5/01/11
Thursday

Module 1.

SYSTEM OF LINEAR EQUATIONS

Gaussian elimination and Gauss Jordan methods, Elementary matrices - permutation matrix - Inverse matrices - system of linear equations - LU factorizations

Row echelon form :-

- All non-zero rows are above any rows of all zeroes
- The leading coefficient (the first nonzero no. from the left, also called the pivot) of a non zero row is always strictly to the right of the leading coefficient of the row above it.
- All entries in a column below a leading entry are zeroes

Reduced row echelon form :-

A matrix is in reduced row echelon form (also called row cannonical form) if it satisfies the following conditions :-

- It is in row echelon form
- Every leading coefficient is 1 and is the only non zero entry in its column

⇒ There are 3 types of solutions

1. Independent
   - Consistent
   - Unique solutions
   - No. of nonzero rows = no. of variables
   - LHS is usually identity matrix
   - No. of equations = no. of variables

2. Dependent
   - Consistent
   - Many solutions
   - Write answer in parametric form
   - More variables than no. of non-zero rows
   - Happens when there are less no. of eqns than no. of variables

3. Inconsistent

- No solution
- A row reduced matrix has row of zeros on left and non-zero element on right

* Rank - No. of non-zero rows in row-echelon form

Rouche's theorem

The system of eqns $AX = B$ is consistent, if and only if the co-efficient matrix A and augment matrix $[A|B]$ are of SAME rank

* If $R(A) \neq R(A|B)$, inconsistent
* If $R(A) = R(A|B) =$ no. of unknowns, Unique
* If $R(A) = R(A|B) <$ no. of unknowns, many sols.

* Two ways of solving linear equations
  1. Direct method
  2. Iterative method

Direct method :-

It uses elementary transformations

method	Initial	Final
Gauss elimination	$AX = B$	$UX = C$
LU decomposition	$AX = B$	$LUX = B$
Gauss-Jordan elimination	$AX = B$	$IX = C$

$U \rightarrow$ Upper Δlar matrix

$L \rightarrow$ Lower Δlar matrix

$I \rightarrow$ Identity matrix

⇒ Gauss elimination :-

- Triangularization
- Back substitution

⊛ Use gauss elimination method to solve

$$9x_1 + 3x_2 + 4x_3 = 7$$
$$4x_1 + 3x_2 + 4x_3 = 8$$
$$x_1 + x_2 + x_3 = 3$$

soln Given,

$$[A/B] = \left[\begin{array}{ccc|c} 9 & 3 & 4 & 7 \\ 4 & 3 & 4 & 8 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

Interchanging R_1 & R_3

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 4 & 3 & 4 & 8 \\ 9 & 3 & 4 & 7 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - 4R_1$ $\quad$ $R_3 \leftrightarrow R_3 - 9R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & -5 & -20 \end{array}\right]$$

$R_3 \leftrightarrow R_3 - 6R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & -5 & -4 \end{array}\right]$$

$$-5x_3 = 4 \Rightarrow \boxed{x_3 = -\frac{4}{5}}$$

$$-x_2 = -4 \Rightarrow \boxed{x_2 = 4}$$

$$x_1 + 4 - \frac{4}{5} = 3$$

$$x_1 + \frac{16}{5} = 3$$

$$x_1 = 3 - \frac{16}{5}$$

$$\boxed{x_1 = -\frac{1}{5}}$$

⊛ solve using Gauss elimination

$$2x_1 - x_2 + x_3 = 4$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$3x_1 + 2x_2 + 2x_3 = 15$$

soln Given,

$$[A/B] = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 4 \\ 4 & 3 & -1 & 6 \\ 3 & 2 & 2 & 15 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - 2R_1 \qquad R_3 \leftrightarrow R_3 - \frac{3}{2}R_1$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 4 \\ 0 & 5 & -3 & -2 \\ 0 & \frac{7}{2} & \frac{1}{2} & 9 \end{array}\right]$$

$R_3 \leftrightarrow R_3 - \frac{7}{10}R_2$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 4 \\ 0 & 5 & -3 & -2 \\ 0 & 0 & \frac{26}{10} & \frac{104}{10} \end{array}\right]$$

$\frac{26}{10}x_3 = \frac{104}{10} \Rightarrow x_3 = \cancel{\frac{104}{26}}^{\,4} \Rightarrow \boxed{x_3 = 4}$

$5x_2 - 3[4] = -2$

$\boxed{x_2 = 2}$

$2x_1 - 2 + 4 = 4 \Rightarrow \boxed{x_1 = 1}$

(*) solve using Gauss elimination

$$2x+y+z=10$$
$$3x+2y+3z=18$$
$$x+4y+9z=16$$

soln Given,

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 10 \\ 3 & 2 & 3 & 18 \\ 1 & 4 & 9 & 16 \end{array}\right]$$

Interchanging R_3 and R_1

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 9 & 16 \\ 3 & 2 & 3 & 18 \\ 2 & 1 & 1 & 10 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - 3R_1$ $\quad$ $R_3 \leftrightarrow R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 9 & 16 \\ 0 & -10 & -24 & -30 \\ 0 & -7 & -17 & -22 \end{array}\right]$$

$-17 + \frac{24\times7}{10}$

$R_3 \leftrightarrow R_3 - \frac{7}{10}R_2$

$\frac{-170+168}{10} = \frac{-2}{10}$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 9 & 16 \\ 0 & -10 & -24 & -30 \\ 0 & 0 & -\frac{1}{5} & -1 \end{array}\right]$$

$-22 - \frac{7\times\cancel{30}}{\cancel{10}}$

$-110-56$

$-\frac{1}{5}z = -1 \quad \boxed{z=5}$

$x - 36 + 45 = 16$

$x + 9 = 16$

$\boxed{x=7}$

$5y + 12\times5 = 15$

$5y = -45 \quad \boxed{y=-9}$

* No. of elementary operations done for matrix of order n is $\boxed{\frac{n^3}{3}}$.

Advantages :-

→ Much less computation required for larger problems

→ Gauss elimination requires $\frac{n^3}{3}$ operations to solve a system of n equations

→ For 8 equations this works out to be 1070 operations (approx.) versus roughly 2.5 million operations for crammer's rule

Disadvantages :-

→ Not quite easy to remember procedure for hand solutions

→ Round off error may become significant but can be partially mitigated by using more advanced technology such as pivoting or scaling

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⇒ Gauss-Jordan method :-

- Reduce the matrix to row reduced echelon form
- Total no. of elementary operations is $\boxed{\frac{n^3}{2}}$

(*) Solve using Gauss-Jordan method.

a) $x+y+z=5$
$2x+3y+5z=8$
$4x+5z=2$

and

b) $x+2y-3z=2$
$6x+3y-9z=6$
$7x+14y-21z=13$

Sol: a) Given,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - 2R_1$ & $R_3 \leftrightarrow R_3 - 4R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$R_3 \leftrightarrow R_3 + 4R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$R_3 \leftrightarrow \frac{R_3}{13}$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$\cancel{C_2 \leftrightarrow C_2 - C_1}$

$$\cancel{\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]}$$

$R_2 \to R_2 - 3R_3$
$R_1 \to R_1 - R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$\cancel{x=7}$ $\cancel{y=4}$
$\cancel{z=-2}$

$R_1 \leftrightarrow R_1 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$x=3$ $y=4$
$z=-2$.

b) Given,

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

$R_3 \leftrightarrow R_3 - 7R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$ All zeros in last row

$\therefore$ The system is inconsistent and there is no solution

⊛ solve. $4y + z = 2$

$2x + 6y - 2z = 3$

$4x + 8y - 5z = 4$

Sol Given,

$$[A|B] = \left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

Interchange R_3 & R_1

$$\left[\begin{array}{ccc|c} 4 & 8 & -5 & 4 \\ 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - \frac{R_1}{2}$.

$$\sim \left[\begin{array}{ccc|c} 4 & 8 & -5 & 4 \\ 0 & 2 & \frac{1}{2} & 1 \\ 0 & 4 & 1 & 2 \end{array}\right]$$

$R_3 \leftrightarrow R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 4 & 8 & -5 & 4 \\ 0 & 2 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_1/4$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -5/4 & 1 \\ 0 & 2 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + 2y - \frac{5}{4}z = 1 \qquad 2y + z/2 = 1$

Let $\boxed{z = t}$

$2y + t/2 = 1$

$2y = 1 - t/2$

$\boxed{y = 1/2 - t/4}$

$x + 1 - t/2 - 5t/4 = 1$

$x = 5t/4 + t/2$

$\boxed{x = \frac{7t}{4}}$

→ Finding Inverse of matrix using GJ method

(*) Find the inverse of $\begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$

Sol: Given,

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & -2 & -1 & 1 & 0 & 0 \\ -4 & 1 & -1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

~~$R_2 \leftrightarrow R_2 + \frac{4}{3}R_1$~~ ~~$R_3 \leftrightarrow R$~~ $R_1 \leftrightarrow R_1 + R_3$ $R_2 \leftrightarrow R_2 + R_3$

$$\sim \left[\begin{array}{ccc|ccc} 5 & -2 & 0 & 1 & 0 & 1 \\ -2 & 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 + 2R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ -2 & 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_2 + 2R_1$ $R_3 \leftrightarrow R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 2 & 5 & 7 \\ 0 & 0 & 1 & -2 & -4 & -5 \end{array}\right]$$

∴ The Inverse matrix of given matrix is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$$

(*) Find Inverse of $\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ Hence solve the systems

$x = A^{-1} \cdot B$

a) $x + 3y + z = -1 \quad 2x - y + z = 1 \quad 3x + y + 2z = 2$

b) $x + 3y + z = 3 \quad 2x - y + z = 1; \quad 3x + y + 2z = 0$

Soln Given,

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$\cancel{R_2 \leftrightarrow R_3}$ $\quad R_2 \leftrightarrow R_2 - 2R_1 \quad R_3 \leftrightarrow R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -7 & -1 & -2 & 1 & 0 \\ 0 & -8 & -1 & -3 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & -8 & -1 & -3 & 0 & 1 \end{array}\right]$$

$R_3 \leftrightarrow R_3 + 8R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 5 & 8 & -7 \end{array}\right]$$

$R_3 \leftrightarrow -R_3 \quad R_1 \leftrightarrow R_1 - R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 6 & 8 & -7 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -5 & -8 & 7 \end{array}\right]$$

$R_1 \rightarrow R_1 - 3R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 5 & -4 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -5 & -8 & 7 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3 & 5 & -4 \\ 1 & 1 & -1 \\ -5 & -8 & 7 \end{bmatrix}$$

a). Given,

$$x + 3y + z = -1$$
$$2x - y + z = 1$$
$$3x + y + 2z = 2$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow \quad X = A^{-1}.B$$

$$X = \begin{bmatrix} 3 & 5 & -4 \\ 1 & 1 & -1 \\ -5 & -8 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$X = \begin{bmatrix} -6 \\ -2 \\ 11 \end{bmatrix}$$

b) Given,

$$x + 3y + z = 3$$
$$2x - y + z = 1$$
$$3x + y + 2z = 0$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$$X = \begin{bmatrix} 3 & 5 & -4 \\ 1 & 1 & -1 \\ -5 & -8 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 14 \\ 4 \\ -23 \end{bmatrix}$$

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* Matrix A and B are said to be row equivalent if either (hence each) can be obtained from the other by a sequence of elementary row operations

* An $n \times n$ matrix is called an elementary matrix

Row operation on I that produces E	Row operation on E that Reproduces I
Multiply row I by $c \neq 0$	Multiply row I by $1/c$
Interchange rows i and j	Interchange rows i and j
Add c times row i to row j	Add -c times row i to row j

⇒ LU-decomposition :-

(*) Find an LU decomposition of $A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}$

Soln Given,

$$A = \begin{bmatrix} 6 & -2 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}. \qquad I = \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

$$\xrightarrow{1/6 R_1} \begin{bmatrix} 1 & -1/3 & 0 \\ 9 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix} \qquad \xrightarrow{6R_1} \begin{bmatrix} 6 & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

$$\xrightarrow[R_3-3R_1]{R_2-9R_1} \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 2 & 1 \\ 0 & 8 & 5 \end{bmatrix} \qquad \xrightarrow[R_3+3R_1]{R_2+9R_1} \begin{bmatrix} 6 & 0 & 0 \\ 9 & * & 0 \\ 3 & * & * \end{bmatrix}$$

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$\xrightarrow{R_1/2}$ $\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 8 & 5 \end{bmatrix}$ $\xrightarrow{2R_2}$ $\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & * & * \end{bmatrix}$

$\xrightarrow{R_3-8R_2}$ $\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$ $\xrightarrow{R_3+8R_2}$ $\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & * \end{bmatrix}$

$\xrightarrow{1.R_3}$ $\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$ $\xrightarrow{!R_3}$ $\begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}$

$$\therefore \quad A = LU = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

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(*) Find an LU decomposition of $A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$

Sol: Given,

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\xrightarrow{\frac{1}{2}R_1}$ $\begin{bmatrix} 1 & 3 & 1 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$ $E_1 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_1^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\xrightarrow{R_2+3R_1} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 4 & 9 & 2 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3-4R_1} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3+3R_2} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \quad E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$A = LU$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & & \\ 0 & 0 & \end{bmatrix}$$

(*) Find an LU decomposition of co-eff. matrix & solve the system $2x+8y=-2$, $-x-y=-2 \Rightarrow x+y=2$

Sol $A = \begin{bmatrix} 2 & 8 \\ 1 & 1 \end{bmatrix}$ $\qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\xrightarrow{R_2 - R_1/2} \begin{bmatrix} 2 & 8 \\ 0 & -3 \end{bmatrix}$ $\qquad E_1 = \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix}$ $\qquad E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$

$\xrightarrow{R_1/2} \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ $\qquad E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$\xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & 4 \\ 0 & -3 \end{bmatrix}$ $\qquad \xrightarrow{R_2+R_1} E_2^{-1} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & -3 \end{bmatrix}$

$AX = B$

$LUX = B$ $\qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad B = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$\Rightarrow 2y_1 = -2$ $\qquad y_1 = -1$

$y_1 + y_2 = 2$ $\qquad y_2 = 3.$

$$\begin{pmatrix} 1 & 4 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$x + 4y = -1$ $\qquad x = 3$

$-3y = 3$ $\qquad y = -1.$

$\therefore \boxed{x = 3} \quad \boxed{y = -1}$

(*) Find an LU decomposition of co-eff. matrix and solve the system $3\times3 \rightarrow \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix}$

Sol Given

$$A = \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1/2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \qquad \xrightarrow{2R_1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3+R_1}\begin{bmatrix}1 & -1 & -1\\0 & -2 & 2\\0 & 4 & 1\end{bmatrix} \qquad \xrightarrow{R_3-R_1}\begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix}$$

$$\xrightarrow{R_3+2R_2}\begin{bmatrix}1 & -1 & -1\\0 & -2 & 2\\0 & 0 & 5\end{bmatrix} \qquad \xrightarrow{R_3-2R_2}\begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\-1 & -2 & 1\end{bmatrix}$$

$$A = \begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\-1 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & -1\\0 & -2 & 2\\0 & 0 & 5\end{bmatrix}$$

$$AX = B$$

$$LUX = B$$

$$\begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\-1 & -2 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & -1\\0 & -2 & 2\\0 & 0 & 5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-4\\-2\\6\end{bmatrix}$$

$$\begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\-1 & -2 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}-4\\-2\\6\end{bmatrix}$$

$2y_1 = -4 \qquad y_1 = -2$

$y_2 = -2$

$2+4+y_3 = 6 \qquad y_3 = 0$

$$\Rightarrow \begin{bmatrix}1 & -1 & -1\\0 & -2 & 2\\0 & 0 & 5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-2\\-2\\0\end{bmatrix} \Rightarrow \boxed{z=0}$$

$\boxed{y=1}$

$x-1=-2$

$\boxed{x=-1}$

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Friday

Module 2

Vector Spaces, the euclidean space & vector space sub space linear combination.

Euclidean Space : R^n / E^n

$R^2 = R \times R = \{(x, y) / x, y \in R\}$ - plane

$R^3 = \{(x, y, z) / x, y, z \in R\}$ - space

R - Real line

$R^n = \{(x_1, x_2, x_3 \cdots x_n) / x_i \in R \ \forall i\}$ - n-dimentional space

$\Rightarrow$ For $(R^n, +)$

$(x_1, x_2, \cdots x_n), (y_1, y_2, \cdots y_n) \in R^n \quad x_i, y_i \in R, \forall i$

- $(x_1, x_2, \cdots x_n) + (y_1, y_2 \cdots y_n) = (x_1+y_1, y_2+x_2 + \cdots x_n+y_n) \in R^n$, $x_i + y_i \in R, \forall i$

$\therefore$ R^n is closed under addition

- $[(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots y_n)] + (z_1, z_2 \cdots z_n) = (x_1+y_1, x_2+y_2 \cdots x_n+y_n) + (z_1, z_2 \cdots z_n)$

$= (x_1+y_1+z_1, \cdots x_n+y_n+z_n) \quad x_i+y_i+z_i \in R$

$= [x_1+(y_1+z_1) \cdots x_n+(y_n+z_n)]$

$= (x_1, x_2, x_3 \cdots x_n) + (y_1+z_1, y_2+z_2, \cdots y_n+z_n)$

$= (x_1, x_2, \cdots x_n) + [(y_1, y_2, \cdots y_n) + (z_1, z_2 \cdots z_n)]$

$\therefore$ R^n is associative under addition

- $\forall (x_1, x_2 \cdots x_n) \in R^n \ \exists (0, 0, \cdots n \text{ times}) \ni (x_1, x_2, \cdots x_n) + (0, 0 \cdots n \text{ times})$
$= (x_1, x_2, \cdots x_n)$

$\therefore$ Identity exists

- $\forall (x_1, x_2 \cdots x_n) \in R^n \ \exists (-x_1, -x_2 \cdots -x_n) \ni (x_1, x_2 \cdots x_n) +$
$(-x_1, -x_2, \cdots -x_n)$
$= (0, 0, 0 \cdots n \text{ times})$

$\therefore$ Additive Inverse exists

- $\forall (x_1, x_2 \cdots x_n)$ & $(y_1, y_2 \cdots y_n)$
$(x_1, x_2, \cdots x_n) + (y_1, y_2, \cdots y_n) = (x_1 + y_1, x_2 + y_2 \cdots x_n + y_n)$
$= (y_1 + x_1, y_2 + x_2, \cdots y_n + x_n)$ [As R is comm under +]

$\therefore R^n$ is commutative under addition

Hence, $\langle R^n, + \rangle$ is an Abelian group.

* For any scalar $c \in R$,
$c(x_1, x_2, \cdots x_n) = (cx_1, cx_2, \cdots cx_n) \in R^n$
as $cx_i \in R$

$\therefore$ closed under scalar multiplication

* for any scalar c and $(x_1, x_2, \cdots x_n), (y_1, y_2, \cdots y_n) \in R^n$
$c[(x_1, x_2, \cdots x_n) + (y_1, y_2, \cdots y_n)] = c[x_1 + y_1, x_2 + y_2, \cdots x_n + y_n]$
$= [c(x_1 + y_1), c(x_2 + y_2), \cdots c(x_n + y_n)]$
$= [cx_1 + cy_1, cx_2 + cy_2, \cdots cx_n + cy_n]$
$= c(x_1, x_2 \cdots x_n) + c(y_1, y_2, \cdots y_n)$

* $(c+d)(x_1, x_2, \dots x_n) = c(x_1, x_2, \dots x_n) + d(x_1, x_2, \dots x_n)$
* $1.(x_1, x_2, \dots x_n) = (x_1, x_2, \dots x_n)$
* $(cd)(x_1, x_2, \dots x_n) = c(d(x_1, x_2, \dots x_n))$

$\Rightarrow$ $\langle R, + \rangle$ is a commutative Group - [satisfy 5 laws]

6. $a, b \in R \Rightarrow ab \in R$ - closure under mul.
7. $a, b, c \in R, \Rightarrow a(bc) = (ab)c$ - Ass. under mul
8. $1 \in R$, $a.1 = 1.a = a$ - Identity

For 8 laws - $\langle R, +, \cdot \rangle$ is a $\boxed{\text{ring}}$.

9. $ab = ba \ \forall \ a, b \in R$. - comm. under mul

If this also satisfies $\langle R, +, \cdot \rangle$ is a $\boxed{\text{Commutative Ring}}$

10. $\forall a \in R \ \exists \frac{1}{a} \in R$; $a \times \frac{1}{a} = \frac{1}{a} \times a = 1$

If this satisfies, $\langle R, +, \cdot \rangle$ is a $\boxed{\text{division Ring}}$

* If both 9, 10 i.e., all the 10 conditions satisfies, then $\langle R, +, \cdot \rangle$ is a commutative div. ring also called $\boxed{\text{Field}}$

$\Rightarrow$ Vector space :-

$V = R^n = \{(x_1, x_2, x_3 \dots x_n) \mid x_i \in R \ \forall i\}$ = Euclidean space

1. $\langle V, + \rangle$ is an abelian group
2. For any scalar c & $u \in V$, $cu \in V$

3. For any scalar c & $u, v \in V$

$$c \cdot (u+v) = cu + cv$$

4. For scalar c and d and a vector $u \in V$

$$(c+d)u = cu + du$$

$$(cd)u = c(du)$$

5. $1 \cdot u = u$

* $V = \{0\}$ is a zero vector space

18/01/17
Wednesday

⇒ Vector subspace :-

W is said to be a vector subspace, if it is a subset of V and W itself is a vector space under the same binary operations addition and scalar multiplications defined on V.

$W \subset V \rightarrow$ W satisfy all 10 laws

Theorem

If W is a set of one or more vectors from a Vector space V, then W is a subspace of V if and only if the following conditions hold.

i) If u, v are vectors in W, then
$u+v \in W$
ii) If c is any scalar and u is a vector in W, then $cu \in W$

Ex: A set of all lines through the origin is a subspace of vector space $V = R^2 = R \times R$

Ex2: Let $W = \{(x,y) / x \geq 0, y \geq 0, x, y \in R\}$ is not a subspace of R^2

sol: Because Inverse of $(x, y) = (-x, -y) \notin W$
$\Rightarrow$ All the laws are not satisfying

$\Rightarrow$ If $Ax = b$ is a system of linear eqn, then each vector x that satisfies this eqn. is called a solution vector of the system

Theorem
If U, W are any 2 subspaces of a vector space V, then the intersection of U and W is also a subspace

Theorem
If $Ax = 0$, is a homogenious linear system of m equations in n unknowns then the set of solution vectors is a subspace of R^n.

Proof:-

$W = \{x / Ax = 0\}$

$x_1, x_2 \in W$

$Ax_1 = 0 \quad Ax_2 = 0$

$A(x_1 + x_2) =$

$Ax_1 + Ax_2 = 0 + 0$

$= 0$

$\Rightarrow x_1 + x_2$ is a soln. vector

$\therefore$ W is closed under addition

$\rightarrow A(cx) = c(Ax)$

$= c(0) = 0$

$\Rightarrow x \in W$, c is any scalar

$cx \in W$

$\therefore$ W is closed under scalar multiplication

24/01/17
Tuesday

Linear combination:-

A vector W is called linear combination of vectors $v_1, v_2 \ldots v_r$ if it can be expressed of the form $W = c_1v_1 + c_2v_2 \ldots + c_rv_r$, where c_1, c_2..etc. are scalars $(r \geq 1)$

Ex: vectors in R^3 are linear combinations of the unit vectors i, j, k $(a\hat{i} + b\hat{j} + c\hat{k})$

Ex2) Every vector $V = (a, b, c) \in R^3$ can be expressed in the form

$$V = a(1,0,0) + b(0,1,0) + c(0,0,1)$$
$$= a\hat{i} + b\hat{j} + c\hat{k}.$$

(*) Let $u = (1, 2, -1)$ $v = (6, 4, 2)$ belonging to R^3. Show that $W = (9, 2, 7)$ is a linear combination of u and v and also that $w' = (4, -1, 8)$ is not a linear comb of u and v

Sol: Given,

$$W = (9, 2, 7)$$

$$(9, 2, 7) = c_1(1, 2, -1) + c_2(6, 4, 2)$$

$$c_1 + 6c_2 = 9$$
$$2c_1 + 4c_2 = 2$$
$$-c_1 + 2c_2 = 7$$

$$8c_2 = 16$$
$$c_2 = 2.$$

$$c_1 + 12 = 9$$
$$c_1 = -3$$

$$-6 + 8 = 2 \checkmark$$

$$\therefore \boxed{c_1 = -3} \boxed{c_2 = 2}$$

$\therefore$ It is a linear combination.

$$W = -3u + 2v.$$

$$w' = (4, -1, 8)$$

$$(4, -1, 8) = c_1(1, 2, -1) + c_2(6, 4, 2)$$

$$c_1 + 6c_2 = 4$$
$$2c_1 + 4c_2 = -1$$
$$-c_1 + 2c_2 = 8$$

$$\left[\begin{array}{cc|c} 1 & 6 & 4 \\ 2 & 4 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 6 & 4 \\ 0 & -8 & -9 \end{array}\right]$$

$-8c_2 = -9 \quad c_2 = 9/8$

$$c_1 = 4 - \cancel{6}^{3}(9/\cancel{8}_{4}) = \frac{16-27}{4} = \frac{-11}{4}$$

$$\frac{11}{4} + \frac{2 \times 9}{\cancel{8}_4} = \frac{20}{4} = 5 \neq 8$$

c_1, c_2 satisfies (1) & (2) but not (3)

$\therefore$ It is not a linear combination

Spanning :

If $v_1, v_2 \cdots v_r$ are vectors in a vector space V, then generally some vectors in V may be a linear combination of $v_1, v_2 \cdots v_r$ and others may not be. If we construct a set W consisting of all those vectors that are linear combinations of $v_1, v_2 \cdots v_r$ then W forms a subspace of V.

Theorem

If $v_1, v_2, v_3 \cdots v_r$ are vectors in a vector space V, then the first conclusion i) the set W of all linear combinations of $v_1 v_2 \cdots v_r$ is a subspace of V.

ii) W is the smallest subspace of V that contains $v_1, v_2 \cdots v_r$.

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Wednesday

Span:

If $S = \{v_1, v_2, \cdots v_r\}$ is a set of vectors in a vector space V, then subspace W of V consisting of all linear combinations of the vectors in S is called the space spanned by $v_1, v_2 \cdots v_r$ and we say that the vectors $v_1, v_2 \cdots v_r$ span W.

- Denote it as $W = span(S)$ or
$W = span\{v_1, v_2 \cdots v_r\}$

Ex: $P_n = span\{1, x, x^2, \cdots x^n\}$
as $p = a_0 + a_1 x + \cdots a_n x^n$, a polynomial

(*) Determine whether $v_1 = (1, 1, 2)$ $v_2 = (1, 0, 1)$ and $v_3 = (2, 1, 3)$ span the vector space R^3.

Sol: Let,

$$b = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$(b_1\ b_2\ b_3) = k_1(1, 1, 2) + k_2(1, 0, 1) + k_3(2, 1, 3)$$

$$b_1 = k_1 + k_2 + 2k_3$$
$$b_2 = k_1 + k_3$$
$$b_3 = 2k_1 + k_2 + 3k_3$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$Ak = B$$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -1 - 1 + 2 = 0$$

$\therefore$ The system is inconsistent & Hence, $v_1\ v_2\ v_3$ do not span R^3

* The system is consistent for all values of $b_1\ b_2\ b_3$ iff the coefficient matrix A has a non-zero determinant

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Theorem:

If $S = \{v_1, v_2 \cdots v_r\}$ and $S_1 = \{w_1, w_2 \cdots w_k\}$ are two sets of vectors in a vector space V then span (S) = span (S_1) iff each vector in S is a linear combination of those in S_1 and vice versa.

⇒ If $S = \{v_1, v_2 - - v_r\}$ is a non empty set of vectors then the vector equations $k_1 v_1 + k_2 v_2 + \cdots k_r v_r = 0$ has at least one solution namely $k_1 = k_2 = \cdots k_r = 0$. If this is the ONLY solution then S is called a linearly independent set.

* If span$(S) = V$, then S spans (generates) V or V is spanned or generated by S.

Eg 1) $v_1 = (2, -1, 0, 3)$ $v_2 = (1, 2, 5, -1)$ and $v_3 = (7, -1, 5, 8)$ then the set $S = \{v_1, v_2, v_3\}$ is linearly dependent as $3v_1 + v_2 + v_3 = 0$.

2) i, j, k $i = (1, 0, 0)$ $j = (0, 1, 0)$ $k = (0, 0, 1)$ are linearly independent

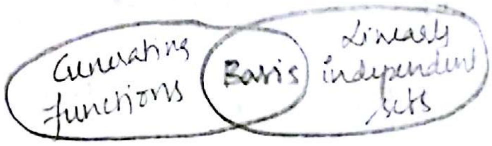
Theorem 2

(A property of linearly dependent sets)

A set $S = \{v_1, v_2, \dots, v_k\}$; $k \geq 2$ is linearly dependent if and only if at least one of the vectors v_i in S can be written as a linear combination of the other vectors in S.

Basis:-

V: a vector space



$S = \{v_1, v_2, \dots v_r\} \subseteq V$

a) S spans V [i.e, span$(S) = V$]

b) S is linearly independent

$\Rightarrow$ S is called a 'basis' for V.

Ex: i) ϕ is a basis for $\{0\}$

ii) The standard basis for R^3: $\{i, j, k\}$

$i = (1, 0, 0)$ $j = (0, 1, 0)$ $k = (0, 0, 1)$

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Tuesday

iii) $P_n(x)$: $\{1, x, x^2, \dots x^n\}$

Ex. $P_3(x)$ $\{1, x, x^2, x^3\}$

$P_n(x)$ — set of all polynomials of degree n

iv) the standard basis for $m \times n$ matrix space:

$$\{E_{ij} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

Ex. 2×2 matrix space

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

v) The standard basis for R^n:

$\{e_1, e_2, \dots e_n\}$ $e_1 = (1, 0 \dots 0)$, $e_2 = (0, 1, \dots 0)$

$\dots$ $e_n (0, 0, \dots 1)$.

Ex: R^4: $\{(1,0,0,0)(0,1,0,0), (0,0,1,0)(0,0,0,1)\}$

Theorem: Uniqueness of basis representation

If $S = \{v_1, v_2 \dots v_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = w \ ;$$

$$\beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n = w$$

then $\alpha_i = \beta_i \ \forall i$

Theorem: Bases and linear dependence

If $S = \{v_1 v_2, \dots v_n\}$ is a basis for a vector space V, then every set containing more than

n vectors in V is linearly dependent.

Theorem:

If the homogeneous system has fewer equations than variables, then it must have infinitely many solutions.

Theorem: Number of vectors in a basis

If a vector space V has one basis with n vectors, then every basis for V has n vectors. (All bases for a finite-dimentional vector space has the same number of vectors)

Finite dimensional:

A vector space V is called finite-dimensional if it has a basis consisting of a finite number of elements

Infinite dimensional:

If a vector space V is not finite dimentional, then it is called infinite dimensional.

Dimension:

The dimension of a finite dimensional vector space V is defined to be the number of vectors in a basis for V.

Ex:-

1) Vector space $\mathbb{R}^n$ $\Rightarrow$ basis $\{e_1, e_2, \cdots, e_n\}$

$\Rightarrow \dim(\mathbb{R}^n) = n$

2) Vector space $M_{m\times n}$ $\Rightarrow$ basis $\{E_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$

$\Rightarrow \dim(M_{m\times n}) = mn.$

3) Vector space $P_n(x)$ $\Rightarrow$ basis $\{1, x, x^2, \cdots x^n\}$

$\Rightarrow \dim(P_n(x)) = n+1$

4) Vector space $P(x)$ $\Rightarrow$ basis $\{1, x, x^2, \cdots\}$

$\downarrow$

$[P_\infty(x)]$ $\Rightarrow \dim(P(x)) = \infty$

1/02/2017
Wednesday

(*) Find the dimension of the subspace

$W = \{(d, c-d, c) \,;\, c, d \in \mathbb{R}\}$

Sol: Given,

$W = \{(d, c-d, c) \,;\, c, d \in \mathbb{R}\}$

$(d, c-d, c) = d(1, -1, 0) + c(0, 1, 1)$

Let $B = \{v_1 = (0, 1, 1),\ v_2 = (1, -1, 0)\}$

B spans W as

every set of W can be written as a

LC of v_1 & v_2

If possible let

$c_1(0,1,1) + c_2(1,-1,0) = (0,0,0)$

$\Rightarrow (c_2, c_1-c_2, c_1) = (0,0,0)$

$\Rightarrow c_2 = 0, c_1 = c_2, c_1 = 0$

$\Rightarrow \{v_1, v_2\}$ are linearly Independent

$\therefore B = \{v_1, v_2\}$ forms a basis

Hence, the dimension of $W = \underline{\underline{\text{"}2\text{"}}}$

(f) Find the dimension of the subspace $W = \{(2b, b, 0) / b \in R\}$

Sol: Given, $W = \{2b, b, 0) / b \in R\}$

$(2b, b, 0) = b(2,1,0)$

$\therefore$ W is generated by $(2,1,0)$

$\therefore B = \{v_1 = (2,1,0)\}$ spans W

If possible let,

$c_1(2,1,0) = (0,0,0)$

$\Rightarrow 2c_1 = 0 \quad c_1 = 0 \Rightarrow c_1 = 0$

$\therefore \{v_1\}$ is linearly independent

$\therefore B = \{v_1\}$ forms a basis.

Hence, the dimension of $W = \underline{\underline{\text{"}1\text{"}}}$

(*) Let W be the subspace of all symmetric 2×2 real matrices. What is the dimension of W.

Sol: Let,

$$W = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \;/\; a, b, c \in R \right\}$$

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ Let $B = \left\{ V_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, V_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, V_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Every set of W can be written as an LC of V_1, V_2, V_3

$\therefore$ B spans W.

If possible, let

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow c_1 = 0, \; c_2 = 0, \; c_3 = 0$

$\Rightarrow V_1, V_2, V_3$ are linearly independent

$\therefore B = \{V_1, V_2, V_3\}$ forms a basis

Hence the dimension of W = "3"

(*) Let H be the set of all vectors of the form $(a-3b, b-a, a, b)$ where a, b are arbitrary scalars. Show that H is a subspace of R^4

Sol: Given,

$H = \{(a-3b, b-a, a, b) / a, b$ are all constants$\}$

$(a-3b, b-a, a, b) = a(1,-1,1,0) + b(-3,1,0,1)$

$= aV_1 + bV_2 \in H$

$\therefore$ H is is a subspace of R^4

* It is enough to prove that the linear combination of a vector W is a closed
  - To prove W is a subspace

$\Rightarrow$ Let,
V be a vector space with dimension n, then the following statements holds,

i) Any L.I. set in V contains atmost 'n' vectors

ii) Any spanning set for V contains atleast 'n' vectors

iii) Any linearly independent set of exactly n vectors is a basis for V

iv) Any spanning set for V consisting of exactly n vectors is a basis for V.

v) Any linearly independent set in V can be extended to a basis for V

vi) Any spanning set for V can be reduced to a basis for V.

* $V = P_2$, $S = \{1+x, 2-x+x^2, -1+3x+x^2\}$ Determine whether S is a basis for V.

Sol Given,

$$V = P_2$$

Let $P_2 = a+bx+cx^2$

$$S = \{1+x,\ 2-x+x^2,\ -1+3x+x^2\}$$

$$a+bx+cx^2 = c_1(1+x) + c_2(2-x+x^2) + c_3(x^2+3x-1)$$

$$\Rightarrow \quad c_1+2c_2-c_3 = a$$

$$c_1 - c_2 + 3c_3 = b$$

$$c_2 + c_3 = c$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 1 & -1 & 3 & b \\ 0 & 1 & 1 & c \end{array}\right]$$

$R_2 \leftrightarrow R_2 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -3 & 4 & b-a \\ 0 & 1 & 1 & c \end{array}\right]$$

$R_3 \leftrightarrow R_3 + \frac{R_2}{3}$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -3 & 4 & b-a \\ 0 & 0 & 7/3 & c+\frac{b-a}{3} \end{array}\right]$$

$$\Rightarrow \quad \frac{7}{3}c_3 = c + \frac{b-a}{3}$$

$$7c_3 = 3c+b-a$$

$$\boxed{c_3 = \frac{3c+b-a}{7}}$$

$$-3c_2+\frac{4}{7}(3c+b-a)=b-a$$

$$-3c_2=b-a-\frac{12}{7}c-\frac{4}{7}b+\frac{4}{7}a$$

$$-3c_2=\frac{3b-3a-12c}{7}$$

$$\boxed{c_2=\frac{a-b+4c}{7}}$$

$$c_1+2\left[\frac{a-b+4c}{7}\right]-\left[\frac{3c+b-a}{7}\right]=a$$

$$c_1+\left[\frac{3a-3b+5c}{7}\right]=a$$

$$c_1=a-\frac{3a-3b+5c}{7}$$

$$c_1=\frac{4a+3b-5c}{7}$$

$$\boxed{c_1=\frac{4a+3b-5c}{7}}$$

$\therefore$ Any P_2 can be written as

$$a+bx+cx^2=c_1(1+x)+c_2(2-x+x^2)+c_3(x^2+3x-1)$$

If possible let,

$$c_1(1+x)+c_2(2-x+x^2)+c_3(x^2+3x-1)=0$$

$$c_1+2c_2-c_3=0$$
$$c_1-c_2+3c_3=0$$
$$c_2+c_3=0$$

$$[A|B]=\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & -1 & 3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - R_1$

$$= \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$R_3 \leftrightarrow 3R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right]$$

$\Rightarrow 7c_3 = 0$

$c_3 = 0$

$-3c_2 + 4(0) = 0$

$\Rightarrow c_2 = 0$

$c_1 + 2(0) - 1(0) = 0$

$\Rightarrow c_1 = 0$

$\therefore c_1 = 0;\ c_2 = 0;\ c_3 = 0$

$\therefore S$ is linearly independent.

$\therefore S = \{1+x,\ 2-x+x^2,\ -1+3x+x^2\}$ forms a basis for 'V'.

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3/02/17
Friday

Module 3

*Let A be an $m \times n$ matrix,

i) The 'row space' of A is the sub space row(A) of R^n spanned by the rows of A.

ii) The 'column space' of A is the sub space col(A) of R^m spanned by the columns of A

⇒ consider the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$

(*) i) Determine whether the vector B is the column space of A $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

ii) Determine whether the vector $W = [4 \ 5]$ is in the row space of A

iii) Describe the row & column space of A.

Soln Given,

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad W = \begin{bmatrix} 4 & 5 \end{bmatrix}$$

i) $$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{array}\right]$$

$$R_3 - 3R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\boxed{c_2 = 2}$$

$$c_1 - c_2 = 1$$

$$\boxed{c_1 = 3}$$

$\therefore$ The system is consistent with $c_1 = 3$ & $c_2 = 2$

$\therefore$ B is in col space of A

ii) To check whether W is in row space

$$\begin{bmatrix} A \\ \hline W \end{bmatrix} = \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 3 & -3 \\ \hline 4 & 5 \end{array}\right]$$

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$$\xrightarrow{\substack{R_3-3R_1 \\ R_4-4R_1}} \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 9 \end{array}\right]$$

$$\xrightarrow{R_4-9R_2} \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

$c_1 = 0 \quad -c_1 + c_2 = 0$

$c_2 = 0.$

The system is consistent, with $c_1 = 4$ $c_2 = 9$

(*) find the basis for the row space of $A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$

Soln Given

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

$$\begin{matrix} R_4 - 4R_1 \\ R_3 + 3R_1 \\ R_2 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 5 & 10 & 1 & 19 \\ 0 & -3 & -6 & -3 & -21 \end{bmatrix}$$

$$R_4 - R_2 \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 5 & 10 & 1 & 19 \\ 0 & 0 & 0 & -2 & -8 \end{bmatrix}$$

$$R_3 \leftrightarrow 3R_3 + 5R_2 \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & -2 & -8 \end{bmatrix}$$

$$R_4 - R_3 \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_1 + \frac{R_2}{3} \quad \sim \begin{bmatrix} 1 & 0 & 1 & \frac{2}{3} & \frac{5}{3} \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_3 \leftrightarrow \frac{R_3}{-2}$$

$$R_2 \leftrightarrow \frac{R_2}{-3} \quad \sim \begin{bmatrix} 1 & 0 & 1 & 2/3 & 5/3 \\ 0 & 1 & 2 & 1/3 & 13/3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to R_2 - R_3/3 \quad \sim \begin{bmatrix} 1 & 0 & 1 & 2/3 & 5/3 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to R_1 - \frac{2}{3}R_3 \quad \sim \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The row reduced form of A is

$$A_R = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

* All the non-zero rows in the row-reduced form of A gives the '<u>row spaces</u>' of A

$\therefore \{(1\,0\,1\,0\,-1), (0\,1\,2\,0\,3), (0\,0\,0\,1\,4)\}$

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<u>Saturday</u>

(*) Find the row space & col space for A.

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ -1 & 1 & -3 & 3 & 4 \\ 0 & -2 & 2 & -4 & 1 \\ 2 & 0 & 4 & -2 & 0 \\ 1 & 0 & 2 & -1 & 1 \end{bmatrix}$$

Find the basis.

<u>Sol</u> Given

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ -1 & 1 & -3 & 3 & 4 \\ 0 & -2 & 2 & -4 & 1 \\ 2 & 0 & 4 & -2 & 0 \\ 1 & 0 & 2 & -1 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 + R_1 \quad R_4 \leftrightarrow R_4 - 2R_1 \quad R_5 \leftrightarrow R_5 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 3 & -3 & 6 & 6 \\ 0 & -2 & 2 & -4 & 1 \\ 0 & -4 & 4 & -8 & -4 \\ 0 & -2 & 2 & -4 & -1 \end{bmatrix}$$

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$R_2 \leftrightarrow R_2/3$ $\cancel{R_3 \leftrightarrow R_3/2}$ $R_4 \leftrightarrow R_4/4$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & -2 & 2 & -4 & 1 \\ 0 & -1 & 1 & -2 & -1 \\ 0 & -2 & 2 & -4 & -1 \end{bmatrix}$$

$R_5 \leftrightarrow R_5 - R_3$ $R_4 \leftrightarrow R_4 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & -2 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 + 2R_2$ $R_5 \leftrightarrow R_5 + 2R_4$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -1 & -2 \\ 0 & 1 & -1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 + 2R_4$ $R_2 \leftrightarrow R_2 - 2R_4$ $\cancel{R_3 \leftrightarrow R_3 - 5R_4}$

$R_3 \leftrightarrow R_3/5$

$R_4 \leftrightarrow R_4 - R_3/5$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Row(A) = { (1 0 2 -1 0), (0 1 -1 2 0) (0 0 0 0 1) }

$\therefore$ This is the basis of row space of A.

Given A = $\begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ -1 & 1 & -3 & 3 & 4 \\ 0 & -2 & 2 & -4 & 1 \\ 2 & 0 & 4 & -2 & 0 \\ 1 & 0 & 2 & -1 & 1 \end{bmatrix}$

The corresponding column of Given A to the column with pivot in RR form gives the basis for col space of A.

$\therefore$ for the given A,

Col(A) = { (1 -1 0 2 1) (2 1 -2 0 0) (2 4 1 0 1) }

(*) Procedure :-

1. Find the reduced row echelon form R(A)
2. Use the nonzero row vectors of R (containing the leading 1's) to form a basis for row(A)
3. Use the column vectors of A that correspond to the columns of R containing the leading 1's (the pivot columns) to form a basis for col(A)
4. Solve the leading variables of $Rx = 0$ in terms of the free variables, set a free variables equal to parameters, substitute back into x and write the result as a linear combination of 'f' vectors where f is the no. of free variables. these f vectors form a basis for null(A).

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* Find the basis for col & row space of

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

Sol Given,

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - 2R_1$ $R_3 \leftrightarrow R_3 + 3R_1$ $R_4 \leftrightarrow R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & -3 & -6 & -1 & -13 \\ 0 & 5 & 10 & 1 & 19 \\ 0 & -3 & -6 & -3 & -21 \end{bmatrix}$$

$R_4 \leftrightarrow R_4 - R_2$ $R_2 \leftrightarrow R_2/(-3)$ $R_3 \leftrightarrow R_3/5$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & 1 & 2 & 1/3 & 13/3 \\ 0 & 1 & 2 & 1/5 & 19/5 \\ 0 & 0 & 0 & -2 & -8 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - R_2$ $R_4 \leftrightarrow R_4/(-2)$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & 1 & 2 & 1/3 & 13/3 \\ 0 & 0 & 0 & -2/15 & -8/15 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_3 \leftrightarrow R_3\left[\frac{-15}{2}\right]$ $\quad R_4 \leftrightarrow R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & 1 & 2 & 1/3 & 13/3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2/3 & 5/3 \\ 0 & 1 & 2 & 1/3 & 13/3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - R_3/3 \quad R_1 \leftrightarrow R_1 - 2/3 R_3$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Row(A) = $\{(1\ 0\ 1\ 0\ {-1})\ (0\ 1\ 2\ 0\ 3)\ (0\ 0\ 0\ 1\ 4)\}$

Col(A) = $\{(1\ 2\ {-3}\ 4)\ (1\ {-1}\ 2\ 1)\ (1\ 1\ {-2}\ 1)\}$

To get null space,

$$Rx = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 - x_5 = 0$$

$$x_2 + 2x_3 + 3x_5 = 0$$

$$x_4 + 4x_5 = 0$$

Let $x_5 = t$, $x_3 = s$

$$\Rightarrow x_1 = t - s$$

$$x_2 = -3t - 2s$$

$$x_4 = -4t$$

$$\therefore x = \begin{bmatrix} t-s \\ -2s-3t \\ s \\ -4t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$\therefore$ The null space of A is

$$\text{null}(A) = \{(-1\ -2\ 1\ 0\ 0)\ (1\ -3\ 0\ -4\ 1)\}$$

7/02/17
Tuesday

* A - $m \times n$ matrix

$$\dim(\text{row}(A)) = \dim(\text{col}(A)) = r = \text{rank}(A)$$

$$\dim(\text{null}(A)) = n - r = \text{Nullity}(A)$$

Theorem - Rank Nullity

If A is a matrix of order $m \times n$, then,

$$\text{rank}(A) + \text{nullity}(A) = n$$

(*) Find the basis for the row space, col space and null space of a matrix

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

Sol Given,

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & 10 & 0 \\ 2 & 6 & 18 & 8 & 6 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - 2R_1$ $\quad$ $R_3 \leftrightarrow R_3 - 2R_1$

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$$\Rightarrow \quad \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 5 & 15 & 10 & 0 \\ 0 & 10 & 18 & 8 & 0 \end{bmatrix}$$

$R_4 \leftrightarrow R_4 - 2R_3 \qquad R_2 \leftrightarrow -R_2$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 5 & 15 & 10. & 0 \\ 0 & 0 & -12 & -12 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_3/5 \qquad R_4 \leftrightarrow R_4/(-12)$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$\cancel{R_4 \leftrightarrow 3R_4 - R_3} \qquad \cancel{R_3 \leftrightarrow 2R_3 + R_1} \qquad R_3 \leftrightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

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$R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 + 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 6 & 4 & 3 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - 6R_3$ $\quad R_2 \leftrightarrow R_2 - 3R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore Row(A) = \{(1\ 0\ 0\ -2\ 3)\ (0\ 1\ 0\ -1\ 0)\ (0\ 0\ 1\ 1\ 0)\}$

$Col(A) = \{(1\ 2\ 0\ 2)\ (-2\ -5\ 5\ 6)\ (0\ -3\ 15\ 18)\}$

To get the null space of A,

$Rx = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 - 2x_4 + 3x_5 = 0$$
$$x_2 - x_4 = 0$$
$$x_3 + x_4 = 0$$

Let $x_4 = t, \quad x_5 = s$

$$\Rightarrow x_1 - 2t + 3s = 0$$
$$x_1 = 2t - 3s$$
$$x_2 = t$$
$$x_3 = -t$$
$$x_4 = t$$
$$x_5 = s$$

$$\therefore x = \begin{bmatrix} 2t-3s \\ t \\ -t \\ t \\ s \end{bmatrix} = s\begin{bmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ The null space of A is

null$(A) = \{(-3\ 0\ 0\ 0\ 1)\ (2\ 1\ -1\ 1\ 0)\}$

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(*) find the row space col space and null space of $A = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Sol Given,

$$A = \begin{bmatrix} 1 & 2 & -1 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & -9 & -1 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 + 9R_3 \quad R_2 \leftrightarrow R_2 - 4R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -64 \\ 0 & 1 & 0 & 31 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 + 64R_4 \quad R_2 \leftrightarrow R_2 - 31R_4 \quad R_3 \leftrightarrow R_3 + 7R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Row $(A) = \{(1\ 0\ 0\ 0)\ (0\ 1\ 0\ 0)\ (0\ 0\ 1\ 0)$
$(0\ 0\ 0\ 1)\}$

Col $(A) = \{(1\ 0\ 0\ 0)\ (2\ 1\ 0\ 0)\ (-1\ 4\ 1\ 0)$
$(5\ 3\ -7\ 1)\}$

To get null(A),

$RX = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 = 0 \quad x_2 = 0 \quad x_3 = 0 \quad x_4 = 0$

$\therefore$ null $(A) = \phi$

(*) Find the null space of $A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$

Also find the rank (A)

Sol Given,

$$A = \begin{bmatrix} 1 & -3 & 2 & 2 & 1 \\ 0 & 3 & 6 & 0 & -3 \\ 2 & -3 & -2 & 4 & 4 \\ 3 & -6 & 0 & 6 & 5 \\ -2 & 9 & 2 & -4 & -5 \end{bmatrix}$$

$R_2 \leftrightarrow R_2/3$

$R_3 \leftrightarrow R_3 - 2R_1$ $\quad$ $R_4 \leftrightarrow R_4 - 3R_1$ $\quad$ $R_5 \leftrightarrow R_5 + 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 3 & 2 & 0 & 2 \\ 0 & 3 & -6 & 0 & 2 \\ 0 & 3 & 6 & 0 & -3 \end{bmatrix}$$

$R_5 \leftrightarrow R_5 - 3R_2$ $\quad$ $R_4 \leftrightarrow R_4 - R_3$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 3 & 2 & 0 & 2 \\ 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - 3R_2$ $\quad$ $R_4 \leftrightarrow R_4/(-8)$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & -4 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - 2R_4$ $\quad$ $R_2 \leftrightarrow R_2 - 2R_4$ $\quad$ $R_3 \leftrightarrow R_3 + 4R_4$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - 3R_2$ $\quad$ $R_3 \leftrightarrow R_4$ $\quad$ $R_3 \leftrightarrow R_3/5$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - 4R_4 \quad R_2 \leftrightarrow R_2 + R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

To get null (A),

$\therefore \quad Rx = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 = 0 \quad x_2 = 0 \quad x_3 = 0 \quad x_4 = 0$

$x_5 = t$

$$\therefore \quad x = t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ null (A) = $\{(0\ 0\ 0\ 0\ 1)\}$

8/02/17
Wednesday

Polynomial

Theorem:
Given any n points in the x-y plane, that have distinct x co-ordinates. There is a unique polynomial of degree n-1 or less whose graph passes through these points

$$y = P(x) = a_0 + a_1x + \dots a_{n-1}x^{n-1}$$

Find a cubic polynomial whose graph passes through (1, 3) (2, -2) (3, -5) and (4, 0).

Sol: Given,
(1, 3) (2, -2) (3, -5) (4, 0)

Let, $y = a_0 + a_1x + a_2x^2 + a_3x^3$

Subst. (1, 3),

$$a_0 + a_1 + a_2 + a_3 = 3$$

Subst. (2, -2)

$$a_0 + 2a_1 + 4a_2 + 8a_3 = -2$$

Subst (3, -5)

$$a_0 + 3a_1 + 9a_2 + 27a_3 = -5$$

subst $(4,0)$

$$a_0 + 4a_1 + 16a_2 + 64a_3 = 0$$

$\therefore$ The equations are,

$$a_0 + a_1 + a_2 + a_3 = 3$$

$$a_0 + 2a_1 + 4a_2 + 8a_3 = -2$$

$$a_0 + 3a_1 + 9a_2 + 27a_3 = -5$$

$$a_0 + 4a_1 + 16a_2 + 64a_3 = 0$$

$$\therefore \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 8 & -2 \\ 1 & 3 & 9 & 27 & -5 \\ 1 & 4 & 16 & 64 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - R_1 \quad R_3 \leftrightarrow R_3 - R_1 \quad R_4 \leftrightarrow R_4 - R_1$

$$\sim \left[\begin{array}{ccccc} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 2 & 8 & 26 & -8 \\ 0 & 3 & 15 & 63 & -3 \end{array}\right]$$

$R_3 \leftrightarrow R_3 - 2R_2 \qquad R_4 \leftrightarrow R_4 - 3R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 0 & 2 & 12 & 2 \\ 0 & 0 & 6 & 42 & 12 \end{array}\right]$$

$R_3 \leftrightarrow R_3/2 \quad R_4 \leftrightarrow R_4/6. \quad R_1 \leftrightarrow R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & -2 & -6 & 8 \\ 0 & 1 & 3 & 7 & -5 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 1 & 7 & 2 \end{bmatrix}$$

$R_4 \leftrightarrow R_4 - R_3. \quad R_2 \leftrightarrow R_2 - 3R_3 \quad R_1 \leftrightarrow R_1 + 2R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 6 & 10 \\ 0 & 1 & 0 & -11 & -8 \\ 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - 6R_4 \quad R_2 \leftrightarrow R_2 + 11R_4 \quad R_3 \leftrightarrow R_3 - 6R_4$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore \quad a_0 = 4 \quad a_1 = 3 \quad a_2 = -5 \quad a_3 = 1$

Hence, $y = 4 + 3x - 5x^2 + x^3$

(*) Find the quadratic polynomial whose graph passes through (1, 1) (2, 2) (3, 5).

Sol Given,

(1, 1) (2, 2) (3, 5)

Let $y = a_0 + a_1 x + a_2 x^2$

Substituting all points,

$$a_0 + a_1 + a_2 = 1$$
$$a_0 + 2a_1 + 4a_2 = 2$$
$$a_0 + 3a_1 + 9a_2 = 5$$

$$\therefore \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 \\ 1 & 3 & 9 & 5 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - R_1$ $\quad R_3 \leftrightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 8 & 4 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - 2R_2$ $\quad R_1 \leftrightarrow R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$R_3 \leftrightarrow R_3/2$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - 3R_3 \quad R_1 \leftrightarrow R_1 + 2R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\therefore \quad a_0 = 2 \quad a_1 = -2 \quad a_2 = 1$

Hence, the polynomial is

$$y = 2 + 2x + x^2$$

(*) Integrate. $\int_0^1 \sin\left(\frac{\pi x^2}{2}\right) dx$

Soln Given,

$(0,1) \rightarrow (0, 0.25),\ (0.25 - 0.5),\ (0.5 - 0.75);\ (0.75 - 1)$

Let $f(x) = \sin\left(\frac{\pi x^2}{2}\right)$

$f(0) = 0$

$f(0.25) = 0.098017$

$f(0.5) =$

(*) Find the polynomial whose graph passes through (-1, -1) (0, 1) (1, 3) (4, -1)

Sol Given,

(-1, -1) (0, 1) (1, 3) (4, -1)

Let

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

Substituting the given points,

$$a_0 - a_1 + a_2 - a_3 = -1$$

$$a_0 = 1$$

$$a_0 + a_1 + a_2 + a_3 = 3$$

$$a_0 + 4a_1 + 16a_2 + 64a_3 = -1$$

Substituting $a_0 = 1$ in the other eqns

$$1 - a_1 + a_2 - a_3 = -1$$

$$a_1 - a_2 + a_3 = 2$$

$$a_1 + a_2 + a_3 = 2$$

$$4a_1 + 16a_2 + 64a_3 = -2$$

$$\therefore \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 4 & 16 & 64 & -2 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - R_1$ $\quad R_3 \leftrightarrow R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 20 & 60 & -10 \end{bmatrix}$$

$R_3 \leftrightarrow R_3/10 \quad R_2 \leftrightarrow R_2/2$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 6 & -1 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - 2R_2 \quad R_1 \leftrightarrow R_1 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & -1 \end{bmatrix}$$

$R_3 \leftrightarrow R_3/6$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/6 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 13/6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/6 \end{bmatrix}$$

$\therefore \ a_1 = 13/6 \quad a_2 = 0 \quad a_3 = -1/6.$

Hence, the polynomial is

$$y = 1 + \frac{13}{6}x - \frac{1}{6}x^3$$

9/02/17
Thursday

(*) Let P_4 be the vector space consisting of all polynomials of degree 4 or less with real coefficients. Let W be the sub space of P_4

$$W = \{ P(x) \in P_4 \mid P(1) + P(-1) = 0 \ \& \ P(2) + P(-2) = 0 \}$$

Find the basis of the sub space W and determine the dimension of W.

Sol: Given,

$$W = \{ P(x) \in P_4 \mid P(1) + P(-1) = 0 \ \& \ P(2) + P(-2) = 0 \}$$

Let $P(x) = W$

& let $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$P(1) + P(-1) = 0 \Rightarrow a_0 + a_1 + a_2 + a_3 + a_4 + a_0 - a_1 + a_2 - a_3 + a_4 = 0$

$\Rightarrow 2[a_0 + a_2 + a_4] = 0$

$\Rightarrow a_0 + a_2 + a_4 = 0$ —— (1)

$P(2) + P(-2) = 0 \Rightarrow a_0 + 4a_2 + 16a_4 = 0$ —— (2)

from (1) & (2)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 4 & 16 & 0 \end{array}\right]$$

$$\xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 15 & 0 \end{bmatrix}$$

$$R_2 \leftrightarrow R_2/3 \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 5 & 0 \end{bmatrix}$$

~~$R_2 \leftrightarrow R_2 - R_1$~~

$$a_2 + 5a_4 = 0$$
$$a_0 + a_2 + a_4 = 0$$

Let $a_4 = t$, $\Rightarrow$ $a_2 = -5a_4$

$a_0 - 5t + t = 0 \Rightarrow a_0 = 4a_4$

$\therefore$ $P(x) = 4a_4 + a_1 x - 5a_4 x^2 + a_3 x^3 + a_4 x^4$

$\Rightarrow P(x) = a_1 x + a_3 x^3 + a_4(4 - 5x^2 + x^4)$

Let, $B = \{x, x^3, 4 - 5x^2 + x^4\}$ generates W.

If possible, let

$c_1(x) + c_2(x^3) + c_3(4 - 5x^2 + x^4) = 0$

comparing co-efficients of x,

$c_1 = 0 = c_2 = c_3$

$\therefore$ B is linearly independent set of vectors

$\therefore$ Basis $= \{x, x^3, 4 - 5x^2 + x^4\}$

Hence, the dimension of W is "$\underline{3}$".

* Find the dimension of the plane $x+2z=0$ in R^3.

Solr Given,

$$x+2z=0$$

GS of $x+2z=0$ is,

let $z=s$, $y=t$ $\Rightarrow x=-2s$

$\therefore$ GS is $(-2s, t, s)$

$(-2s, t, s) = s(-2,0,1) + t(0,1,0)$

Let $B = \{v_1 = (-2,0,1), v_2 = (0,1,0)\}$

forms a basis for W.

* If $\det\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \neq 0$, they are linearly independent

$\Rightarrow$ Extend the basis of W to R^3,

Solr The standard basis of $R^3 = \{(1\,0\,0)(0\,1\,0)(0\,0\,1)\}$

Add this standard bases to bases of W.

Let $S = \{\underset{v_1}{(0,1,0)}\ \underset{v_2}{(-2,0,1)}\ \underset{v_3}{(1,0,0)}, \underset{v_4}{(0,0,1)}\}$

Here, v_2 is dependent on v_3 and v_4

$\therefore$ Let $S = \{(0\,1\,0)(-2\,0\,1)(1\,0\,0)\}$

to verify linear independency

$$\det|S| = \begin{vmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = -1 + 1 - 1 = -1 \neq 0$$

$\therefore$ Linearly independent.

Also,

$S_1 = \{(0\ 1\ 0)\ (-2\ 0\ 1)\ (0\ 0\ 1)\}$

By verifying, these 3 are also linearly independent.

$\therefore$ S, S_1, both will be the extended bases of W to R^3

(*) Let V and W be subspaces of R^2 spanned by $(1,1)$ and $(1,2)$ respectively. Find vectors $v \in V$, $w \in W$ so that $v+w = (2,-1)$

Sol Given,

$v+w = (2,-1)$

Let $s(1,1) + t(1,2) = (2,-1)$

$(s+t \quad s+2t) = (2,-1)$

$s+t = 2$

$\Rightarrow$ $s+2t = -1$

$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 2 & -1 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

$\therefore \quad t = -3$

$s - 3 = 2 \qquad s = 5$

$\therefore \quad \boxed{s = 5} \quad \boxed{t = -3}$

$\therefore \quad v = s(1, 1)$

$\Rightarrow \boxed{v = (5, 5)}$

$w = t(1, 2)$

$\Rightarrow \boxed{w = (-3, -6)}$

(*) Extend $\{1+x, 1-x\}$ to a basis of P_2

Sol) Given,

$B = \{1+x, 1-x\}$

The standard basis of $P_2 = \{1, x, x^2\}$

To extend B,

Let $S = \{1+x, 1-x, 1, x, x^2\}$

Here v_1 and v_2 are linearly dependent on v_3 and v_4

$\therefore$ Let $S = \{1+x, 1-x, x^2\}$

Because,

$\frac{1}{2}(1+x+1-x) = 1$ $\therefore \{1+x, 1-x, 1\}$ are l.d

$\frac{1}{2}(1+x-1+x) = x$ $\therefore \{1+x, 1-x, x\}$ are l.d

$\therefore S = \{1+x, 1-x, x^2\}$

forms the basis of P_2

11/02/17
Saturday

(*) Decompose the matrix

$$\begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix}$$ into L·U triangular matrices

and use the decomposition to solve $6x+18y+3z=3$

$2x+12y+z=19$ and $4x+15y+3z=0$

Sol Given,

$$A = \begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1/6 \sim \begin{bmatrix} 1 & 3 & 1/2 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix} \qquad 6R_1 \sim \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 3 & 1/2 \\ 0 & 6 & 0 \\ 4 & 15 & 3 \end{bmatrix} \qquad \xrightarrow{R_2+2R_1} \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\xrightarrow{R_3 - 4R_1} \begin{bmatrix} 1 & 3 & 1/2 \\ 0 & 6 & 0 \\ 0 & 3 & 1 \end{bmatrix} \qquad \xrightarrow{R_3 + 4R_1} \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2/2} \begin{bmatrix} 1 & 3 & 1/2 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \xrightarrow{R_3 + R_2/2} \begin{bmatrix} 6 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 1/2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2/6} \begin{bmatrix} 1 & 3 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \xrightarrow{6R_2} \begin{bmatrix} 6 & 0 & 0 \\ 2 & 6 & 0 \\ 4 & 1/2 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2/6} \begin{bmatrix} 1 & 3 & 1/2 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \qquad \xrightarrow{6R_2} \begin{bmatrix} 6 & 0 & 0 \\ 2 & 6 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 3 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \xrightarrow{R_3 + 3R_2} \begin{bmatrix} 6 & 0 & 0 \\ 2 & 6 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

$$\therefore \quad A = LU = \begin{bmatrix} 6 & 0 & 0 \\ 2 & 6 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given, $Ax = \begin{bmatrix} 3 \\ 19 \\ 0 \end{bmatrix}$

$$\therefore \begin{bmatrix} 6 & 0 & 0 \\ 2 & 6 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 19 \\ 0 \end{bmatrix}$$

Let $Ux = Y$

$$\Rightarrow \begin{bmatrix} 6 & 0 & 0 \\ 2 & 6 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 19 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 6y_1 = 3 \qquad y_1 = 1/2$$

$$1 + 6y_2 = 19 \qquad y_2 = 3$$

$$2 + 9 + y_3 = 0 \qquad y_3 = -11$$

$$\therefore \begin{bmatrix} 1 & 3 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3 \\ -11 \end{bmatrix}$$

$$\Rightarrow \quad z = -11$$

$$y = 3$$

$$x + 9 - 11/2 = 1/2$$

$$x = -9 + 6 = -3$$

$$\therefore \quad \boxed{x = -3} \quad \boxed{y = 3} \quad \boxed{z = -11}$$

* Let $V = \{(a_1, a_2) / a_1, a_2 \in R\}$. Define addition of elements of V coordinatewise and for $(a_1, a_2) \in V$ and $c \in R$.

$$c(a_1, a_2) = \begin{cases} (0,0) & \text{if } c = 0 \\ (ca_1, \frac{a_2}{c}) & \text{if } c \neq 0 \end{cases}$$

Check whether V is a vector space over R or not. Justify your answer

Soln Let $u = (a_1, a_2)$ $c, d \in R$

$(c+d)u = (c+d)(a_1, a_2)$

$= \left((c+d)a_1, \frac{a_2}{c+d}\right)$ [by given] $c+d \neq 0$

$cu + du = c(a_1, a_2) + d(a_1, a_2)$

$= (ca_1, \frac{a_2}{c}) + (da_1, \frac{a_2}{d})$

$= \left[(c+d)a_1, a_2(1/c + 1/d)\right]$ $c \neq 0$, $d \neq 0$

$\therefore$ LHS $\neq$ RHS

Property (8) is not satisfied here

Hence it is <u>not</u> a vector space.

(1) (*) Find the inverse of the matrix $A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$ Using elementary row transformation.

(2) (*) Suppose A and its row reduced echelon form are given as

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ -1 & 1 & -3 & 3 & 4 \\ 0 & -2 & 2 & -4 & 1 \\ 2 & 0 & 4 & -2 & 0 \\ 1 & 0 & 2 & -1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i) Find the basis for col space of A
ii) Row space
iii) Null space
iv) Find rank and nullity of A.

(1) Sol Given

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \to R_1 - R_2 \quad \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 2 & 0 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & -2 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 + R_3 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -2 & 3 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 + R_3 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -2 & 3 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$R_3 \to R_3 - \cancel{2}R_2/2 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & -2 & 3 & 1 \\ 0 & 0 & 1 & -1 & -3/2 & 1/2 \end{array}\right]$

$R_2 \to R_2/2 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 3/2 & 1/2 \\ 0 & 0 & 1 & -1 & -3/2 & 1/2 \end{array}\right]$

$\therefore$ Inverse of the given matrix A is

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 1/2 \\ -1 & -3/2 & 1/2 \end{bmatrix}$$

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(2)
Soln Given,

$$RREF = \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i) Basis for col(A)

$$B = \{ (1\ -1\ 0\ 2\ 1), (2\ 1\ -2\ 0\ 0), (2\ 4\ 1\ 0\ 1) \}$$

ii) Basis for row(A)

$$B = \{ (1\ 0\ 2\ -1\ 0)(0\ 1\ -1\ 2\ 0)(0\ 0\ 0\ 0\ 1) \}$$

iii) Basis for null(A)

Let $RX = 0$

$$\begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_3 = 0 \quad x_2 + 2x_4 = 0 \quad x_1 - x_4 = 0$

Let $x_4 = t \quad x_5 = s$

$\Rightarrow x_2 = -2t \quad x_1 = t$

$\Rightarrow (x_1\ x_2\ x_3\ x_4\ x_5) = (t\ -2t\ 0\ t\ s)$

$$\therefore x = t\begin{bmatrix}1\\-2\\0\\1\\0\end{bmatrix} + s\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$$

$$\therefore null(A) = \{(1\ -2\ 0\ 1\ 0)\ (0\ 0\ 0\ 0\ 1)\}$$

iv) To find the rank and nullity of A

$rank(A) = dim(row(A)) = dim(col(A))$

$= 3.$

$Nullity(A) = dim(null(A)) = n - r$

$= 2.$

15/02/17

① * Consider all 4×4 matrices where the sum of all elements in the perimeter of the matrix is equal to half the sum of the four elements in the inner position. Show that this set is a vector space

Prove all 10 laws

Hint: $V = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \middle/ \begin{array}{l} a_{11}+a_{12}+a_{13}+a_{14}+a_{24}+a_{34} \\ +a_{44}+a_{43}+a_{42}+a_{41}+a_{31}+a_{21} \\ = \frac{1}{2}\{a_{22}+a_{23}+a_{32}+a_{33}\} \end{array} \right\}$

② * In P_5 consider the subset of all polynomials which are in the form $(x^2-6x+3)g(x)$, for some polynomial $g(x)$ show that this is a vector subspace of P_5.

Hint: $W \subset P_5$,

$W = \{(x^2-6x+3)(ax^3+bx^2+cx+d)\}$

Prove $\alpha u_1 + \beta u_2 \in W$ where $u_1, u_2 \in W$, $\alpha, \beta \in F$

③ * Show that the set $\{(x+1)^2, (x+2)^2, x^2-1\}$ is a basis for P_2 and write down the co-ordinates of $4x^2-3x-7$ w.r.t this basis.

Hint: ① span & ②

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④ ✱ Determine which sets are vectorspaces under the given operations. For those that are not vector spaces, list all axioms that fail to hold

a) The set of all pairs of real no.s of the form $(1, x)$ with the operations,

$(1, y) + (1, y') = (1, y+y')$ and $k(1, y) = (1, ky)$

Hint: All 10 laws, it is a vector space

b) The set of all triples of real numbers with addition defined by $(x, y, z) + (u, v, w) = (z+w, y+v, x+u)$ and standard scalar multiplication

c) The set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with r.t. matrix addition and scalar mul.

d) The set of all +ve real numbers with the operations $x+y = xy$ and $kx = x^k$

⑤ ✱ If $V = R^n$, then check whether $W = \{(x_1, x_2 \dots x_n) \mid x_1 + x_2 + \dots x_n = 1, x_i \in R\}$ is a subspace of V.

Hint: * find $1 \cdot \alpha \in W$ such that $W \neq \phi$

* $\alpha u + \beta v \in W$

⑥ ✱

⑦ ✱

⑧ ✱

⑨ ✱

⑥ * check whether the following set $W = \{ f(x) \in R[x] \mid f(2) = 0 \}$ is a subspace of $R[x]$.
↳ step function

⑦ * Show that the given sets of functions are linearly independent in the vector space $C[-\pi, \pi]$
↳ ?

i) $\{1, x, x^2, x^3, x^4\}$

ii) $\{1, e^x, e^{2x}, e^{3x}\}$

⑧ * Find the basis for the row space, col space and null space for each of the following

i) $A = \begin{bmatrix} 1 & 2 & 1 & 5 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix}$

ii) $\begin{bmatrix} 0 & 1 & -1 & -2 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ 2 & 1 & -1 & 8 & 3 \\ 0 & 0 & -2 & 2 & 1 \\ 3 & 5 & -5 & 5 & 10 \end{bmatrix}$

⑨ * In the 3-space R^3, let W be the set of all vectors (x, y, z) that satisfy the equation, $x - y - z = 0$. Prove that W is subspace of R^3. Find a basis for the subspace W.

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9 sol: Given,

W is a set of all vectors with (x, y, z) that satisfy the equation $x-y-z=0$

Consider u_1 and u_2

$u_1 = (x_1, y_1, z_1) \Rightarrow x_1-y_1-z_1=0$ — ①

$u_2 = (x_2, y_2, z_2) \Rightarrow x_2-y_2-z_2=0$ — ②

Linear combination of u_1 and u_2

$\Rightarrow c_1u_1+c_2u_2$ ∵ where c_1, c_2 are scalars

$\Rightarrow c_1(x_1, y_1, z_1) + c_2(x_2, y_2, z_2)$

Let $(X, Y, Z) = (c_1x_1, c_1y_1, c_1z_1) + (c_2x_2, c_2y_2, c_2z_2)$

$= (c_1x_1+c_2x_2, c_1y_1+c_2y_2, c_1z_1+c_2z_2)$

considering the condition,

$x-y-z=0$

$X-Y-Z = (c_1x_1+c_2x_2) - (c_1y_1+c_2y_2) - (c_1z_1+c_2z_2)$

$= (c_1x_1-c_1y_1-c_1z_1) + (c_2x_2-c_2y_2-c_2z_2)$

$= c_1(x_1-y_1-z_1) + c_2(x_2-y_2-z_2)$

$= c_1(0) + c_2(0)$ [From ① & ②

$= 0$

$\therefore x - y - z = 0$

$\Rightarrow c_1 u_1 + c_2 u_2 \in W$.

$\therefore W$ is a subspace of R^3

To find the basis for W,

we have $x - y - z = 0$

$x = y + z$

Let $y = s$ $z = t$

$\Rightarrow x = s + t$.

$\therefore (s+t, s, t) = s(1, 1, 0) + t(1, 0, 1)$

$\therefore$ Basis for $W = \{(1, 1, 0), (1, 0, 1)\}$

8 soln

i) Given,

$$A = \begin{bmatrix} 1 & 2 & 1 & 5 \\ 2 & 4 & 3 & 0 \\ 1 & 2 & -1 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - 2R_1$ $R_3 \leftrightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 0 & -5 & -10 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 / (-5)$ $R_3 \leftrightarrow R_3 / (-2)$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ RREF of $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$row(A) = \{(1\ 2\ 0\ 3)\ (0\ 0\ 1\ 2)\}$

$col(A) = \{(1\ 2\ 1)\ (1\ -3\ -1)\}$

Let $RX = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_3 + 2x_4 = 0 \qquad x_1 + 2x_2 + 3x_3 = 0$

Let $x_2 = s \quad x_3 = t$

$x_4 = -t/2 \qquad x_2 = s$

$\Rightarrow x_1 = -2s - t \qquad x_3 = t$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s-3t \\ s \\ t \\ -t/2 \end{bmatrix}$$

$$= s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 1 \\ -1/2 \end{bmatrix}$$

null$(A) = \{(-2\ 1\ 0\ 0)\ (-3\ 0\ 1\ -1/2)\}$

ii) Given,

$$A = \begin{bmatrix} 0 & 1 & -1 & -2 & 1 \\ 1 & 1 & -1 & 3 & 1 \\ 2 & 1 & -1 & 8 & 3 \\ 0 & 0 & -2 & 2 & 1 \\ 3 & 5 & -5 & 5 & 10 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 2 & 1 & -1 & 8 & 3 \\ 0 & 0 & -2 & 2 & 1 \\ 3 & 5 & -5 & 5 & 10 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - 2R_1 \qquad R_5 \leftrightarrow R_5 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 2 & -2 & -4 & 7 \end{bmatrix}$$

$R_5 \leftrightarrow R_5 + 2R_3$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & -1 & 1 & 2 & 1 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 3 & 1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - R_2$ $\quad R_3 \to R_3/2$ $\quad R_3 \leftrightarrow R_5 - 9R_3$

$R_3 \leftrightarrow R_5$ $\quad R_5 \leftrightarrow R_4$

$R_4 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 \to R_4/9$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - R_4$ $\quad R_3 \leftrightarrow R_3 - R_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_3/(-1)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 + R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \text{row}(A) = \{(1\ 0\ 0\ 5\ 0)\ (0\ 1\ 0\ -3\ 0)$
$(0\ 0\ 1\ -1\ 0)\ (0\ 0\ 0\ 0\ 1)\}$

$\therefore \text{Col}(A) = \{(0\ 1\ 2\ 0\ 3)\ (1\ 1\ 1\ 0\ 5)$
$(-1\ -1\ -1\ -2\ -5)\ (3\ 5\ -5\ 5\ 10)\}$

To find null(A),

$Rx = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_5 = 0 \quad x_3 - x_4 = 0 \Rightarrow \cancel{x_3 = 0}$

$x_2 - 3x_4 = 0 \Rightarrow x_2 = 0 \quad x_1 + 5x_4 = 0 \ \cancel{\Rightarrow x_1 = 0}$

$x_4 = t$

$\therefore \text{null}(A) = \{(-5, 3, 1, 1, 0)\}$

~~{Soln}~~

~~i) Given, $\{1, x, x^2, x^3, x^4\}$~~

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Note:-

1. Suppose if 0 is one of the vectors, $v_1, v_2 \ldots v_m$, say $v_1 = 0$. Then the vectors must be linearly dependent as we have the following linear combination where the co-efficient of $v_1 \neq 0$.,

$$1 \cdot v_1 + 0v_2 + \ldots 0v_m = 0.$$

2. Suppose v is a non-zero vector, then v by itself is linearly independent.

3. Suppose two of the vectors, $v_1, v_2 \ldots v_m$ are equal or one is a scalar multiple of the other, then the vectors must be linearly dependent.

4. Two vectors v_1 and v_2 are linearly dependent if and only if one of them is a multiple of the other.

5. If $\{v_1, v_2 \cdots v_m\}$ is linearly independent, then any re-arrangement of the vectors is also linearly independent.

6. If a set of vectors S is linearly independent, then any subset of S is linearly independent. Alternatively if S contains a linearly dependent subset, then S is linearly dependent.

(*) Determine which of the following list of vectors is linearly independent

i) $\{(1, 2, 0, -1, 5), (0, 0, 0, 0, 0), (15, 6, 2, -17, 0)\}$

Sol: Here, one of the vectors is zero vector.

Hence this set of vectors is linearly dependent.

ii) $\{(5, 7)\}$

Sol: This set has only one non-zero vector $(5, 7)$.

Hence this set is linearly independent.

iii) $\{(3,1,4), (-2,2,5), (3,0,4) (2\ -1\ -2)\}$

Solb As the dimension of R^3 is 3, and the given set has got 4 elements, (1 more than the dimension), The set of vectors become linearly dependent

iv) $\{(1\ 0\ 0\ 0)\ (1\ 1\ 0\ 0)\ (1\ 1\ 1\ 0)\ (1\ 1\ 1\ 1)\}$

Sol: None of the vectors in the given set can be expressed as a linear combination of the other vectors.

∴ Linearly independent

*v) $\{(1\ 2\ 3)\ (3\ 2\ 1)\ (2\ 1\ 3)\}$

Sol: finding the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5 - 2(7) + 3(-1) \neq 0$$

∴ Linearly independent

Sums and Direct sums :-

Sums:-

⇒ Let U and W be subsets of a vector space V. The sum of U and W is written as

$$U+W = \{ v : v = u+w,\ u \in U,\ w \in W \}$$

⇒ Suppose U and W are subspaces of V, then U+W is a subspace of V and also U∩W is also a subspace of V.

⇒ Suppose U and W are finite dimensional subspace of V, then U+W has a finite dimension

$$\boxed{\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)}$$

Example 1:

⇒ Let $V = M_{22}$, the vector space of 2×2 matrices., Let U consists of those matrices whose 2nd row is '0'. and W consists of those matrices whose 2nd col is '0'.

Then,
$$U = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\} \quad W = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \right\}$$

$$U+W = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \right\}$$

Here,

$$U = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$B_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

$\therefore \dim(U) = 2$

$$W = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$B_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

$\therefore \dim(W) = 2$

$$U \cap W = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

$$B_3 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$$

* $\therefore \dim(U \cap W) = 1$

$$\therefore \dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$
$$= 2 + 2 - 1$$
$$= 3.$$

→ **Direct sums:-**

The vector space V is said to be the direct sum of its subspaces U and W, defined by

$\boxed{V = U \oplus W}$

if every $v \in V$ can be written in one and only way as $v = u + w$, where $u \in U$ $w \in W$

Theorem:
the vector space V is the directsum of subspaces, U and W, if and only if,

$V = U + W$ and $U \cap W = \phi$

* consider the following subspaces of R^5

$U = \text{span}(u_1, u_2, u_3)$ $W = \text{span}(w_1, w_2, w_3)$ where

$u_1 = (1\ 3\ -2\ 2\ 3)$ $u_2 = (1\ 4\ -3\ 4\ 2)$ $u_3 = (2\ 3\ -1\ -2\ 9)$

$w_1 = (1\ 3\ 0\ 2\ 1)$ $w_2 = (1\ 5\ -6\ 6\ 3)$ $w_3 = (2\ 5\ 3\ 2\ 1)$

find a basis for $U + W$. and also find the dimension

Sol Given,

U, W are subspaces of R^5

$u_1\ u_2\ u_3$ spans U

w_1, w_2, w_3 spans W

writing a 6×5 matrix,

$$A = \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 9 \\ 1 & 3 & 0 & 2 & 1 \\ 1 & 5 & -6 & 6 & 3 \\ 2 & 5 & 3 & 2 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_2 - R_1 \quad R_3 \leftrightarrow R_3 - 2R_1 \quad R_4 \leftrightarrow R_4 - R_1 \quad R_5 \leftrightarrow R_5 - R_1$

$R_6 \leftrightarrow R_6 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 3 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 2 & -4 & 4 & 0 \\ 0 & -1 & 7 & -2 & -5 \end{bmatrix}$$

$R_3 \leftrightarrow R_3 - 3R_2 \quad R_4 \leftrightarrow R_4/2 \quad R_5 \leftrightarrow R_5/2$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & -1 & 7 & -2 & -5 \end{bmatrix}$$

Interchanging the rows,

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & -1 & 7 & -2 & -5 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 \leftrightarrow R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 5 & 0 & -5 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_5 \leftrightarrow 5R_5 - R_4$, $R_4 \leftrightarrow R_4/5$

$$\sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - 3R_3$ $R_2 \leftrightarrow R_2 - R_3$

$$\sim \begin{bmatrix} 1 & 0 & 4 & -4 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 \leftrightarrow R_4 - R_2$ & $R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & 4 & -4 & 3 \\ 0 & 1 & -2 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_1 - 4R_3$ $\quad R_2 \leftrightarrow R_2 + 2R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -4 & 7 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The basis for $U+W$ is.

$\{(1\ 0\ 0\ -4\ 7)\ (0\ 1\ 0\ 2\ -2)\ (0\ 0\ 1\ 0\ -1)\}$

Dimension of $U+W$ is "$\underline{\underline{3}}$".

Polynomials as Vector spaces :-

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Let P_n denote the set of all polynomials of degree ≤ 1 with real co-efficients, Any such polynomial can be uniquely expressed in the form $a_0 + a_1 t + a_2 t^2 \ldots a_n t^n$. This means the set $\{1, t, t^2 \ldots t^n\}$ forms a basis.

Thus the dimension of P_n is $n+1$.

theorem:

Any set of polynomials, all having different degrees is linearly independent

⇒ Let $V \subset P_3$ be the collection of all polynomials having 5 as a root. If $f(t)$ and $g(t)$ have 5 as a root, then.

$af(t) + bg(t)$ also has 5 as a root.

⇒ clearly V is a proper vector subspace of P_3. and so the dimension of V is less than 4.

⇒ clearly $(t-5)$, $(t-5)^2$, $(t-5)^3$ are all in P_3 and they all have 5 as a root.

$\Rightarrow$ By the above theorem, as they have different degrees they are linearly independent.

$\Rightarrow$ Hence, $\{(t-5)(t-5)^2(t-5)^3\}$ forms a basis for V

Theorem

In the vector space P_n of dimension $n+1$, the set of all polynomials V having a specific no. 'r' as a root is a vector subspace of dimension n and a basis for this is
$$\{(t-r)(t-r)^2 \cdots (t-r)^n\}$$

(*) Let $W \subset P_3$ be the set of all polynomials having 3 as a root. Find a basis for W which contains the polynomials t^2-6t+5 and t^3-5t^2-t+5.

Sol:- $\Rightarrow$ The first polynomial has 2, 3 as its roots while the second has 1, -1 and 3 as roots. Therefore, they both belong to W.

→ Note that they are not scalar multiples of each other ∴ The two vectors are linearly independent

→ By the prev. theorem, we know that W has dimension '3'. So, we need to find one more polynomial in P_3 with 3 as a root and L.I to the two given polynomials

(or it can be 't' itself)

→ For example, let us take a quadratic polynomial

$(t-3)(t-a) = t^2-(a+3)t+3a$

If these 3 polynomials must be L.I.,

the coefficient matrix of 3 polynomials must be non-zero.

Consider, (0 1 0) in first row, (1 −5 1) in 2nd row, (−6, −1, −(a+5)) in 3rd row, (5 5 5a) in 4th

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -5 & 1 \\ -6 & -1 & -(a+5) \\ 5 & 5 & 5a \end{bmatrix}$$

[write column wise]

We need to find 'a' such that the 3 columns of matrix are linearly independent

This means, some 3×3 formed out of it should be non-zero.

→ If $a = 1$, then the value of determinant under consideration will be '0'

→ Therefore, ~~a≠0 and~~ a can be anything other than 1.

Let,

$$\begin{vmatrix} 1 & -5 & 1 \\ -6 & -1 & -(a+3) \\ 5 & 5 & 3a \end{vmatrix} = -3a+5a+15+5[-18a+5a+15] + [-30+5]$$

$$= 2a+15-65a+75-25$$

$$= -63a+65 \neq 0$$

$$a \neq \frac{65}{63}$$

→ a can be anything other than 65/63, so let $a = 5$.

$\therefore (t-5)(t-3) \Rightarrow t^2 - 8t + 15$

$\therefore$ Basis for $\mathbb{R}^3$, that contains the given polynomials is

$\{ t^2 - 6t + 5,\ t^3 - 5t - t + 5,\ t^2 - 8t + 15 \}$.

(1) Consider the vectorspace $P_3(t)$ of polynomials of degree ≤ 3.

i) Show that $\{(t-1)^3, (t-1)^2, (t-1), 1\}$ forms a basis for $P_3(t)$

ii) Find the co-ordinate vector of v where $v = 3t^3+4t^2+2t-5$ relative to this basis.

$[v] = [c_1\ c_2\ c_3\ c_4]$

3
13
19
4

(2) Find the co-ordinate vector of A which is $\begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix}$ in the real vector space $M = M_{2\times2}$ relative to:

i) The basis $S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

ii) w.r.t the standard basis.

1sol Given,

$$P_3(t) = a_0 + a_1x + a_2x^2 + a_3x^3.$$

i) Given,

$\{(t-1)^3, (t-1)^2, (t-1), 1\}$

To find whether it is a basis of $P_3(t)$

As the degree of each element is different,

all are linearly independent

thus given seat forms a basis for $P_3(t)$

ii) Given,

$$V = 3t^3+4t^2+2t-5$$

To find the coordinate vector of V

Let

$$(3t^3+4t^2+2t-5) = c_1(t-1)^3 + c_2(t-1)^2 + c_3(t-1) + c_4$$

$$\Rightarrow c_1 = 3.$$

$$3t^3+4t^2+2t-5 = c_1(t^3-1-3t^2+3t) + c_2(t^2+1-2t) + c_3(t-1) + c_4$$

$$= c_1t^3 + t^2(-3c_1+c_2) + t(+3c_1-2c_2-c_3) - c_1+c_2-c_3+c_4$$

$$\Rightarrow c_1 = 3; \quad -3c_1+c_2 = +4 \qquad +3c_1-2c_2-c_3 = 2$$

$$\Rightarrow c_2 = +13 \qquad \Rightarrow +9-26-c_3 = 2$$

$$c_3 = 19$$

$$-c_1+c_2-c_3+c_4 = -5$$

$$-3+13-19+c_4 = -5$$

$$c_4 = 4$$

$$\therefore [V] = [c_1 \quad c_2 \quad c_3 \quad c_4]$$

$$= [3 \quad 13 \quad 19 \quad 4]$$

Soln
Given $A = \begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix}$

i) Given
$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix} = c_1\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_4\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix} = \begin{bmatrix} c_1+c_2+c_3+c_4 & c_1-c_2-c_3 \\ c_1+c_2 & c_1 \end{bmatrix}$$

$\Rightarrow c_1 = -7 \quad c_2 = 11$

$c_3 = -21 \qquad c_4 = 2-11+7+21$

$c_4 = 19$

$\therefore [V] = [-7 \quad 11 \quad -21 \quad 19]$

ii) standard basis
$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 4\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-7)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore [V] = [2 \quad 3 \quad 4 \quad -7]$

PROBLEMS ON VECTOR SPACES

Solutions:-

5 Soln

Given, $V = R^n$

$$W = \{(x_1, x_2, x_3 \cdots x_n) \mid x_1 + x_2 + \cdots x_n = 1, x_i \in R\}$$

Let a vector $u = (x_1\ x_2 \cdots x_n)$. $u \in W$.

$v = (y_1, y_2 \cdots y_n)$ $v \in W$

Let c_1, c_2 be constants

$$c_1 u + c_2 v$$
$$= c_1 (x_1\ x_2 \cdots x_n) + c_2 (y_1, y_2 \cdots y_n)$$
$$= (c_1 x_1 + c_2 y_1\ \ c_1 x_2 + c_2 y_2 \cdots c_1 x_n + c_2 y_n)$$

Here as $x_1 + x_2 + \cdots x_n = 1$ and $y_1 + y_2 + \cdots y_n = 1$

obviously

$$c_1 x_1 + c_2 y_1 + c_1 x_2 + c_2 y_2 + \cdots c_1 x_n + c_2 y_n$$
$$= c_1 (x_1 + x_2 + \cdots x_n) + c_2 (y_1 + y_2 + \cdots y_n)$$
$$= c_1 + c_2 \neq 1$$
$$\therefore c_1 u + c_2 v \notin W.$$

$\therefore$ Not subspace

Sol's

a) Given,

set of all pairs of real nos of the form, $(1, x)$. , say W.

$+ :$ $(1,y) + (1,y') = (1, y+y')$

$* :$ $k(1,y) = (1, ky)$

Let $u = (1,x)$ $v = (1,y)$ $u,v \in W$

c, d be constants

$\therefore cu+dv = c(1,x) + d(1,y)$

$= (1, cx) + (1, dy)$

$= (1, cx+dy) \in W$

$\therefore$ $cu+dv$ is closed

Hence it is a vector ~~sub~~space

Under addition,

1. closure

Let $u = (1, x_1)$ $v = (1, x_2)$

$u+v = (1, x_1+x_2) \in W$

$\therefore$ It is closed

2. Associative

$u = (1, x_1)$ $v = (1, x_2)$ $w = (1, x_3)$

$(u+v)+w = [(1, x_1) + (1, x_2)] + (1, x_3)$

$= (1, x_1+x_2) + (1, x_3)$

$= (1, x_1+x_2+x_3)$

$= (1, x_1) + (1, x_2+x_3)$

$= (1, x_1) + [(1, x_2) + (1, x_3)]$

$= u + [v+w]$

$\Rightarrow$ It is associative

3. Identity

Let $e = (1, e)$, $u = (1, x)$

$u + e = u$

$\Rightarrow (1, x) + (1, e) = (1, x)$

$(1, x+e) = (1, x) \rightarrow e = 0$

$\therefore (1, 0)$ is identity $\in W$

4. Inverse

Let $u' = (1, x')$ $u = (1, x)$

$u + u' = e$

$\Rightarrow (1, x') + (1, x) = (1, 0)$

$\Rightarrow x' = -x$

$\therefore (1, -x)$ is inverse $\in W$.

5. Commutative

$u + v = v + u$

$\therefore$ commutative

$\Rightarrow$ W is an abelian group — ①

$\Rightarrow$ Let c be scalar, , $u=(1,x)$

$cu = c(1,x) = (1, cx) \in W$ — ②

$\Rightarrow$ Let c,d be scalars $u=(1,x)$.

$(c+d)u = cu+du$

$\Rightarrow (c+d)(1,x) = c(1,x) + d(1,x)$

$(1, (c+d)x) = (1, (c+d)x)$

LHS = RHS — ③

$\Rightarrow$ c be scalar, $u=(1,x_1)$ $v=(1,x_2)$

$c(u+v) = cu+cv$

$\Rightarrow c(1, x_1+x_2) = c(1, cx_1) + (1, cx_2)$

$\Rightarrow (1, c(x_1+x_2)) = (1, c(x_1+x_2))$

LHS = RHS — ④

$\Rightarrow$ Let c,d be scalars $u=(1,x_1)$.

$c(du) = (cd)u$

$c(1, dx_1) = cd(1, x_1)$

$(1, cdx_1) = (1, cdx_1)$

LHS = RHS — ⑤

$\Rightarrow$ Also $1u = u$ — ⑥

From ⑥ equations,

W is a vector space

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b) Given,

$W = \{$set of all triples of real numbers$\}$

Addition defined by,

$(x, y, z) + (u, v, w) = (z+w, y+v, x+u)$

1. It will be abelian group ✓

2. $c(u+v) = cu + cv$

3. $(c+d)u = ~~cu+d~~\ cu + du$ ✓

All satisfies

It is a vector space

c) $W = \{$ set of all 2×2 matrices of the form $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \}$

for matrix addition

$\begin{bmatrix} a_1+a_2 & 2 \\ 2 & b_1+b_2 \end{bmatrix} \notin W.$

$\therefore$ Not vector space

d) set of all real no-s

with $x+y = xy$ $\quad kx = x^k$

It is not satisfied for,

$c(x+y) = cx + cy$

$\Rightarrow cxy \neq c^2xy$

$\therefore$ Not vector space....

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1/03/17
Wednesday

Module 4
Linear transformation and applications

Linear transformation :-
$T: V \to W$ is a linear transformation if for all vectors $u \,\&\, v \in V$ and for all scalars c,

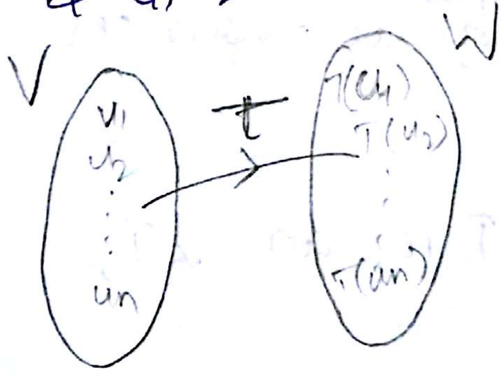
$$T(u+v) = T(u) + T(v)$$
$$T(cu) = cT(u)$$

where V and W are vector spaces

⇒ This can be extended to any no. of vectors and scalars, i.e.,
$T: V \to W$ is a linear transformation iff

$$T(c_1u_1 + c_2u_2 + c_3u_3 + \dots c_nu_n) = c_1T(u_1) + c_2T(u_2) + \dots c_nT(u_n)$$

where $u_1, u_2, u_3 \dots u_n \in V$
& $c_1, c_2, c_3 \dots c_n$ are scalars



Eg

a) Every matrix transformation is a linear transformation ie,

If A is an $m\times n$ matrix, then the transformation $T_A : R^n \to R^m$ is a linear transformation defined by

$$T_A(x) = Ax, \quad x \in R^n$$

Pf: To prove that T is a LT,

$$T_A(c_1x_1 + c_2x_2) = c_1T_A(x_1) + c_2T_A(x_2), \quad x_1, x_2 \in R^n$$

$c_1, c_2 \to$ Scalars

$$LHS = T_A(c_1x_1 + c_2x_2)$$

$$= A(c_1x_1 + c_2x_2)$$

$$= Ac_1x_1 + Ac_2x_2$$

$$= c_1(Ax_1) + c_2(Ax_2)$$

$$= c_1T_A(x_1) + c_2T_A(x_2)$$

Hence the proof

b) Define $T : M \to N$ by $T(A) = A^T$ where M and N are set of all square matrices of order n

Pf: To prove that T is an LT,

Given, both M, N are set of all square matrices of order n,

so let $T: M \to M$

$T(A) = A^T$

Consider, $T(c_1A_1 + c_2A_2)$, c_1 & c_2 are scalars

$A_1, A_2 \in M$

$= (c_1A_1 + c_2A_2)^T$

$= (c_1A_1)^T + (c_2A_2)^T$ $[\because (A+B)^T = A^T + B^T]$

$= c_1A_1^T + c_2A_2^T$

$= c_1T(A_1) + c_2T(A_2)$

$=$ RHS.

Hence the proof.

c) Let D be the differential operator defined by $\mathcal{D}: \mathcal{D} \to F \ni \mathcal{D}(f) = f'$. Show that $\mathcal{D}$ is a linear transformation.

Pf: Let $f, g \in \mathcal{D}$, c_1, c_2 are scalars

$D(c_1f + c_2g)$

$= (c_1f + c_2g)'$

$= (c_1f)' + (c_2g)'$

$= c_1f' + c_2g'$

$= c_1D(f) + c_2D(g)$

Hence the proof

d) Define $S : \mathcal{C}[a,b] \to R$ defined by.

$$S(f) = \int_a^b f(x)\,dx$$

Prf: $\boxed{\mathcal{C}[a,b] \to \text{set of all continuous functions b/w } a \text{ and } b.}$

Consider, $f, g \in \mathcal{C}[a,b]$, c_1, c_2 are scalars

$$S(c_1 f + c_2 g) = \int_a^b [c_1 f(x) + c_2 g(x)]\,dx$$

$$= \int_a^b c_1 f(x)\,dx + \int_a^b c_2 g(x)\,dx$$

$$= c_1 \int_a^b f(x)\,dx + c_2 \int_a^b g(x)\,dx$$

$$= c_1 S(f) + c_2 S(g)$$

Hence the proof.

(*) check whether the following are linear transformations $T : M_{22} \to R \ni T(A) = \det|A|$

Sol: Consider $A_1, A_2 \in M_{22}$ c_1, c_2 are scalars

$$T(c_1 A_1 + c_2 A_2) = \det(c_1 A_1 + c_2 A_2)$$

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$\neq \det(c_1A_1) + \det(c_2A_2)$

Let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$|A_1| = 0$ $|A_2| = 0$ But $|A_2 + A_1| = 1$

$|A_1| + |A_2| = 0$

$\therefore |A_2 + A_1| \neq |A_1| + |A_2|$

Hence it is NOT a linear Transformation

(*) check, $T: R \to R \Rightarrow T(x) = 2^x$

Sol: consider, $x_1, x_2 \in R$. c_1, c_2 are scalars

$T(c_1x_1 + c_2x_2) = 2^{(c_1x_1 + c_2x_2)}$

$= 2^{c_1x_1} * 2^{c_2x_2}$

$= (2^{x_1})^{c_1} \times (2^{x_2})^{c_2}$

$= [T(x_1)]^{c_1} [T(x_2)]^{c_2}$

$\neq c_1T(x_1) + c_2T(x_2)$

Except when ~~~~ $x_1 = y_1 = 1$

$\therefore$ NOT an LT

(*) check $T: R \to R \Rightarrow T(x) = x + 1$

Sol: Consider, $x_1, x_2 \in R$ c_1, c_2 are scalars

$T(c_1x_1 + c_2x_2) = (c_1x_1 + c_2x_2) + 1$

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or, $T(x) = x+1$

$T(y) = y+1$

$T(x+y) = x+y+1$

$T(x)+T(y) = x+y+2$

$\therefore T(x+y) \neq T(x)+T(y)$

$\therefore$ Hence, it is NOT an LT

3/02/17

Friday

Matrix of the Linear Transformation:

$$\begin{array}{ccc} v & \xrightarrow{T} & T(v) = w \\ \downarrow & & \downarrow \\ [v]_B & \xrightarrow{T_A} & A[v]_B = [T(v)]_C \end{array}$$

Let V and W be two finite dimensional vector spaces with bases B and C respectively where $B = \{v_1, v_2 \dots v_n\}$. If $T: V \to W$ is a linear transformation, then the $m \times n$ matrix A is defined by

$$A := \left[[T(v_1)]_C, [T(v_2)]_C, \dots [T(v_n)]_C \right]$$

satisfies $A[v]_B = [T(v)]_C$. for every $v \in V$

The matrix A is called the matrix of T w.r.t bases B and C

7) Let $T: R^3 \to R^2$ be the LT defined by

~~T(x,y,z)~~ $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-2y \\ x+y-3z \end{pmatrix}$ Let $B = \{e_1, e_2, e_3\}$

$C = \{e_2, e_1\}$ be bases for R^3 & R^2. Find the matrix of T w.r.t B & C and verify the theorem $A[V]_B = [T(v)]_C$ for the vector $v = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$.

Sol Given,

$B = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$C = \left\{ e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

$$T(e_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 1+0-0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T(e_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0-2 \\ 0+1-0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$T(e_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

$$\cancel{A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 0 & -3 \end{bmatrix}}$$

$$A_{C \leftarrow B} = 2 \times 3$$

$$= \left[[T(e_1)]_C , [T(e_2)]_C , [T(e_3)]_C \right]$$

$$A = \begin{pmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{pmatrix}$$

<u>Verification</u>

$$A \cdot [V]_B = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

$$T(V) = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$$

$$[T(V)]_C = \begin{pmatrix} 10 \\ -5 \end{pmatrix}$$

$$\therefore A[V]_B = [T(V)]_C$$

Hence the proof

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(7) Let $\mathcal{D} : P_3 \to P_2$ be the differential operator defined by $\mathcal{D}(P(x)) = P'(x)$. Let B be equal to $B = \{1, x, x^2, x^3\}$ and $C = \{1, x, x^2\}$ be bases for P_3 and P_2 respectively

a) Find the matrix A of D. w.r.t B & C

b) Find the matrix A' of D w.r.t B' & C where $B' = \{x^3, x^2, x, 1\}$

c) Using a ~~power~~ k, compute $\mathcal{D}(5 - x + 2x^3)$ and $\mathcal{D}(a + bx + cx^2 + dx^3)$ to verify, the theorem $A[V]_B = [T(v)]_C$

Sol a) Given, $B = \{1, x, x^2, x^3\}$ $C = \{1, x, x^2\}$

$\mathcal{D}(1) = 0$

$\mathcal{D}(x) = 1$

$\mathcal{D}(x^2) = 2x$

$\mathcal{D}(x^3) = 3x^2$

$A = [[T(v_1)]_C, [T(v_2)]_C \ldots [T(v_n)]_C]$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

b) Given

$B' = \{x^3, x^2, x, 1\}$

$C = \{1, x, x^2\}$

$D(x^3) = 3x^2$

$D(x^2) = 2x$

$D(x) = 1$

$D(1) = 0$

$\cancel{A'} =$

$$\begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

c) Given

$D(5, -x + 2x^3)$

$D(a + bx + cx^2 + dx^3)$

$D(5 - x + 2x^3) = -1 + 6x^2$

$D(a + bx + cx^2 + dx^3) = b + 2cx + 3dx^2$

where,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$A \cdot [V]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}$$

$$= [D(5 - x + 2x^3)]_C$$

same way

$$A [V']_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$= \begin{bmatrix} b \\ 2c \\ 3d \end{bmatrix}$$

$$= [D(a + bx + cx^2 + dx^3)]_C$$

Hence, the proof of

$$A [V]_B = [T(V)]_C.$$

(*) Let $T: P_2 \to P_2$ be the L.T, defined by

$T(P(x)) = P(2x-1)$

a) find the matrix of T, w.r.t the standard basis (E) *

b) Compute $T(3+2x-x^2)$, indirectly using (a)

Sol Given,

$T: P_2 \to P_2$

$T(P(x)) = P(2x-1)$

$E = \{1, x, x^2\}$

a) $A = [[T(v_1)]_c, [T(v_2)]_c \cdots [T(v_n)]_c]$

$A = [T]_{C \leftarrow B}$

$A = [[T(1)]_c, [T(x)]_c, [T(x^2)]_c]$

$T(1) = 1$

$T(x) = 2x-1$

$T(x^2) = (2x-1)^2 = 4x^2+1-4x$

$$\therefore A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

b) Given, $T(3+2x-x^2) = 3+2(2x-1) - (2x-1)^2$

we have.

$$A[v]_B = [T(v)]_C$$

Let $3+2x-x^2 = v$

$$\Rightarrow A[v]_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 8 \\ -4 \end{bmatrix} = [T(v)]_C$$

$$T(v) = 8x - 4x^2$$

$$\therefore T(3+2x-x^2) = 8x - 4x^2$$

(*) D be the vector space of all differentiable functions. Consider the subspace W of D given by $W = span\{e^{3x}, xe^{3x}, x^2e^{3x}\}$. Since the set $B = \{e^{3x}, xe^{3x}, x^2e^{3x}\}$ is linearly independent, it is the basis for W.

a) Show that the differential operator D, maps W onto itself

b) find the matrix of D w.r.t B.

c) Compute the derivative of $5e^{3x} + 2xe^{3x} - x^2e^{3x}$ indirectly using (b) and verify it using the theorem,

$$[D(f(x))]_B = D_B [f(x)]_B$$

Sol Given,

$$W = span(e^{3x}, xe^{3x}, x^2e^{3x})$$

(*) take any $u = ae^{3x} + bxe^{3x} + cx^2e^{3x} \in W$ find $D(u)$

Basis $B = \{e^{3x}, xe^{3x}, x^2e^{2x}\}$

$D(e^{3x}) = 3e^{3x}$

$D(xe^{3x}) = x.3e^{3x} + e^{3x}$

$D(x^2e^{3x}) = x^2.3e^{3x} + e^{3x}2x$

$(3a+b)e^{3x} + (3b+2c)xe^{3x} + 3cx^2e^{3x}$

Clearly D on any element of W is spanned by $\{e^{3x}, xe^{3x}, x^2e^{3x}\}$ also the above 3 are independent

$\therefore$ It maps a VS whose basis is $\{e^{3x}, xe^{3x}, x^2e^{3x}\}$

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It is nothing but W.

$\therefore D : W \to W$.

D maps W onto itself

Hence the proof

b) To find the matrix,

we have

$$D(e^{3x}) = 3e^{3x}$$

$$D(xe^{3x}) = 3xe^{3x} + e^{3x}$$

$$D(x^2e^{3x}) = 3x^2e^{3x} + 2xe^{3x}$$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

c) To compute, $D(5e^{3x} + 2xe^{3x} - x^2e^{3x})$

consider

$$D_B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Let $f(x) = 5e^{3x} + 2xe^{3x} - x^2e^{3x}$

$$[f(x)]_B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$D_B[f(x)]_B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 17 \\ 4 \\ -3 \end{bmatrix}$$

$$\left[\mathcal{D}(f(x))\right]_B = ?$$

$$\mathcal{D}(f(x)) = \mathcal{D}\left(5e^{3x}+2xe^{3x}-x^2e^{3x}\right)$$

$$= 15e^{3x}+2\left(3xe^{3x}+e^{3x}\right) - \left(3x^2e^{3x}+2xe^{3x}\right)$$

$$= 17e^{3x}+4xe^{3x}-3x^2e^{3x}$$

$$\therefore \left[\mathcal{D}(f(x))\right]_B = \begin{bmatrix} 17 \\ 4 \\ -3 \end{bmatrix}$$

LHS = RHS.

$$\therefore \left[\mathcal{D}(f(x))\right]_B = \mathcal{D}_B\left[f(x)\right]_B$$

Hence the proof

9/03/17
Thursday

Change of basis matrix

(*) Let $V = P_2$, and bases $B = \{1, 1+x, 1+x+x^2\}$ and $C = \{2+x+x^2, x+x^2, x\}$. Verify whether B and C are Basis for V, find the change of basis matrix from B to C.

Use it to calculate the change of basis matrix from C to B.

Sol: Given,

$$B = \{1, 1+x, 1+x+x^2\}$$

$$C = \{2+x+x^2, x+x^2, x\}$$

$$1 = c_1(2+x+x^2) + c_2(x+x^2) + c_3 x$$

$$1 = 2c_1 + x^2(c_1+c_2) + x(c_1+c_2+c_3)$$

$$\Rightarrow \quad c_1+c_2+c_3 = 0$$

$$c_1 = \frac{1}{2}$$

$$c_2 = -\frac{1}{2}, \quad c_3 = 0$$

$$\therefore \ \mathcal{D}(1) = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix} = [1]_C$$

$$1+x = c_1(2+x+x^2) + c_2(x+x^2) + c_3 \cdot x$$

$$\Rightarrow \quad 2c_1 = 1 \qquad c_1+c_2+c_3 = 1$$

$$c_1+c_2 = 0$$

$$\rightarrow c_2 = -\frac{1}{2} \quad c_1 = \frac{1}{2} \qquad c_3 = 1$$

$$\mathcal{D}(1+x) = \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} = [1+x]_C$$

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$$1+x+x^2 = c_1(2+x+x^2) + c_2(x+x^2) + c_3 x$$

$$\Rightarrow \quad 2c_1 = 1 \qquad c_1+c_2+c_3 = 1$$

$$c_1 + c_2 = 1$$

$$c_1 = \frac{1}{2} \Rightarrow c_2 = 1/2 \qquad c_3 = 0$$

$$\therefore \mathcal{D}(1+x+x^2) = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix} = [1+x+x^2]_C$$

Change of Basis matrix from B to C is

$$P_{C\leftarrow B} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\boxed{P_{B\leftarrow C} = (P_{C\leftarrow B})^{-1}}$$

To find the inverse of $P_{C\leftarrow B}$,

$$[A|I] = \left[\begin{array}{ccc|ccc} 1/2 & 1/2 & 1/2 & 1 & 0 & 0 \\ -1/2 & -1/2 & 1/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \leftrightarrow 2R_1 \quad R_2 \leftrightarrow 2R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_2 + R_1 \quad R_3 \leftrightarrow R_2 + R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 0 \\ -1 & 0 & 1 & 0 & 2 & 1 \end{array}\right]$$

$R_3 \leftrightarrow R_3 + R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 0 \\ 0 & 1 & 2 & 2 & 2 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_2 + R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 4 & 4 & 4 & 1 \\ 0 & 1 & 2 & 2 & 2 & 1 \end{array}\right]$$

$R_3 \leftrightarrow R_3 - R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 4 & 4 & 4 & 1 \\ 0 & 0 & 2 & 2 & 2 & 0 \end{array}\right]$$

$R_1 \leftrightarrow -(R_2 - R_1) \quad R_3 \leftrightarrow R_3/2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & -2 & -4 & -1 \\ 0 & 1 & 4 & 4 & 4 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - 4R_3$

$$\sim \left[\begin{array}{cccccc} 1 & 0 & -3 & -2 & -4 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$R_1 \leftrightarrow R_1 + 3R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Hence $P_{B\leftarrow C}$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(*) Find the change of basis matrices, $P_{C\leftarrow B}$ and $P_{B\leftarrow C}$ for the basis $B = \{1, x, x^2\}$ and $C = \{1+x, x+x^2, 1+x^2\}$. of P_2 and find the co-ordinate vector of $p(x) = 1+2x-x^2$ w.r.t C.

Sol) Given,

$B = \{1, x, x^2\}$

$C = \{1+x,\ x+x^2,\ 1+x^2\}$

$1 = c_1(1+x) + c_2(x+x^2) + c_3(1+x^2)$

$\Rightarrow c_1 + c_3 = 1$

$c_1 + c_2 = 0$ $\Rightarrow$ $c_1 - c_3 = 0$

$c_2 + c_3 = 0$ $\quad c_1 + c_3 = 1$

$2c_1 = 1$

$c_1 = \frac{1}{2},\ c_3 = 1/2$

$c_2 = -1/2$

$$\therefore [1]_c = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$x = c_1(1+x) + c_2(x+x^2) + c_3(1+x^2)$

$c_1 + c_3 = 0 \quad c_1 + c_2 = 1 \quad c_2 + c_3 = 0$

$\Rightarrow c_1 = c_2 \Rightarrow c_1 = 1/2 \quad c_2 = 1/2 \quad c_3 = -1/2$

$$\Rightarrow [x]_c = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$x^2 = c_1(1+x) + c_2(x+x^2) + c_3(1+x^2)$

$c_1 + c_3 = 0 \quad c_2 + c_1 = 0 \quad c_2 + c_3 = 1$

$$[x^2]_c = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$P_{C\leftarrow B} = \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix}$$

To find $P_{B\leftarrow C}$

$$[P_{C\leftarrow B} \mid I] = \left[\begin{array}{ccc|ccc} 1/2 & 1/2 & -1/2 & 1 & 0 & 0 \\ -1/2 & 1/2 & 1/2 & 0 & 1 & 0 \\ 1/2 & -1/2 & 1/2 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_1/2 \quad R_2 \leftrightarrow R_2/2 \quad R_3 \leftrightarrow R_3/2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 2 & 0 & 0 \\ -1 & 1 & 1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 & 0 & 2 \end{array}\right]$$

$R_2 \leftrightarrow R_2 + R_1 \quad R_3 \leftrightarrow R_3 - R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 \\ 0 & -2 & 2 & -2 & 0 & 2 \end{array}\right]$$

$R_3 \leftrightarrow R_3 + R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 & 2 \end{array}\right]$$

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$R_2 \leftrightarrow R_2/2 \quad R_3 \leftrightarrow R_3/2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & \phi & 4 \end{array}\right]$$

$R_1 \leftrightarrow R_1 - R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_1 + R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$P_{B\leftarrow C} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

To find co-ord vector of $P(x) = 1+2x-x^2$ wrt C,

$[P(x)]_C = A\,[P(x)]_B$ where $A = P_{C\leftarrow B}$

$$[P(x)]_B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

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$$\therefore [P(x)]_C = [P_{C\leftarrow B}][P(x)]_B$$

$$= \begin{bmatrix} 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$[P(x)]_C = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

(*) In $M_{2\times 2}$ let B be the Basis $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

and C be $C = \{A, B, C, D\}$ where,

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

find the change of basis matrix $P_{C\leftarrow B}$ and verify that $[x]_C = P_{C\leftarrow B}[x]_B$ where

$X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Sol Given,

$$B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$$

$C' = \{A, B, C, D\}$

To find $P_{C \leftarrow B}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow \quad c_1 = 1, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0$

$$[E_{11}]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\cancel{c_1 = 0}$

$c_2 + c_3 + c_4 = 1$

$c_1 + c_2 + c_3 + c_4 = 0$

$c_3 + c_4 = 0, \quad c_4 = 0$

$\Rightarrow c_3 = 0$

$\Rightarrow c_1 + c_2 = 0$

$c_2 = 1 \quad c_1 = -1$

$$\therefore [E_{12}]_C = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow \quad c_3 + c_4 = 1 \qquad c_1 + c_2 + c_3 + c_4 = 0$

$c_2 + c_3 + c_4 = 0$

$\Rightarrow \quad c_2 = -1$

$c_4 = 0$

$\Rightarrow \quad c_3 = 1 \qquad \Rightarrow \quad c_1 = 0$

$$\sim [E_{21}]_c = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow \quad c_4 = 1$

$c_3 + c_4 = 0 \quad \Rightarrow \quad c_3 = -1$

$c_2 + c_3 + c_4 = 0 \quad \Rightarrow \quad c_2 = 0 \quad \Rightarrow \quad c_1 = 0$

$$\therefore \quad [E_{22}]_c = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore P_{C \leftarrow B} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Given,

$X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow$ ~~writing it as~~ $\cancel{X \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}$

$[X]_B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

To find, $[X]_C = P_{C \leftarrow B} [X]_B$

$$P_{C \leftarrow B}[X]_B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$P_{C \leftarrow B}[X]_B \cancel{[X]_C} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

To find, $[X]_C$,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = C_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + C_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + C_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + C_4 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

we get $C_1 = -1 \quad C_2 = -1 \quad C_3 = -1 \quad C_4 = 4$

Hence $[X_B]_C = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$

Hence, $[X]_C = P_{C \leftarrow B}[X]_B$

Hence the verification

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10/03/17
Friday

The KERNEL and RANGE of an LT

Let $T: V \to W$ be a linear transformation. The set of all vectors $v \in W$ such that $T(v) = 0$ is called the kernel of T. It is denoted by $\ker(T)$.

$$\ker(T) = \{v \in V / T(v) = 0\}$$

The range of an LT $T: V \to W$ is a subset of W consisting of all transformed vectors from V denoting range of T by $Rng(T)$.

$$Rng(T) = \{T(v) / v \in V\}$$

(*) Determine $\ker(T)$ for the LT, $T: C^2(I) \to C^0(I)$ defined by $T(y) = y'' + y$.

Sol) Given,

$$T: C^2(I) \to C^0(I)$$

$$T(y) = y'' + y$$

$C^2(I) \to$ set of all integers that are continuous even when differentiated twice

To get $\ker(T)$,

$T(y) = 0$

$\Rightarrow y'' + y = 0$

$AE \Rightarrow m^2 + 1 = 0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm i = \alpha \pm i\beta$

CF of $y_c = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$CS = CF$

$y = c_1\cos x + c_2\sin x \quad [\alpha = 0, \beta = 1]$

$\therefore \ker(T) = c_1\cos x + c_2\sin x$

Note :-

→ If $T : R^m \to R^n$ is an LT with matrix A, then $\ker(T)$ is a solution set of the homogeneous linear system. $AX = 0$, nothing but,

$$\boxed{\ker(T) = \text{null}(A)}$$

→ The range of T is given by $Rng(T) = \{AX : X \in R^n\}$

$$\boxed{Rng(T) = \text{col}(A)}$$

(*) If $T: R^3 \to R^2$ be the LT with matrix $A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \end{bmatrix}$. Determine Ker(T) & Rng(T)

Sol: Given,

$$T: R^3 \to R^2$$

$$A = \begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & -10 \end{bmatrix}$$

RR form of A,

$$R_2 \leftrightarrow R_2 + 2R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore R.R_A = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad \boxed{Range(T) = Col(A)}$$

$$Col(A) = \{(1 \ -2)\} \Rightarrow Rng(T) = \left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$$

To get Ker(A)

$$RR_A X = 0$$

$$\begin{bmatrix} 1 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\approx \left[\begin{array}{ccc|c} 1 & -2 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

~~Let~~ $\Rightarrow x - 2y + 5z = 0$

$y = t, \quad z = s$

$x = 2t - 5s$

$$\therefore \begin{bmatrix} 2t-5s \\ t \\ s \end{bmatrix} = t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ null(A) $= [(2\ 1\ 0), (-5\ 0\ 1)]$

We know $\boxed{\text{Ker}(T) = \text{null}(A)}$

$$\therefore \text{Ker}(T) = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \right\}$$

<u>Theorem</u>

If $T: V \to W$ is a LT, then Ker(T) is the subspace of V, Rng(T) is the subspace of W.

* Rng(T) = col(A) because of the fact that A is made up of $[T(v_1)]_c, [T(v_2)]_c \ldots [T(v_n)]_c$ as its <u>columns</u>.

✻ Find the Kernel & Range of S and their dimensions for the LT, $S: M_2(R) \to M_2(R)$ Defined by $S(A) = A - A^T$

Soln Given,

$$S: M_2(R) \to M_2(R)$$

$$S(A) = \cancel{A} - A^T$$

$$\ker(S) = \{A \in M_2(R) / S(A) = 0\}$$

$$= \{A \in M_2(R) / A - A^T = 0\}$$

$$= \{A \in M_2(R) / A = A^T\}$$

$$= \{\text{symmetric matrices} \in M_2(R)\}$$

$$= \left\{\begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_2(R)\right\}$$

Basis $= \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

Hence, $\ker(S) = \text{span}\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$

$\text{Dim}(\ker(S)) = 3.$

To find the range of S.

$$Rng(S) = \{ S(A) / A \in M_2(R) \}$$

$$= \{ A - A^T / A \in M_2(R) \}$$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix} \Big/ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R) \right\}$$

$$= \left\{ \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix} \Big/ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R) \right\}$$

Basis $= \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

$\therefore$ Rang $(S) = \text{span} \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

Dim $(Rng(S)) = 1.$

14/3/17
Tuesday

Inverrtible Linear transformations

Let $T: V \to W$ be a linear transformations, between n-dimensional vector spaces, V and W. Let B and C be bases for V and W respectively, Then T is invertible iff $[T]_{C \leftarrow B}$ is inveetible.

In this case,

$$\left([T]_{C \leftarrow B}\right)^{-1} = [T^{-1}]_{B \leftarrow C}$$

(*) Let the LT, $T: R_2 \to P_1$ be defined by,

$T\begin{pmatrix} a \\ b \end{pmatrix} = a + (a+b)x$ $B = \{(1,0), (0\ 1)\}$

$C = \{1, x\}$. Find the inverse of the given LT, $T^{-1}: P_1 \to R_2$

sol Given,

$$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$$

$$B = \{(1,0), (0,1)\}$$

$$C = \{1, x\}$$

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 + (1+0)x = 1+x$$

$$\left[T\begin{bmatrix}1\\0\end{bmatrix}\right]_c = \begin{bmatrix}1\\1\end{bmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = x$$

$$\left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_c = \begin{bmatrix}0\\1\end{bmatrix}$$

$$[T]_{C\leftarrow B} = \left[\left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_c, \left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_c\right]$$

$$= \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$$

$[T]_{C\leftarrow B}$ is invertible as its det $\neq 0$.

$$\boxed{A^{-1} = \frac{1}{|A|}\,\mathrm{Adj}\,A}$$

To find, $[T^{-1}]_{B\leftarrow C}$

$$= \frac{1}{1}\begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}$$

$$[T^{-1}]_{B\leftarrow C} = \begin{bmatrix}1 & 0\\-1 & 1\end{bmatrix}$$

To find $T^{-1}: P_1 \to \mathbb{R}_2$

every polynomial in P_1 = $a+bx$

$T^{-1}(a+bx)$ w.r.t. B, is,

$$\left[T^{-1}(a+bx)\right]_B = (T^{-1})_{B\leftarrow C}\left[(a+bx)\right]_C$$

$$\left[T^{-1}(a+bx)\right]_B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$\left[T^{-1}(a+bx)\right]_B = \begin{bmatrix} a \\ b-a \end{bmatrix}$$

(*) Show that the differential operator restricted to the subspace $W = \text{span}(e^{3x}, xe^{3x}, x^2e^{3x})$ of $\mathcal{D}$ is invertible and use this fact to find the integral, $\int x^2 e^{3x}\,dx$

Sol: Given,

Basis of W,

$$B = \{e^{3x}, xe^{3x}, x^2e^{3x}\}$$

$$\mathcal{D} : W \to W.$$

$$\mathcal{D}(e^{3x}) = 3e^{3x}$$

$$\left[\mathcal{D}(e^{3x})\right]_B = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathcal{D}(xe^{3x}) = 3xe^{3x} + e^{3x}.$$

$$\left[\mathcal{D}(xe^{3x})\right]_B = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$\mathcal{D}(x^2e^{3x}) = 3x^2e^{3x} + 2xe^{3x}$

$$\left[\mathcal{D}(xe^{3x})\right]_B = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ Matrix of $\mathcal{D}: W \to W$,

$$\left[\mathcal{D}_{B\leftarrow B}\right] = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$\mathcal{D}_{B\leftarrow B}$ is invertible as $\det \neq 0$.

To find the inverse of $\left[\mathcal{D}_{B\leftarrow B}\right]$,

consider,

$$\left[\mathcal{D}_{B\leftarrow B} \mid I\right] = \left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \leftrightarrow R_3/3$

$$\sim \left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

$R_2 \leftrightarrow R_2/3$

$$\sim \left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

$R_2 \leftrightarrow R_2 - \frac{2}{3}R_3$

$$\sim \left[\begin{array}{ccc|ccc} 3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/3 & -2/9 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

$R_1 \leftrightarrow R_1/3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1/3 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/3 & -2/9 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

$R_1 \leftrightarrow R_1 - 1/3 R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & -1/9 & 2/27 \\ 0 & 1 & 0 & 0 & 1/3 & -2/9 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}\right]$$

$$\therefore [D_B^{-1}] = \begin{bmatrix} 1/3 & -1/9 & 2/27 \\ 0 & 1/3 & -2/9 \\ 0 & 0 & 1/3 \end{bmatrix}$$

To find the integral, $\int x^2 e^{3x} dx$

$$[D_B^{-1}] = [D_B^{-1}][D]_B$$

$$[D]_B = \begin{bmatrix} 1/3 & -1/9 & 2/27 \\ 0 & 1/3 & -2/9 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[D]_B = \begin{bmatrix} 2/27 \\ -2/9 \\ 1/3 \end{bmatrix}$$

$$\therefore \int x^2 e^{3x} dx = \frac{2}{27}e^{3x} - \frac{2}{9}xe^{3x} + \frac{1}{3}x^2 e^{3x}$$

Similarity Transformations

Let A and B be $n \times n$ matrices, we say that A is similar to B. if there is an invertible $n \times n$ matrix P such that $\boxed{P^{-1}AP = B}$.

We write, $A \sim B$ (A is similar to B)

* Let A, B and C be 3 $n \times n$ matrices. Then

a) $A \sim A$ — reflexive

b) $A \sim B \Rightarrow B \sim A$ — symmetric

c) $A \sim B, B \sim C \Rightarrow A \sim C$ - transitive

Hence, called an Equivalence Relation

* If A, B are $n \times n$ matrices with $A \sim B$. Then

→ $|A| = |B|$

→ A is invertible iff B is invertible

→ A and B have the same RANK

→ A and B have the same characteristic polynomial

→ They have same Eigen values

16/03/17
Thursday

<u>Theorem</u>

Let V be a finite dimensional vector space with bases B and C. Let $T : V \to V$ be a linear transformation. Then,

$$[T]_C = P^{-1} [T]_B P$$

where P is change of basis matrix, $P_{B \leftarrow C}$

$$P^{-1} \to P_{C \leftarrow B}$$

$$\Rightarrow [T]_C = P_{C \leftarrow B} [T]_B P_{B \leftarrow C}$$

(*) Let $T : R_2 \to R_2$ be defined by $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+3y \\ 2x+2y \end{bmatrix}$

If possible, find a bases C for R_2 such that, the matrix of T w.r.t C is diagonal

* <u>Diagonalization</u>

Let V be a vector space & $T : V \to V$ be a linear transformation. Then T is called diagonalizable if there is a bases C for V such that the matrix $[T]_C$ is a diagonal matrix

~~Soln~~ Let, $B = \{(1,0)\ (0,1)\}$

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}$$

$$[T]_B = \begin{bmatrix}1 & 3\\2 & 2\end{bmatrix}$$

To find $P_{B\leftarrow C}$

$\boxed{\text{characteristic eqn} \Rightarrow \lambda^2 - (\text{trace})\,\lambda + \det = 0}$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

$\lambda = 4$; $\lambda = -1$ → Eigen values

$$\therefore D = \begin{bmatrix}4 & 0\\0 & -1\end{bmatrix}$$

$\boxed{A - \lambda I = 0}$

To find Eigen vectors:

$$\begin{bmatrix}1-\lambda & 3\\2 & 2-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Let $\lambda = -1$

$$\begin{bmatrix}2 & 3\\2 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$2x_1 + 3x_2 = 0$$

$2x_1 + 3x_2 = 0$

Let $x_2 = t$

$x_1 = \frac{-3}{2} t$

$x_2 = -2$

$x_1 = +3$

$\Rightarrow \begin{bmatrix} +3 \\ -2 \end{bmatrix}$

Let $\lambda = 4$

$$\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3x_1 + 3x_2 = 0$

$x_1 - x_2 = 0$

$x_1 = t \ ; \ x_2 = t$

$x_1 = 1 \ ; \ x_2 = 1 \quad \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$X_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad X_2 = \begin{bmatrix} +3 \\ -2 \end{bmatrix}$

$$P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$$

To find P^{-1}

$$P^{-1} = \frac{1}{-5} \begin{bmatrix} -2 & -3 \\ -1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{5} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

from theorem,

$$[T]_C = \frac{-1}{5}\begin{bmatrix}-2 & -3\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\ 1 & -2\end{bmatrix}$$

$$= \frac{-1}{5}\begin{bmatrix}-2 & -3\\ -1 & 1\end{bmatrix}\begin{bmatrix}4 & -3\\ 4 & 2\end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix}-20 & 0\\ 0 & 5\end{bmatrix}$$

$$[T]_C = \begin{bmatrix}4 & 0\\ 0 & -1\end{bmatrix} = D$$

$$\Rightarrow \quad T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}4\\4\end{pmatrix} = 4\begin{pmatrix}1\\1\end{pmatrix} + 0\begin{pmatrix}3\\-2\end{pmatrix}$$

$$T\begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}-3\\2\end{pmatrix} = 0\begin{pmatrix}1\\1\end{pmatrix} - 1\begin{pmatrix}3\\-2\end{pmatrix}$$

$$[T]_C = D$$

$$C = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}3\\-2\end{pmatrix}\right\}$$

(*) Let $T : P_2 \to P_2$ be a LT defined by
$T(P(x)) = P(2x-1)$.

a) Find the matrix T w.r.t the bases
$B = \{1+x, 1-x, x^2\}$

b) Show that T is diagonalizible and find a bases C for P_2 such that the matrix $[T]_C$ is the diagonal matrix

21/03/17
Tuesday

Sol:- Given,

$$T : P_2 \to P_2$$

$$T(P(x)) = P(2x-1)$$

a) To find $[T]_B$,

$$T(1+x) = 1+(2x-1) = 2x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$T(1-x) = 1-(2x-1) = 2-2x = [$$

$$T(x^2) = \cancel{P}(2x-1)^2 = 4x^2+1-4x$$