Chi-square tests

There are three kinds of χ^2-tests (chi-squared) we want to look at.

1. A χ^2-test for homogeneity

2. A χ^2-test for association/independence

3. A χ^2 goodness-of-fit test

In general, χ^2-tests let us investigate the relationship between categorical variables. For instance, the χ^2-test for homogeneity is a test we can use to determine whether the probability distributions for two different groups are homogeneous with respect to some characteristic. Whereas the χ^2-test for association lets us determine whether two variables are related in the same group. And the χ^2 goodness-of-fit test is used to determine whether data fits a specified distribution.

We'll cover all three of these χ^2-tests in much more detail in this section. But first, we need to remember our conditions for sampling, since we'll often be using samples to represent populations in χ^2 problems.

Conditions for inference

Sometimes we'll have a χ^2 problem where someone's taking a sample of a population. When sampling occurs, in order for us to be able to use a χ^2 -test, we need to be able to meet typical sampling conditions:

1. **Random**: Any sample we're using needs to be taking randomly.

2. **Large counts**: Each expected value (more on "expected value" later) that we calculate needs to be 5 or greater.

3. **Independent**: We should be sampling with replacement, but if we're not, then the sample we take shouldn't be larger than 10% of the total population.

4. **Categorical**: The variables that we study are categorical.

If any of these conditions aren't met, we can't use a χ^2 testing method. But assuming we can meet these three conditions, then each of these three χ^2 -tests looks pretty much the same.

χ^2-test for homogeneity

When we use a χ^2-test for homogeneity, we're trying to determine whether the distributions for two variables are similar, or whether they differ from each other.

Let's use an example. Let's say that we take a sample of male students and a sample of female students and ask all of them whether they prefer cats, dogs, or some other animal as pets.

	Cats	Dogs	Other	Totals
Male	19	34	22	75
Female	26	28	6	60
Totals	45	62	28	135

This is a χ^2-test for homogeneity because we're sampling from two different groups (males and females) and comparing their probability distributions.

So in this study we have two variables, gender and pet, and we want to know whether gender affects pet preference, or if pet preference is not affected at all by gender. If pet preference isn't affected by gender, then the distribution of pet preference for males should be the same or similar to the distribution of pet preference for females. If gender does affect pet preference, then the distributions will be different (they won't be homogeneous).

To test this, we want to state the null hypothesis, which will be that gender doesn't affect pet preference.

H_0: gender doesn't affect pet preference

H_a: pet preference is affected by gender

The totals in the table margins let us determine the overall probability of being male or female in this study, regardless of pet preference, and the overall probability of preferring cats, dogs, or some other pet, regardless of gender.

If the distributions for pet preference for males and females are homogeneous, then we should be able to use these marginal probabilities to predict the expected number of students who should be in each cell. If the actual value is very different than the predicted value based on the marginal probabilities, that difference tells us that pet preference might be affected by gender, and therefore that the distributions of pet preference for males and females might not be homogeneous.

In this survey, the overall probability of being male is $75/135 \approx 0.56$, and the overall probability of liking cats is $45/135 \approx 0.33$. Which means that if the distributions for males and females are homogeneous, then $33\,\%$ of $56\,\%$ of the participants should be male and prefer cats. The value that *should* be in each cell if the distributions are homogeneous is called the **expected value** for that cell.

A simple way to find the expected value for each cell is to multiply the row and column totals and then divide by the overall total.

Expected Male-Cats: $(75 \cdot 45)/135 = 25$

Expected Male-Dogs: $(75 \cdot 62)/135 \approx 34.4$

Expected Male-Other: $(75 \cdot 28)/135 \approx 15.6$

Expected Female-Cats: $(60 \cdot 45)/135 = 20$

Expected Female-Dogs: $(60 \cdot 62)/135 \approx 27.6$

Expected Female-Other: $(60 \cdot 28)/135 \approx 12.4$

Then we can put each **actual value**, which is the count we got for each category when we surveyed the students, next to the (expected value) in an updated table.

	Cats	Dogs	Other	Totals
Male	19 (25.0)	34 (34.4)	22 (15.6)	75
Female	26 (20.0)	28 (27.6)	6 (12.4)	60
Totals	45	62	28	135

If we've done this correctly, the expected values should still sum to the same row and column totals.

The χ^2 distribution and degrees of freedom

This is where the χ^2 part comes in. We use the χ^2 formula to compare the actual and expected value in each cell, and then we sum all those values together to get χ^2.

$$\chi^2 = \sum \frac{(\text{observed} - \text{expected})^2}{\text{expected}}$$

$$\chi^2 = \frac{(19 - 25.0)^2}{25} + \frac{(34 - 34.4)^2}{34.4} + \frac{(22 - 15.6)^2}{15.6}$$

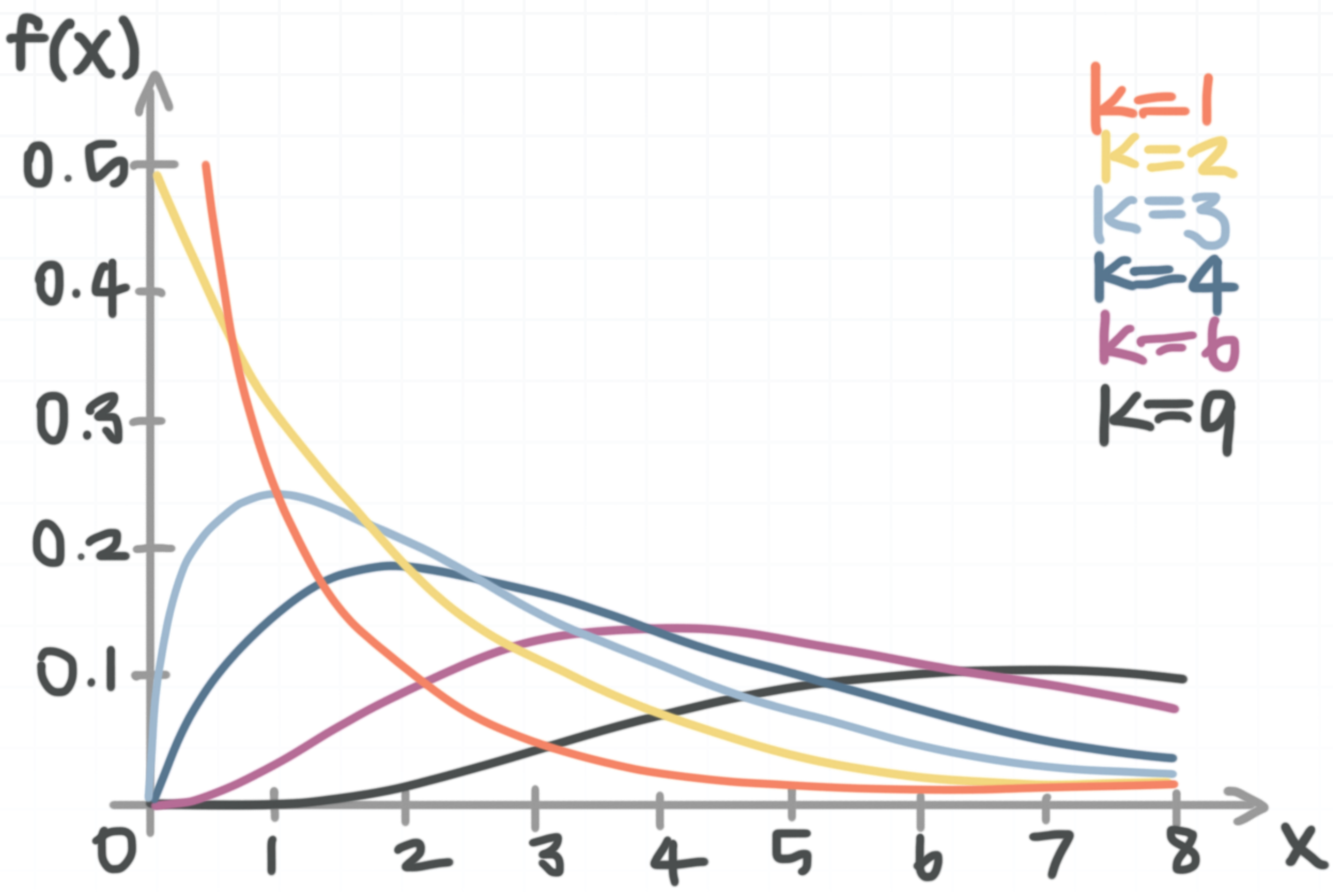
$$+ \frac{(26 - 20.0)^2}{20.0} + \frac{(28 - 27.6)^2}{27.6} + \frac{(6 - 12.4)^2}{12.4}$$

$$\chi^2 = 1.44 + 0.00 + 2.63 + 1.80 + 0.01 + 3.30$$

$$\chi^2 = 9.18$$

In general, the larger the value of χ^2, the more likely it is that the variables affect each other (not homogeneous).

But to determine whether or not we can reject the null hypothesis, we need to look up the χ^2 value and the degrees of freedom in the χ^2-table. Similar to the normal- and t-distributions, χ^2 has its own probability distribution that looks like this:



There's a distinct χ^2-distribution for each degree of freedom. So we can see the χ^2-distribution for df $= 1$ in red, the χ^2-distribution for df $= 2$ in yellow, the χ^2-distribution for df $= 3$ in light blue, etc.

Of course, like the t-table for t-distributions, the χ^2-table for χ^2-distributions includes a degrees of freedom value.

The degrees of freedom is always the minimum number of data points we'd need to have all of the information in the table. So for example in a χ^2-test for homogeneity, degrees of freedom is given by

$$\text{df} = (\text{number of rows} - 1)(\text{number of columns} - 1)$$

When we calculate degrees of freedom, we need to make sure to only include rows and columns from the body. We should never include the total column or the total row. So for the table in the example we've been working with, we get

$$\text{df} = (2-1)(3-1)$$

$$\text{df} = (1)(2)$$

$$\text{df} = 2$$

It makes sense that $\text{df} = 2$ for this example, because if we have any two pieces of information from the table, we can always figure out the rest of the information.

For instance, if we only know Male-Cats and Male-Dogs, we can find the rest of the missing values.

	Cats	Dogs	Other	Totals
Male	19	34		75
Female				60
Totals	45	62	28	135

Taking Cats-Total minus Male-Cats gives Female-Cats; taking Dogs-Total minus Male-Dogs gives Female-Dogs; taking Male-Total minus Male-Cats minus Male-Dogs gives Male-Other, and then once we have Male-Other, taking Other-Total minus Male-Other gives Female-Other.

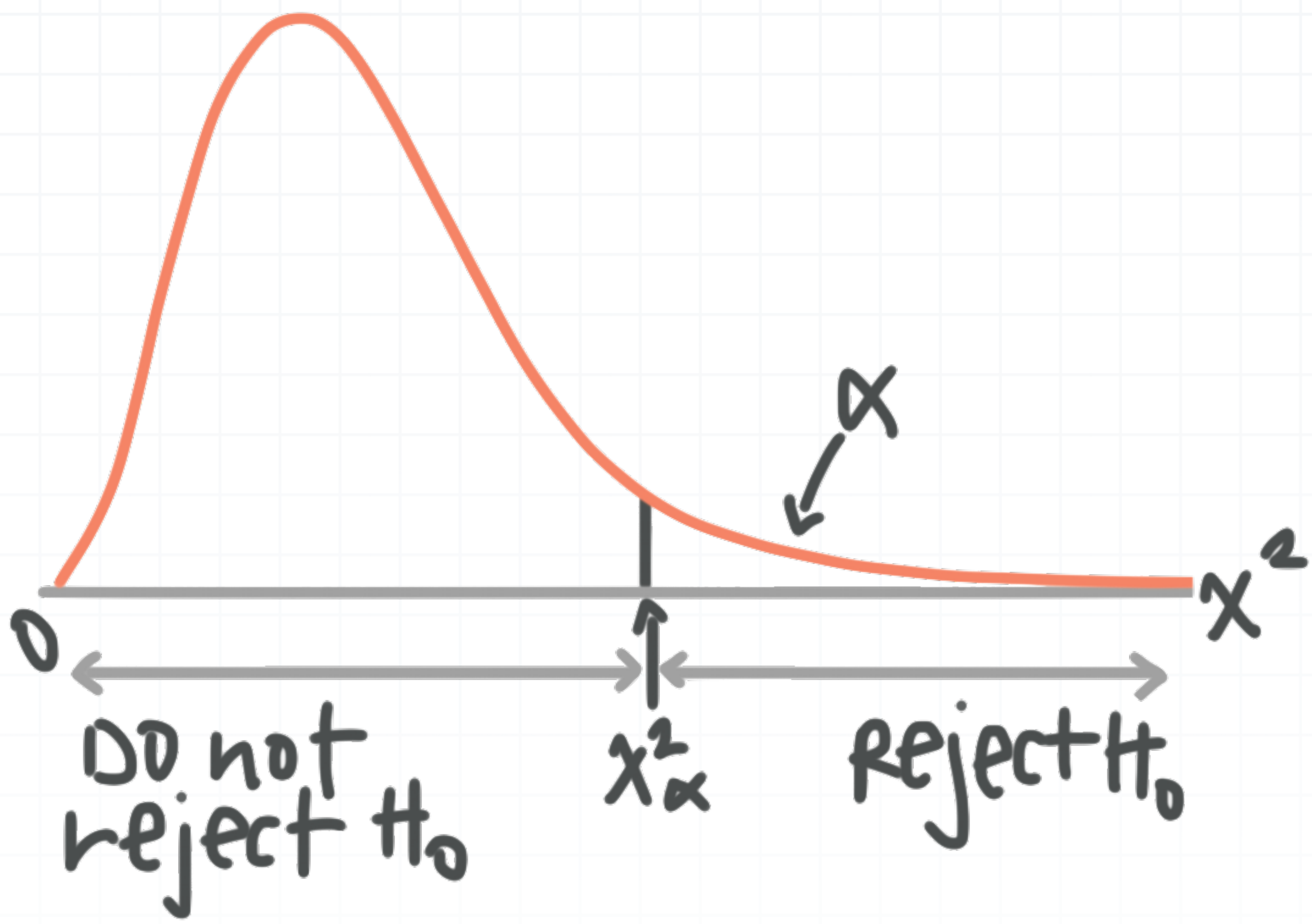
But back to the example, looking up $\chi^2 = 9.18$ and $\text{df} = 2$ in the χ^2-table,

	Upper-tail probability p											
df	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.02**	**0.01**	**0.005**	**0.0025**	**0.001**	**0.0005**
1	1.32	1.64	2.07	2.71	3.81	5.02	5.41	6.63	7.88	9.14	10.83	12.12
2	2.77	3.22	3.79	4.61	5.99	7.38	**7.82**	**9.21**	10.60	11.98	13.82	15.20
3	4.11	4.64	5.32	6.25	7.81	9.35	9.84	11.34	12.84	14.32	16.27	17.73

puts us between a p-level of 0.02 and 0.01, closer to 0.01. So we could have picked an alpha value as high as $\alpha = 0.02$ and still rejected the null hypothesis, concluding that we've found support for the alternative hypothesis that gender affects pet preference (or that pet preference is affected by gender). But if we had picked an alpha value of $\alpha = 0.01$, we'd be unable to reject the null hypothesis.

So for this particular example, we can reject the null hypothesis and conclude that there's a relationship between gender and pet preference (at $\alpha = 0.02$).

In general, we usually look at the χ^2-table to determine whether the χ^2-value falls into the region of rejection. We set χ^2_α as the critical value that corresponds to our level of significance α, but the value of χ^2_α will be determined not only by α, but also by the degrees of freedom. Once we've calculated our χ^2-value, we compare it to χ^2_α. If $\chi^2 > \chi^2_\alpha$, we reject the null hypothesis, and if $\chi^2 < \chi^2_\alpha$, we fail to reject the null hypothesis.



For example, if we'd chosen $\alpha = 0.02$, our critical value χ^2_α at df $= 2$ would be $\chi^2_\alpha = 7.82$, and because $9.18 > 7.82$, we would've rejected the null hypothesis.

χ^2-test for association/independence

In a χ^2-test for association or independence, we're sampling from one group (instead of from two, like we did with the homogeneity test), but we're thinking about two different categorical variables for that same group.

In the example that follows, we're taking a sample of employees at our company (sampling from one group), and looking at their eye color and handedness (two variables about them), so see whether or not eye color and handedness are associated.

Example

We want to know whether eye color and handedness are associated in employees of our $15,000$-person company, so we take a random sample of company employees and ask them their eye color, and whether they're left- or right-handed.

	Brown	Blue	Green	Hazel	Totals
Left-handed	72	36	20	12	140
Right-handed	460	215	130	55	860
Totals	532	251	150	67	1,000

Use a χ^2-test for independence with $\alpha = 0.05$ to say whether or not eye color and handedness are associated.

Start by computing expected values.

Expected Left-Brown: $(140 \cdot 532)/1,000 = 74.48$

Expected Left-Blue: $(140 \cdot 251)/1,000 = 35.14$

Expected Left-Green: $(140 \cdot 150)/1,000 = 21.00$

Expected Left-Hazel: $(140 \cdot 67)/1,000 = 9.38$

Expected Right-Brown: $(860 \cdot 532)/1,000 = 457.52$

Expected Right-Blue: $(860 \cdot 251)/1,000 = 215.86$

Expected Right-Green: $(860 \cdot 150)/1,000 = 129.00$

Expected Right-Hazel: $(860 \cdot 67)/1,000 = 57.62$

Then fill in the table.

	Brown	Blue	Green	Hazel	Totals
Left observed	72	36	20	12	140
Left expected	74.48	35.14	21.00	9.38	140
Right observed	460	215	130	55	860
Right expected	457.52	215.86	129.00	57.62	860
Totals	532	251	150	67	1,000

Now we'll check our sampling conditions. The problem told us that we took a random sample, and all of our expected values are at least 5 (Left-Hazel has the smallest expected value at 9.38), so we've met the random sampling and large counts conditions.

And even though we're sampling without replacement, there are $15,000$ employees in our company and we're sampling less than 10% of them ($1,000$ is less than 10% of $15,000$), so we've met the independence condition as well.

We'll state the null hypothesis.

H_0: eye color and handedness are independent (not associated)

H_a: eye color and handedness aren't independent (they're associated)

Calculate χ^2.

$$\chi^2 = \frac{(72-74.48)^2}{74.48} + \frac{(36-35.14)^2}{35.14} + \frac{(20-21.00)^2}{21.00} + \frac{(12-9.38)^2}{9.38}$$

$$+\frac{(460-457.52)^2}{457.52} + \frac{(215-215.86)^2}{215.86} + \frac{(130-129.00)^2}{129.00} + \frac{(55-57.62)^2}{57.62}$$

$$\chi^2 = 0.082578 + 0.021047 + 0.047619 + 0.731812$$

$$+0.013443 + 0.003426 + 0.007752 + 0.119132$$

$$\chi^2 \approx 1.03$$

The degrees of freedom are

$$\text{df} = (\text{number of rows} - 1)(\text{number of columns} - 1)$$

$$\text{df} = (2-1)(4-1)$$

$$\text{df} = (1)(3)$$

$$\text{df} = 3$$

With $\text{df} = 3$ and $\chi^2 \approx 1.03$, the χ^2-table gives

	Upper-tail probability p											
df	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.02**	**0.01**	**0.005**	**0.0025**	**0.001**	**0.0005**
2	2.77	3.22	3.79	4.61	5.99	7.38	7.82	9.21	10.60	11.98	13.82	15.20
3	4.11	4.64	5.32	6.25	**7.81**	9.35	9.84	11.34	12.84	14.32	16.27	17.73
4	5.39	5.99	6.74	7.78	9.49	11.14	11.67	13.28	14.86	16.42	18.47	20.00

It's clear that $\chi^2 \approx 1.03 < \chi^2_\alpha = 7.81$. Therefore, we fail to reject the null hypothesis, which means we can't see an association between eye color and handedness, and we therefore assume that those two variables are not associated, which means they're independent.

χ^2 goodness-of-fit test

In a χ^2-test for homogeneity, we sample from two different groups and compare whether their probability distributions are similar, and in a χ^2-test for independence, we sample from one group and try to determine the association between two variables about this one group.

But in a χ^2 goodness-of-fit test, we have a data table that gives some value or values, contingent upon some other condition, and we test to see whether the sample results are consistent with the values in the data table we've been given.

For instance, in the example that follows, we're working with a contingency table where we're given cupcake sales by month. Which means we're looking at cupcake sales, contingent upon the month of the year.

We're not sampling from two different groups so it doesn't make sense to use a χ^2-test for homogeneity, and we're not sampling from one group to look at the relationship between multiple variables of theirs, so it doesn't make sense to use a χ^2-test for association, either.

Instead, we'll use a χ^2 goodness-of-fit test.

Example

A small cupcake company wants to know if their sales are affected by month. They've recorded actual sales over the past year and collected the data in a table. Use a χ^2-test with $\alpha = 0.05$ to say whether the company's cupcake sales are affected by month.

Month	J	F	M	A	M	J	J	A	S	O	N	D	Total
Sales	60	80	65	70	80	100	140	120	90	90	60	65	1,020

The null hypothesis would be that sales aren't affected by month.

H_0: sales aren't affected by month (the month doesn't affect sales)

H_a: sales are affected by month (the month does affect sales)

If sales aren't affected by month, then they should be evenly distributed over each month, which means sales each month should be

$$\text{expected monthly sales} = \frac{1{,}020 \text{ total sales}}{12 \text{ months}} = 85 \text{ sales per month}$$

Which means we can expand the table to show observed versus expected sales.

Month	J	F	M	A	M	J	J	A	S	O	N	D	Total
Observed	60	80	65	70	80	100	140	120	90	90	60	65	1,020
Expected	85	85	85	85	85	85	85	85	85	85	85	85	1,020

Now that we have actual and expected values, we can calculate χ^2.

$$\chi^2 = \frac{(60-85)^2}{85} + \frac{(80-85)^2}{85} + \frac{(65-85)^2}{85} + \frac{(70-85)^2}{85}$$

$$+\frac{(80-85)^2}{85} + \frac{(100-85)^2}{85} + \frac{(140-85)^2}{85} + \frac{(120-85)^2}{85}$$

$$+\frac{(90-85)^2}{85} + \frac{(90-85)^2}{85} + \frac{(60-85)^2}{85} + \frac{(65-85)^2}{85}$$

$$\chi^2 = \frac{625}{85} + \frac{25}{85} + \frac{400}{85} + \frac{225}{85}$$

$$+\frac{25}{85} + \frac{225}{85} + \frac{3{,}025}{85} + \frac{1{,}225}{85}$$

$$+\frac{225}{85} + \frac{225}{85} + \frac{625}{85} + \frac{400}{85}$$

$$\chi^2 = \frac{7{,}250}{85} \approx 85.29$$

In a problem like this one, degrees of freedom is simply given by $n-1$. Because looking at the body of the table, there are 12 pieces of information in the body (one for each month), and we would need 11 of these values in order to be able to figure out the 12th. So we can only be missing one pieces of data, and degrees of freedom is therefore $n-1=12-1=11$. Look up $\chi^2 \approx 85.29$ and df $=11$ in the χ^2-table.

	Upper-tail probability p											
df	0.25	0.20	0.15	0.10	0.05	0.025	0.02	0.01	0.005	0.0025	0.001	0.0005
10	12.55	13.44	14.53	15.99	18.31	20.48	21.16	23.21	25.19	27.11	29.59	31.42
11	13.70	14.63	15.77	17.28	19.68	21.92	22.62	24.72	26.76	28.73	31.26	33.14
12	14.85	15.81	16.99	18.55	21.03	23.24	24.05	26.22	28.30	30.32	32.91	34.82

It's clear that $\chi^2 \approx 85.29 > \chi^2_\alpha = 19.68$, and there's a massive difference between the actual and expected values, which tells us that cupcake sales are very likely affected by month of the year. Which means we can reject the null hypothesis that sales aren't affected by month of the year, and conclude that cupcake sales are affected by month.

Let's walk through another example with a contingency table that looks a little different.

Example

Marla owns a dog walking service in which she employees 5 of her friends as dog-walkers. She created a table to record the number of friends she believes are walking dogs for her at any given time.

Number of walkers working	1	2	3	4	5
Percent of the time	15%	25%	30%	20%	10%

To test her belief, she took a random sample of 100 times and recorded the number of dog walkers working at that time.

Number of walkers working	1	2	3	4	5
Times	15	16	37	24	8

Say whether a χ^2-test can be used to say whether her findings disagree with her belief. If a χ^2-test is valid, say whether her findings support her belief with 95% confidence.

Since Marla is sampling, we need to meet three conditions if she's going to use a χ^2-test.

First, the sample needs to be random, and we were told in the problem that she took a random sample.

Second, every expected value needs to be at least 5. To find the expected values, we multiply her expected percentages by the total number of samples, 100.

Walkers working	1	2	3	4	5
Observed	15	16	37	24	8
Expected	15%*100=15	25%*100=25	30%*100=30	20%*100=20	10%*100=10

The smallest of these expected values is 10, which is greater than 5, so we've met the large counts condition.

Third, Marla isn't sampling with replacement, so the sample can't be more than 10% of the total population. It's safe to assume that Marla could continue taking an infinite number of samples at any given time, in theory gathering hundreds or thousands of samples as her business continues, so 100 samples shouldn't violate the independence condition.

Therefore, it's appropriate for her to use a χ^2-test. She'll state the null hypothesis that her model for the number of friends walking dogs at any given time is correct (her model matches the actual counts that she collected).

To compute χ^2, she'll use the actual and expected values.

$$\chi^2 = \frac{(15-15)^2}{15} + \frac{(16-25)^2}{25} + \frac{(37-30)^2}{30} + \frac{(24-20)^2}{20} + \frac{(8-10)^2}{10}$$

$$\chi^2 = \frac{0}{15} + \frac{81}{25} + \frac{49}{30} + \frac{16}{20} + \frac{4}{10}$$

$$\chi^2 = \frac{911}{150} \approx 6.07$$

With 5 possibilities for number of walkers, there are 4 degrees of freedom, so look up df $= 4$ with $\chi^2 \approx 6.07$.

In order for Marla to reject the null hypothesis at 95% confidence, she would have needed to surpass a value of 9.49 to be above the 5% threshold.

	Upper-tail probability p											
df	**0.25**	**0.20**	**0.15**	**0.10**	**0.05**	**0.025**	**0.02**	**0.01**	**0.005**	**0.0025**	**0.001**	**0.0005**
3	4.11	4.64	5.32	6.25	7.81	9.35	9.84	11.34	12.84	14.32	16.27	17.73
4	5.39	5.99	6.74	7.78	**9.49**	11.14	11.67	13.28	14.86	16.42	18.47	20.00
5	6.63	7.29	8.12	9.24	11.07	12.83	13.39	15.09	16.75	18.39	20.52	22.11

However, $\chi^2 \approx 6.07 < \chi^2_\alpha = 9.49$. Therefore, Marla cannot reject the null hypothesis, which means she can't conclude that her model is incorrect. In other words, her findings are consistent enough with her model that she can continue to use it.