Fourier Analysis and its Application to Roth's theorem

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Table of Contents

Fourier transform in $\mathbb{F}_p^n$

The Fourier transform of $f : \mathbb{F}_p^n \to \mathbb{C}$ is a function $\widehat{f} : \mathbb{F}_p^n \to \mathbb{C}$ defined by setting, for each $r \in \mathbb{F}_p^n$,

$$\widehat{f}(r) := \mathbb{E}_{x \in \mathbb{F}_p^n} f(x)\omega^{-r \cdot x} = \frac{1}{p^n} \sum_{x \in \mathbb{F}_p^n} f(x)\omega^{-r \cdot x}$$

where

- $\omega = \exp(2\pi i/p)$
- $r \cdot x = r_1 x_1 + \cdots + r_n x_n$.

Note: $\widehat{f}(0) = \mathbb{E}f$

Fourier inversion formula

Let $f : \mathbb{F}_p^n \to \mathbb{C}$. For every $x \in \mathbb{F}_p^n$,

$$f(x) = \sum_{r \in \mathbb{F}_p^n} \widehat{f}(r)\omega^{r \cdot x}$$

Parseval's identity

Given $f, g : \mathbb{F}_p^n \to \mathbb{C}$, we have

$$\mathbb{E}_{x \in \mathbb{F}_p^n} f(x)\overline{g(x)} = \sum_{r \in \mathbb{F}_p^n} \widehat{f}(r)\overline{\widehat{g}(r)}$$

In particular, as a special case $(f = g)$,

$$\mathbb{E}_{x \in \mathbb{F}_p^n} |f(x)|^2 = \sum_{r \in \mathbb{F}_p^n} |\widehat{f}(r)|^2$$

Definition (3-AP density)

Given $f, g, h : \mathbb{F}_p^n \to \mathbb{C}$, we write

$$\Lambda(f, g, h) := \mathbb{E}_{x,y} f(x)g(x+y)h(x+2y)$$

and

$$\Lambda_3(f) := \Lambda(f, f, f)$$

Proposition (Fourier and 3-AP)

Let p be an odd prime. If $f, g, h : \mathbb{F}_p^n \to \mathbb{C}$, then

$$\Lambda(f, g, h) = \sum_r \widehat{f}(r)\widehat{g}(-2r)\widehat{h}(r).$$

Lemma (3-AP counting lemma)

Let $f : \mathbb{F}_3^n \to [0,1]$. Then

$$\left|\Lambda_3(f) - (\mathbb{E}f)^3\right| \le \max_{r \neq 0} |\widehat{f}(r)| \|f\|_2^2.$$

Proof

Using Fourier 3-AP Proposition

$$\Lambda_3(f) = \sum_r \widehat{f}(r)^3 = \widehat{f}(0)^3 + \sum_{r \neq 0} \widehat{f}(r)^3$$

Since $\mathbb{E}f = \widehat{f}(0)$, we have

$$\left|\Lambda_3(f) - (\mathbb{E}f)^3\right| \le \sum_{r \neq 0} |\widehat{f}(r)|^3 \le \max_{r \neq 0} |\widehat{f}(r)| \cdot \sum_r |\widehat{f}(r)|^2 = \max_{r \neq 0} |\widehat{f}(r)| \|f\|_2^2.$$

last step is by Plancherel theorem.

Roth's theorem in $\mathbb{F}_3^n$

Theorem (Roth's theorem in $\mathbb{F}_3^n$ or Cap set problem)

Every 3-AP-free subset of $\mathbb{F}_3^n$ has size $O\left(3^n/n\right)$.

- In an abelian group, a set A is said to be **3-AP-free** if A does not have three distinct elements of the form $x, x+y, x+2y$.
- The **cap set problem** is to determine the size of the largest 3-AP-free subset in $\mathbb{F}_3^n$.

Proof Strategy

- If A is pseudo-random, which means that all its Fourier coefficients are small, Then there is a counting lemma which shows that A has lots of 3-AP.
- If A is not pseudo-random, then A has a large Fourier coefficient.

The strategy for Roth's theorem is the **density increment argument**. Given $A \subset \mathbb{F}_3^n$, we use the following strategy:

1. 3-AP free $\Rightarrow \exists$ a large Fourier coefficient.
2. A large Fourier coefficient $\Rightarrow$ We can find hyper-plane where density of A will increase.
3. Iterate the Density Increment Lemma.

Step 1. *A 3-AP-free set has a large Fourier coefficient*

3-AP-free implies large Fourier coefficient

Let $A \subset \mathbb{F}_3^n$ and $\alpha = |A|/3^n$. If A is 3 -AP-free and $3^n \geq 2\alpha^{-2}$, then there is $r \neq 0$ such that $\left|\widehat{1_A}(r)\right| \geq \alpha^2/2$.

Proof

Using the fact

$$\Lambda_3\left(1_A\right) = 3^{-2n}\left|\left\{(x, y, z) \in A^3 : x + y + z = 0\right\}\right|$$

Also,

$$\mathbb{E}1_A = |A|/3^n = \alpha \text{ and } \left\|1_A\right\|_2^2 = \mathbb{E}1_{A^2} = \alpha$$

Since A is 3-AP-free,

$$\Lambda_3(A) = |A|/3^{2n} = \alpha/3^n$$

By the 3-AP counting lemma,

$$\alpha^3 - \frac{\alpha}{3^n} = (\mathbb{E}1_A)^3 - \Lambda_3(1_A) \leq \max_{r\neq 0} \left|\widehat{1}_A(r)\right| \|1_A\|_2^2 = \max_{r\neq 0} \left|\widehat{1}_A(r)\right| \alpha.$$

By the hypothesis $3^n \geq 2\alpha^{-2}$,

$$\max_{r\neq 0} \left|\widehat{1}_A(r)\right| \geq \alpha^2/2.$$

So there is some $r \neq 0$ with $\left|\widehat{1}_A(r)\right| \geq \alpha^2/2$.

Step 2. *A large Fourier coefficient implies density increment on some hyperplane*

Large Fourier coefficient implies density increment

Let $A \subset \mathbb{F}_3^n$ with $\alpha = |A|/3^n$. Suppose $\left|\widehat{1_A}(r)\right| \geq \delta > 0$ for some $r \neq 0$. Then A has density at least $\alpha + \delta/2$ when restricted to some hyperplane.

Proof

We have

$$\widehat{1_A}(r) = \mathbb{E}_x 1_A(x)\omega^{-r\cdot x} = \frac{\alpha_0 + \alpha_1\omega + \alpha_2\omega^2}{3}$$

where $\alpha_0, \alpha_1, \alpha_2$ are densities of A on the cosets of $r^\perp$.
Claim: To show that one of $\alpha_0, \alpha_1, \alpha_2$ is significantly larger than α.
We have

$$\alpha = (\alpha_0 + \alpha_1 + \alpha_2)/3 \tag{1}$$

By the triangle inequality,

$$\begin{aligned}3\delta &\le \left|\alpha_0 + \alpha_1\omega + \alpha_2\omega^2\right| \\ &= \left|(\alpha_0 - \alpha) + (\alpha_1 - \alpha)\,\omega + (\alpha_2 - \alpha)\,\omega^2\right| \\ &\le |\alpha_0 - \alpha| + |\alpha_1 - \alpha| + |\alpha_2 - \alpha| \\ &= \sum_{j=0}^{2} (|\alpha_j - \alpha| + (\alpha_j - \alpha)) \quad \text{[by equation 1]}.\end{aligned}$$

By pigeonhole principle, there exists j such that $|\alpha_j - \alpha| + (\alpha_j - \alpha) \ge \delta$. Note

$$|t| + t = \begin{cases} 2t & t > 0 \\ 0 & t \le 0 \end{cases}$$

So we get $\alpha_j - \alpha \ge \delta/2$.

Density Increment Lemma

Let $A \subset \mathbb{F}_3^n$ and $\alpha = |A|/3^n$. If A is 3 -AP-free and $3^n \geq 2\alpha^{-2}$, then A has density at least $\alpha + \alpha^2/4$ when restricted to some hyperplane.

Step 3. *Iterate the density increment*

- We start with a 3-AP-free $A \subset \mathbb{F}_3^n$.
- Let $V_0 := \mathbb{F}_3^n$ with density $\alpha_0 := \alpha = |A|/3^n$.
- Repeatedly apply **density increment lemma**.
- After i rounds,
  - We restrict A to a hyper plane (with $V_0 \supset V_1 \supset \cdots$).
  - Let $\alpha_i = |A \cap V_i| \,/\, |V_i|$ be the density of A in V_i.
  - As long as $2\alpha_i^{-2} \leq |V_i| = 3^{n-i}$, we can apply density increment lemma to obtain a V_{i+1} with density increment

$$\alpha_{i+1} \geq \alpha_i + \alpha_i^2/4$$

Claim:

$$\alpha_i \geq \alpha + \frac{i\alpha^2}{4} \tag{2}$$

- Each round, α_i increases by at least $\alpha^2/4$.
- So it takes $\leq \lceil 4/\alpha \rceil$ initial rounds for α_i to double.
- Once $\alpha_i \geq 2\alpha$, it then it takes $\leq \lceil 1/\alpha_i \rceil \leq \lceil 1/\alpha \rceil$ additional round for the density to double again.
- And so on: the k-th doubling time is at most $\left\lceil 4^{2-k}/\alpha \right\rceil$.
- Since the density is always at most 1.
- The density can double at most $\log_2(1/\alpha)$ times.

- So the total number of rounds is at most

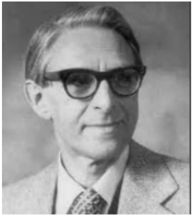
$$\sum_{j \leq \log_2(1/\alpha)} \left\lceil \frac{4^{2-j}}{\alpha} \right\rceil = O\left(\frac{1}{\alpha}\right)$$

- Suppose the process terminates after m steps with density α_m.
- Then, we check the hypothesis of Density increment lemma, $3^{n-m} < 2\alpha_m^{-2} \leq 2\alpha^{-2}$
- So $n \leq m + \log_3(2/\alpha^2) \leq O(1/\alpha)$.
- Thus $\alpha = O(1/n)$.
- Equivalently, $|A| = \alpha N = O\left(\frac{3^n}{n}\right)$.

Hence, this completes the proof of Roth's theorem in $\mathbb{F}_3^n$.

Table of Contents

Roth's Theorem in the Integers



Roth's theorem 1953

Every 3-AP-free subset of $[N] = \{1, \ldots, N\}$ has size $O(N/\log\log N)$.

- An important difference between $\mathbb{F}_3^n$ and $\mathbb{Z}$ is that $\mathbb{Z}$ has no sub-spaces.
- Instead, we will proceed in $\mathbb{Z}$ by restricting to sub-progressions.

Table of Contents

History

- In 1982, Brown and Buhler $r_3\left(\mathbb{F}_3^n\right) = o\left(3^n\right)$.
- In 1995 , Roy Mesuhlam $r_3\left(\mathbb{F}_3^n\right) = O\left(\frac{3^n}{n}\right)$.
- In 2012, Bateman and Katz improve bound to $O\left(3^n/n^{1+\epsilon}\right)$.
- In 5 May 2016, Croot-Lev-Pach $r_3\left(\mathrm{Z}_4^n\right) = O\left(3.61^n\right)$.
- In 12 May 2016, Ellenberg and Gijswijt started working together.

- In 30 May 2016, Ellenberg and Gijswijt published (ArXiv) paper with useful discussions with Terence Tao, Tim Gowers, and Lev.
- In 2017, Ellenberg and Gijswijt developed a new technique based on the polynomial method to prove that $r_3\left(\mathbb{F}_3^n\right) = O\left(2.756^n\right)$.
- In 2004, Eden the best known lower bound $O\left(2.21^n\right)$.

Theorem (Cap set upper bound)

Every 3-AP-free subset of $\mathbb{F}_3^n$ has size $O\left(2.76^n\right)$.

Definition (Slice rank)

A function $F : A \times A \times A \to \mathbb{F}$ is said to have **slice rank 1** if it can be written as

$$u(x)v(y,z), \quad u(y)v(x,z), \quad \text{or} \quad u(z)v(x,y)$$

for some nonzero functions $u : A \to \mathbb{F}$ and $v : A \times A \to \mathbb{F}$.
The **slice rank of a function** $F : A \times A \times A \to \mathbb{F}$ is the minimum r so that F can be written as a sum of r slice rank 1 functions, i.e

$$F(x,y,z) = \sum_{i=1}^{r_1} u_i(x)v_i(y,z) + \sum_{i=r_1+1}^{r_2} u_i(y)v_i(x,z) + \sum_{i=r_2+1}^{r} u_i(z)v_i(x,y).$$

Tao's blog

We will use a formulation that appear on Tao's blog [1]
Let $A \subseteq \mathbb{F}_3^n$ be 3 -AP- free. Then we have identity

$$\delta_{0^n}(x + y + z) = \sum_{a \in A} \delta_a(x)\delta_a(y)\delta_a(z)$$

Note:

- Above hold $x + y + z = 0$ iff $z - y = y - x$ in $\mathbb{F}_3^n \implies x, y, z$ in A.P.
- This possible only when $x = y = z = a$ (trivial A.P) for some a $\in \mathbb{F}_3^n$..

Connection between slice rank and the cap-set problem

- Suppose that A is a subset of $\mathbb{F}_3^n$ that contain $x = y = z = a$ for some a $\in \mathbb{F}_3^n$ with $x + y + z = 0$.
- Then if x, y, z belong to A and are not all the same, we must have that $x + y + z \neq 0$ and hence that there exists i such that $x_i + y_i + z_i \neq 0$.
- In $\mathbb{F}_3$, one has

$$1 - x^2 = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

- This can be written in a polynomial form as

$$1 - (x_i + y_i + z_i)^2 = 0.$$

- Now let $F : A^3 \to \mathbb{F}_3$ be defined by $F(x, y, z) = \delta_{0^n}(x + y + z)$. Then

$$F(x, y, z) = \prod_{i=1}^{n} \left(1 - (x_i + y_i + z_i)^2\right) \tag{3}$$

for every $(x, y, z) \in |A|^3$.

Slice rank of a diagonal

Suppose $F : A \times A \times A \to \mathbb{F}$ satisfies $F(x, y, z) \neq 0$ if and only if $x = y = z$. Then F has slice rank $|A|$.

Lemma (Upper bound on the slice rank of $1_{x+y+z=0}$)

Define $F : A^3 \to \mathbb{F}_3$ by $F(x, y, z) = \delta_{0^n}(x + y + z)$. Then the slice rank of F is at most

$$3 \sum_{\substack{a,b,c\geq 0 \\ a+b+c=n \\ b+2c\leq 2n/3}} \frac{n!}{a!b!c!}$$

Proof

- $$F(x, y, z) = \prod_{i=1}^{n} \left(1 - (x_i + y_i + z_i)^2\right)$$
- If we expand the right-hand side, we obtain a polynomial in $3n$ variables with degree $2n$.

Proof

- This is a sum of monomials, each of the form

$$x_1^{i_1} \cdots x_n^{i_n} y_1^{j_1} \cdots y_n^{j_n} z_1^{k_1} \cdots z_n^{k_n},$$

 where $i_1, i_2, \ldots, i_n, j_1, \ldots, j_n, k_1, \ldots, k_n \in \{0, 1, 2\}$.
- For each term, by the pigeonhole principle, at least one of $i_1 + \cdots + i_n, j_1 + \cdots + j_n, k_1 + \cdots + k_n$ is at most $2n/3$.
- So we can split these summands into three sets:

$$\begin{aligned}\prod_{i=1}^{n} \left(1 - (x_i + y_i + z_i)^2\right) = &\sum_{i_1 + \cdots + i_n \leq \frac{2n}{3}} x_1^{i_1} \cdots x_n^{i_n} f_{i_1,\ldots,i_n}(y, z) \\ &+ \sum_{j_1 + \cdots + j_n \leq \frac{2n}{3}} y_1^{j_1} \cdots y_n^{j_n} g_{j_1,\ldots,j_n}(x, z) \\ &+ \sum_{k_1 + \cdots + k_n \leq \frac{2n}{3}} z_1^{k_1} \cdots z_n^{k_n} h_{k_1,\ldots,k_n}(x, y)\end{aligned}$$

- The number of summands in the first sum is precisely the number of triples of non-negative integers a, b, c with $a + b + c = n$ and $b + 2c \leq 2n/3$.
- *Note:* a, b, c correspond to the numbers of i_* 's that are equal to $0, 1, 2$ respectively
- Hence, the slice rank of F is at most

$$3 \sum_{\substack{a,b,c \geq 0 \\ a+b+c=n \\ b+2c \leq 2n/3}} \frac{n!}{a!b!c!}$$

A trinomial coefficient estimate

For every positive integer n,

$$\sum_{\substack{a,b,c\geq 0\\ a+b+c=n\\ b+2c\leq 2n/3}} \frac{n!}{a!b!c!} \leq 2.76^n.$$

Proof

Let $x \in [0,1]$.

$$\left(1+x+x^2\right)^n = \sum_{i=0}^{2n}\left(\sum_{\substack{a+b+c=n\\ b+2c=i}} \frac{n!}{a!b!c!}\right) x^i$$

$$
\begin{aligned}
\frac{\left(1+x+x^2\right)^n}{x^{\frac{2n}{3}}} &= \sum_{i=0}^{2n} \left(\sum_{\substack{a+b+c=n \\ b+2c=i}} \frac{n!}{a!b!c!} \right) x^{i-\frac{2n}{3}} \\
&\geq \sum_{i=0}^{2n/3} \sum_{\substack{a+b+c=n \\ b+2c=i}} \frac{n!}{a!b!c!}
\end{aligned}
\tag{4}
$$

- By deleting contributions x^i with $i > 2n/3$ and using $1 \leq x^{i-2n/3}$ whenever $i \leq 2n/3$, we have

$$
\sum_{\substack{a,b,c \geq 0 \\ a+b+c=n \\ b+2c \leq 2n/3}} \frac{n!}{a!b!c!} \leq \frac{\left(1+x+x^2\right)^n}{x^{2n/3}}
$$

- Setting $x = 0.6$ shows that the left-hand side sum is $\leq (2.76)^n$.

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Thank you