CONTROL ENGINEERING LAB

WINTER SEMESTER 2022

GROUP - 18

Aman Saini (2020EEB1155)
Anirudh Sharma (2020EEB1158)
Ansaf Ahmad (2020EEB1160)

OBJECTIVE:

We are given a non-linear system representing an overhead crane, we are required to analyse the stability features by observing the movement of eigenvalues across the s-plane.

INTRODUCTION:

An overhead crane is represented by the following system of differential equations:

$$[m_L + m_C] \cdot \ddot{x}_1(t) + m_L l \cdot [\ddot{x}_3(t) \cdot cosx_3(t) - \dot{x}_3^2(t) \cdot sinx_3(t)] = u(t)$$

$$m_L[\ddot{x}_1(t) \cdot cosx_3(t) + l \cdot \ddot{x}_3(t)] = -m_L g \cdot sinx_3(t)$$

Here, $\boldsymbol{m_C}$ = *mass of trolley; (10 kg)*

$\boldsymbol{m_L}$ = *Mass of hook and load; the hook mass is 10 kg; load can be varied from 0 to several hundred kg*

l = *Rope length; 1m or higher, const. for a given operation*

g = *Acceleration due to gravity*

Variables for the problem include:

INPUT: *u: Force in Newtons, applied to the trolley*

OUTPUT: *y: Position of load in meters,* $y(t) = x_1(t) + l \cdot sinx_3(t)$

STATES:

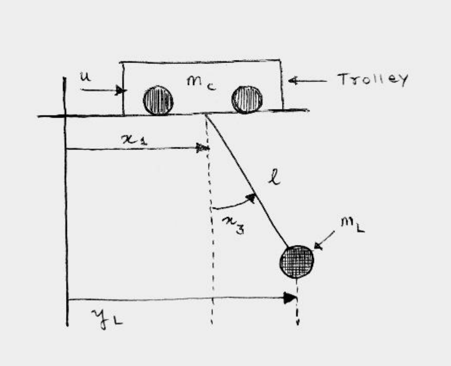
x_1: *Position of trolley in meters*

x_2: *Speed of trolley in m/s*

x_3: *Rope angle in rads*

x_4: *Angular speed of rope in rad/s*

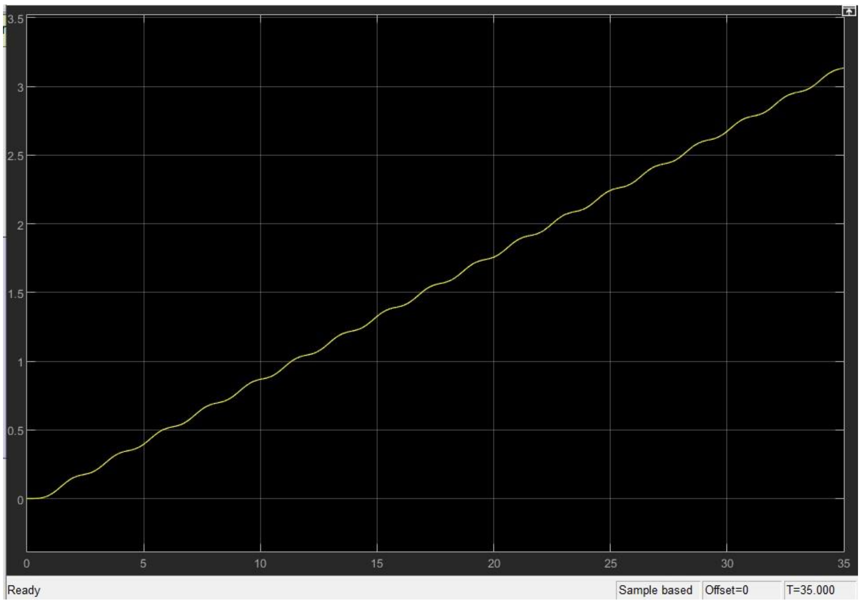
The operation and physical representation of states can be illustrated using the following figure:



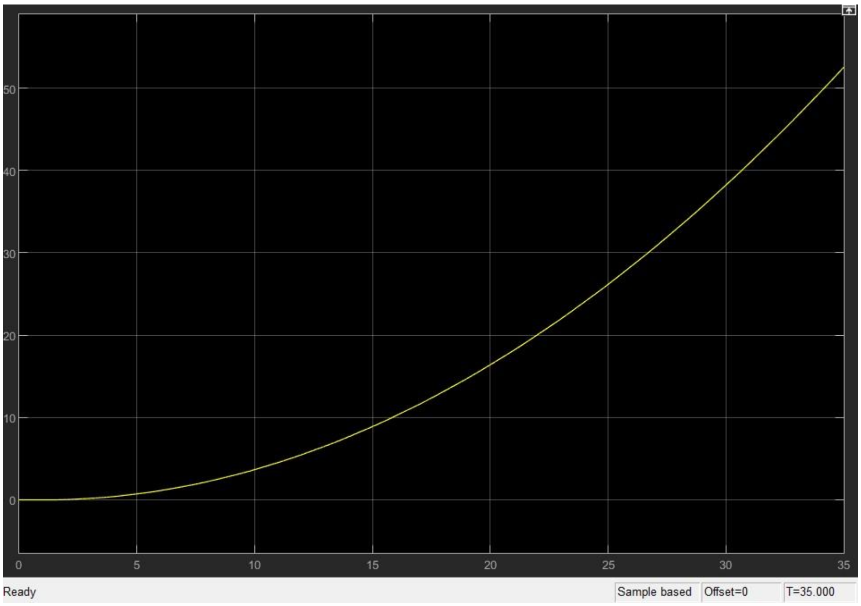
We first analyze the response of given non-linear system to impulse and step inputs using a Simulink model. The results we get from the simulating the system allows us to make the following observations:

1.) The given equations represent the system in its **Ideal State** i.e.
  - Trolley moves along the track without any friction or slip.
  - There is no damping of the pendulum oscillations.
  - Rope has no mass and elasticity.
  - Load is a point mass.

The following plots were obtained on providing impulse and step as inputs to the system.



Impulse response of the system



Step response of the system

The output of the system increases almost linearly (impulse response) and quadratically (step response) with constant undamped oscillations. Also, on plotting states for step and impulse inputs, alongside varying mass load and length of the rope, the amplitude of oscillations shown by angular position and velocity is very small (which is usually true for load transportation). Therefore, it makes sense to linearize the system by considering **x_3, x_4 = 0**. This approximation leads to negligible error (order of 10^{-4}) in the system response.

The above approximations are themselves enough to linearize the system and obtain differential equations and **A, B and C matrices** to represent state space (which is obviously in terms of system variables). The following Diff. Equations and matrices are obtained (taking $cos x_3(t)$ = 1; $sin x_3(t)$ = $x_3(t)$; $\dot{x}_3^2(t)$ = 0):

$$[m_L + m_C] \cdot \dot{x}_2(t) \; + \; m_L l \cdot [\dot{x}_4(t)] \; = \; u(t)$$

$$m_L[\dot{x}_2(t) \; + \; l \cdot \dot{x}_4(t)] \; = \; -m_L g x_3(t)$$

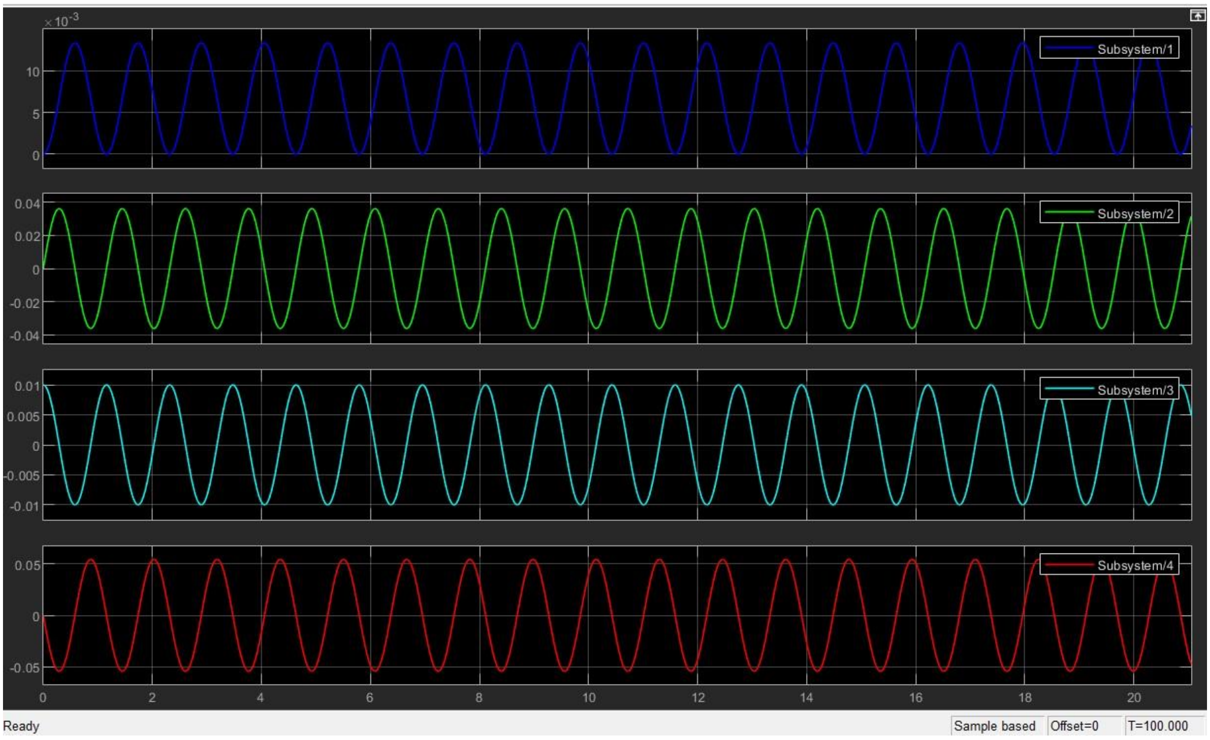
$$A \; = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{m_L}{m_C} g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\dfrac{(m_L \; + \; m_C)g}{m_C l} & 0 \end{bmatrix} B \; = \begin{bmatrix} 0 \\ \dfrac{1}{m_C} \\ 0 \\ -\dfrac{1}{m_C l} \end{bmatrix} C \; = \; [1 \quad 0 \quad l \quad 0]$$

Thus, we obtain a linear system model which is approximates the given non-linear system with a pretty good accuracy for small values of states **x_3** and **x_4** (which is generally true for the system for variety of loads, lengths of rope, and bounded inputs).

EQUILIBRIUM POINTS and STABILITY ANALYSIS:

The system is in equilibrium, when each of its states and output (which only depends on system states in our case) remains constant with time, this implies that for system to be in equilibrium states $x_2 = (\dot{x}_1)$ and $x_4 = (\dot{x}_3)$ are required to be zero, and it is also clear that system cannot maintain a constant value of x_3 (angular position) with zero input therefore we conclude that $x_3 = 0$ for equilibrium, only arbitrary value that we have (for equilibrium condition) is x_1. Thus, we identify the equilibrium points as: **{x_1, 0, 0, 0}**, where x_1 is arbitrary, this is also a state of global stability as it represents the point of minimum energy, but because the model in no way includes any term representing dissipation of energy or non-ideality, therefore without non-zero input the system never returns to that state.

It is also clear that these equilibrium points correspond to unstable equilibria. A simple impulse input would be enough to change the state of system and system would not return to that equilibrium point without a non-zero input. As it is obvious till now that the model represents an ideal system, therefore the states in no way return to globally and locally stable points on their own (because the energy of the system is not lost through non-ideality), so we cannot choose that criterion to define globally and marginally stable points.



The response of system states (x_1 x_2 x_3 x_4 respectively) to initial conditions

(It is noticeable that there is no decay of oscillations)

We can define the globally and locally stable points in terms of energy associated with those states.

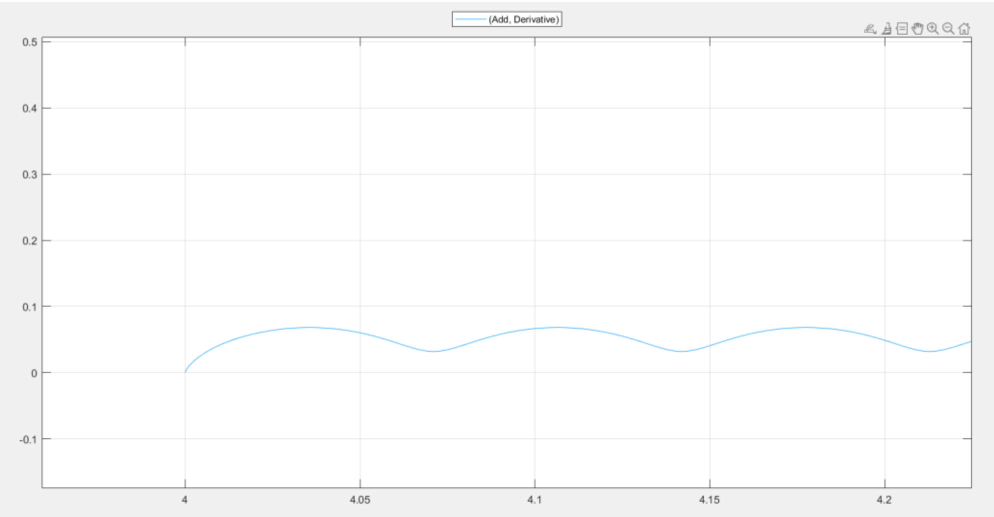
$$(\text{K.E} + \text{P.E} = \frac{1}{2}(m_c + m_l) \cdot x_2^2 + \frac{1}{2}(m_l) \cdot l^2 \cdot x_4^2 + mgl(1 - \cos x_3)),$$

and whether the states (and hence the output) starting on a stable point diverge to infinity in absence of any input.

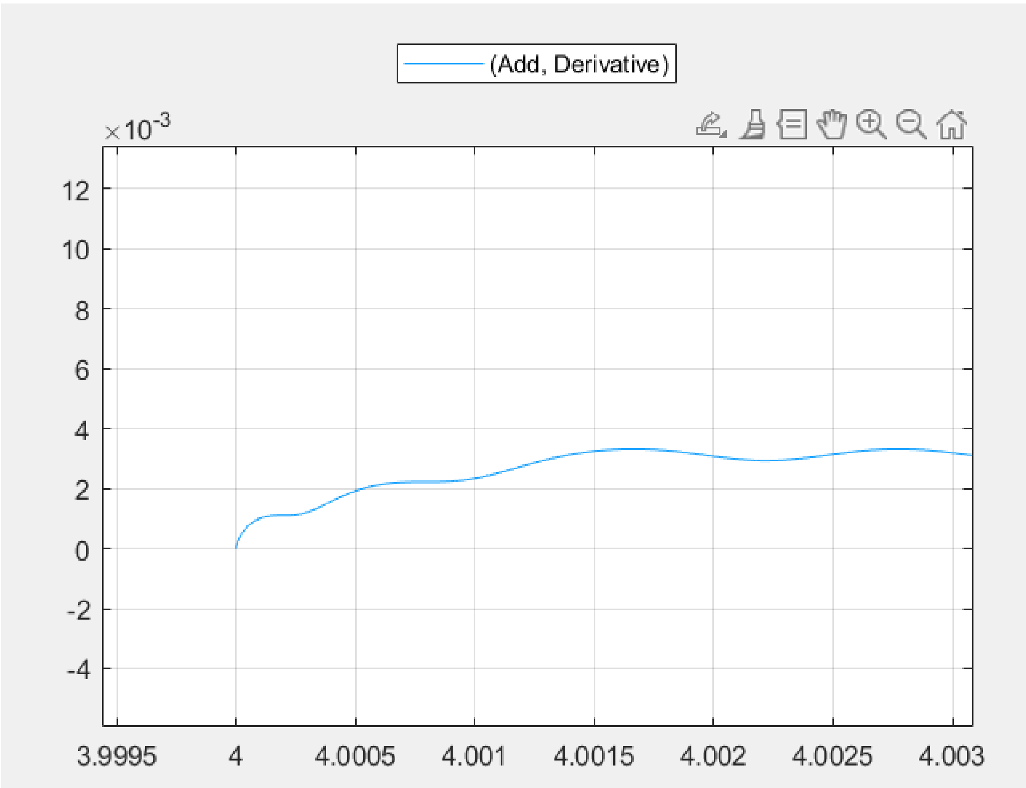
We get **[x_1, 0, 0, 0]** as globally stable point and **[x_1, 0, x_3, 0]** as locally stable point. One important thing to note here is that for both globally and locally stable points, net momentum of the system is 0, which is the main reason why the states and output of the system remains bounded.

We further plotted the phase-plot between $\dot{y}$ (dy/dt) and y (output) and varied the parameters m_L and l and give the discrete impulse as input for all the cases plotted below.

Variation of m_L

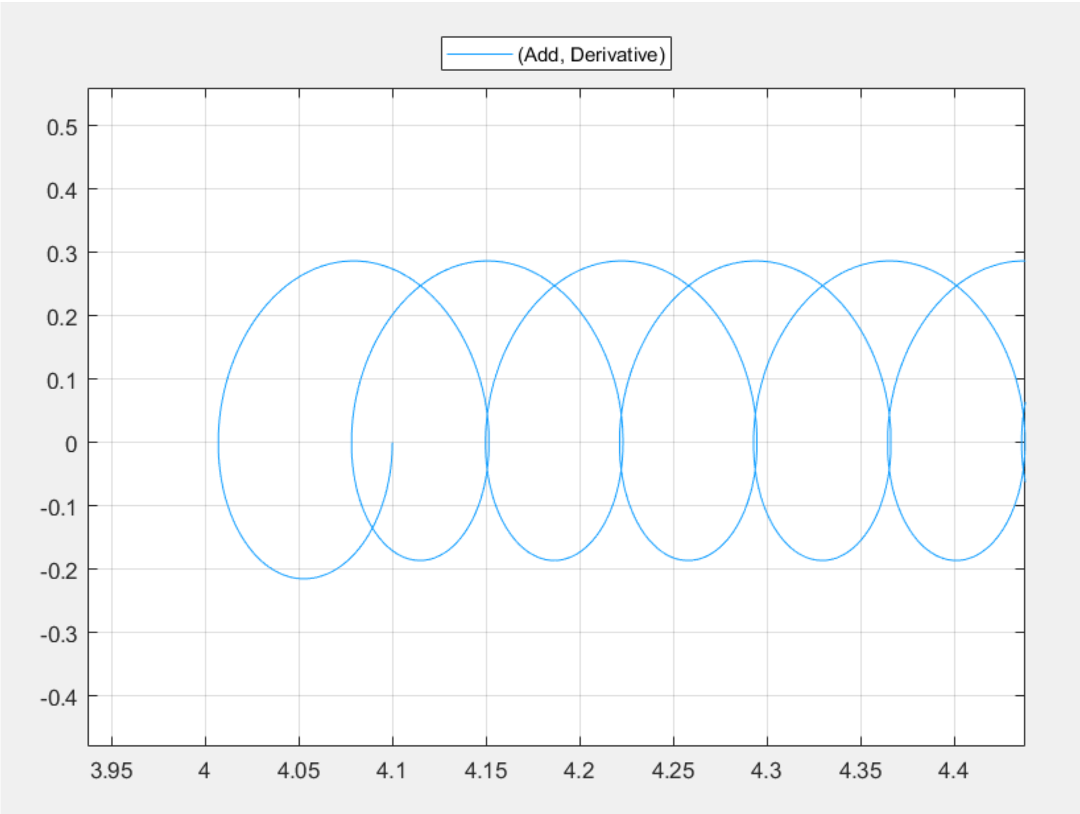


M = 0, l = 1

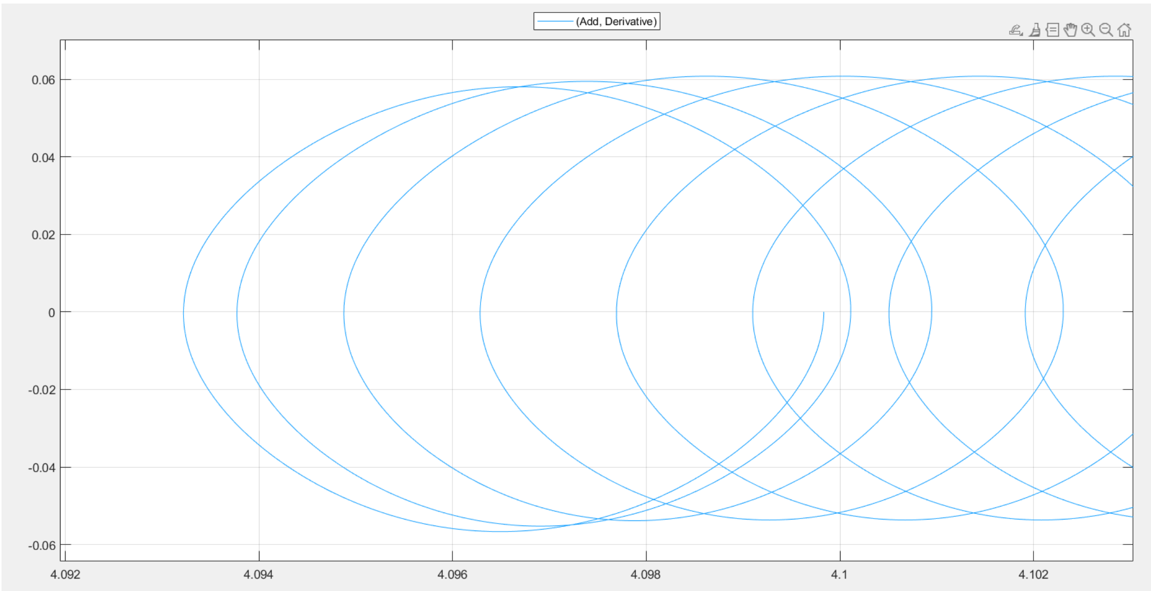


M = 300, l = 1

We observed that increasing the mass m_L decreases the rate of change of output, and hence increases the stability of the system. We obtain no points where $\dot{y} = 0$ hence, we obtain no equilibrium points. When we provide an initial condition of non-zero and small state $x_3 = 0.1$, we repeat the above procedure of parameter variation.



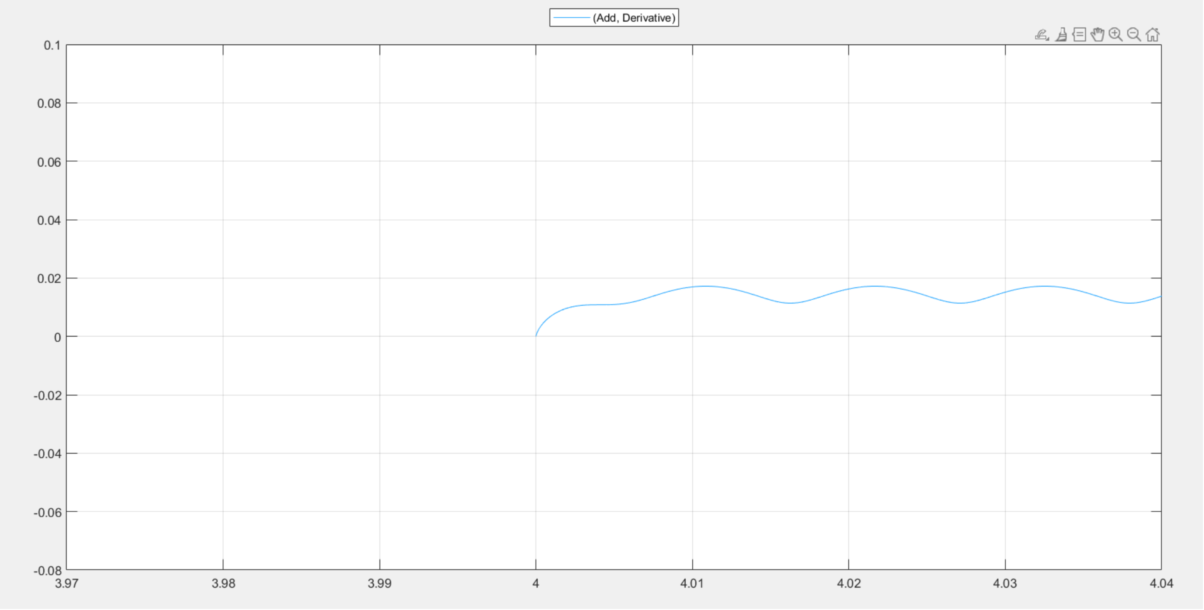
M = 0, l = 1, $x_3 = 0.1$ (initially)



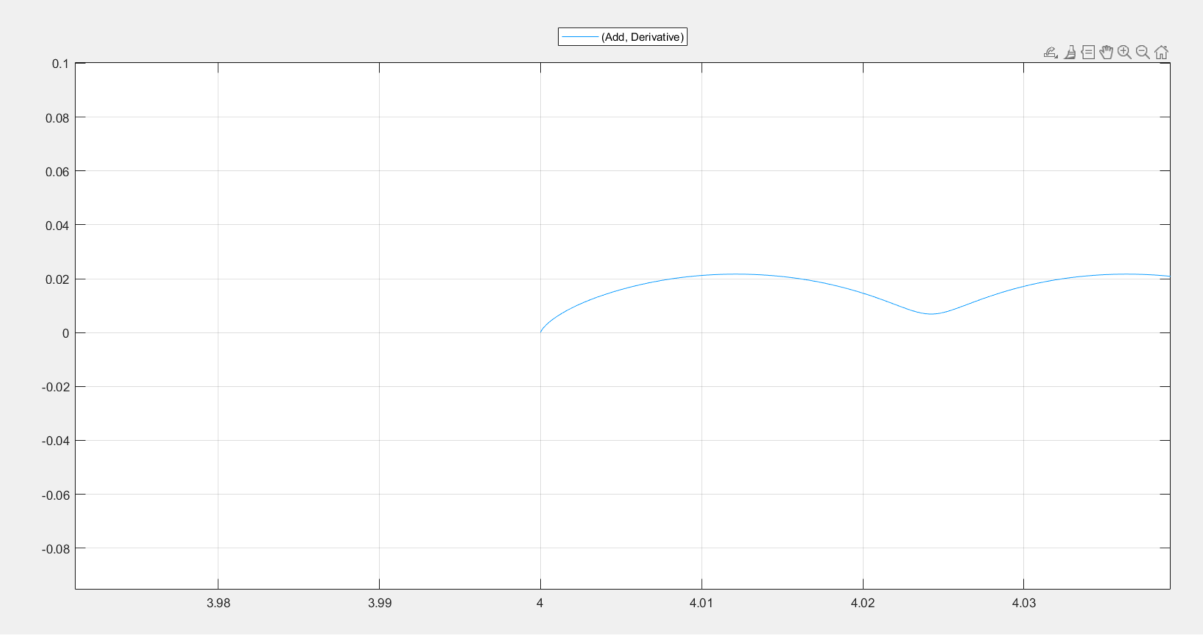
M = 300, l = 1, $x_3 = 0.1$ (initially)

When we provide an initial value to x_3 we get equilibrium points $\dot{y} = 0$, we observe that these equilibrium points come closer when mass is increased.

Variation of l

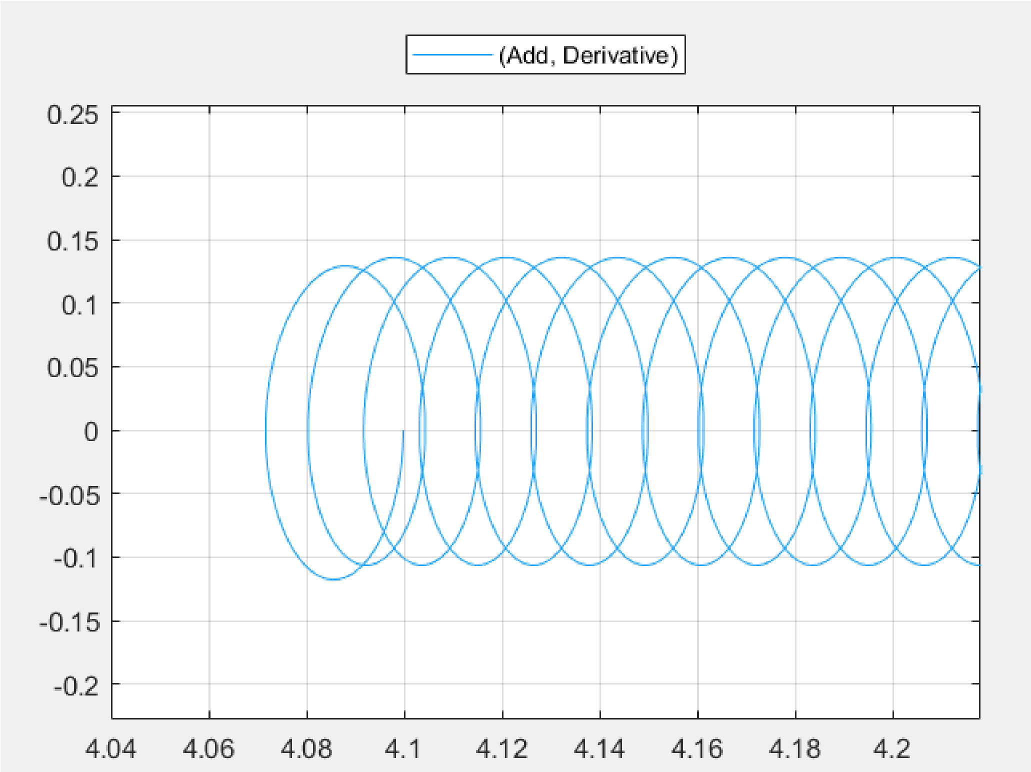


M = 50, l = 1

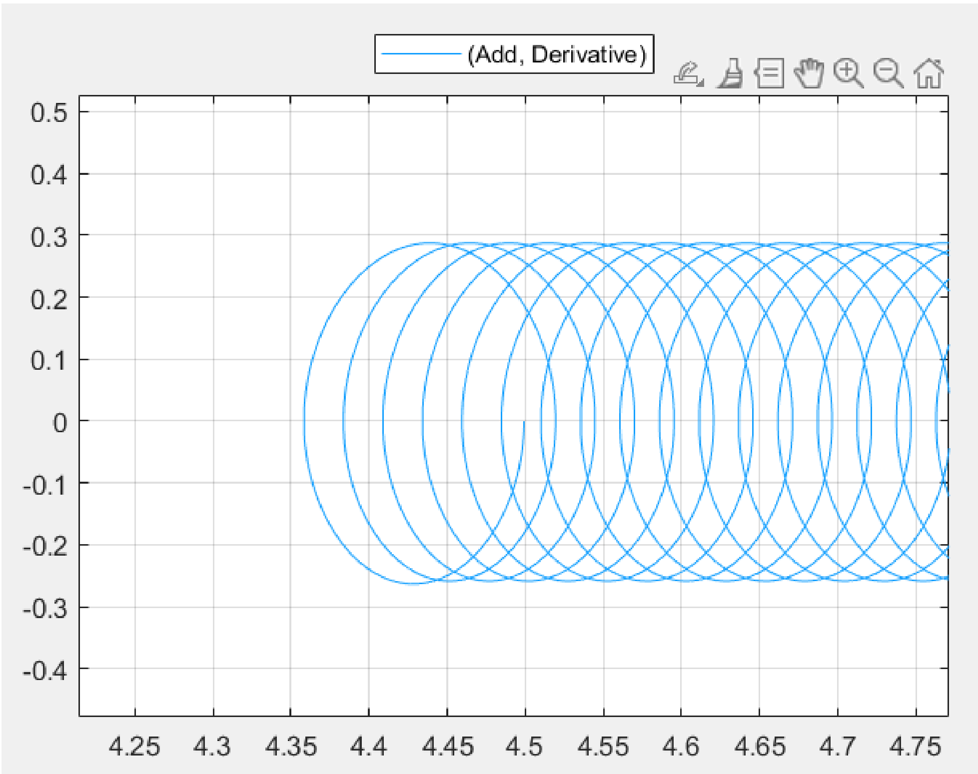


M = 50, l = 5

When we increase the length of the rope, keeping the load constant, the rate of change of output y slightly increases, and we obtain no equilibrium points for initial condition of $x_3 = 0$.



M = 50, l = 1, x_3 *= 0.1*



M = 50, l = 5, x_3 *= 0.1*

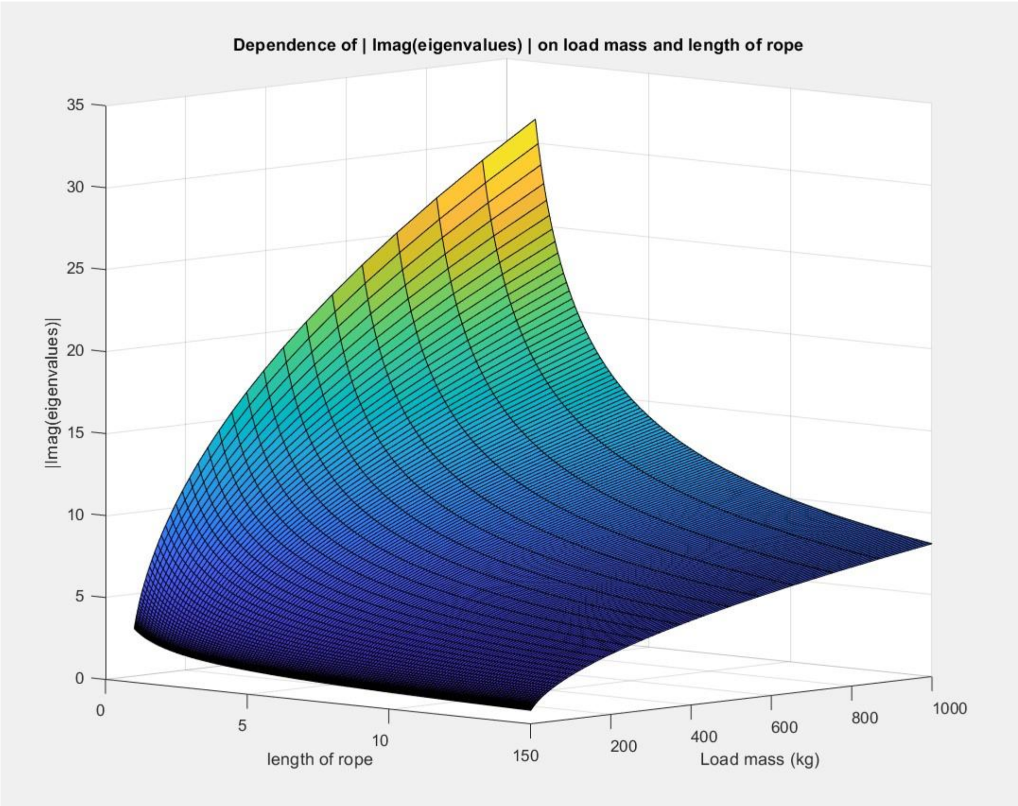
Again, we repeat the above procedure of parameter variations, and we provide an initial condition $x_3 = 0.1$. We observe that we get equilibrium points $\dot{y} = 0$ and these are attained more slowly when length of rope is increased.

ANALYSIS OF EIGEN-VALUES:

It is clear from the matrices obtained after linearizing system that the eigen values depend on the load and length of the rope, on solving for the eigen values we see that two of the eigenvalues always lie on s-plane origin, and other 2 eigen values are always complex conjugate and lie on imaginary axis. This implies that the given system in unstable for every state vector (because 2 of the eigen values (corresponding to state x1 and x2) lie on the origin), this leads to linear increase/ decrease in position of trolley for impulse inputs. Using analytic method, we obtain the following function for the imaginary part of complex conjugate eigen values:

$$s_{3,4} = \pm j \cdot \sqrt{\frac{g}{l}(1 + \frac{m_L}{m_C})}$$

Using the above relation, we can obtain the plot for dependence of eigenvalues on load and length of rope.



Now as we increase the load mass or decrease the length of the rope, this results in an increase in imaginary part of the complex conjugate eigenvalues, which increases the frequency of the sustained oscillations.

We compare the above results with the change in eigen values and observe that **increasing the imaginary part of Eigen Value (increases load mass or decreasing the length of the rope) decreases the rate of encountering of equilibrium points and vice-versa.**

We also observe that increasing the mass (increasing the imaginary part of the eigen values) leads to decrease in rate of change of output, and hence decreases the sway of the system while increasing the length (decreasing the eigen values) increases the sway of the system.

CONCLUSIONS:

We linearized and analyzed the stability of the given non-linear system, and located the local and global stability points, we further looked at how the eigen-values (and hence the system response) depends on the load mass and length of the rope.

MATLAB SCRIPT USED:

```
g = 9.8;
ml = logspace(0, 3, 50);
l = [1:0.1:15];
for i = 1:50
    for j = 1:141
        A = [0 1 0 0;
            0 0 (ml(i)*g)/10 0;
            0 0 0 1;
            0 0 -(10 + ml(i)) *g/(10*l(j)) 0];
        B = [0;
            1/10;
            0;
            -1/(10*l(j))];
        C = [1 0 l(j) 0];
        D = [0];
        sys = ss(A, B, C, D);
        eig(sys);
        eigen(i, j) = abs(imag(ans(4)));
    end
end
surf(l, ml, eigen);
```