optimalAmountIn For Profitable Arbitrage Between Two AMMs with xy=k

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For a successful arbitrage between two pools following the equation *xy = k*, factors such as slippage, reserves, and amountIn must be carefully considered. Arbitrary inputs will not yield the expected results when the reserves and fee parameters differ between the two pools. To address this, *optimalAmountIn* can be derived based on the reserves and fee structures using the pool equations. The approach involves expressing *amountIn* in terms of the reserves and then finding its local maximum through derivatives to determine *optimalAmountIn*.

*optimalAmountIn, d*Ya =*

$$d^*Ya = \frac{X_aX_bY_aY_b}{k} \pm \frac{\sqrt{Y_a^2X_b^2(dYa^2-1)-(1-f)^2X_aX_bY_aY_b}}{k}$$

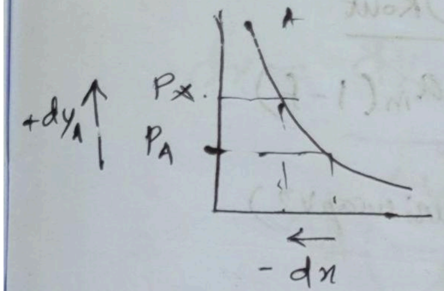
This equation represents the optimal amount of TOKENB (`$$d^*Ya$$`) to input for maximizing profit in the arbitrage scenario between DEXA and DEXB.

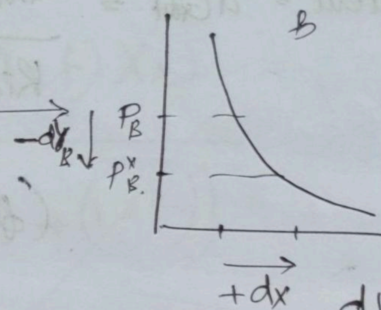
Parameters:

- X_a: Reserve of TOKENA on DEXA
- X_b: Reserve of TOKENA on DEXB
- Y_a: Reserve of TOKENB on DEXA
- Y_b: Reserve of TOKENB on DEXB
- k: Constant from the xy=k formula
- f: Swap fee
- dYa: Amount of TOKENB used in the transaction (note: this seems to be a placeholder for a specific value or condition in your original equation)

Here is the derivation:

Optimal Amount In





$dy_A \rightarrow$ Amount of token in.

Arbitrage Profit

$$F(dy_A) = dy_B - dy_A$$

$dy_B \rightarrow$ Amount of token Out.

$$F'(d^*y_A) = (d'y_B - d'y_A) = 0$$

$d^*y_A \rightarrow$ optimal amount in.

$$= \underline{d'y_B} - 1 = 0$$

$f \rightarrow$ swap fee.
$0 < f \leq 1$

$$dy_A^* = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$X_A \rightarrow$ AMM A Reserve Out

$Y_A \rightarrow$ AMM A Reserve In.

$a = k^2$

$b = 2 k Y_A X_B$

$X_B \rightarrow$ AMM B Reserve In

$c = (Y_A X_B)^2 - (1-f)^2 X_A Y_A X_B Y_B$

$k = (1-f) X_B + (1-f)^2 X_A$

$Y_B \rightarrow$ AMM B Reserve Out

~~Y_A~~

$$\text{swapAmountOut} = a_{out} = \frac{a_{in}(1-f)R_{out}}{R_{in} + a_{in}(1-f)},$$

(from uniswap v2)

AMM-A

$$dx_{A} = \frac{dY_A(1-f)\cancel{X}_A}{Y_A + (1-f)dY_A} \quad \text{— (1)}$$

AMM-B

$$dy_B = \frac{dx(1-f)Y_B}{X_B + (1-f)\cancel{d}Y_B} \quad \text{— (2)}$$

Profit = F = $(dY_B - dY_A)$

From (1) and (2)

$$dY_B = \frac{dx(1-f)\,Y_B}{X_B + dx(1-f)}$$

But,

$$dx = \frac{dY_A(1-f)\,X_A}{Y_A + (1-f)\,dY_A}$$

So,

$$dY_B = \frac{\dfrac{dY_A(1-f)\,X_A}{Y_A+(1-f)dY_A}(1-f)\,Y_B}{X_B + \dfrac{dY_A(1-f)X_A}{Y_A+(1-f)dY_A}(1-f)}$$

$$dY_B = \cfrac{\cfrac{dY_A (1-f)^2 X_A Y_B}{\cancel{Y_A + (1-f)\, dY_A}}}{\cfrac{X_B Y_A + (1-f) X_B\, dY_A + dY_A (1-f)^2 X_A}{\cancel{Y_A + (1-f)\, dY_A}}}$$

$$dY_B = \frac{dY_A (1-f)^2 X_A Y_B}{X_B Y_A + (1-f) X_B\, dY_A + dY_A (1-f)^2 X_A}$$

$$F(dY_A) = dY_B - dY_A$$

$$\frac{d(F)}{d(dY_A)} = F' = \frac{dY_B}{d(dY_A)} - 1 = 0 \text{ for maxima.}$$

$F'(dY_A) = d'Y_B - 1$

$$dY_B = \frac{dY_A(1-f)^2 X_A Y_B}{X_B Y_A + (1-f) X_B dY_A + dY_A (1-f)^2 X_A} = \frac{f}{g}$$

$$d'Y_B = \frac{f'g - fg'}{g^2}$$

$f' \Rightarrow (1-f)^2 X_A Y_B$

$g' \Rightarrow \frac{d}{dY_A}\left(X_B Y_A + (1-f) X_B dY_A\right) + dY_A (1-f^2) X_A$

$= 0 + (1-f) X_B + (1-f)^2 X_A$

$\cancel{d'Y_B} = \cancel{(1-f)^2 X_A Y_B \left((1-f) X_B dY_A + dY_A (1-f)^2 X_A\right)}$

$$d'Y_B = \frac{f'g - g'f}{g^2}$$

$f'g - g'f \Rightarrow$

$= (1-f)^2 X_A Y_B \left[X_B Y_A + (1-f) X_B dY_A + dY_A (1-f)^2 X_A\right]$

$- \left[(1-f) X_B + (1-f)^2 X_A\right]\left[dY_A (1-f)^2 X_A Y_B\right]$

$= (1-f)^2 X_A X_B Y_A Y_B + \cancel{(1-f)^3 X_A X_B Y_B dY_A} + \cancel{(1-f)^4 X_A^2 Y_B dY_A}$

$- \cancel{(1-f)^3 X_A X_B Y_B dY_A} - \cancel{(1-f)^4 X_A^2 Y_B dY_A}$.

$f'g - g'f = (1-f)^2 X_A X_B Y_A Y_B$ ——— (3)

$g^2 = \left[X_B Y_A + dY_A \left[\underbrace{(1-f) X_B + (1-f)^2 X_A}_{k}\right]\right]^2$

$= \cancel{(X_B Y_A)^2 + d^2Y_A^2 \left((1-f) X_B + (1-f)^2 X_A\right)^2}$

$\cancel{+ 2 X_B Y_A dY_A \left[(1-f) X_B + (1-f)^2 X_A\right]}$

let $k = (1-f) X_B + (1-f)^2 X_A$

$g^2 = \left[X_B Y_A + dY_A k\right]^2$

$= X_B^2 Y_A^2 + 2k\, dY_A X_B Y_A + dY_A^2 k^2$

$$d'Y_B = \frac{f'g - g'f}{g^2} = \frac{(1-f)^2 X_A X_B Y_A Y_B}{X_B^2 Y_A^2 + 2k\, dY_A X_B Y_A + dY_A^2 k^2}$$

$d'Y_B - 1 = 0$

$\Rightarrow d'Y_B = 1$

$\Rightarrow (1-f)^2 X_A X_B Y_A Y_B = X_B^2 Y_A^2 + 2k\, dY_A X_B Y_A + dY_A^2 k^2$

$$\underbrace{k^2}_{a} dY_A^2 + \underbrace{2k X_B Y_A}_{b} dY_A + \underbrace{(Y_A X_B)^2 - (1-f)^2 X_A X_B Y_A Y_B}_{c} = 0$$

$$dY_A = d^*Y_A = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2kX_BY_A dY_A \pm \sqrt{(2kX_BY_AdY_A)^2 - 4 \times k^2\left(Y_A X_B^2 - (1-f)^2 X_AX_BY_AY_B\right)}}{2\times k^2}$$

$$= \frac{-2kX_BY_AdY_A \pm \sqrt{4k^2X_B^2Y_A^2dY_A^2 - 4k^2Y_A^2X_B^2 - 4k^2(1-f)^2X_AX_BY_AY_B}}{2k^2}$$

$$= \frac{-2kX_BY_AdY_A \pm \sqrt{4k^2Y_A^2X_B^2(dY_A^2-1) - 4k^2(1-f)^2X_AX_BY_AY_B}}{2k^2}$$

$$\frac{-\cancel{2}kX_AX_BY_AY_B \pm \cancel{2}k\sqrt{Y_A^2X_B^2(dY_A^2-1) - (1-f)^2X_AX_BY_AY_B}}{\cancel{2}k^2}$$

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dY_A^* = Optimal Amount In

$$\frac{X_AX_BY_AY_B}{k} \pm \frac{\sqrt{Y_A^2X_B^2(dY_A^2-1) - (1-f)^2X_AX_BY_AY_B}}{k}$$

where $k = (1-f)X_B + (1-f)^2X_A$

The same approach can be extended to any AMMs like DLMMs,CPMMs etc..

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