Binomial random variables

Remember that “bi” means two, so a **binomial variable** is a variable that can take on exactly two values. A coin is the most obvious example of a binomial variable because flipping a coin can only result in two values: heads or tails. On the other hand, “rolling a die until a 3 appears” can’t be represented by a binomial random variable. We’ll learn more about why when we look at the required characteristics of a binomial random variable.

The two outcomes that the binomial random variable can take do not have to be equally probable. The probability of getting heads when we flip a fair coin is a binomial random variable where the probability of “success” is 50%. But the probability of choosing a girl from our math class when we randomly choose one student might be a binomial random variable where the probability of “success” is 70% (if 70% of the students in our class are female).

Binomial random variables

In order for a variable X to be a **binomial random variable**,

- each trial must be independent,
- each trial can be called a “success” or “failure,”
- there are a fixed number of trials, and
- the probability of success on each trial is constant.

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Let's think about X as how many heads we get when we flip a coin 10 times.

- Each trial is independent, because the result of one flip doesn't affect the result of any other flip.

- Each trial can be called a success or failure because "heads" is a success and "tails" is a failure.

- We're flipping the coin 10 times, so there are a fixed number of trials.

- The probability of getting heads on the first flip is 50%, the probability of getting heads on the second flip is 50%, the probability of getting heads on the third flip is 50%, etc. The probability of heads in each trial is constant.

So if X is how many heads we get when we flip a coin 10 times, then we could call X a binomial random variable. On the other hand, let's think about J as how many jacks we get when we pull 3 cards from a deck without replacing the card after each pull.

- Each trial is not independent, because the result of the first pull affects the result of every pull thereafter. If we get a jack on the first pull, then the probability we get a jack on the second pull is $3/51$. But if we get something other than a jack on the first pull, then the probability we get a jack on the second pull is $4/51$.

- Each trial can be called a success or failure because a jack is a success and anything else is a failure.

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- We're pulling a card 3 times, so there are a fixed number of trials.

- The probability of getting a jack on the first pull is $4/52$. But the probability of getting a jack on the second pull changes depending on what we got on the first pull. The probability of pulling a jack in each trial is not constant.

Since it fails two of the four conditions, J cannot be called a binomial random variable.

Binomial probability

In binomial probability questions, we're often asked to figure out the probability that we get an exact number of "successes," assuming we perform a specific number of independent trials. The formula we'll use for this is

$$P(k \text{ successes in } n \text{ attempts}) = \binom{n}{k} p^k (1-p)^{n-k}$$

where the binomial coefficient

$$\binom{n}{k}$$

is the combination ${}_nC_k$, p is the probability of a success, k is the exact number of times we want the success, and n is the total number of independent trials we'll run.

Example

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Let's say there are three marbles in a bag: 2 are green and 1 is red. We're going to do 5 trials where we pull a marble, note the color, and then replace the marble. What is the probability that we get the red marble exactly 3 times?

First, we'll confirm that this is a binomial random variable.

1) Since we're replacing each marble we pull before pulling another, each pull is an independent trial.

2) We can classify a red marble as a success, and green marble as a failure.

3) There are a fixed number of trials: 5.

4) The probability of success (getting a red marble) is constant through each trial, since we're replacing the marbles. Since all four conditions are met, this is a binomial random variable.

We're trying to pull a red marble exactly 3 times in 5 pulls. To solve this problem, we need to first figure out how many possible combinations we can do this in.

$$_5C_3 = \binom{5}{3} = \frac{5!}{3!(5-3)!}$$

$$_5C_3 = \binom{5}{3} = \frac{5!}{3!2!}$$

$$_5C_3=\binom{5}{3}=\frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot2\cdot1}$$

$$_5C_3=\binom{5}{3}=\frac{5\cdot4}{2\cdot1}$$

$$_5C_3=\binom{5}{3}=\frac{20}{2}$$

$$_5C_3=\binom{5}{3}=10$$

Then to find the probability that we get exactly 3 reds on 5 pulls, we say that f is the probability of pulling a red marble, and therefore that the probability of pulling 3 red marbles in 5 pulls is

$$P(k \text{ successes in } n \text{ attempts})=\binom{n}{k}p^k(1-p)^{n-k}$$

$$P(3 \text{ reds in } 5 \text{ pulls})=(10)\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2$$

$$P(3 \text{ reds in } 5 \text{ pulls})=\frac{40}{243}\approx 16.5\,\%$$

In other words, we have approximately a $16.5\,\%$ chance of getting exactly 3 red marbles when we pull a random marble from this bag 5 times.

Probability distributions for binomial random variables

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We can also create a probability distribution for binomial random variables. Using our example of pulling a marble 5 times, where we have a $1/3$ probability of pulling a red marble and $2/3$ probability of pulling a green marble on each pull, we could calculate the following probabilities.

$$P(0 \text{ red in } 5 \text{ pulls}) = \binom{5}{0} 0.33^0 0.67^5 = (1)(1)(0.67^5) \approx 0.1350$$

$$P(1 \text{ red in } 5 \text{ pulls}) = \binom{5}{1} 0.33^1 0.67^4 = (5)(0.33^1)(0.67^4) \approx 0.3325$$

$$P(2 \text{ red in } 5 \text{ pulls}) = \binom{5}{2} 0.33^2 0.67^3 = (10)(0.33^2)(0.67^3) \approx 0.3275$$

$$P(3 \text{ red in } 5 \text{ pulls}) = \binom{5}{3} 0.33^3 0.67^2 = (10)(0.33^3)(0.67^2) \approx 0.1613$$

$$P(4 \text{ red in } 5 \text{ pulls}) = \binom{5}{4} 0.33^4 0.67^1 = (5)(0.33^4)(0.67^1) \approx 0.0397$$

$$P(5 \text{ red in } 5 \text{ pulls}) = \binom{5}{5} 0.33^5 0.67^0 = (1)(0.33^5)(0.67^0) \approx 0.0039$$

We could then plot this probability distribution to get a picture of the probability.

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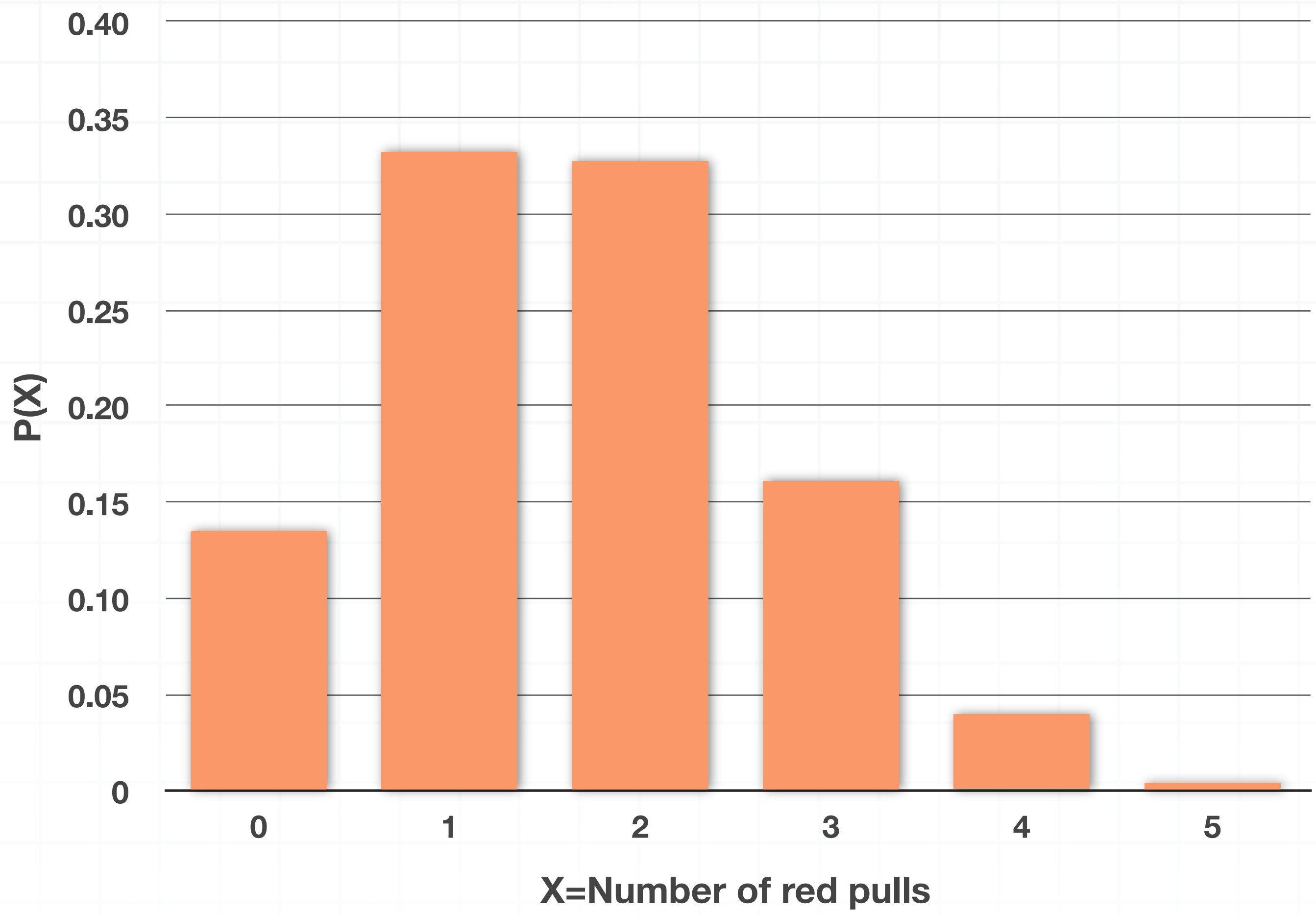

This probability distribution is a visual representation of the fact that we're most likely to get 1 or 2 red marbles when we pull 5 times from the bag of marbles. We're much less likely to get 4 or 5 red marbles when we pull 5 times.