AE675: Introduction to FEM

Take-home mid-semester examination

Time: 24 HRS.

(28/2/23 – 1/3/23
9 PM   9 PM)

Full Marks: 30

Q1 For the domain $\Omega = [0,1]$, the following differential equation is given:

$$-\frac{d}{dx}\left(x^{\theta}\frac{du}{dx}\right) = 10, \quad 0 < x < 1 \; ; \quad \theta = 0.7$$

with $u(0) = 0$, $\left(x^{\theta}\frac{du}{dx}\right)\Big|_{x=1} = 5$.

Answer the following:

(1) Using the approach followed in class, develop the weak formulation for this problem.

(2) Prove that for $\{\varphi_i\}_{i=1}^{N}$ with $\varphi_i(x) = x^{i-1}$, a unique solution exists to this problem (i.e. for the weak solution)

(3) Determine the EXACT solution $u(x)$ to this problem by solving the differential equation.

(4) If $u_N(x) = \sum_{i=1}^{N} a_i \varphi_i(x)$, then show that the error $e_N(x)$ is "orthogonal" to $u_N(x)$ in $[0,1]$.

(5) Can you prove if $\|u_N\|^2 \geq \|u\|^2$ or $\|u_N\|^2 \leq \|u\|^2$ where $\|\cdot\|$ is the usual norm discussed in class.

(6) Obtain $u_3(x)$ and plot $u(x), u_3(x)$ versus x. Also plot $u_5(x)$ vs x on the same graph (use MATLAB).

[2×5+5 = 15 pts]

Q2 It is desired that FEM be used to solve this problem. Consider the 2 - element mesh with $X_1 = 0$, $X_2 = h_1$, $X_3 = 1$. Answer the following :

(1) Discuss the "goodness", or lack of it, of the $U_N(x)$ obtained in 1 (6). Does the observation create a case for a better approximation ?

(2) Let $h_1 = 1/2$ and $p = 1$ (linear basis functions) be used. How does this $U_3(x)$ compare with $U_3(x)$ obtained in 1 (6)? Comment on result.

(3) Let $h_1 = 0.1$, and $p = 1$. Now compute the new $U_3(x)$. Comment on this result. Is it better ?

(4) Now it is desired to use a LINEAR ($p = 1$) approximation in $(0, h_1)$ and a CUBIC ($p = 3$) approximation in $(h_1, 1)$. Suggest how you will construct the BASIS functions. Develop the basis functions and plot them.

(5) Use matlab to SOLVE the problem posed in 2 (4). Plot the $U_N(x)$ (what will be N ?) against $u(x)$, Comment on your result.

[5x3 = 15pts]