Bayes' theorem

Bayes' theorem, also known as Bayes' law or Bayes' rule, tells us the probability of an event, given prior knowledge of related events that occurred earlier. Bayes' theorem is

$$P(A\,|\,B) = \frac{P(B\,|\,A) \cdot P(A)}{P(B)}$$

To simplify Bayes' theorem problems, it can be really helpful to create a tree diagram. If we're ever having trouble figuring out a conditional probability problem, a tree diagram is a great tool to fall back on, because it shows all of the sample space of the problem.

Example

We have two dice. One is fair, and the other one is weighted to land on 6, $50\,\%$ of the time. There's an equal probability for the other five faces on the biased die. Without knowing which one we're choosing, we pick one of the dice, roll it, and get a 6. What is the probability that we rolled the biased die?

Since we're looking for the probability that a die is biased given that we already rolled a 6, we can say that we're looking for $P(\text{Biased}\,|\,6)$. In this case, $P(A) = P(\text{Biased})$ and $P(B) = P(6)$. So we need to find the following values and put them into a formula:

- $P(6\,|\,\text{Biased})$

- $P(\text{Biased})$

- $P(6)$

We know from the problem that $P(6\,|\,\text{Biased}) = 1/2$. And since there are two die and each choice is equally likely $P(\text{Biased}) = 1/2$.

The probability of rolling a 6 is the probability of choosing the biased die and rolling a 6 or the probability of choosing the fair die and rolling a 6.

Let's first find the probability that the die is biased and we roll a 6. $P(\text{Biased}) = 1/2$ and the probability of a 6 on the biased die is $50\,\%$. So the probability of biased and 6 is

$$P(\text{Biased and } 6) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

Now let's find the probability that the die is fair and we roll a 6.

$$P(\text{Fair and } 6) = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right) = \frac{1}{12}$$

Therefore, the probability of rolling a 6 is

$$P(6) = \frac{1}{4} + \frac{1}{12}$$

$$P(6) = \frac{3}{12} + \frac{1}{12}$$

$$P(6) = \frac{4}{12}$$

$$P(6) = \frac{1}{3}$$

Putting these values into Bayes' theorem, we get

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$P(A|B) = \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{3}}$$

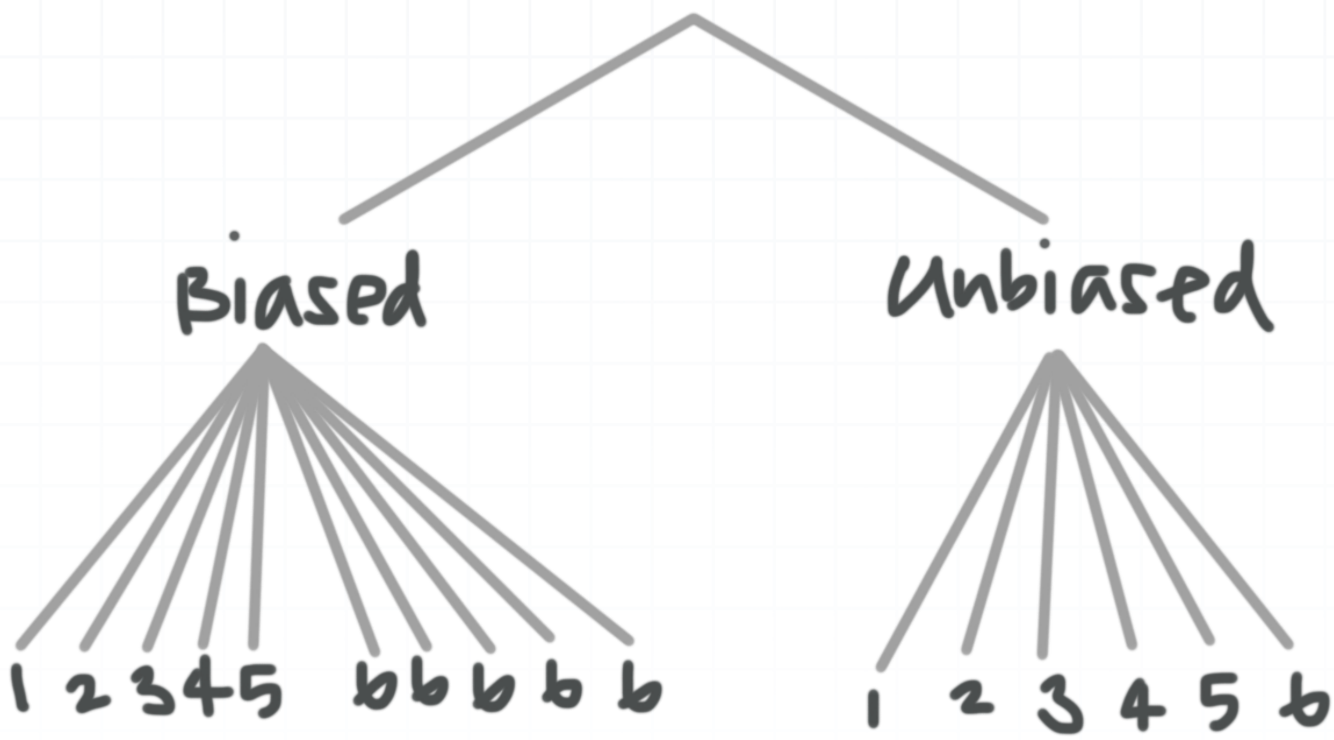
$$P(A|B) = \frac{\frac{1}{4}}{\frac{1}{3}}$$

$$P(A|B) = \frac{1}{4} \cdot \frac{3}{1}$$

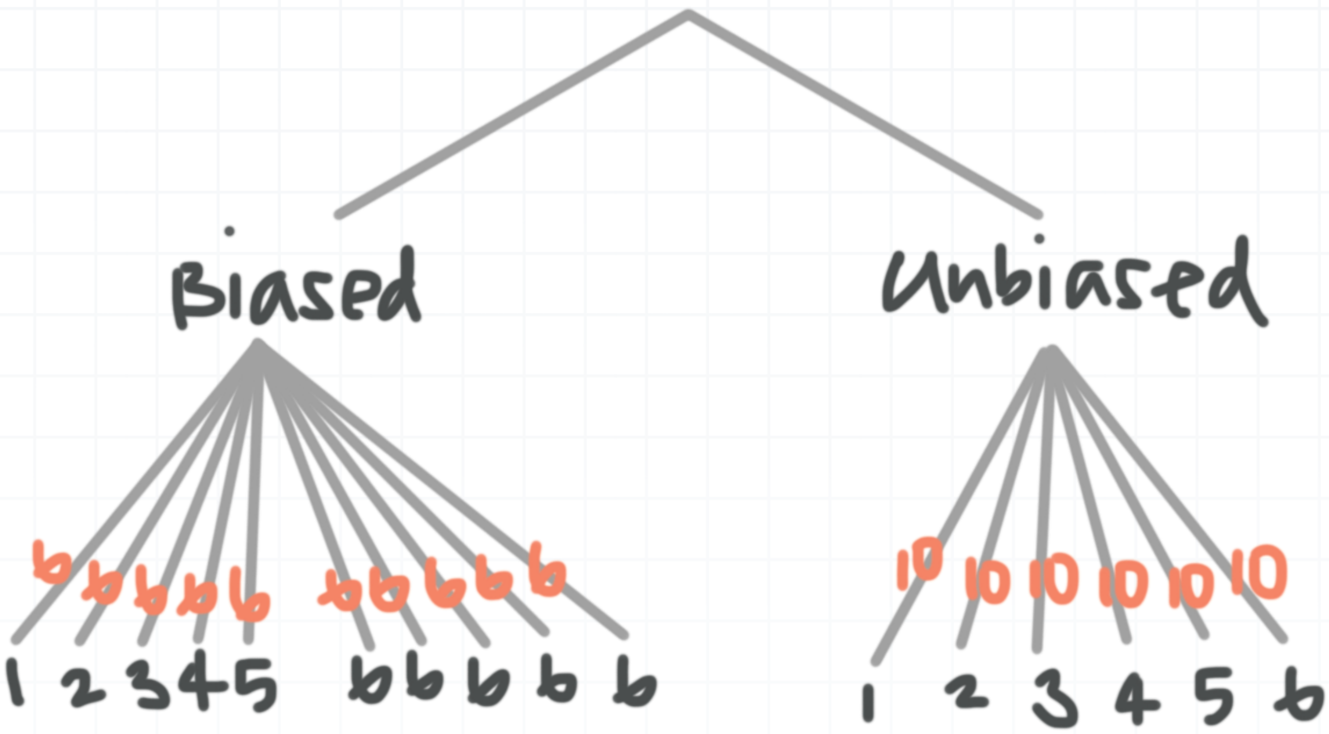
$$P(A|B) = \frac{3}{4}$$

Let's look at how we would do this with a tree diagram. We need to show branches in our tree for every event in our problem. First, we picked a die. It can either be the biased die or the unbiased die, and we have an equal chance of picking each one.

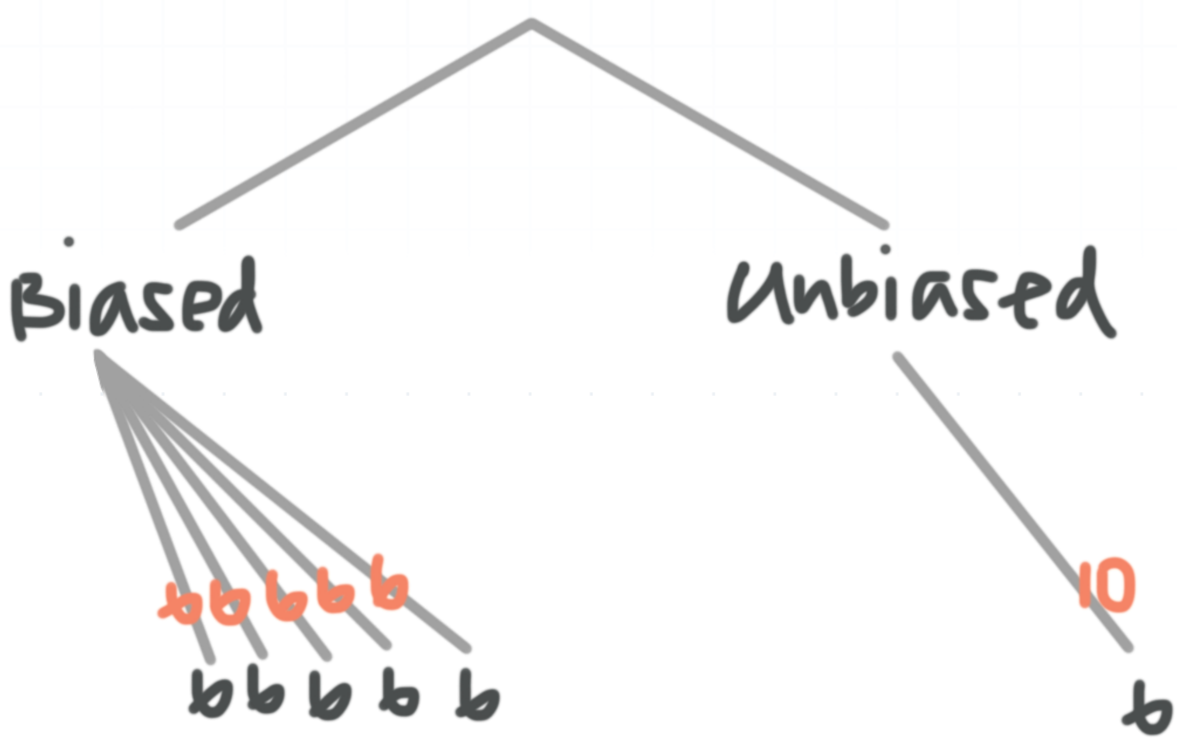
Each die can lead to six possible outcomes, so we'll show these in our tree. We also know that the biased die will land on 6, 50% of the time, and that all five other outcomes are equally likely, which means we could think of this as 5 possible outcomes leading to 6, and one possible outcome leading to each of the other numbers.



But we always need to make sure our tree is balanced. We can't have 10 branches coming down from the biased side, and 6 branches coming down from the unbiased side. We have to have an equal number of branches on both sides, so we'll scale up both sides to the least common multiple, which is 60. That means we need to scale up all the biased branches by 6 and all the unbiased branches by 10.



The next part of the problem tells us that we rolled one die and got a 6. So we'll trim all the branches of our tree that lead to results that didn't occur.



Now we want to know the probability that we rolled the biased die. Remember that each of the branches on the biased side represents 6 possibilities, so there are 30 branches coming from the biased side. The branch on the unbiased side represents 10 possibilities, so there are 10 branches coming from the unbiased side. Therefore, the probability that we rolled the biased die is all the outcomes coming from the biased side, 30, divided by all of the branches in total, 40:

$$P(\text{Biased})=\frac{30}{30+10}=\frac{30}{40}=75\,\%$$