Optimization Techniques
Paper Code – BMS-09
Lecture – 02(Unit -1)
Topic-Multiple Variables Optimization

Dr. Ram Keval

Mathematics and Scientific Computing Deptt.

M. M. M. University of Technology Gorakhpur

Unit-01

Classical Optimization Techniques: Single variable optimization, Multi-variable with no constraints. Non-linear programming: One Dimensional Minimization methods. Elimination methods: Fibonacci method, Golden Section method

Unit-02

Unit-02

Linear Programming: Constrained Optimization Techniques: Simplex method, Solution of System of Linear Simultaneous equations, Revised Simplex method, Transportation problems, Karmarkar’s method, Duality Theorems, Dual Simplex method, Decomposition principle.

MULTIVARIABLE OPTIMIZATION WITH NO CONSTRAINTS

Theorem 2.3 Necessary Condition If $f(\mathbf{X})$ has an extreme point (maximum or minimum) at $\mathbf{X} = \mathbf{X}^*$ and if the first partial derivatives of $f(\mathbf{X})$ exist at $\mathbf{X}^*$, then

$$\frac{\partial f}{\partial x_1}(\mathbf{X}^*) = \frac{\partial f}{\partial x_2}(\mathbf{X}^*) = \cdots = \frac{\partial f}{\partial x_n}(\mathbf{X}^*) = 0$$

Theorem 2.4 Sufficient Condition A sufficient condition for a stationary point $\mathbf{X}^*$ to be an extreme point is that the matrix of second partial derivatives (Hessian matrix) of $f(\mathbf{X})$ evaluated at $\mathbf{X}^*$ is (i) positive definite when $\mathbf{X}^*$ is a relative minimum point, and (ii) negative definite when $\mathbf{X}^*$ is a relative maximum point.

Note: A matrix **A** will be positive definite if all its eigenvalues are positive; that is, all the values of λ that satisfy the determinantal equation

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

should be positive. Similarly, the matrix [**A**] will be negative definite if its eigenvalues are negative.

Another test that can be used to find the positive definiteness of a matrix **A** of order n involves evaluation of the determinants as

$$A = \quad |a_{11}|,$$

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

$$A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{32} \end{vmatrix}, \ldots,$$

$$A_n = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

The matrix **A** will be positive definite if and only if all the values A_1, A_2, A_3, . . . , A_n are positive.

The matrix **A** will be negative definite if and only if the sign of A_ j is (–1)^j for j = 1, 2, . . . , n [Means (–1)^j * A_ j are positive].

If some of the A_ j are positive and the remaining A_ j are zero, the matrix **A** will be positive semidefinite.

Q.

Find the extreme points of the function

$$f(x_1, x_2) = x_1^3 + x_2^3 + 2x_1^2 + 4x_2^2 + 6$$

Ans

$$\frac{\partial f}{\partial x_1} = 3x_1^2 + 4x_1$$

$$\frac{\partial f}{\partial x_2} = 3x_2^2 + 8x_2$$

now for maxima | minima

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$$

$$\Rightarrow 3x_1^2 + 4x_1 = 0 \Rightarrow x_1(3x_1 + 4) = 0 \Rightarrow x_1 = 0 \text{ or } x_1 = -4/3$$

$$\Rightarrow 3x_2^2 + 8x_2 = 0 \Rightarrow x_2(3x_2 + 8) = 0 \Rightarrow x_2 = 0 \text{ or } x_2 = -8/3$$

So. extreme point $(x^*) = (x_1^*, x_2^*)$

$(0,0)$, $\left(0, -\frac{8}{3}\right)$, $\left(-\frac{4}{3}, 0\right)$, $\left(-\frac{4}{3}, -\frac{8}{3}\right)$

$f = f(x_1, x_2)$

$$H(x^*) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

$f = f(x_1, x_2, x_3)$

$$H(x^*) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_3 \partial x_2} \\ \partial^2 f / \partial x_1 \partial x_3 & \partial^2 f / \partial x_2 \partial x_3 & \partial^2 f / \partial x_3^2 \end{bmatrix}$$

So, in our problem

$$\frac{\partial^2 f}{\partial x_1^2} = 6x_1 + 4 \quad , \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0, \quad \frac{\partial^2 f}{\partial x_2^2} = 6x_2 + 8$$

So, Hessian Matrix,

$$H(x^*) = \begin{bmatrix} 6x_1+4 & 0 \\ 0 & 6x_2+8 \end{bmatrix}$$

$$A_1 = |6x_1+4| = 6x_1+4, \quad A_2 = \begin{vmatrix} 6x_1+4 & 0 \\ 0 & 6x_2+8 \end{vmatrix}$$

$$= (6x_1+4)(6x_2+8)$$

at $(0,0)$, $A_1 = (6x_1+4)_{(0,0)} = 4$

at $(0,0)$, $A_2 = (6x_1+4)(6x_2+8)_{(0,0)} = 32$

we can see here $A_1 > 0$, $A_2 > 0 \Rightarrow H(x^0)_{(0,0)}$ is +ve definite.

$\Rightarrow (0,0)$ is a relative minimum point and minimum value $(0,0)$, $f(x_1, x_2) = 06$.

at $\left(0, -\frac{8}{3}\right)$, $A_1 = 4$, | $A_2 = 4\left[\left(6x - \frac{8}{3}\right) + 8\right]$

$= 4[-8] = -32$

$(-1)^j A_j > 0$, $j=1$, $-A_1 > 0 \Rightarrow A_1 < 0$

-ve definite

$j=2$, $(-1)^2 A_2 > 0 \Rightarrow A_2 > 0$

$j=3$, $(-1)^3 A_3 > 0 \Rightarrow A_3 < 0$

simi, $A_4 > 0$

so, at $(0, -\frac{8}{3})$, this point is indefinite $\Rightarrow$ saddle point.

similarly $(-\frac{4}{3}, 0)$, $A_1 = (6x_1+4)\Big|_{(-\frac{4}{3},0)} = -6 \times \frac{4}{3} + 4$

$= -4$

at $(-\frac{4}{3}, 0)$, $A_2 = (6x_1+4)(6x_2+8)\Big|_{-4/3,0} = -4 \times 8 = -32$

at $(-\frac{4}{3}, 0)$, this point is also indefinite $\Rightarrow$ saddle point.

$\left(-\frac{4}{3}, -\frac{8}{3}\right)$ now check your self.

summery

Point $\mathbf{X}$	Value of J_1	Value of J_2	Nature of $\mathbf{J}$	Nature of $\mathbf{X}$	$f(\mathbf{X})$
$(0, 0)$	$+4$	$+32$	Positive definite	Relative minimum	6
$(0, -\frac{8}{3})$	$+4$	-32	Indefinite	Saddle point	418/27
$(-\frac{4}{3}, 0)$	-4	-32	Indefinite	Saddle point	194/27
$(-\frac{4}{3}, -\frac{8}{3})$	-4	$+32$	Negative definite	Relative maximum	50/3