Fourier analysis and it's applications

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Definition

- Given an integrable function $f : [0, L] \to \mathbb{C}$, we define the **Fourier series of f** as

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{\frac{2\pi i n x}{L}}$$

where

$$\hat{f}(n) := \frac{1}{L} \int_0^L f(x) e^{-2\pi i n x/L} dx = a_n$$

denotes the **n-th Fourier coefficient** of f for $n \in \mathbb{N}$.

- The N^{th} **partial sum** of the Fourier series of f, for N a positive integer is given by

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x/L}.$$

[1]E.M. Stein and R. Shakarchi: Fourier Analysis: An Introduction

Theorem

Let f be an integrable function on the circle with $f \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$.
Then we have:
(i) **Mean-square convergence** of the Fourier series

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta) - S_N(f)(\theta)|^2 \, d\theta \to 0 \quad \text{as } N \to \infty$$

(ii) **Parseval's identity**

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta = \|f\|^2.$$

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Theorem

Let f be an integrable function on the circle. Suppose f is differentiable at θ_0. Then $\lim_{N\to\infty} S_N(f)(\theta_0) = f(\theta_0)$.

Construction of a continuous function with diverging Fourier series

- **Theorem fails**, if the differentiability assumption is replaced by the **weaker assumption of continuity**.
- Construction is based on **"Breaking of symmetry"** in the partial sum.
- When we break the symmetry, that is, when we split the Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ into the two pieces

$$\sum_{n\geq 0} a_n e^{in\theta} \quad \text{and} \quad \sum_{n<0} a_n e^{in\theta}.$$

- Consider **Sawtooth function** after re-scaling

$$f(\theta) = \begin{cases} -i(\pi + \theta) & \text{if } -\pi < \theta < 0 \\ i(\pi - \theta) & \text{if } 0 < \theta < \pi \end{cases}$$

.

- The fourier series of sawtooth function is given by $f(\theta) \sim \sum_{n \neq 0} \frac{e^{in\theta}}{n}$.
- Consider the series

$$\sum_{n=-\infty}^{-1} \frac{e^{in\theta}}{n}.$$

- **Note:** above is not fourier series of Riemann integrable function.

- For each $N \geq 1$ we define the following two functions on $[-\pi, \pi]$,

$$f_N(\theta) = \sum_{1 \leq |n| \leq N} \frac{e^{in\theta}}{n} \quad \text{and} \quad \tilde{f}_N(\theta) = \sum_{-N \leq n \leq -1} \frac{e^{in\theta}}{n}.$$

Lemma

1. $\left|\tilde{f}_N(0)\right| \geq c \log N$,
2. $f_N(\theta)$ is uniformly bounded in N and θ.

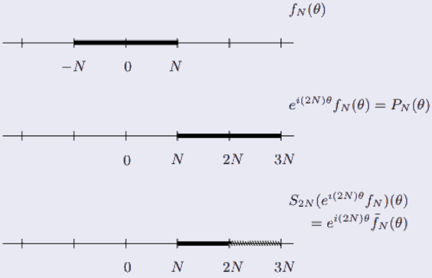
- Now, we define the following two function by **shifting frequency of f_N and $\tilde{f}_N$ by $2N$ units**, we define

$$P_N(\theta) = e^{i(2N)\theta} f_N(\theta) \quad \text{and} \quad \tilde{P}_N(\theta) = e^{i(2N)\theta} \tilde{f}_N(\theta).$$

- Now the coefficients of P_N are non-vanishing for $N \leq n \leq 3N$, $n \neq 2N$.
- Whereas coefficients of $\tilde{P}_N$ are non-vanishing for only when $N \leq n \leq 2N - 1$.
- $\left|\tilde{P}_N(\theta)\right| = \left|\tilde{f}_N(\theta)\right| \Rightarrow \left|\tilde{P}_N(0)\right|$ is **badly behaved**.

Lemma

$$S_M(P_N) = \begin{cases} P_N & \text{if } M \geq 3N \\ \tilde{P}_N & \text{if } M = 2N \\ 0 & \text{if } M < N \end{cases}$$

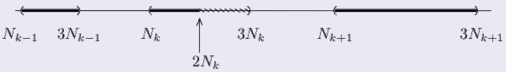


Breaking symmetry in Lemma

- Finally, we **need to find a convergent series** of positive terms $\sum \alpha_k$ and a sequence of integers $\{N_k\}$ which increases rapidly enough so that:
  1. $N_{k+1} > 3N_k$,
  2. $\alpha_k \log N_k \to \infty$ as $k \to \infty$.
- We choose $\alpha_k = 1/k^2$ and $N_k = 3^{2^k}$ which are easily seen to satisfy the above criteria.
- Finally, our desired function is

$$g(\theta) = \sum_{k=1}^{\infty} \alpha_k P_{N_k}(\theta).$$

- Continuity of g follows from absolute convergence of $\sum_{k=1}^{\infty} \alpha_k P_{N_k}(\theta)$, the series above converges uniformly to a continuous periodic function.



Symmetry broken in the middle interval $(N_k, 3N_k)$

- With the choice of $N_k's$,one can verity that

$$|S_{2N_m}P_{N_k}(0)| = \begin{cases} 0 & k > m \\ O(1) & k < m \end{cases}$$

- However, by our lemma we get

$$|S_{2N_m}(g)(0)| \geq c\alpha_m \log N_m + O(1) \to \infty \quad \text{ as } m \to \infty.$$

- So the partial sums of the Fourier series of g at 0 are not bounded.
- Hence we proved the divergence of the Fourier series of g at $\theta = 0$.

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Definition : Equidistribution

A sequence of numbers $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ in $[0,1)$ is said to be equidistributed if for every interval $(a,b) \subset [0,1)$,

$$\lim_{N\to\infty} \frac{\#\left\{1 \leq n \leq N : \xi_n \in (a,b)\right\}}{N} = b - a$$

where $\#A$ denotes the cardinality of the finite set A.

Theorem

If γ is irrational, then the sequence of fractional parts $\langle\gamma\rangle, \langle 2\gamma\rangle, \langle 3\gamma\rangle, \dots$ is equidistributed in $[0, 1)$.

Lemma

If f is continuous and periodic of period 1 , and γ is irrational, then

$$\frac{1}{N}\sum_{n=1}^{N} f(n\gamma) \to \int_0^1 f(x)dx \quad \text{as } N \to \infty.$$

Proof of theorem

- Fix $(a, b) \subset [0, 1)$ Define $\chi_{(a,b)}(x)$

$$\chi_{(a,b)}(x) = \begin{cases} 1 & \text{if } x \in (a, b) \\ 0 & \text{if } x \in [0, 1) \setminus (a, b) \end{cases}$$

- We may extend this function to $\mathbb{R}$ by periodicity (period 1).
- Then, as a consequence of the definitions, we find that

$$\#\{1 \leq n \leq N : \langle n\gamma \rangle \in (a, b)\} = \sum_{n=1}^{N} \chi_{(a,b)}(n\gamma)$$

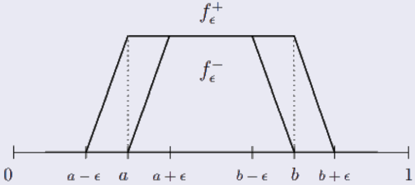
and the theorem can be reformulated as the statement that

$$\frac{1}{N} \sum_{n=1}^{N} \chi_{(a,b)}(n\gamma) \to \int_0^1 \chi_{(a,b)}(x)dx, \quad \text{as } N \to \infty$$

Translation from number theory to analysis

- Choose two continuous periodic functions f_ϵ^+ and f_ϵ^- of period 1 which approximate $\chi_{(a,b)}(x)$ on $[0,1)$ from above and below.
- In particular, $f_\epsilon^-(x) \leq \chi_{(a,b)}(x) \leq f_\epsilon^+(x)$, and

$$b - a - 2\epsilon \leq \int_0^1 f_\epsilon^-(x)dx \quad \text{and} \quad \int_0^1 f_\epsilon^+(x)dx \leq b - a + 2\epsilon$$



Approximations of $\chi_{(a,b)}(x)$

- If $S_N = \frac{1}{N}\sum_{n=1}^{N} \chi_{(a,b)}(n\gamma)$, then we get

$$\frac{1}{N}\sum_{n=1}^{N} f_\epsilon^-(n\gamma) \leq S_N \leq \frac{1}{N}\sum_{n=1}^{N} f_\epsilon^+(n\gamma)$$

- $$b - a - 2\epsilon \leq \liminf_{N\to\infty} S_N \quad \text{and} \quad \limsup_{N\to\infty} S_N \leq b - a + 2\epsilon.$$

- Since this is true for every $\epsilon > 0$, the limit $\lim_{N\to\infty} S_N$ exists and must equal $b - a$.

Weyl's criterion

A sequence of real numbers $\xi_1, \xi_2 \ldots$ in $[0, 1)$ is equidistributed **if and only if** for all integers $k \neq 0$ one has

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \xi_n} \to 0, \qquad \text{as } N \to \infty.$$

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DFT

Suppose $z = (z(0), \ldots, z(N-1)) \in \ell^2(\mathbb{Z}_N)$. For $m = 0, 1, \ldots, N-1$, define

$$\hat{z}(m) = \sum_{n=0}^{N-1} z(n) e^{-2\pi imn/N}$$

Let

$$\hat{z} = (\hat{z}(0), \hat{z}(1), \ldots, \hat{z}(N-1))$$

Then $\hat{z} \in \ell^2(\mathbb{Z}_N)$. The map $\hat{}: \ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)$, which takes z to $\hat{z}$, is called the discrete Fourier transform.

The DFT can be represented by a matrix,because the map taking z to $\hat{z}$ is a linear transformation.

[2] Michael W Frazier: An introduction to wavelets through linear algebra.

DFT in matrix form

$$\hat{z} = W_N z$$

where, W_N be the matrix $[w_{mn}]_{0\leq m,n\leq N-1}$ such that $\omega_N^{mn} = e^{2\pi imn/N}$ and

$$W_N = \begin{bmatrix} 1 & 1 & 1 & 1 & . & . & 1 \\ 1 & \omega_N & \omega_N^2 & \omega_N^3 & . & . & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \omega_N^6 & . & . & \omega_N^{2(N-1)} \\ 1 & \omega_N^3 & \omega_N^6 & \omega_N^9 & . & . & \omega_N^{3(N-1)} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \omega_N^{3(N-1)} & . & . & \omega_N^{(N-1)(N-1)} \end{bmatrix}.$$

How Many Complex Multiplications Are Required?

- Each inner product requires N complex multiplications.
  - There are N inner products.
- Hence we require N^2 multiplications.
- However, the first row and first column are all $1\,\mathrm{s}$, and should not be counted as multiplications.
  - There are $2N - 1$ such instances.
- Hence, the number of complex multiplications is $N^2 - 2N + 1$ i.e., $(N-1)^2$.

Lemma (FFT)

Suppose $M \in \mathbb{N}$, and $N = 2M$. Let $z \in \ell^2(\mathbb{Z}_N)$. Define $u, v \in \ell^2(\mathbb{Z}_M)$ *by*

$$u(k) = z(2k) \quad \text{for} \quad k = 0, 1, \ldots, M-1$$

and

$$v(k) = z(2k+1) \text{ for } \quad k = 0, 1, \ldots, M-1$$

Then for $m = 0, 1, \ldots, M-1$

$$\hat{z}(m) = \hat{u}(m) + e^{-2\pi i m/N}\hat{v}(m) \tag{1}$$

Also, for $m = M, M+1, M+2, \ldots, N-1$, let $\ell = m - M$. Note that the corresponding values of ℓ are $\ell = 0, 1, \ldots, M-1$. Then

$$\hat{z}(m) = \hat{z}(\ell + M) = \hat{u}(\ell) - e^{-2\pi i \ell/N}\hat{v}(\ell). \tag{2}$$

Proof

- For any $m = 0, 1, \ldots, N-1$,

$$\hat{z}(m) = \sum_{n=0}^{N-1} z(n) e^{-2\pi i m n / N}$$

- Then for $m = 0, 1, \ldots, M-1$

$$\begin{aligned}
\hat{z}(m) &= \sum_{k=0}^{M-1} z(2k) e^{-2\pi i 2km/N} + \sum_{k=0}^{M-1} z(2k+1) e^{-2\pi i (2k+1)m/N} \\
&= \sum_{k=0}^{M-1} u(k) e^{-2\pi i km/(N/2)} + e^{-2\pi i m/N} \sum_{k=0}^{M-1} v(k) e^{-2\pi i km/(N/2)} \\
&= \sum_{k=0}^{M-1} u(k) e^{-2\pi i km/M} + e^{-2\pi i m/N} \sum_{k=0}^{M-1} v(k) e^{-2\pi i km/M} \\
&= \hat{u}(m) + e^{-2\pi i m/N} \hat{v}(m).
\end{aligned}$$

- For $m = M, M+1, \ldots, N-1$. By writing $m = \ell + M$ for $\ell = 0, 1, \ldots, M-1$, we get

$$\begin{aligned}
\hat{z}(m) &= \sum_{k=0}^{M-1} z(2k) e^{-2\pi i 2km/N} + \sum_{k=0}^{M-1} z(2k+1) e^{-2\pi i(2k+1)m/N} \\
&= \sum_{k=0}^{M-1} u(k) e^{-2\pi ik(\ell+M)/M} + e^{-2\pi i(\ell+M)/N} \sum_{k=0}^{M-1} v(k) e^{-2\pi ik(\ell+M)/M} \\
&= \sum_{k=0}^{M-1} u(k) e^{-2\pi ik\ell/M} - e^{-2\pi i\ell/N} \sum_{k=0}^{M-1} v(k) e^{-2\pi ik\ell/M} \\
&= \hat{u}(\ell) - e^{-2\pi i\ell/N} \hat{v}(\ell) = \hat{z}(\ell + m)
\end{aligned}$$

- Since the exponential $e^{-2\pi ikl/M}$ are periodic with period M, and $e^{-2\pi iM/N} = e^{-\pi i} = -1$ for $N = 2M$.

Complexity

- $$\#_N \leq 2\#_M + M \tag{3}$$

  where, $\#_N$, for any positive integer N, to be the least number of complex multiplications required to compute the DFT of a vector of length N.
- The most favorable case is when $N = 2^n$.

If $N \neq 2^n$, then it is harmless to pad it with some extra zeros at the end until it has length $N = 2^n$.

Lemma

Suppose $N = 2^n$ for some $n \in \mathbb{N}$. Then

$$\#_N \leq \frac{1}{2} N \log_2 N.$$

Applications of FFT

IDFT

- $$\check{w}(n) = \frac{1}{N}\hat{w}(N-n)$$
- FFT algorithm can be used to compute the IDFT quickly also, in at most $(N/2)\log_2 N$ steps if $N = 2^n$.

Convolution

- $$z * w = (\hat{z}\hat{w})^{\vee}$$
- If $z, w \in \ell^2(\mathbb{Z}_N)$, for $N = 2^n$, it takes at most $N\log_2 N$ multiplications to compute $\hat{z}$ and $\hat{w}$.
- N multiplications to compute $\hat{z}\hat{w}$, and at most $(N/2)\log_2 N$ multiplications to take the IDFT of $\hat{z}\hat{w}$.
- Thus overall, $N + (3N/2)\log_2 N$ multiplications to compute $z * w$.

Thank you