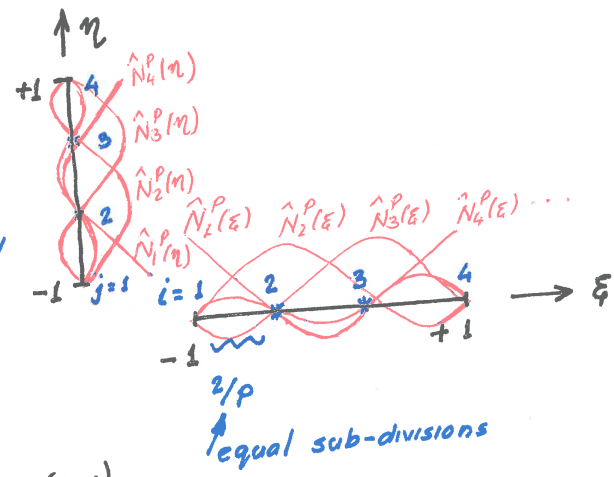
Note: As $|J(\xi,\eta)|$ varies with (ξ,η) we have to be watchful that $|J| > 0$ strictly everywhere in an element.

<u>APPROXIMATION</u> : Tensor-product family of shape functions

Order p in $\eta \Rightarrow (p+1)$ equally spaced points in $-1 \leq \eta \leq 1$, denoted by $j = 1, 2, \cdots, (p+1)$

1-D Lagrangian shape functions in η as:

$\hat{N}_j^p(\eta)$, $j = 1, 2, \cdots, (p+1)$



order p in ξ $\Rightarrow (p+1)$ points, equally spaced, in $-1 \leq \xi \leq 1$

Points denoted as $i = 1, 2, \cdots, p+1$

1-D Lagrangian shape functions in ξ as:

$\hat{N}_i^p(\xi)$, $i = 1, 2, \cdots, (p+1)$

<u>2D shape functions</u>:

$$\hat{N}_k^p(\xi,\eta) = \hat{N}_i^p(\xi)\,\hat{N}_j^p(\eta) \quad ; \; i = 1, 2, \cdots (p+1); \; j = 1, 2, \cdots, (p+1)$$

where $k = \big((j-1)(p+1) + i\big)$; $k = 1, 2, \cdots, NDOF$

$$NDOF = (p+1)^2$$

$$T_h = \sum_{k=1}^{NDOF} \alpha_k \, \hat{N}_k^p(\xi,\eta)$$

~ All the others things, e.g. element stiffness matrix, load vector calculation/formulation follow the development earlier

Some salient points:

- Numerical integration is relatively easy here as the integration points and weights can be taken as a TENSOR product of 1D-points (and weights) in ξ and η directions.

  * If $\frac{\partial N_i^p}{\partial x} \cdot \frac{\partial N_j^p}{\partial x}$; $\frac{\partial N_i^p}{\partial y} \frac{\partial N_j^p}{\partial x}$; ... are the terms of the integrand in master element calculation, then integrand is of order $(p-1)$ in ξ and $(p-1)$ in η

  For load vector, integrand will be of order $(p+1)$ in both ξ and η if r is constant

$$\Rightarrow \frac{p+1+1}{2} \leq n_\xi^{int}, \; n_\eta^{int} \approx \frac{p}{2} + 1$$

$\therefore$ for $p = 1 \Rightarrow$ 2-pt. rule in each ξ, η.

for $p = 2 \Rightarrow$ 2-pt. " " " " "

The integration points (q-point rule in each of ξ, η):

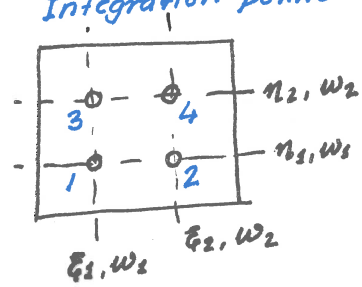
$$(\xi_k, \eta_k) = (\xi_i, \eta_j)$$

$$w_k = w_i \cdot w_j$$

where $k = (j-1)q + i$; $i, j = 1, 2, \ldots, q$

* $k = 1, 2, \ldots, q \times q$

Integration points



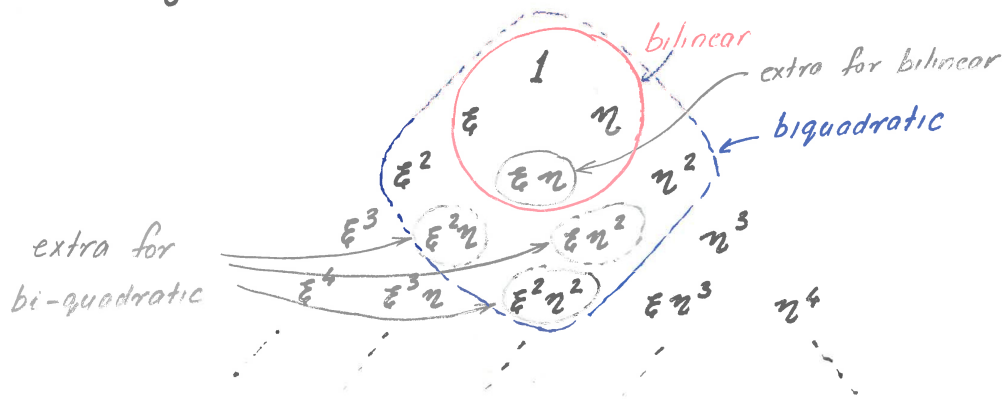
(2x2 point set)

- What about completeness of tensor product elements?

  The bi-p element covers MORE than p, i.e. it has

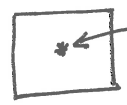
$$r = (p+1)^2 - \frac{(p+1)(p+2)}{2} \quad \text{extra terms} \quad \left[\begin{array}{l} r = 1 \text{ for } p = 1 \; ; \\ r = 3 \text{ for } p = 2 \\ r = 6 \text{ for } p = 3, \ldots \end{array} \right]$$

- It is easy to show that



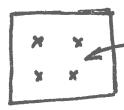
- The derivatives, i.e. $\frac{\partial u_h}{\partial x}$, $\frac{\partial u_h}{\partial y}$ are SUPERCONVERGENT at the Gauss integration points.

E.g. bilinear approximation



1 x 1 pt. superconvergence

biquadratic $\Rightarrow$



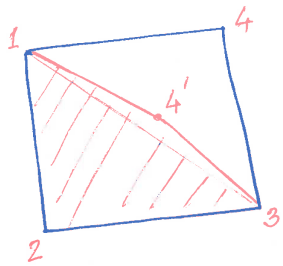
2 x 2 superconvergence points

* In triangles we do not have such points.

→ IS EVERYTHING GOOD IN QUADRILATERALS ?

Not so. They can suffer from degeneracy.

E.g.



- Quadrilateral 1-2-3-4' is heavily distorted, hence $|J| \approx 0$ at certain points.
- This can destroy the super convergence property.

On a special request !! (from a student)

SERENDIPITY ELEMENTS

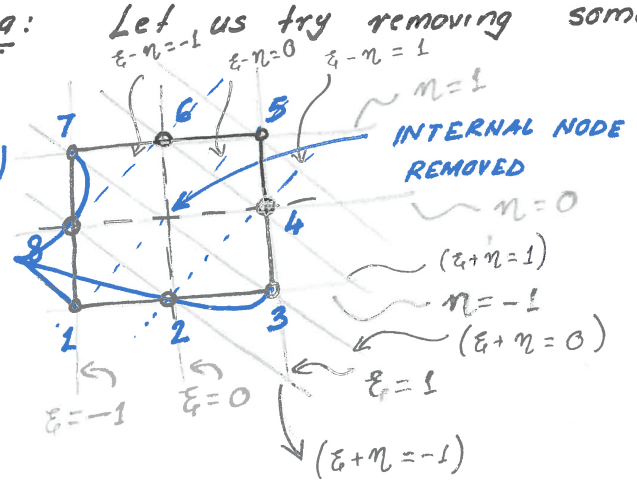
~ In L6-27 we noted that tensor product elements give us $(p+1)^2 - \frac{(p+1)(p+2)}{2} = (p+1)\left[(p+1) - \frac{(p+2)}{2}\right]$

$= \frac{(p+1)(p)}{2}$ more basis functions than required for completeness.

– This means that we have to use this many more degrees of freedom per element ⇒ OVERALL INCREASED COST OF COMPUTATION !! Can we reduce this ? YES

Idea: Let us try removing some "solution nodes".

$p = 2$ (Quadratic)



$\hat{N}_1^{s,(2)}(\xi,\eta) \rightarrow$ has to be zero at nodes 2 to 8 ⇒ is zero on lines $(\eta-1)=0$; $(\xi-1)=0$; $(\xi+\eta+1=0)$

$$\Rightarrow \hat{N}_1^{s,(2)}(\xi,\eta) = C_1(\xi-1)(\eta-1)(\xi+\eta+1)$$

$$\hat{N}_1^{s,(2)}(-1,-1) = 1 \Rightarrow C_1 = -1/4$$

$$\hat{N}_2^{s,(2)} = C_2(\xi+1)(\eta-1)(\xi-1), \quad \hat{N}_2^{s,(2)}(0,-1) = 1 \Rightarrow C_2 = \frac{1}{2}$$

$$\hat{N}_3^{s,(2)} = C_3(\xi-\eta-1)(\eta-1)(\xi+1)$$

$$\hat{N}_4 = C_4(\eta-1)(\eta+1)(\xi+1)$$

$$\hat{N}_5 = C_5(\xi+\eta-1)(\xi+1)(\eta+1)$$

$$\hat{N}_6 = C_6(1+\xi)(1+\eta)(1-\xi)$$

$$\hat{N}_7 = C_7(\xi-\eta)(\eta+1)(\xi-1)$$

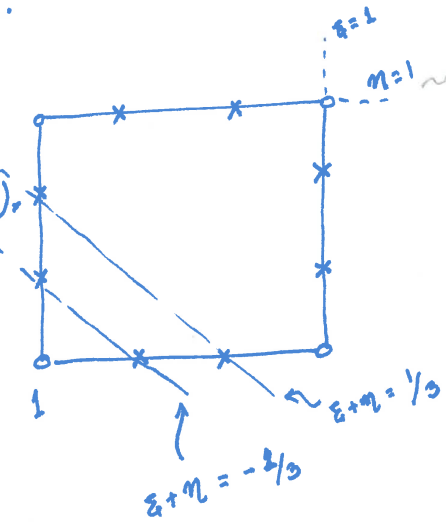
$$\hat{N}_8 = C_8(\eta+1)(\eta-1)(\xi-1)$$

* Product of equations of 3 lines !

What about cubic serendipity ?

$$\hat{N}_1^{s,(3)}(\xi,\eta) = C_1(\xi+\eta+1/3)(\xi+\eta-1/3)(\xi-1)(\eta-1)$$

PRODUCT OF EQUATIONS OF 4 LINES !



12 nodes > 10 required for completeness

Continue with this logic to get the other shape functions !

How many shape functions do we have this way ?

$$4\times(p-1)+4 = 4p$$

How many more than minimum required ?

$$4p - \frac{(p+1)(p+2)}{2} = \frac{8p - p^2 - 3p - 2}{2} = \frac{5p - p^2 - 2}{2}$$

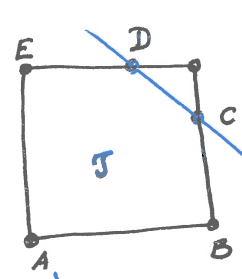
Is this ever ≤ 0 ?

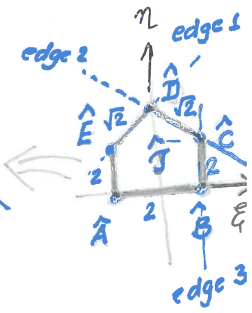
$$-p^2+5p-2 = 0 \Rightarrow$$

$$p^2-5p+2 = 0 \Rightarrow p = \frac{5}{2} \pm \sqrt{\frac{25-8}{2}} = \frac{5}{2} \pm \sqrt{\frac{17}{2}} \approx 5 \quad (<5)$$

$\therefore$ For $p \leq 4$, this is OK, but for $p \geq 5$ you cannot really construct serendipity elements !!

Q: Can we create a pentagon (i.e. a cut square) ?





Pentagon ABCDE :

Hint: $\hat{N}_A(\xi,\eta) = C_A \times (\text{eqn. of edge 1}) \times (\text{eqn. of edge 2}) \times (\text{eqn. of edge 3})$

Continue this way for higher order SERENDIPITY !

Lecture 23 (Contd.)

J

Higher Order Approximation $\rightarrow$ Pascal's Triangle in 2D

$$
\begin{array}{lll}
1 & & p=0 \\
x \quad y & & p=1 \\
x^2 \quad xy \quad y^2 & & p=2 \\
x^3 \quad x^2y \quad xy^2 \quad y^3 & & p=3 \\
\vdots & &
\end{array}
$$

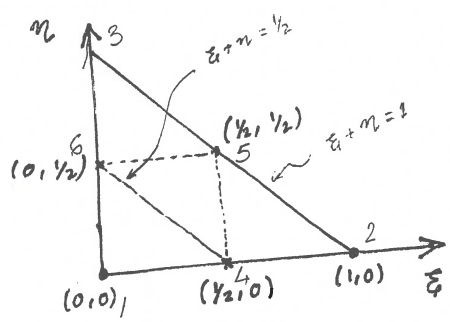
Total of $(p+1)(p+2)/2$ monomials in a polynomial of order p. $(3, 6, 10, 15, 21, \ldots)$

$\uparrow$ $p=1$, $\uparrow$ $=2$, $\uparrow$ $=3$, $\uparrow$ $=4$, $\uparrow$ $=5$

$\rightarrow$ For completeness these elements have to be covered for a given p.

$\rightarrow$ Master Triangle, Lagrangian interpolation

$p = 2$



$\hat{N}_1(\xi,\eta) = C_1(\xi+\eta-1/2)(2\xi+\eta-1)$

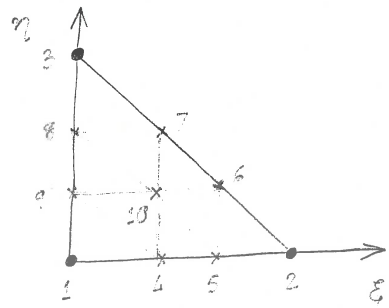
$\hat{N}_2(\xi,\eta) = C_2(\xi)(2\xi-1)$

$\hat{N}_3(\xi,\eta) = C_3\,\eta(2\eta-1/2)$

$\hat{N}_4(\xi,\eta) = C_4\,\xi(2\eta-1)$

$\hat{N}_5(\xi,\eta) = C_5(\xi)(\eta)$

$\hat{N}_6(\xi,\eta) = C_6(\eta)(\xi+\eta-1)$



$p = 3$ (10 points)

$$\hat{N}_1(\xi,\eta) = C_1\left(\xi+\eta-\tfrac{1}{3}\right)\left(\xi+\eta-\tfrac{2}{3}\right)(\xi+\eta-1)$$

$\vdots$

$$\hat{N}_{10}(\xi,\eta) = C_{10}\,\xi\,\eta\,(\xi+\eta-1)$$

→ $$u_{FE}\big|_{J}(x,y) = \hat{u}_{FE}\big|_{J}(\xi,\eta) = \sum_{i=1}^{(p+1)(p+2)/2} u_i^J\,\hat{N}_i(\xi,\eta)$$

→ Remember that the mapping is still linear, i.e. $x = \sum_{i=1}^{3} x_i \hat{M}_i$

$$y = \sum_{i=1}^{3} y_i \hat{M}_i$$

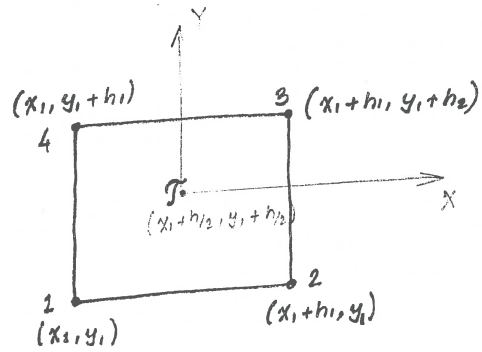
where $\hat{M}_i(\xi,\eta)$ are the LINEAR Lagrangian functions.

→ Note that on each edge of element J, these shape functions reduce to one-dimensional shape functions of order p (i.e. each edge has $(p+1)$ functions).

OTHER ELEMENTS ?

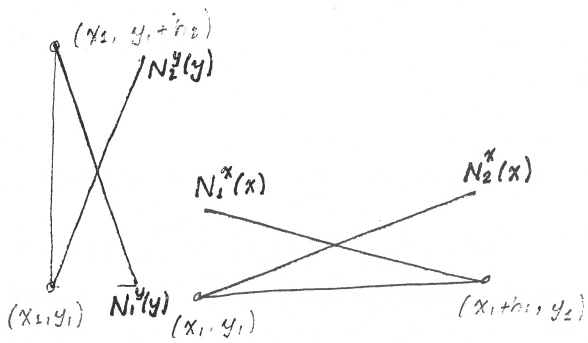
LECTURE 24

→ The quadrilateral / rectangular element



→ 2 perpendicular edges.

→ Why not take the 1D-shape functions along each edge and take tensor product to get the 2D-shape functions



$$N_1(x,y) = N_1^x N_1^y$$

$$N_2(x,y) = N_2^x N_1^y$$

$$N_3(x,y) = N_2^x N_2^y$$

$$N_4(x,y) = N_1^x N_2^y$$

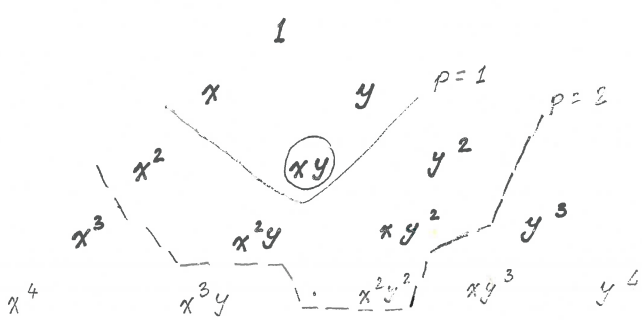
→ Tensor product family of shape functions

$$N_1^x(x) = \frac{(x_1+h_1-x)}{h_1} \qquad N_2^x(x) = \frac{(-x_1+x)}{h_1}$$

$$N_1^y(y) = \frac{(y_1+h_2-y)}{h_2} \qquad N_2^y(y) = \frac{(y-y_1)}{h_2}$$

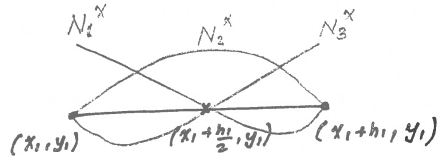
$$\Rightarrow N_1(x,y) = \frac{(x_1+h_1-x)(y_1+h_2-y)}{h_1 h_2} = \frac{\overbrace{(x_1+h_1)(y_1+h_2)}^{\text{constant}} + \overbrace{xy}^{\text{bilinear!}} - \underbrace{x(y_1+h_2)}_{\text{linear in } x} - \underbrace{y(x_1+h_1)}_{\text{linear in } y}}{h_1 h_2}$$

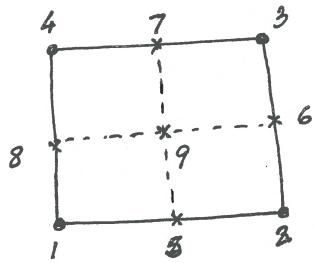
Bilinear

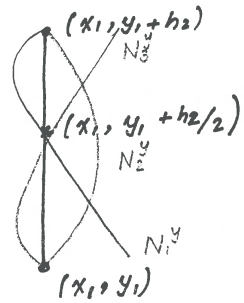


→ These shape functions account for all the linear terms PLUS the bilinear term from the quadratic

Biquadratic ($p=2$)







$N_1(x,y) = N_1^x N_1^y$

$N_2(x,y) = N_3^x N_1^y$

$N_3(x,y) = N_3^x N_3^y$

$N_4(x,y) = N_1^x N_3^y$

$N_5(x,y) = N_2^x N_1^y$

$N_6(x,y) = N_3^x N_2^y$

$N_7(x,y) = N_2^x N_3^y$

$N_8(x,y) = N_1^x N_2^y$

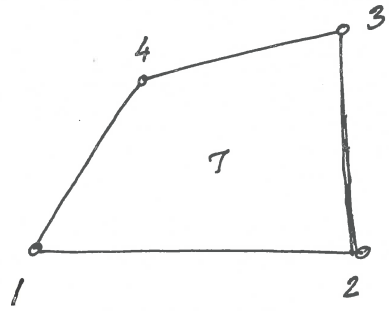
$N_9(x,y) = N_2^x N_2^y$

→ also contain x^2y xy^2 x^2y^2 terms in addition to the quadratic.

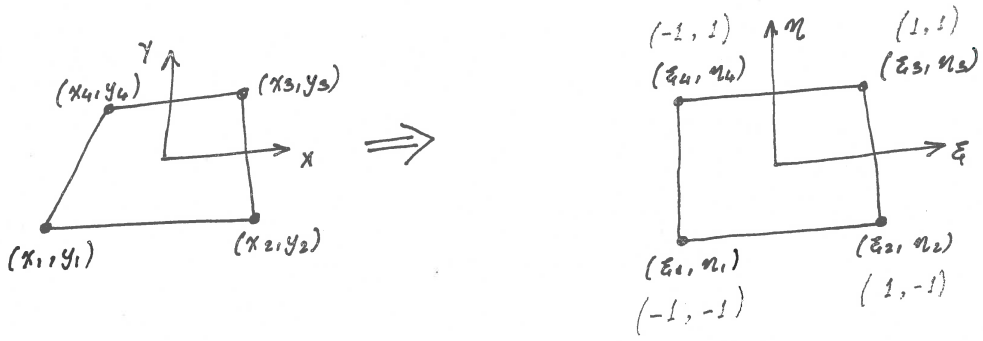
→ Satisfies completeness but does a little more → price to be paid is that number of unknowns in a FE representation increases

→ Generic Quadrilateral

→ Now how do we define the shape functions?



→ Map to a master element (<u>master square</u>)



Linear Map ⇒

$$x(\xi, \eta) = x_1 M_1(\xi, \eta) + x_2 M_2(\xi, \eta) + x_3 M_3(\xi, \eta) + x_4 M_4(\xi, \eta)$$

$$y(\xi, \eta) = \sum_{i=1}^{4} y_i M_i(\xi, \eta)$$

$$M_1(\xi, \eta) = \frac{1}{4}(1-\xi)(1-\eta) \qquad M_2(\xi, \eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$M_3(\xi, \eta) = \frac{1}{4}(1+\xi)(1+\eta) \qquad M_4(\xi, \eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

bilinear functions defined on the master square.

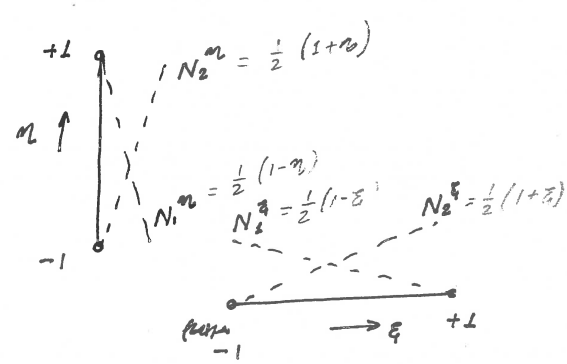
→ Now define the tensor product family of shape functions over this master element in the usual way, i.e.

$$N_1(\xi, \eta) = N_1^{\xi} N_1^{\eta}$$

$$N_2(\xi, \eta) = N_2^{\xi} N_1^{\eta}$$

$$N_3(\xi, \eta) = N_2^{\xi} N_2^{\eta}$$

$$N_4(\xi, \eta) = N_1^{\xi} N_2^{\eta}$$



→ Given the 1D - Lagrangian shape functions, the 2D tensor product shape functions can be generated for any p.

MAPPING, DERIVATIVE COMPUTATION

$$x = x_1 \frac{1}{4}(1-\xi)(1-\eta) + x_2 \frac{1}{4}(1+\xi)(1-\eta) + x_3 \frac{1}{4}(1+\xi)(1+\eta) + x_4 \frac{1}{4}(1-\xi)(1+\eta)$$

$$\frac{\partial x}{\partial \xi} = -\frac{x_1(1-\eta)}{4} + \frac{x_2(1-\eta)}{4} + \frac{x_3(1+\eta)}{4} - \frac{x_4(1+\eta)}{4}$$

$$= \frac{\overbrace{(x_2 - x_1 + x_3 - x_4)} + \eta\overbrace{(x_1 - x_2 + x_3 - x_4)}}{4} \Leftarrow \text{linear in } \eta$$

$$= A_1 + A_2 \eta$$

$$\frac{\partial x}{\partial \eta} = -x_1 \frac{(1-\xi)}{4} - x_2 \frac{(1+\xi)}{4} + x_3 \frac{(1+\xi)}{4} + x_4 \frac{(1-\xi)}{4}$$

$$= \frac{(x_3 + x_4 - x_1 - x_2) + \xi(x_1 - x_2 + x_3 - x_4)}{4} \Leftarrow \text{linear in } \xi$$

$$= B_1 + B_2 \xi$$

Similarly, $\frac{\partial y}{\partial \xi}$ and $\frac{\partial y}{\partial \eta}$ can be found by substituting x above by y

$$\Rightarrow \begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{bmatrix} \partial x/\partial \xi & \partial x/\partial \eta \\ \partial y/\partial \xi & \partial y/\partial \eta \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix}$$

$$\Rightarrow |J| = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} = (A_1 + A_2 \eta)(D_1 + D_2 \xi) - (B_1 + B_2 \xi)(C_1 + C_2 \eta)$$

$$= A_1 D_1 + A_2 D_1 \eta + A_1 D_2 \xi + A_2 D_2 \xi \eta - B_1 C_1 - B_1 C_2 \eta - B_2 C_1 \xi - B_2 C_2 \xi \eta$$

* Note that for a rectangular element, $x_1 = x_4$, $x_2 = x_3$
$y_1 = y_2$, $y_3 = y_4$

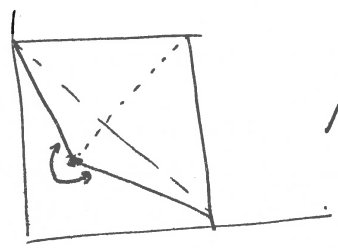
$\Rightarrow$ $A_2, B_2, B_1 \equiv 0$; $C_1, C_2, D_2 = 0$ $[A_1, D_1 \neq 0]$

$\Rightarrow$ $|J| = A_1 D_1 - B_1 C_1 = \frac{h_1}{2} \times \frac{h_2}{2} = \frac{h_1 h_2}{4}$ ← physical element area

4 ← master element area

When does it fail ?

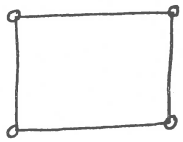
When $|J| \leq 0$

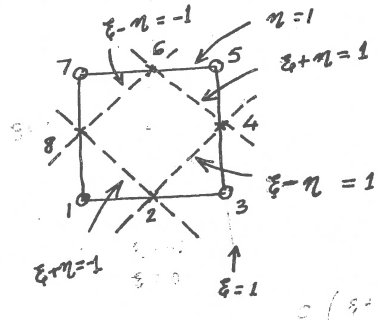


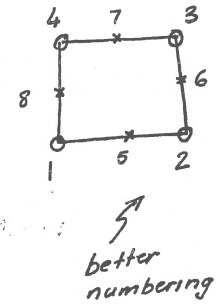
Non Convex Element

Lecture 25

Serendipity family of shape functions :



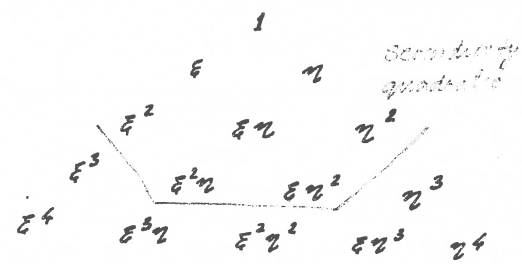




$$N_1 = c(\xi+\eta+1)(\eta-1)(\xi-1)$$

$$= c(\xi+\eta+1)(\xi\eta-\xi-\eta+1) = c_1(\xi^2\eta-\xi^2-\xi\eta+\xi+\xi\eta^2-\xi\eta-\eta^2+\eta+\xi\eta-\xi-\eta+1)$$

does not have $\underline{\underline{\xi^2\eta^2}}$



$$N_1(-1,-1) = 1 = c_1(-1)(-2)(-2) \Rightarrow c = -\frac{1}{4}$$

$$N_2(\xi,\eta) = c_2(\xi+1)(\xi-1)(\eta-1)$$

$$N_2(0,-1) = 1 = c_2(1)(-1)(-2) \Rightarrow c_2 = 1/2$$

$$N_3(\xi,\eta) = c_3(\xi-\eta-1)(\xi+1)(\eta-1)$$

$$N_3(1,-1) = 1 = c_3(1)(2)(-2) \Rightarrow c_3 = -1/4$$

$$N_4(\xi,\eta) = c_4(\eta-1)(\xi+1)(\eta+1)$$

$$N_4(1,0) = 1 = c_4(-1)(2)(1) \Rightarrow c_4 = -\frac{1}{2}$$

$$N_5(\xi,\eta) = c_5(\xi+\eta-1)(\xi+1)(\eta+1)$$

$$N_5(1,1) = 1 = c_5(1)(2)(2) \Rightarrow c_5 = 1/4$$

$$N_6(\xi,\eta) = c_6(\xi+1)(\eta+1)(\xi-1)$$

$$N_6(0,1) = 1 = c_6(1)(2)(-1) \Rightarrow c_6 = -1/2$$

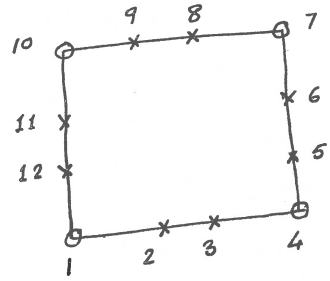
$$N_7(\xi,\eta) = c_7(\xi-\eta+1)(\eta+1)(\xi-1)$$

$$N_7(-1,1) = 1 = c_7(-1)(2)(-2) \Rightarrow c_7 = 1/4$$

$$N_8(\xi,\eta) = c_8(\eta+1)(\xi-1)(\eta-1)$$

$$N_8(-1,0) = 1 = c_8(1)(-2)(-1) \Rightarrow c_8 = 1/2$$

→ Again, on each edge the shape functions should be equivalent to 1D shape functions.

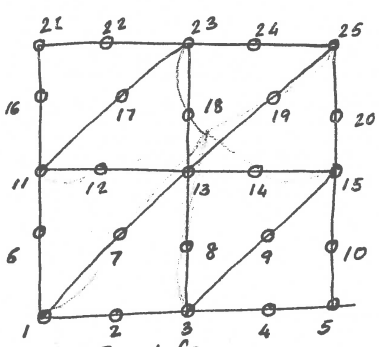


⇝ same job as the quadratics

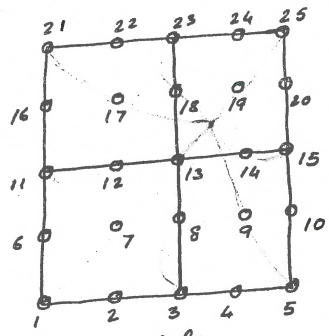
⇝ 12 functions

→ Saving in terms of number of unknowns in the problem

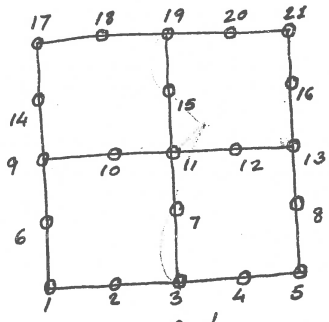
<u>The final form</u>



25 dofs



25 dofs



21 dofs!

$$u_{FE}(x,y) = \sum_{i=1}^{NDOFS} u_i \, \phi_i(x,y)$$

$$u_{FE}\Big|_{T} = \sum_{i=1}^{NDOFEL} u_i^T N_i^T(x,y)$$

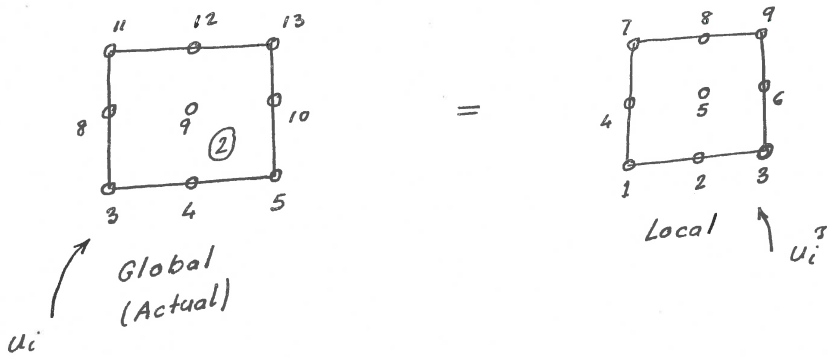
$$\Leftrightarrow \sum_{i=1}^{NDOFEL} u_i^T \hat{N}_i(\xi,\eta)$$

NDOFEL = $(p+1)(p+2)/2$ → triangles

$(p+1)^2$ ⟶ tensor product quads.

$4p$ → serendipity quad.

$u_i^{\mathcal{T}} \rightarrow u_i$



Local	Global
1	3
2	4
3	5
4	8
5	9
6	10
7	11
8	12
9	13

~~global to local enumeration~~

~~→ iddofs ( )~~

→ <u>Element Calculations</u>

<u>Stiffness</u>

$$K_{ij} = \int_{\mathcal{T}} \left(\frac{\partial N_i^{\mathcal{T}}}{\partial x} \frac{\partial N_j^{\mathcal{T}}}{\partial x} + \frac{\partial N_i^{\mathcal{T}}}{\partial y} \frac{\partial N_j^{\mathcal{T}}}{\partial y} \right) dA \equiv \int_{\hat{\mathcal{T}}} \left(\frac{\partial \hat{N}_i}{\partial x} \frac{\partial \hat{N}_j}{\partial x} + \frac{\partial \hat{N}_i}{\partial y} \frac{\partial \hat{N}_j}{\partial y} \right) |J| \, d\hat{A}$$

$$= \int_{\xi=-1}^{1} \int_{\eta=-1}^{1} \left(\frac{\partial \hat{N}_i}{\partial x} \frac{\partial \hat{N}_j}{\partial x} + \frac{\partial \hat{N}_i}{\partial y} \frac{\partial \hat{N}_j}{\partial y} \right) |J| \, d\xi \, d\eta$$

$$\frac{\partial \hat{N}_i}{\partial x} = \frac{\partial \hat{N}_i}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial \hat{N}_i}{\partial \eta} \frac{\partial \eta}{\partial x} \quad ; \quad \frac{\partial \hat{N}_i}{\partial y} = \frac{\partial \hat{N}_i}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial \hat{N}_i}{\partial \eta} \frac{\partial \eta}{\partial y}$$

$$\begin{Bmatrix} dx \\ dy \end{Bmatrix} = \begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix} \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} \Rightarrow \begin{Bmatrix} d\xi \\ d\eta \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} y_{,\eta} & -x_{,\eta} \\ -y_{,\xi} & x_{,\xi} \end{bmatrix} \begin{Bmatrix} dx \\ dy \end{Bmatrix}$$

$$\Rightarrow \frac{\partial \xi}{\partial x} = \frac{y_{,\eta}}{|J|} \; ; \; \frac{\partial \xi}{\partial y} = \frac{-x_{,\eta}}{|J|} \; ; \; \frac{\partial \eta}{\partial x} = \frac{-y_{,\xi}}{|J|} \; ; \; \frac{\partial \eta}{\partial y} = \frac{x_{,\xi}}{|J|}$$

The desired expressions had been derived earlier

< Remember: For both families the mapping is the <u>same</u> >

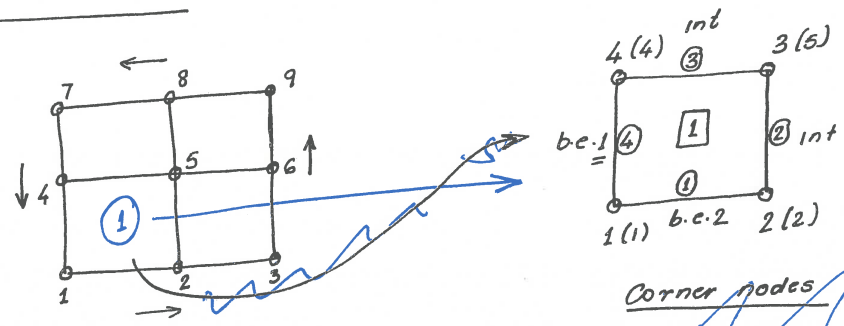
→ All the above metrics are now functions of ξ, η

→ $|J| > 0$ implies these metrics do not blow up!

Body Force Terms

$$F_i^I = \int_I r\, N_i^I\, dA = \int_{\hat{I}} \hat{r}\, \hat{N}_i\, |J|\, d\xi\, d\eta$$

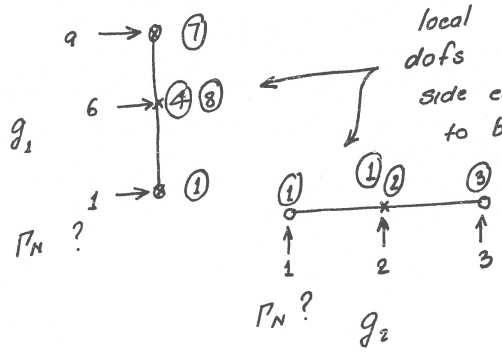
Boundary Traction



~~Corner nodes 1, 2, 3, 4~~

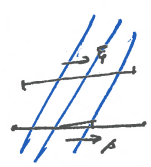
~~information on nodes ( )~~

~~array~~



local dofs on the side edges have to be identified

$$F_i^I = F_i^I + \int_{\partial I_j} g\, N_i^I\, ds =$$



$$\int_{\bar{\xi}=-1}^{1} \hat{g}(\bar{\xi})\, \hat{\bar{N}}_i\, |\bar{J}|\, d\bar{\xi}$$

Take edge 1 → $\eta = -1$

$\bar{N}_i(\xi) = \hat{N}_i(\xi, -1)$ ← projection of shape fn. on edge

or $\int_{\xi, \eta = -1}^{1}$

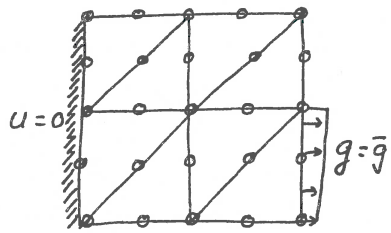
$$ds = \sqrt{dx^2 + dy^2}$$

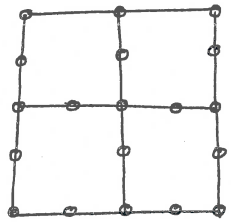
$$= \underbrace{\sqrt{\left(\frac{\partial x}{\partial \xi}\Big|_{\eta=-1}\right)^2 + \left(\frac{\partial y}{\partial \xi}\Big|_{\eta=-1}\right)^2}}_{|j(\xi,-1)|}\, d\xi$$

$$\int_{\xi=-1}^{1} g(x(\xi,-1), y(\xi,-1))\, \overset{\bar{N}_i(\xi)}{\downarrow}\, |j(\xi,-1)|\, d\xi$$

~~$\hat{g}(\xi)\, \hat{\bar{N}}_i(\xi)\, |j|\, d\xi$~~

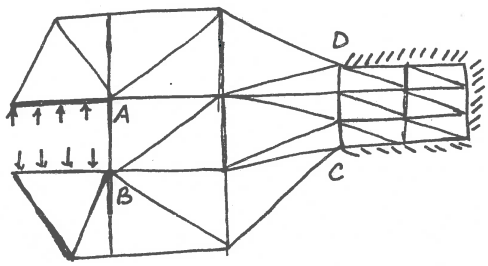
Lecture 26





→ Solution is smooth

→ FEM solution not exact at nodes



→ Domain has reentrant corners (angles more than 180°)

$$e = u - u_{FE} \quad \Rightarrow \quad B(u,u) = \int_A \left[\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial x} + \left(a_{12}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial y}\right] dA$$

$\Rightarrow B(e,e) = B(u-u_{FE},\, u-u_{FE}) = B(u,u) - B(u_{FE}, u_{FE})$ when only homogeneous Dirichlet B.C. on Γ_D.

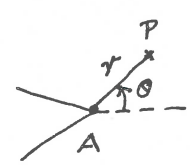
$$\|u\|_{E(\Omega)} = \sqrt{B(u,u)}, \quad \|e\|_{E(\Omega)} = \sqrt{B(e,e)} \leq Ch^{\mu}$$

$$\mu = \min(\alpha, p)$$

where $\alpha = \min(\alpha_A, \alpha_B, \alpha_C, \alpha_D)$ and

$$u(r,\theta) \approx \underbrace{\sum A_i r^{\alpha} f(\theta,\alpha)}$$

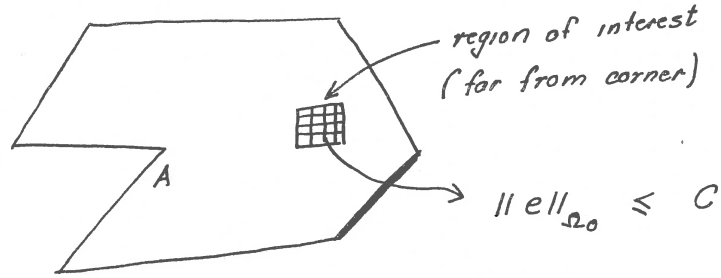
Asymptotic representation in the neighborhood of the reentrant corner



Polar representation

→ Convergence achieved by refining mesh uniformly with p fixed

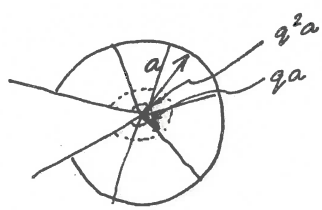
Accuracy hampered by presence of the corners.



$$\|e\|_{\Omega_0} \leq C h^{\mu_1}$$

$$\boxed{\mu_1 = \min(p, 2\alpha)}$$

When $p < \alpha/2$, then the effect of the corner far-away dominates. Thus, we cannot just refine mesh in vicinity of region of interest only. Mesh has to be suitably refined near the corner to capture the singular behavior better, → graded refinement at corner (geometric)



Then, with proper geometric refinement, we get

$$\|e\|_{E(\Omega)} \leq C N^{-p/2}$$

and

$$\|e\|_{E(\Omega_0)} \leq C_1 h^p$$

p - version → i.e. raise p only → exponential convergence for smooth solutions.

→ Same problem for domains with corners

Post - Processing :

Lecture 28

Plane Stress and Plane Strain problems (2D - elasticity)

Let us take plane stress problem for isotropic elasticity :

$$\varepsilon_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right)$$

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right)$$

$$\gamma_{xz} = \frac{1}{G}\sigma_{xz} \quad ; \quad \gamma_{yz} = \frac{1}{G}\sigma_{yz} \; ; \quad \gamma_{xy} = \frac{1}{G}\sigma_{xy}$$

$$G = \frac{1}{2(1+\nu)}$$

Plane Stress $\Rightarrow$ $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$

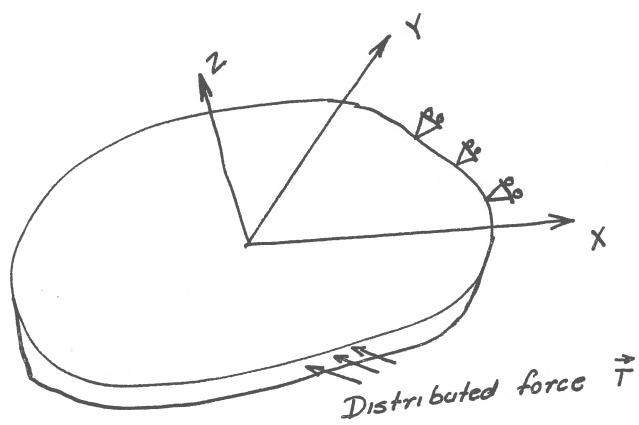
$$\Rightarrow \quad \sigma_{xx} = \frac{E}{(1-\nu^2)}\left(\varepsilon_{xx} + \nu\,\varepsilon_{yy}\right)$$

$$\sigma_{yy} = \frac{E}{(1-\nu^2)}\left(\varepsilon_{yy} + \nu\,\varepsilon_{xx}\right)$$

$$\sigma_{xy} = G\,\gamma_{xy}$$

→ Thin domain

→ In-plane load

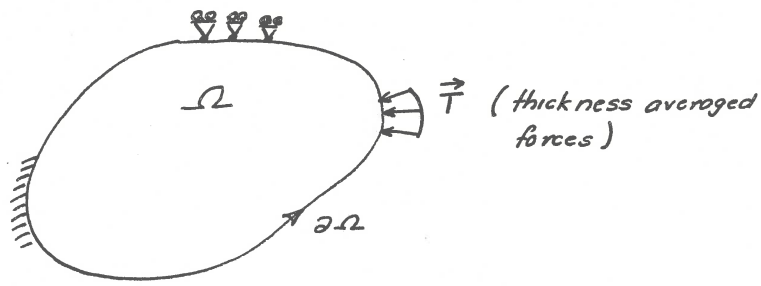


State of stress is uniform in z, and function of x, y only

$\Rightarrow$ State of strain also function of (x, y)

$\Rightarrow$ Displacement also only function of (x, y)

$\Rightarrow$ $u(x,y)$, $v(x,y)$, enough to characterize state of stress.



<u>Plane Strain</u> $\rightarrow$ $h \gg$ cross-sectional dimensions

$\Rightarrow$ ε_{zz}, γ_{xz}, $\gamma_{yz} \approx 0$

$$\Rightarrow \quad \sigma_{xx} = \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \varepsilon_{xx} + \frac{\nu E}{(1+\nu)(1-2\nu)} \varepsilon_{yy}$$

$$\sigma_{yy} = \frac{\nu E}{(1-2\nu)(1+\nu)} \varepsilon_{xx} + \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \varepsilon_{yy}$$

$$\sigma_{xy} = G \gamma_{xy}$$

We are interested only in $\underbrace{u(x,y),\ v(x,y)}_{u_1(x,y),\ u_2(x,y)}$ $(w = 0)$

$\boxed{\text{2 unknown variables}}$

Equilibrium Equations :

$$r_1 \rightarrow \sigma_{xx,x} + \sigma_{xy,y} + f_x = 0 \quad \text{in } \Omega$$

$$r_2 \rightarrow \sigma_{xy,x} + \sigma_{yy,y} + f_y = 0 \quad \text{in } \Omega$$

Weak Formulation : Virtual or test vector (w_1, w_2)

$$\int_\Omega \vec{r} \cdot \vec{w} \, d\Omega = 0 \quad \Rightarrow \quad \int_\Omega (r_1 w_1 + r_2 w_2) \, d\Omega = 0$$

$$\Rightarrow \int_\Omega \left\{ (\underline{\underline{\sigma_{xx,x}}} + \underline{\underline{\sigma_{xy,y}}} + f_x) w_1 + (\underline{\underline{\sigma_{xy,x}}} + \underline{\underline{\sigma_{yy,y}}} + f_y) w_2 \right\} d\Omega = 0$$

2nd derivatives of u & v involved

So weaken the requirement by integration by parts once,

$$\Rightarrow \quad \sigma_{xx,x} w_1 = (\sigma_{xx} w_1)_{,x} - \sigma_{xx} w_{1,x}$$

$$\sigma_{xy,y} w_1 = (\sigma_{xy} w_1)_{,y} - \sigma_{xy} w_{1,y}$$

$$\sigma_{xy,x} w_2 = (\sigma_{xy} w_2)_{,x} - \sigma_{xy} w_{2,x}$$

$$\sigma_{yy,y} w_2 = (\sigma_{yy} w_2)_{,y} - \sigma_{yy} w_{2,y}$$

gives :

$$\int_\Omega \left(\sigma_{xx} w_{1,x} + \sigma_{xy} (w_{2,x} + w_{1,y}) + \sigma_{yy} w_{2,y} \right) d\Omega =$$

$$\int_\Omega (f_x w_1 + f_y w_2) \, d\Omega + \int_\Omega \left\{ (\sigma_{xx} w_1 + \sigma_{xy} w_2)_{,x} + (\sigma_{xy} w_1 + \sigma_{yy} w_2)_{,y} \right\}$$

Gauss Divergence Theorem gives:

$$\int_\Omega (f)_{,x}\, d\Omega = \int_{\partial\Omega} f\, n_1\, ds \quad ; \quad \int_\Omega (f)_{,y}\, d\Omega = \int_{\partial\Omega} f\, n_2\, ds$$

$\Rightarrow$ we get

$$\int_\Omega (\sigma_{xx} w_1 + \sigma_{xy} w_2)_{,x}\, d\Omega = \int_{\partial\Omega} (\sigma_{xx} w_1 + \sigma_{xy} w_2)\, n_1\, d\Omega$$

$$\int_\Omega (\sigma_{xy} w_1 + \sigma_{yy} w_2)_{,y}\, d\Omega = \int_{\partial\Omega} (\sigma_{xy} w_1 + \sigma_{yy} w_2)\, n_2\, d\Omega$$

$\Rightarrow$ The integral expression becomes:

virtual work done by existing stress

$$\int_\Omega \{ \sigma_{xx}(\vec{u})\, \varepsilon_{xx}(\vec{w}) + \sigma_{yy}(\vec{u})\, \varepsilon_{yy}(\vec{w}) + \sigma_{xy}(\vec{u})\, \gamma_{xy}(\vec{w}) \}\, d\Omega$$

$$= \int_\Omega (f_x w_1 + f_y w_2)\, d\Omega + \int_{\partial\Omega} \underbrace{(\sigma_{xx} n_1 + \sigma_{xy} n_2)}_{T_1} w_1\, ds + \overbrace{(\sigma_{xy} n_1 + \sigma_{yy} n_2)}^{T_2} w_2\, ds$$

Virtual Work done by external forces

Equivalent Form : $\underbrace{B(\vec{u}, \vec{w})}_{\text{bilinear form}} = \underbrace{F(\vec{w})}_{\text{linear functional}}$

$$B(\vec{u} + \alpha \vec{v}, \vec{w}) = B(\vec{u}, \vec{w}) + \alpha B(\vec{v}, \vec{w})$$

$$F(\alpha \vec{w}) = \alpha F(\vec{w})$$

$$B(\vec{u}, \vec{w}) = B(\vec{w}, \vec{u}) \longleftarrow \text{symmetric}$$

Lecture 29

Take the following choice of $\vec{w} = \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \begin{Bmatrix} \phi_i \\ 0 \end{Bmatrix}, \begin{Bmatrix} 0 \\ \phi_i \end{Bmatrix}$

$$\vec{u}_{FE} = \begin{Bmatrix} u_{1,FE} \\ u_{2,FE} \end{Bmatrix} = \begin{bmatrix} \phi_1 & 0 & \phi_2 & 0 & \phi_3 & 0 & \cdots \\ 0 & \phi_1 & 0 & \phi_2 & 0 & \phi_3 & \cdots \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \vdots \end{Bmatrix}$$

$$\Rightarrow [\varepsilon_{FE}] = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u_1/\partial x \\ \partial u_2/\partial y \\ \partial u_1/\partial y + \partial u_2/\partial x \end{Bmatrix} = \underbrace{\begin{bmatrix} \phi_{1,x} & 0 & \phi_{2,x} & 0 & \cdots \\ 0 & \phi_{1,y} & 0 & \phi_{2,y} & \cdots \\ \phi_{1,y} & \phi_{1,x} & \phi_{2,y} & \phi_{2,x} & \cdots \end{bmatrix}}_{[B]} \{\alpha\}$$

$$\Rightarrow \{\sigma_{FE}\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = [C][B]\{\alpha\}$$

For the particular choice of $\{w\}$, we get equations $(2i-1)$ and $(2i)$ as:

Eqn. $(2i-1)$ $\rightarrow$ $\{w\} = \begin{Bmatrix} \phi_i \\ 0 \end{Bmatrix} \Rightarrow \varepsilon_{xx}(\vec{w}) = \phi_{i,x}, \quad \varepsilon_{yy}(\vec{w}) = 0,$
$\gamma_{xy}(\vec{w}) = \phi_{i,y}$

$$\Rightarrow \int_\Omega \begin{Bmatrix} \phi_{i,x} & 0 & \phi_{i,y} \end{Bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} dV = \int_\Omega f_x \phi_i \, dA + \int_{\partial\Omega} T_1 \phi_i \, ds$$

Eqn. $(2i)$ $\rightarrow$ $\{w\} = \begin{Bmatrix} 0 \\ \phi_i \end{Bmatrix} \Rightarrow \varepsilon_{xx}(\vec{w}) = 0, \quad \varepsilon_{yy}(\vec{w}) = \phi_{i,y},$
$\gamma_{xy}(\vec{w}) = \phi_{i,x}$

$\Rightarrow$ we get

$$\int_{\Omega} \{ 0 \quad \phi_{i,y} \quad \phi_{i,x} \} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} dA = \int_{\Omega} f_y \phi_i \, dA + \int_{\partial\Omega} T_2 \phi_i \, ds$$

Both eqns. ~~having~~ together are:

this corresponds to columns $2i-1, 2i$ of $[B]$ matrix

$$\left(\int_{\Omega} \begin{bmatrix} \phi_{i,x} & 0 & \phi_{i,y} \\ 0 & \phi_{i,y} & \phi_{i,x} \end{bmatrix} [C][B] \, dA \right) \{\alpha\} = \begin{Bmatrix} \int_{\Omega} f_x \phi_i \, dA + \int_{\partial\Omega} T_1 \phi_i \, ds \\ \int_{\Omega} f_y \phi_i \, dA + \int_{\partial\Omega} T_2 \phi_i \, ds \end{Bmatrix}$$

Assemble all the rows for all the i's to get

$[K]$

$$\left(\int_{\Omega} [B]^T [C] [B] \, dA \right) \{\alpha\} = \{F\}$$

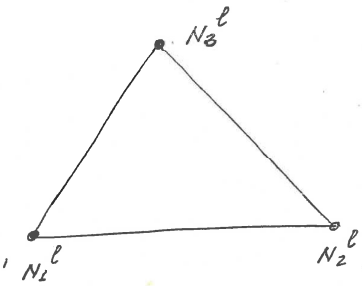
popularly known as $[B]^T[D][B]$ matrix

$$[K] = \sum_{\ell=1}^{NEL} [K^{\ell}]$$

$[K^{\ell}]$ corresponds to the basis functions that are non-zero in the element ℓ, written in terms of the element shape functions as:

At the element level :

N_i^{ℓ} , $i = 1, 2, \ldots, (p+1)(p+2)/2$



$$\Rightarrow \quad \vec{u}_{FE}\Big|_{\ell} = \begin{bmatrix} N_1^{\ell} & 0 & N_2^{\ell} & 0 & \cdots \\ 0 & N_1^{\ell} & 0 & N_2^{\ell} & \cdots \end{bmatrix}_{2\times\bar{N}} \{\alpha^{\ell}\}_{\bar{N}\times 1}$$

$$\bar{N} = \frac{(p+1)(p+2)}{2} \times 2$$ ← 2 d.o.f's per "node"

$$\Rightarrow \quad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \underbrace{\begin{bmatrix} N_{1,x}^{\ell} & 0 & N_{2,x}^{\ell} & 0 & \cdots \\ 0 & N_{1,y}^{\ell} & 0 & N_{2,y}^{\ell} & \cdots \\ N_{1,y}^{\ell} & N_{1,x}^{\ell} & N_{2,y}^{\ell} & N_{2,x}^{\ell} & \cdots \end{bmatrix}}_{[B^{\ell}]} \{\alpha^{\ell}\}$$

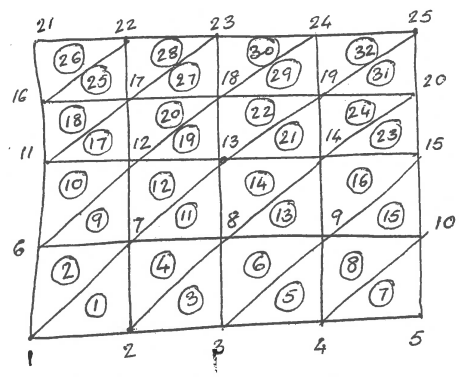
$\Rightarrow$ By analogy,

$$[K^{\ell}] = \int_{\ell} [B^{\ell}]^T [c] [B^{\ell}] \, dA$$

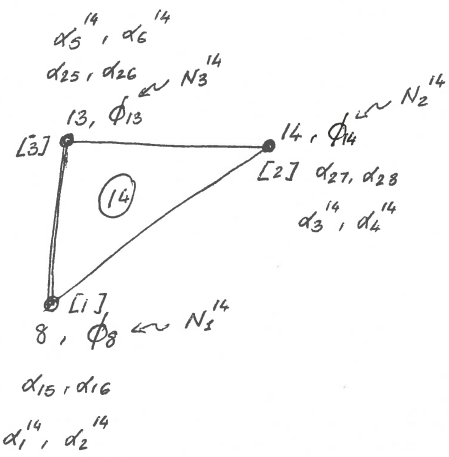
$$\{F^{\ell}\} = \begin{Bmatrix} \int_{\ell} f_x N_1^{\ell} \, dA + \int_{\partial\ell\cap\partial\Omega} T_1 N_1^{\ell} \, ds \\ \int_{\ell} f_y N_1^{\ell} \, dA + \int_{\partial\ell\cap\partial\Omega} T_2 N_1^{\ell} \, ds \\ \vdots \end{Bmatrix}$$

ASSEMBLY

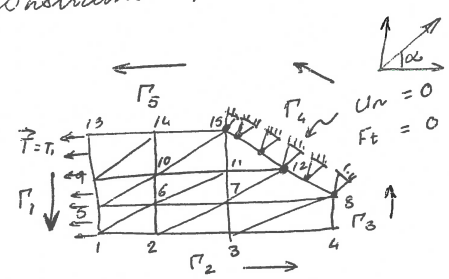
$N_i^{\ell} \rightarrow \phi_k$ correspondence has to be set



Let $p = 1$ (piecewise linear)

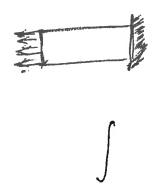


How to impose rotated constraints ?



$$\Rightarrow \int_A \{\varepsilon(\vec{w})\}^T \{\sigma(\vec{u})\}\, dA$$

$$= \int_A \vec{f} \cdot \vec{w}\, dA + \int_{\Gamma_1} (-T_1 w_1)\, dA + 0$$



Now $u_n = 0$ on $\Gamma_4 \Rightarrow \sigma_n = 0$ on Γ_4

~~Alrodeve~~ Along Γ_4, $\vec{u}_{FE}\big|_{\Gamma_4} = \begin{Bmatrix} \alpha_{15}\phi_8 + \alpha_{23}\phi_{12} + \alpha_{29}\phi_{15} \\ \alpha_{16}\phi_8 + \alpha_{24}\phi_{12} + \alpha_{30}\phi_{15} \end{Bmatrix}$

$$\Rightarrow u_n\big|_{\Gamma_4} = (\alpha_{15}\phi_8 + \alpha_{23}\phi_{12} + \alpha_{29}\phi_{15})\cos\alpha + (\alpha_{16}\phi_8 + \alpha_{24}\phi_{12} + \alpha_{30}\phi_{15})\sin\alpha = 0$$

$$\Rightarrow \underline{\alpha_{15}\cos\alpha + \alpha_{16}\sin\alpha} = 0 \;;\quad \alpha_{23}\cos\alpha + \alpha_{24}\sin\alpha = 0 \;;\quad \alpha_{29}\cos\alpha + \alpha_{30}\sin\alpha = 0$$

Time Dependent Problems

Parabolic Problems: $-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + c_1 \dot{u} = f(x,t)$

with (a) $u(x,t)$ or $EA\frac{\partial u}{\partial x}$ given at the nodes $\leftarrow$ B.C.

(b) $u(x,0)$ given $\leftarrow$ I.C.

Hyperbolic: $-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) + c_1 \ddot{u} = f(x,t)$

with I.C. $\longrightarrow$ $\underbrace{u(x,0),\ \dot{u}(x,0)}_{\text{given}}$

Semi discrete formulation:

$\rightarrow$ Use standard principle of virtual work to get to weak form.

$\rightarrow$ The weak form has the temporal parts present.

$\rightarrow$ Handle temporal part by finite difference in time.

$$\int_0^L \left(a\frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + c_1 \dot{u} w + c_2 \ddot{u} w \right) dx = \int_0^L f w \, dx + P(t)\, w(t)\Big|_L$$

$$u_{FE}(x,t) = \sum_{i=1}^{N} \alpha_i(t)\, \phi_i(x) \quad \leftarrow \text{separation of variables}$$

$$\Rightarrow \boxed{[K]\{\alpha\} + [C]\{\dot{\alpha}\} + [M]\{\ddot{\alpha}\} = \{F\}}$$

where

$$K_{ij} = \int_0^L a\, \phi_i \phi_j \, dx \quad ; \quad C_{ij} = \int_0^L c_2 \phi_i \phi_j \quad ;$$

$$M_{ij} = \int_0^L c_2 \phi_i \phi_j \, dx$$

<u>Parabolic Problem</u> : $[M] = [0]$

$$\Rightarrow \quad [C]\{\dot{\alpha}\} + [K]\{\alpha\} = \{F\}$$

with $\{\alpha(0)\}$ obtained from initial conditions.

i.e. $u(x,0)$ is given. Use interpolation to get $\alpha_i(0)$

α – family of approximations, i.e.

given solution at time t, we need solution at $t+\Delta t$

$$\Rightarrow \quad (1-\alpha)\{\dot{a}\}\big|_t + \alpha\{\dot{a}\}\big|_{t+\Delta t} = \frac{\{a\}_{t+\Delta t} - \{a\}_t}{\Delta t} \quad \text{for } 0 \le \alpha \le 1$$

~~now, $\{a\}_{t+\Delta t} = \{a\}_t + \{\Delta a\}_t$~~

Now, the equations are valid at both times t & $t+\Delta t$

$$\Rightarrow \quad [M]\{\dot{a}\}_t + [K]_t\{a\}_t = \{F\}_t$$

$$[M]\{\dot{a}\}_{t+\Delta t} + [K]_{t+\Delta t}\{a\}_{t+\Delta t} = \{F\}_{t+\Delta t}$$

$$[M]\left(\{\dot{a}\}_{t+\Delta t} - \{\dot{a}\}_t\right) + [K]\left(\underbrace{\{a\}_{t+\Delta t} - \{a\}_t}\right) = \left(\{F\}_{t+\Delta t} - \{F\}_t\right)$$

$$\{\dot{a}\} = \frac{\{a_{t+\Delta t}\} - \{a\}_t}{\Delta t\ \alpha} - \frac{(1-\alpha)}{\alpha}\{\dot{a}_t\}$$

This gives

$$[\bar{K}]\{a\}_{t+\Delta t} = [\bar{\bar{K}}]\{a\}_t + \{\bar{F}\}_{t+\Delta t}$$

$\{\bar{F}\}_{t+\Delta t} \rightarrow \Delta t\left(\alpha\{F\}_{t+\Delta t} + (1-\alpha)\{F\}_t\right)$

$[\bar{\bar{K}}] \rightarrow [M] - (1-\alpha)\Delta t\,[K]_t$

$[\bar{K}] \rightarrow [M] + \alpha\,\Delta t\,[K]_{t+\Delta t}$

$$\alpha = \begin{cases} 0 \rightarrow & \text{forward difference (or Euler) scheme (conditionally stable), order of accuracy} = O(\Delta t) \\ \frac{1}{2} \rightarrow & \text{Crank - Nicolson scheme (stable), } O(\Delta t^2) \\ \frac{2}{3} \rightarrow & \text{Galerkin scheme (stable), } O(\Delta t^2) \\ 1 \rightarrow & \text{backward difference scheme (stable), } O(\Delta t) \end{cases}$$

* For $\alpha = 0$, $[\bar{K}] = [M]$ and for diagonal $[M]$ the solution is EXPLICIT

* $[M]$ from FE is consistent mass matrix $\rightarrow$ not diagonal.

STABILITY & Accuracy

Unstable $\rightarrow$ truncation + round-off errors grow with time (unbounded)

Stable $\rightarrow$ Errors are bounded

Consistent $\rightarrow$ Truncation + round-off errors $\rightarrow 0$ as $\Delta t \rightarrow 0$.

Accuracy $\rightarrow$ How close is soln. to actual one

Convergent $\rightarrow$ given $u(t_0)$, $u_{FE}(t_0) \rightarrow u(t_0)$ as $\Delta t \rightarrow 0$.

Conditionally stable when for $\Delta t > \Delta t_{cr}$, the approx. is unstable. For $\alpha < 1/2$, α family is conditionally stable with $\Delta t < \Delta t_{cr} = \dfrac{2}{(1-2\alpha)\lambda}$

largest eigenvalue of F.E. mesh.

Hyperbolic

$$[M]\{\ddot{a}\} + [K]\{a\} = \{F\}$$

$$\{a\}_{n+1} = \{a\}_n + \Delta t\,\{\dot{a}\}_n + \tfrac{1}{2}\Delta t^2\,\{\ddot{a}\}_{n+\gamma}$$

$$\{\dot{a}\}_{n+1} = \{\dot{a}\}_n + \{\ddot{a}\}_{n+\alpha}\,\Delta t$$

$$\{\ddot{a}\}_{n+\theta} = (1-\theta)\{\ddot{a}\}_n + \theta\{\ddot{a}\}_{n+1}$$

Newmark family (α, γ, β to be chosen)

$\gamma = 2\beta$

$$\Delta t \leq \Delta t_{cr} = \frac{1}{\sqrt{\frac{1}{2}\omega_{max}^2(\alpha-\gamma)}}$$

(ω_{max} = max. natural freq. of system)

for schemes with $\gamma < \alpha$, $\alpha \geq 1/2$

Choices

$\alpha = 1/2, \quad \gamma = 2\beta = 1/2$ const. - avg. acc. scheme (stable)

$\alpha = 1/2, \quad \gamma = 2\beta = 1/3$ linear acc. method (cond. stable)

$\alpha = 1/2, \quad \gamma = 2\beta = 0$ central difference method (cond. stable)

$\alpha = 3/2, \quad \gamma = 2\beta = 8/5$ Galerkin method (stable)

$\alpha = 3/2, \quad \gamma = 2\beta = 2$ backward diff. method (stable)

$$\Downarrow$$

$$[\hat{K}]_{n+1} \{a\}_{n+1} = \{\hat{F}\}_{n+1}$$

$$[\hat{K}]_{n+1} = [K]_{n+1} + a_3 [M]_{n+1}$$

$$\{\hat{F}\}_{n+1} = \{F\}_{n+1} + [M]_{n+1} \left(a_3 \{a\}_n + a_4 \{\dot{a}\}_n + a_5 \{\ddot{a}\}_n \right)$$

$$a_3 = \frac{2}{\gamma (\Delta t)^2}, \quad a_4 = \frac{2}{\gamma \Delta t}, \quad a_5 = \frac{1}{\gamma} - 1$$

$$\{\ddot{a}\}_0 = [M]^{-1} \left(\{F\}_0 - [K]\{a\}_0 \right)$$

$$\{\ddot{a}\}_{n+1} = a_3 \left(\{a\}_{n+1} - \{a\}_n \right) - a_4 \{\dot{a}\}_n - a_5 \{\ddot{a}\}_n$$

$$\{\dot{a}\}_{n+1} = \{\dot{a}\}_n + a_2 \{\ddot{a}\}_n + a_1 \{\ddot{a}\}_{n+1}$$

$$a_1 = \alpha \Delta t, \quad a_2 = (1-\alpha) \Delta t$$