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What Is Operations Research ?

1.1 INTRODUCTION : THE HISTORICAL DEVELOPMENT

In order to understand 'what Operations Research (OR)* is today,' we must know something of its history and evolution. The main origin of Operations Research was during the Second World-War. At that time, the military management in England called upon a team of scientists to study the strategic and tactical problems related to air and land defence of the country. Since they were having very limited military resources, it was necessary to decide upon the most effective utilization of them, *e.g.* the efficient ocean transport, effective bombing, etc.

During World-War II, the Military Commands of U.K. and U.S.A. engaged several inter-disciplinary teams of scientists to undertake scientific research into strategic and tactical military operations. Their mission was to formulate specific proposals and plans for aiding the Military Commands to arrive at the decisions on optimal utilization of scarce military resources and efforts, and also to implement the decisions effectively. The OR teams were not actually engaged in military operations and in fighting the war. But, they were only advisors and significantly instrumental in winning the war to the extent that the scientific and systematic approaches involved in OR provided a good intellectual support to the strategic initiatives of the military commands. Hence OR can be associated with ***"an art of winning the war without actually fighting it"***.

As the name implies, 'Operations Research' (sometimes abbreviated OR) was apparently invented because the team was dealing with *research* on (military) *operations*. The work of this team of scientists was named as *Operational Research* in England.

The encouraging results obtained by the British OR teams quickly motivated the United States military management to start with similar activities. Successful applications of the U.S. teams included the invention of new fight patterns, planning sea mining and effective utilization of electronic equipment. The work of OR team was given various names in the United States : ***Operational Analysis, Operations Evaluation, Operations Research, Systems Analysis, Systems Evaluation, Systems Research, Systems Analysis, Systems Evaluation, Systems Research, and Management Science.*** The name Operations Research was and is the most widely used so we shall also use it here.

Following the end of war, the success of military teams attracted the attention of *Industrial* managers who were seeking solutions to their complex executive-type problems. The most common problem was : what methods should be adopted so that the total cost is minimum or total profits maximum ? The first mathematical technique in this field (called the *Simplex Method* of linear programming) was developed in 1947 by American mathematician, **George B. Dantzig**. Since then, new techniques and applications have been developed through the efforts and cooperation of interested individuals in academic institutions and industry both.

Today, the impact of OR can be felt in many areas. A large number of management consultimg firms are currently engaged in OR activities. Apart from military and business applications, the OR activities include transportation system, libraries, hospitals, city planning, financial institutions, etc. Many of the Indian industries making use of OR activity are : ***Delhi Cloth Mills, Indian Railways, Indian Airlines, Defence Organizations, Hindustan Lever, Tata Iron & Steel Co., Fertilizer Corporation of India***, etc.

In business and other organizations, OR scientists and specialists always remain enagaged in the background. But, they help the top management officials and other line managers in doing their 'fighting' job better.

* The short word **'OR'** for **'Operations Research'** should not be confused with the word **'or'** throughout the book.

While making use of the techniques of OR, a mathematical model of the problem is formulated. This model is actually a simplified representation of the problems in which only the most important features are considered for reasons of simplicity. Then, an *optimal* or *most favourable* solution is found. Since the model is an idealized in stead of exact representation of real problem, the optimal solution thus obtained may not prove to be the best solution to the actual problem. Although, extremely accurate but highly complex mathematical models can be developed, but they may not be easily solvable. So from both the cost-minimising and mathematical simplicity point of view, it seems beneficial to develop a less accurate but simpler model, and to find a sequence of solutions consisting of a series of increasingly better approximations to the actual course of action. Thus, the apparent weaknesses in the initial solution are used to suggest improvements in the model, its input-data, and the solution procedure. A new solution is thus obtained and the process is repeated until the further improvements in the succeeding solutions become so small that it does not seem economical to make further improvements.

If the model is carefully formulated and tested, the resulting solution should reach to be good approximation to the ideal course of action for the real problem. Although, we may not get the best answers, but definitely we are able to find *the bad answers where worse exist.* Thus operations research techniques are always able to save us from worse situations of practical life.

Q. 1. Comment the following statements : **[Rewa (Maths.) 93]**

(i) O.R. is the art of winning war without actually fighting it.

(ii) O.R. is the art of finding bad answers where worse exist.

2. What is O.R. ? **[Garhwal 97, 96; Meerut (IPM) 90]**

3. Enumerate six applications of Operations Research (O.R.) and describe one briefly. **[IGNOU 2001 (June)]**

1.2 THE NATURE AND MEANING OF 'OR' **[IPM (PGDBA)* 82, 81; Meerut (Math.) 82]**

'OR' has been defined so far in various ways and it is perhaps still too young to be defined in some authoritative way. So it is important and interesting to give below a few opinions about the definition of OR which have been changed according to the development of the subject.

1. OR is a scientific method of providing executive departments with a *quantitative basis* for *decision* regarding the operations under their control. ***—Morse and Kimbal (1946)***
2. OR is a scientific method of providing executive with an *analytical* and *objective* basis for decisions. ***—P.M.S. Blackett (1948)***
3. The term 'OR' has hitherto-fore been used to connate various attempts to study operations of war by scientific methods. From a more general point of view, OR can be considered to be an attempt to study those *operations of modern society which involved organizations of men or of men and machines.* ***—P.M. Morse (1948)***
4. OR is the application of *scientific methods, techniques* and *tools* to problems involving the *operations of systems* so as to provide these in control of the operations with *optimum solutions* to the problem. ***—Churchman, Acoff, Arnoff (1957)***
5. OR is the art of giving *bad answers* to problems to which otherwise *worse answers* are given. ***—T. L. Saaty (1958)***
6. OR is a management *activity* pursued in two complementary ways—one half by the free and bold *exercise of commonsense* untrammelled by any routine, and other half by the application of a repertoire of *well established precreated methods* and techniques. ***—Jagjit Singh* (1968)**
7. OR is the *attack* of modern methods on *complex problems* arising in the *direction and management* to large systems of men, machines, materials, and money in industry, business and defence. The distinctive approach is to developed a *scientific model* of the system, incorporating measurements of factors such as chance and risk with which to predict and *compare* the outcomes of alternative *decisions, strategies* or

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* Wherever the name of the examination is not mentioned in the University Examination references, it should be understood M.A./M.Sc. throughout the book.
* IPM = Institute of Productivity Management. *PGDBA* = Post-Graduate Diploma in Business Administration.
* The symbol **Q.** will stand for '**EXAMINATION QUESTIONS**' throughout the book.

controls. The purpose is to help management to determine its policy and actions scientifically.

—***Operations Research Quarterly*** **(1971)**

8. Operations Research is the art of winning war without actually fighting it.
9. OR is an applied decision theory. It uses any *scientific mathematical or logical means* to attempt to cope with the problems that confront the executive when he tries to achieve a through going rationality in dealing with his decision problems. —***Miller and Starr.***
10. OR is a scientific approach to problem solving for executive management. —***H.M. Wagner***
11. OR is an aid for the executive in making his decisions by providing him with the needed quantitative information based on the scientific method of analysis. —***C. Kittel***
12. OR is the systematic method oriented study of the basic structure, characteristics, functions and relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decision making. —***E.L. Arnoff & M.J. Netzorg***
13. OR is the application of *scientific methods* to problems arising from operations involving *integrated* systems of *men, machines and materials*. It normally utilizes the knowledge and skill of an inter-disciplinary research team to provide the managers of such systems with optimum operating solutions. —***Fabrycky and Torgersen***
14. OR is an experimental and applied science devoted to observing, understanding and predicting the behaviour of purposeful man–machine systems and OR workers are actively engaged in applying this knowledge to practical problems in business, government, and society. —***OR Society of America***
15. OR is the application of scientific method by inter-disciplinary teams to problems involving the controls of organized (man-machine) systems so as to provide solutions which *best serve the purpose of the organization as a whole*. —***Ackoff & Sasieni, (1968)***
16. OR utilizes the planned approach (*updated scientific method)* and an *inter-disciplinary* team in order to represent complex functional relationships as mathematical models for purpose of providing a *quantitative basis* for decision making and *uncovering new problems* for quantitative analysis. —***Thieanf and Klekamp*** **(1975)**

Comments on definitions of OR :

From all above opinions, we arrive at the conclusion that whatever else 'OR' may be, it is certainly concerned with optimization problems. ***A decision, which taking into account all the present circumstances can be considered the best one, is called an optimal decision.*** **(Note)**

There are three main reasons for why most of the definitions of Operations Research are not satisfactory.

(i) First of all, Operations Research is not a science like any well-defined physical, biological, social phenomena. While chemists know about atoms and molecules and have theories about their interactions; and biologists know about living organisms and have theories about vital processes, *operations researchers* do not claim to know or have theories about operations. Operations Research is not a scientific research into the control of operations. It is essentially a collection of mathematical techniques and tools which in conjunction with a system approach are applied to solve practical decision problems of an *economic* or *engineering* nature. Thus it is very difficult to define Operations Research precisely.

(ii) Operations Research is inherently inter-disciplinary in nature with applications not only in military and business but also in medicine, engineering, physics and so on. Operations Research makes use of experience and expertise of people from different disciplines for developing new methods and procedures. Thus, inter-disciplinary approach is an important characteristic of Operations Research which is not included in most of its definitions. Hence most of the definitions are not satisfactory.

(iii) Most of the definitions of Operations Research have been offered at different times of development of 'OR' and hence are bound to emphasise its only one or the other aspect.

For example, 8th of the above definitions is only concerned with war alone. First definition confines 'OR' to be a scientific methodology applied for making operational decisions. It has no concern about the characteristics of different operational decisions and has not described how the scientific methods are applied in complicated situations. Many more definitions have been given by various authors but most of them fail to consider all basic characteristics of 'OR'. However, with further development of 'OR' perhaps more precise definitions should be forthcoming.

Q. 1. (a) Give any three definitions of Operations Research and explain them. [Meerut (IPM) 91; Meerut (O.R.) 90]
(b) Give reasons for : why most of the definitions of Operations Research are not satisfactory.
2. Discuss the three approaches of MIS development. [CA (May) 2000]
3. What are the pre-requisites of a computer based MIS ? [MCI 2000]

1.3 MANAGEMENT APPLICATIONS OF OPERATIONS RESEARCH

Some of the areas of management decision making, where the 'tools' and 'techniques' of O R are applied, can be outlined as follows :

1. **Finance–Budgeting and Investments**
   (i) Cash-flow analysis, long range capital requirements, dividend policies, investment portfolios.
   (ii) Credit policies, credit risks and delinquent account procedures.
   (iii) Claim and complaint procedures.
2. **Purchasing, Procurement and Exploration**
   (i) Rules for buying, supplies and stable or varying prices.
   (ii) Determination of quantities and timing of purchases.
   (iii) Bidding policies.
   (iv) Strategies for exploration and exploitation of raw material sources.
   (v) Replacement policies.
3. **Production Management**
   (i) ***Physical Distribution***
      (a) Location and size of warehouses, distribution centres and retail outlets.
      (b) Distribution policy.
   (ii) ***Facilities Planning***
      (a) Numbers and location of factories, warehouses, hospitals etc.
      (b) Loading and unloading facilities for railroads and trucks determining the transport schedule.
   (iii) ***Manufacturing***
      (a) Production scheduling and sequencing.
      (b) Stabilization of production and employment training, layoffs and optimum product mix.
   (iv) ***Maintenance and Project Scheduling***
      (a) Maintenance policies and preventive maintenance.
      (b) Maintenance crew sizes.
      (c) Project scheduling and allocation of resources.
4. **Marketing**
   (i) Product selection, timing, competitive actions.
   (ii) Number of salesman, frequency of calling on accounts per cent of time spent on prospects.
   (iii) Advertising media with respect to cost and time.
5. **Personnel Management**
   (i) Selection of suitable personnel on minimum salary.
   (ii) Mixes of age and skills.
   (iii) Recruitment policies and assignment of jobs.
6. **Research and Development**
   (i) Determination of the areas of concentration of research and development.
   (ii) Project selection.
   (iii) Determination of time cost trade-off and control of development projects.
   (iv) Reliability and alternative design.

From all above areas of applications, we may conclude that OR can be widely used in taking timely management decisions and also used as a corrective measure. The application of this tool involves certain data and not merely a personality of decision maker, and hence we can say : ***OR has replaced management by personality.***

Q. 1. "Operations Research replaces Management by personality." Discuss.
2. Explain applications of O.R. in Industry. [Garhwal 97; Karnataka 95]
3. Describe the various approaches used for development of MIS. [MCI 2000]

1.4 MODELLING IN OPERATIONS RESEARCH

Definition. *A model in the sense used in OR is defined as a representation of an actual object or situation. It shows the relationships (direct or indirect) and inter-relationships of action and reaction in terms of cause and effect.*

Since a model is an abstraction of reality, it thus appears to be less complete than reality itself. For a model to be complete, it must be a representative of those aspects of reality that are being investigated.

The main objective of a model is to provide means for analysing the behaviour of the system for the purpose of improving its performance. Or, if a system is not in existence, then a model defines the ideal structure of this future system indicating the functional relationships among its elements. The reliability of the solution obtained from a model depends on the validity of the model in representing the real systems. A model permits to 'examine the behaviour of a system without interfering with ongoing operations.

Models can be classified according to following characteristics :

1. Classification by Structure

(i) ***Iconic models.*** *Iconic models* represent the system as it is by scaling it up or down (*i.e.*, by enlarging or reducing the size). In other words, it is an image.

For example, a toy airplane is an iconic model of a real one. Other common examples of it are : photographs, drawings, maps etc. A model of an atom is scaled up so as to make it visible to the naked eye. In a globe, the diameter of the earth is scaled down, but the globe has approximately the same shape as the earth, and the relative sizes of continents, seas, etc., are approximately correct.

The iconic model is usually the simplest to conceive and the most specific and concrete. Its function is generally descriptive rather than explanatory. Accordingly, it cannot be easily used to determine or predict what effects many important changes on the actual system.

(ii) ***Analogue models.*** The models, in which one set of properties is used to represent another set of properties, are called *analogue models.* After the problem is solved, the solution is reinterpreted in terms of the original system.

For example, graphs are very simple analogues because distance is used to represent the properties such as : *time, number, per cent, age, weight,* and many other properties. Contour-lines on a map represent the rise and fall of the heights. In general, analogues are less specific, less concrete but easier to manipulate than are iconic models.

(iii) ***Symbolic (Mathematical) models.*** The *symbolic or mathematical model* is one which employs a set of mathematical symbols (*i.e., letters, numbers,* etc.) to represent the decision variables of the system. These variables are related together by means of a mathematical equation or a set of equations to describe the behaviour (or properties) of the system. The solution of the problem is then obtained by applying well-developed mathematical techniques to the model.

The symbolic model is usually the easiest to manipulate experimentally and it is most general and abstract. Its function is more often explanatory rather than descriptive.

2. Classification by Purpose

Models can also be classified by purpose of its utility. *The purpose of a model may be* descriptive, predictive or prescriptive.

(i) ***Descriptive models.*** A descriptive model simply describe some aspects of a situation based on observations, survey, questionnaire results or other available data. The result of an oppenion poll represents a descriptive model.

(ii) ***Predictive models.*** Such models can answer 'what if' type of questions, *i.e.* they can make predictions regarding certain events. For example, based on the survey results, television networks such models attempt to explain and predict the election results before all the votes are actually counted.

(iii) ***Prescriptive models.*** Finally, when a predictive model has been repeatedly successful, it can be used to prescribe a source of action. For example, linear programming is a prescriptive (or normative) model because it prescribes what the managers ought to do.

3. Classification by Nature of Environment

These are mainly of two types :

(i) ***Deterministic models.*** Such models assume conditions of complete certainty and perfect knowledge. For example, linear programming, transportation and assignment models are deterministic type of models.

(ii) ***Probabilistic (or Stochastic) models.*** These types of models usually handle such situations in which the consequences or payoff of managerial actions cannot be predicted with certainty. However, it is possible to forecast a pattern of events, based on which managerial decisions can be made. For example, insurance companies are willing to insure against risk of fire, accidents, sickness and so on, because the pattern of events have been compiled in the form of probability distributions.

4. Classification by Behaviour

(i) ***Static models.*** These models do not consider the impact of changes that takes place during the planning horizon, *i.e.* they are independent of time. Also, in a static model only one decision is needed for the duration of a given time period.

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(ii) ***Dynamic models.*** In these models, *time* is considered as one of the important variables and admit the impact of changes generated by time. Also, in dynamic models, not only one but a series of interdependent decisions is required during the planning horizon.

5. Classification by Method of Solution

(i) ***Analytical models.*** These models have a specific mathematical structure and thus can be solved by known analytical or mathematical techniques. For example, a *general linear programming* model as well as the specially structured *transportation* and *assignment models are analytical models.*

(ii) ***Simulation models.*** They also have a mathematical structure but *they cannot be solved* by purely using the 'tools' and 'techniques' of mathematics. A simulation model is essentially computer assisted experimentation on a mathematical structure of a real time structure in order to study the system under a variety of assumptions.

Simulation modelling has the advantage of being more flexible than mathematical modelling and hence can be used to represent complex systems which otherwise cannot be formulated mathematically. On the other hand, simulation has the disadvantage of not providing general solutions like those obtained from successful mathematical models.

6. Classification by Use of Digital Computers

The development of the digital computer has led to the introduction of the following types of modelling in OR.

(i) ***Analogue and Mathematical models combined.*** Sometimes analogue models are also expressed in terms of mathematical symbols. Such models may belong to both the types (ii) and (iii) in **classification 1** above.

For example,. **simulation model** is of analogue type but mathematical formulae are also used in it. Managers very frequently use this model to '*simulate*' their decisions by summarizing the activities of industry in a scale-down period.

(ii) ***Function models.*** Such models are grouped on the basis of the function being performed.

For example, a function may serve to acquaint to scientist with such things as–tables, carrying data, a blue-print of layouts, a program representing a sequence of operations (like in computer programming).

(iii) ***Quantitative models.*** Such models are used to measure the observations.

For example, degree of temperature, yardstick, a unit of measurement of length value, etc.

Other examples of quantitative models are : (i) *transformation models* which are useful in converting a measurement of one scale to another (*e.g.*, Centigrade *vs* Fahrenheit conversion scale), and (ii) the *test models* that act as 'standards' against which measurements are compared (*e.g.*, business dealings, a specified standard production control, the quality of a medicine).

(iv) ***Heuristic models.*** These models are mainly used to explore alternative strategies (courses of action) that were overlooked previously, whereas mathematical models are used to represent systems possessing well-defined strategies. Heuristic models do not claim to find the best solution to the problem.

Q. 1. Model building is the essence of the 'O.R. approach'. Discuss.

2. Discuss in detail the three types of models with special emphasis on the important logical properties and the relationship the three types bear to each other and to modelled entities. **[Meerut (OR) 90]**

3. What is meant by a mathematical model of real situation ? Discuss the importance of models in the solution of Operational Research probems ?

4. What is a model ? Discuss various classification schemes of models. **[Agra 95, 94; C.A. (May) 92; Meerut (IPM) 90]**

1.5 PRINCIPLES OF MODELLING

Let us now outline general principles useful in guiding to formulate the models within the context of O R. The model building and their users both should be consiously aware of the following ***Ten*** principles :

1. ***Do not build up a complicated model when simple one will suffice.*** Building the strongest possible model is a common guiding principle for mathematicians who are attempting to extend the theory or to develop techniques that have wide applications. However, in the actual practice of building models for specific purposes, the best advice is to "keep it simple".
2. ***Beware of molding the problem to fit the technique.*** For example, an expert on linear programming techniques may tend to view every problem he encounters as required in a linear programming solutions. In fact, not all optimization problems involve only linear functions. Also, not all OR problems involve optimization. As a matter of fact, not all real-world problems call for *operations research* ! Of course, every one search reality in his own terms, so the field of OR is not unique in this regard. Being human, we rely on the methods we are most comfortable in using and have been most successful within the past. We are certainly not able to use techniques in which we have no competence, and we cannot hope to be competent in all techniques. We must divide OR experts into three main categories :
   (i) Technique developers, (ii) Teachers, and (iii) Problem solvers.
   In particular, one should be ready to tolerate the behaviour "I have found a cure but I am trying to search a disease to fit it" among *technique developers* and *teachers*.
3. ***The deduction phase of modelling must be conducted rigorously.*** The reason for requiring rigorous deduction is that one wants to be sure that if model conclusions are inconsistent with reality, then the defect lies in the assumptions. One application of this principle is that one must be extremely careful when programming computers. Hidden "bugs" are specially dangerous when they do not prevent the program from running but simply produce results which are not consistent with the intention of the model.
4. ***Models should be validated prior to implimentation.*** For example, if a model is constructed to forecast the monthly sales of a particular commodity, it could be tested using historical data to compare the forecasts it would have produced to the actual sales. In case, if the model cannot be validated prior to its implementation, then it can be implemented in phases for validation. For example, a new model for inventory control may be implemented for a certain selected group of items while the older system is retained for the majority of remaining items. If the model proves successful, more items can be placed within its range. It is also worthnoting that real things change in time. A highly satisfactory model may very well degrade with age. So periodic re-evaluation is necessary.
5. ***A model should never be taken too literally.*** For example, suppose that one has to construct an elaborate computer model of Indian economy with many competent researchers spending a great deal of time and money in getting all kinds of complicated interactions and relationships. Under such circumstances, it can be easily believed as if the model duplicates itself the real system. This danger continues to increase as the models become larger and more sophisticated, as they must deal with increasingly complicated problems.
6. ***A model should neither be pressed to do, nor criticized for failing to do that for which it was never intended.*** One example of this error would be the use of forecasting model to predict so far into the future that the data on which the forecasts are based have no relevance. Another example is the use of certain network methods to describe the activities involved in a complex project. A model should not be stretched beyond its capabilities.
7. ***Beware of over-selling a model.*** This principle is of particular importance for the OR professional because most non-technical benefactors of an operations researcher's work are not likely to understand his methods. The increased technicality of one's methods also increases the burden of responsibility on the OR. professional to distinguish clearly between his role as model manipulator and model interpreter. In those cases where the assumptions can be challenged, it would be dishonest to use the model.
8. ***Some of the primary benefits of modelling are associated with the process of developing the model.*** It is true in general that a model is never as useful to anyone else as it is to those who are involved in building it up. The model itself never contains the full knowledge and understanding of the real system that the builder must acquire in order to successfully model it, and there is no practical way to convey this knowledge and understanding properly. In some cases, the sole benefits may occur while the model is

being developed. In such cases, the model may have no further value once it is completed. An example of this case might occur when a small group of people attempts to develop a formal plan for some subject. The plan is the final model, but the real problem may be to agree on 'what the objectives ought to be'.

9. ***A model cannot be any better than the Information that goes into it.*** Like a computer program, a model can only manipulate the data provided to it; it cannot recognize and correct for deficiencies in input. Models may *condense* data or *convert* it to more useful forms, but they do not have the capacity to generate it. In some situations it is always better to gather more information about the system instead of exerting more efforts on modern constructions.

10. ***Models cannot replace decision makers.*** The purpose of OR models should not be supposed to provide "Optimal solutions" free from human subjectivity and error. OR models can aid decision makers and thereby permit better decisions to be made. However, they do not make the job of decision making easier. Definitely, the role of experience, intuition and judgement in decision making is undiminished.

1.6 APPROXIMATIONS (SIMPLIFICATIONS) OF 'OR' MODLES

While constructing a model, two conflicting objectives usually strike in our mind :

(i) The model should be as accurate as possible.

(ii) It should be as easy as possible in solving.

Besides, the management must be able to understand the solution of the model and must be capable of using it. So the reality of the problem under study should be simplified to the extent when there is no loss of accuracy.

The model can be simplified by :

(i) omitting certain variable *(ii)* changing the nature of variables *(iii)* aggregating the variables *(iv)* changing the relationship between variables, and *(v)* modifying the constraints, etc.

1.7 GENERAL METHODS FOR SOLVING 'OR' MODLES

Generally, three types of methods are used for solving OR models.

Analytic Method. If the OR model is solved by using all the tools of classical mathematics such as : differential calculus and finite differences available for this task, then such type of solutions are called *analytic solutions*. Solutions of various inventory models are obtained by adopting the so called analytic procedure.

Iterative Method. If classical methods fail because of complexity of the constraints or of the number of variables, then we are usually forced to adopt an iterative method. Such a procedure starts with a trial solution and a set of rules for improving it. The trial solution is then replaced by the improved solution, and the process is repeated until either no further improvement is possible or the cost of further calculation cannot be justified.

Iterative method can be divided into three groups :

(a) After a finite number of repetitions, no further improvement will be possible.

(b) Although successive iterations improve the solutions, we are only guaranteed the solution as a limit of an infinite process.

(c) Finally, we include the *trial* and *error* method which, however, is likely to be lengthy, tedious, and costly even if electronic computers are used.

The Monte-Carlo Method. The basis of so called Monte-Carlo technique is random sampling of variable's values from a distribution of that variable. Monte-Carlo refers to the use of sampling methods to estimate the value of non-stochastic variables. The following are the main steps of Monte-Carlo method :

Step1. In order to have a general idea of the system, we first draw a *flow diagram* of the system.

Step 2. Then, we take correct sample observations to select some suitable model for the system. In this step, we compute the probability distributions for the variables of our interest.

Step 3. We, then, convert the probability distributions to a cumulative distribution function.

Step 4. A sequence of random numbers is now selected with the help of random number tables.

Step 5. Next, we determine the sequence of values of variables of interest with the sequence of random numbers obtained in **step 4**.

Step 6. Finally, we construct some standard mathematical function to the values obtained in ***step 5.***

Q. 1. State the different types of models used in OR. Explain briefly the general methods for solving these O.R. models. [Agra 95]

2. Write briefly about the following :
(i) Iconic models (ii) Analogue models (iii) Mathematical models [Meerut (MCA III) May 2000]

The following interesting example will make the above procedure clear.

Illustration of Monte-Carlo Technique

Example. *A bombing mission is sent to bomb an important factory, which is rectangular in shape and has the dimensions 250 by 500 feet. The bombers will drop 10 bombs altogether, from high altitude, all aimed at the geometric centre of the plant. We assume that the bombing run is made parallel to the long dimension of the plant, that the deviation of the impact point from the aiming point is normal with mean zero and standard deviation 200 feet in each dimension, and that these two deviations are independent random variables. Use Monte-Carlo sampling to estimate the expected number of bomb-hits, and compare your result with the exact value.*

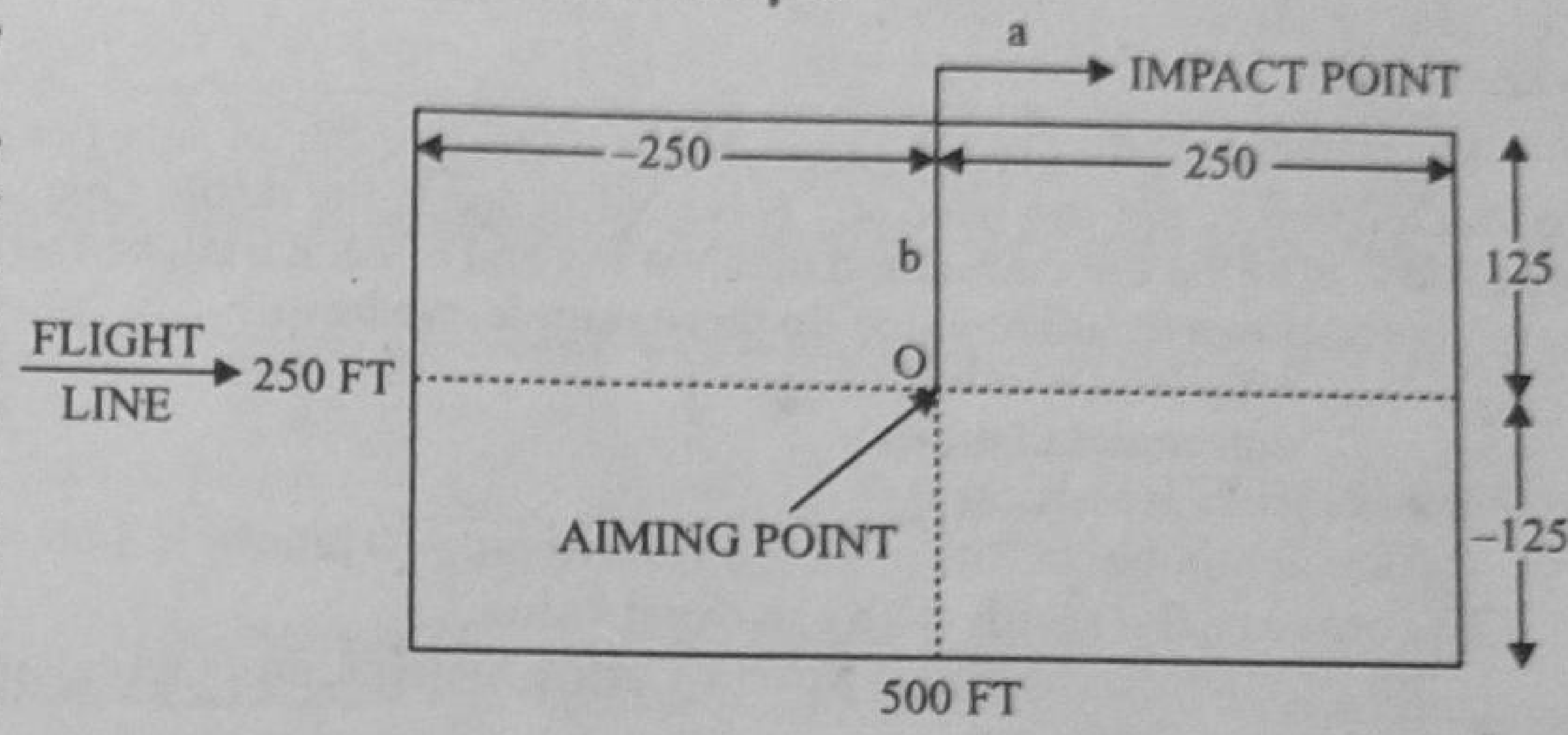


Fig. 1.1.

Solution. Let a be the horizontal deviation, and b be the vertical deviation, as shown in the Fig. 1.1.
The bomb will strike if the following conditions are satisfied :

$$-250 \le a \le 250, \quad -125 \le b \le 125, \quad \text{...(1·1)}$$

otherwise the bomb will miss the target.

If we put $x = a/200$, $y = b/200$, so that x and y will be the corresponding deviates read from a random number table, then the condition (1·1) for hitting the target becomes :

$$-250 \le 200x \le 250, \quad -125 \le 200y \le 125 \quad \text{...(1·2a)}$$

or

$$-1{\cdot}250 \le x \le 1{\cdot}250, \quad -0{\cdot}625 \le y \le 0{\cdot}625 \quad \text{...(1·2b)}$$

Results for the first three trials are given in Table 1·1 below.

Table 1·1

Bomb	*x*	*y*	*Result*
Trial 1, four hits			
1	– 0·291	1·221	Miss
2	– 2·828	– 0·439	Miss
3	0·247	1·291	Miss
4	– 0·584	0·541	**Hit***
5	– 0·446	– 2·661	Miss
6	– 2·127	0·665	Miss
7	0·656	0·340	**Hit***
8	1·041	0·008	**Hit***
9	0·899	0·110	**Hit***
10	– 1·114	1·297	Miss
Trial 2, two hits			
1	1·119	0·004	**Hit***
2	– 0·792	– 1·275	Miss
3	0·063	– 1·793	Miss
4	0·484	– 0·986	Miss
5	1·045	– 2·363	Miss
6	– 0·084	– 0·880	Miss
7	– 0·086	– 0·158	**Hit***
8	0·427	– 0·831	Miss
9	– 0·528	– 0·833	Miss
10	– 1·433	– 1·345	Miss

Trial 3, four hits	x	y	*Result*
1	−2·015	−0·594	Miss
2	−0·623	−1·047	Miss
3	−0·699	−1·347	Miss
4	0·481	0·996	Miss
5	0·586	−1·023	Miss
6	0·579	0·551	Hit*
7	0·120	0·418	Hit*
8	0·191	0·074	Hit*
9	0·071	0·524	Hit*
10	−3·001	0·479	Miss

These three trials give 3·33 as the average number of hits per mission. Many more trials should be conducted before we can have any real confidence in the result. One way of estimating : how many trials are necessary, is to list the cumulated mean at the end of each trial, and to stop the trials when the mean seems to have settled down to stable value. In this example, we have

after trial number	:	1	2	3
cummulated mean	:	4	3	3·33,

so that more trials are necessary.

The mean number of hits in a mission dropping 10 bombs is 3·69.

To compare the result with the exact value.

In this problem, unlike most Monte-Carlo problems, an exact calculation of the answer is much easier than the Monte Carlo calculation.

The probability of a hit with a single bomb is

$$\left[\int_{-1\cdot250}^{1\cdot250} f(x)\,dx\right]\times\left[\int_{-0\cdot625}^{0\cdot625} f(x)\,dx\right] = 2\cdot789 \times 0\cdot468 \text{ (from the table of the normal integers)}$$

$$= 0\cdot369.$$

Thus the previous value 3·69 is ten times of this value.

Advantages :

1. These methods avoid unnecessary expenses and difficulties that arise during the trial and error experimentation.
2. By this technique, we find the solution of much complicated mathematical expression which is not possible by any other method.

Disadvantages :

1. This technique does not give optimal answers to the problems. The good results are obtained only when the sample size is quite large.
2. The computations are much complicated even in simple cases.
3. It is a costly procedure for obtaining a solution of any related problem.

Q. 1. Write a short note on Monte-Carlo Technique and their usefulness in real life situations. **[Meerut (Stat.) 98]**

2. Describe the use of Monte-Carlo methods in sampling experiments. Illustrate with possible examples.

1.8 MAIN CHARACTERISTICS (FEATURES) OF OPERATIONS RESEARCH

The main characteristics of OR are as follows :

1. Inter-disciplinary team approach. In OR, the optimum solution is found by a team of scientists selected from various disciplines such as mathematics, statistics, economics, engineering, physics, etc.

For example, while investigating the inventory management in a factory, perhaps we may require an engineer who knows the functions of various items of stores. We also require a cost accountant and a mathematician-cum-statistician. Each member of such OR team is benefitted by the view points of others, so that the workable solution obtained through such collaborative study has a greater chance of acceptance by management.

Furthermore, an OR team required for a big organization may include a statistician, an economist, a mathematician, one or more engineers, a psychologist, and some supporting staff like computer programmers, etc. A mathematician or a probabilist can apply his tools in a plant problem only if he gets to understand some of the physical implications of the plant from an engineer. Otherwise, he may give such a solution which may not be possible to apply.

2. Wholistic approach to the system. The most of the problems tackled by OR have the characteristic that OR tries to find the *best* (*optimum*) decisions relative to largest possible portion of the total organization. The nature of organization is essentially immaterial.

For example, in attempting to solve a maintenance problem in a factory, OR tries to consider how this affects the production department as a whole. If possible, it also tries to consider how this effect on the production department in turn affects other department and the business as a whole. It may even try to go further and investigate how the effect on this particular business organization in turn affects the industry as a whole, etc. Thus OR attempts to consider inter-actions or chain of effects as far out as these effects are significant.

3. Imperfectness of solutions. By OR techniques, we cannot obtain perfect answers to our problems but, only the quality of the solution is improved from worse to bad answers.

4. Use of scientific research. OR uses techniques of scientific research to reach the optimum solution.

5. To optimize the total output. OR tries to optimize total return by maximizing the profit and minimizing the cost or loss.

Q. 1. Give the main characteristics of Operations Research.
2. Define OR and discuss its characteristics and limitations. [C.A. (May) 92]

1.9 MAIN PHASES OF OPERATIONS RESEARCH STUDY

About fourty years ago, it would have been difficult to get a single operations-researcher to describe a procedure for conducting OR project. The procedure for an OR study generally involves the following major phases :

Phase I : Formulating the problem. Before proceeding to find the solution of a problem, first of all one must be able to formulate the problem in the form of an appropriate model. To do so, the following information will be required.

(i) Who has to take the *decision* ?
(ii) What are the *objectives* ?
(iii) What are the ranges of *controlled variables* ?
(iv) What are the uncontrolled variables that may affect the possible solutions ?
(v) What are the restrictions or constraints on the variables ?

Since wrong formulation cannot yield a right decision (solution), one must be considerably careful while execution this phase.

Phase II : Constructing a mathematical model. The second phase of the investigations is concerned with the reformulation of the problem in an appropriate form which is convenient for analysis. The most suitable form for this purpose is to construct a mathematical model representing the system under study. It requires the identification of both *static* and *dynamic* structural elements. A mathematical model should include the following three important basic factors :

(i) *Decision variables and parameters,* (ii) *Constraints or Restrictions,* (iii) *Objective function.*

Phase III : Deriving the solutions from the model. This phase is devoted to the computation of those values of decision variables that maximize (or minimize) the objective function. Such solution is called an *optimal solution* which is always in the best interest of the problem under consideration. The general techniques for deriving the solution of OR model are discussed in the following sections and further details are given in the text.

Phase IV : Testing the model and its solution (updating the model). After completing the model, it is once again tested as a whole for the errors if any. A model may be said to be valid if it can provide a reliable prediction of the system's performance. A good practitioner of Operations Research realises that his model be applicable for a longer time and thus he updates the model time to time by taking into account the past, present and future specifications of the problem.

Phase V : Controlling the solution. This phase establishes controls over the solution with any degree of satisfaction. The model requires immediate modification as soon as the controlled variables (one or more) change significantly, otherwise the model goes out of control. As the conditions are constantly changing in the world, the model and the solution may not remain valid for a long time.

Phase VI : Implementing the solution. Finally, the tested results of the model are implemented to work. This phase is primarily executed with the cooperation of Operations Research experts and those who are responsible for managing and operating the systems.

Q. 1. Discuss the various phases in solving an OR problem. **[IGNOU 2001; C.A. (Nov.) 92; Meerut (IPM) 90]**

2. What are various phases of O.R. problems ? Explain them briefly. **[VTU (BE Mech.) 2003]**

3. Give the different phases of Operations Research, and explain their significance in decision making. **[Meerut (Stat.) 98, 90; Karnataka (B.E.) 95; C.A. (Nov.) 89]**

4. Explain the steps involved in the solution of an Operations Research problem. **[IGNOU 2001]**

5. What is an operations research? Discuss the various phases in solving an OR problem. **[AIMS (B.E.) Bangalore 2002]**

1.10 THE TERMS : 'TOOLS', 'TECHNIQUES' AND 'METHODS'

We now carefully differentiate the terms : 'tools', 'techniques' and 'methods' which are frequently used in science. It is evident that a table of random numbers is a *tool* of science. The way in which this tool is used is called a *technique.* The research plan which involves the use of ***Monte-Carlo*** procedure and the table of random numbers is called a *method* of science. Similarly, calculus is a scientific *tool*; employing calculus to find an optimum value of a variable in a mathematical model of a system is a scientific *technique*; and the plan of utilizing a mathematical model to optimize a system is a scientific *method.*

1·10-1 Scientific Method in Operations Research

The scientific method in OR study generally involves the three phases : (i) *the judgement phase*, (ii) *the research phase*, and (iii) *the action phase.*

Of these three, the *research phase* is the largest and longest, but the remaining two are just as important as they provide the basis for an implementation of the research.

The judgment phase includes :

(i) A determination of the operation.
(ii) The establishment of the objectives and values related to the operation.
(iii) The determination of the suitable measures of effectiveness.
(iv) Lastly, the formulation of the problems relative to the objectives.

The research phase utilizes :

(i) Observations and data collection for a better understanding of what the problem is.
(ii) Formulation of hypothesis and models.
(iii) Observation and experimentation to test the hypothesis on the basis of additional data.
(iv) Analysis of the available information and verification of the hypothesis using pre-established measures of effectiveness.
(v) Predictions of various results from the hypothesis, generalization of the result and consideration of alternative methods.

The action phase :

OR consists of making recommendations for decision process by those who first posed the problem for consideration, or by anyone in a position to make a decision influencing the operation in which the problem occured.

Q. 1. Discuss Scientifc Method in O R. **[Meerut (O.R.) 90]**

2. Enumerate the approach, technique and tools used in operations research. You may list as many as possible but focus on 4 tools and detail the appropriate computer hardware, software and application programs. **[IGNOU 2001, 99, 96]**

1.11 SCOPE OF OPERATIONS RESEARCH

In its recent years of organized development, OR has entered successfully many different areas of research for military, government and industry. The basic problem in most of the developing countries in Asia and Africa is to remove *poverty* and *hunger* as quickly as possible. So there is a great scope for economists, statisticians, administrators, politicians and the technicians working in a team to solve this problem by an OR approach. Besides this, OR is useful in the following various important fields.

1. In Agriculture. With the explosion of population and consequent shortage of food, every country is facing the problem of—

(i) optimum allocation of land to various crops in accordance with the climatic conditions; and
(ii) optimum distribution of water from various resources like canal for irrigation purposes.

Thus there is a need of determining best policies under the prescribed restrictions. Hence a good amount of work can be done in this direction.

2. In Finance. In these modern times of economic crisis, it has become very necessary for every government to have a careful planning for the economic development of her country. OR-techniques can be fruitfully applied :

(i) to maximize the per capita income with minimum resources;
(ii) to find out the profit plan for the company;
(iii) to determine the best replacement policies, etc.

3. In Industry. If the industry manager decides his policies (not necessarily optimum) only on the basis of his past experience (without using OR techniques) and a day comes when he gets retirement, then a heavy loss is encountered before the Industry. This heavy loss can immediately be compensated by newly appointing a young specialist of OR techniques in *business management*. Thus OR is useful to the *Industry Director* in deciding optimum allocation of various limited resources such as men, machines, material, money, time, etc., to arrive at the optimum decision.

4. In Marketing. With the help of OR techniques a *Marketing Administrator* (Manager) can decide :

(i) where to distribute the products for sale so that the total cost of transportation etc. is minimum,
(ii) the minimum per unit sale price,
(iii) the size of the stock to meet the future demand,
(iv) how to select the best advertizing media with respect to time, cost, etc.
(v) how, when, and what to purchase at the minimum possible cost ?

5. In Personnel Management. A personnel manager can use OR techniques :

(i) to appoint the most suitable persons on minimum salary,
(ii) to determine the best age of retirement for the employees,
(iii) to find out the number of persons to be appointed on full time basis when the workload is seasonal (not continuous).

6. In Production Management. A production manager can use OR techniques :

(i) to find out the number and size of the items to be produced;
(ii) in scheduling and sequencing the production run by proper allocation of machines;
(iii) in calculating the optimum product mix; and
(iv) to select, locate, and design the sites for the production plants.

7. In L.I.C. OR approach is also applicable to enable the L.I.C. offices to decide :

(i) what should be the premium rates for various modes of policies,
(ii) how best the profits could be distributed in the cases of with profit policies ? etc.

Finally, we can say : wherever there is a problem, there is OR. The applications of OR cover the whole extent of any thing. A recent publication of the OR society contains a summary of the applications of OR. The reader wishing more details on applications may consult the publication : ***'Progress in OR'* Vol. 2 by *Hertz., D.B.* and *R.T. Eddison*.**

Q. 1. Define O.R. and discuss its scope. **[Meerut (Stat.) 98; Garhwal 96; Kanpur 96; Rewa (Maths.) 93; Rohil. 93, 92]**
2. What are the areas of applications of O.R., **[Meerut (Maths) 91]**
3. (a) Explain the meaning, scope and methodology of O.R. **[VTU (BE Mech.) 2002]**
(b) Discuss the significance and scope of Operations Research in modern management.
4. Write a critical essay on the definition and scope of Operations Research. **[JNTU (B. Tech) 2002; Virbhadrah 2000]**

1.12 ROLE OF OPERATIONS RESEARCH IN DECISION-MAKING

The Operations Research may be regarded as a tool which is utilized to increase the effectiveness of management decisions. In fact, OR is the objective suppliment to the subjective feeling of the administrator (decision-maker). Scientific method of OR is used to understand and describe the phenomena of operating system. OR models explain these phenomena as to what changes take place under altered conditions, and control these predictions against new observations. For example, OR may suggest the best locations for factories, warehouses as well as the most economical means of transportation. In marketing, OR may help in indicating the most profitable type, use and size of advertising compaigns subject to the financial limitations.

The advantages of OR study approach in business and management decision making may be classified as follows :

1. Better Control. The management of big concerns finds it much costly to provide continuous executive supervisions over routine decisions. An OR approach directs the executives to devote their attention to more pressing matters. For example, OR approach deals with production scheduling and inventory control.

2. Better Co-ordination. Sometimes OR has been very useful in maintaining the law and order situation out of chaos. For example, an OR based planning model becomes a vehicle for coordinating marketing decisions with the limitations imposed on manufacturing capabilities.

3. Better System. OR study is also initiated to analyse a particular problem of decision making such as establishing a new warehouse. Later, OR approach can be further developed into a system to be employed repeatedly. Consequently, the cost of undertaking the first application may improve the profits.

4. Better Decisions. OR models frequently yield actions that do improve an intuitive decision making. Sometimes, a situation may be so complicated that the human mind can never hope to assimilate all the important factors without the help of OR and computer analysis.

In the present text, we restrict ourselves to discuss the problems on : ***Inventory control, Replacement, Queues, Linear programming, Goal Programming, Transportation, Assignment, Games theory, Sequencing, Dynamic programming, Information theory, PERT/CPM, Simulation, and Decision theory*** etc.

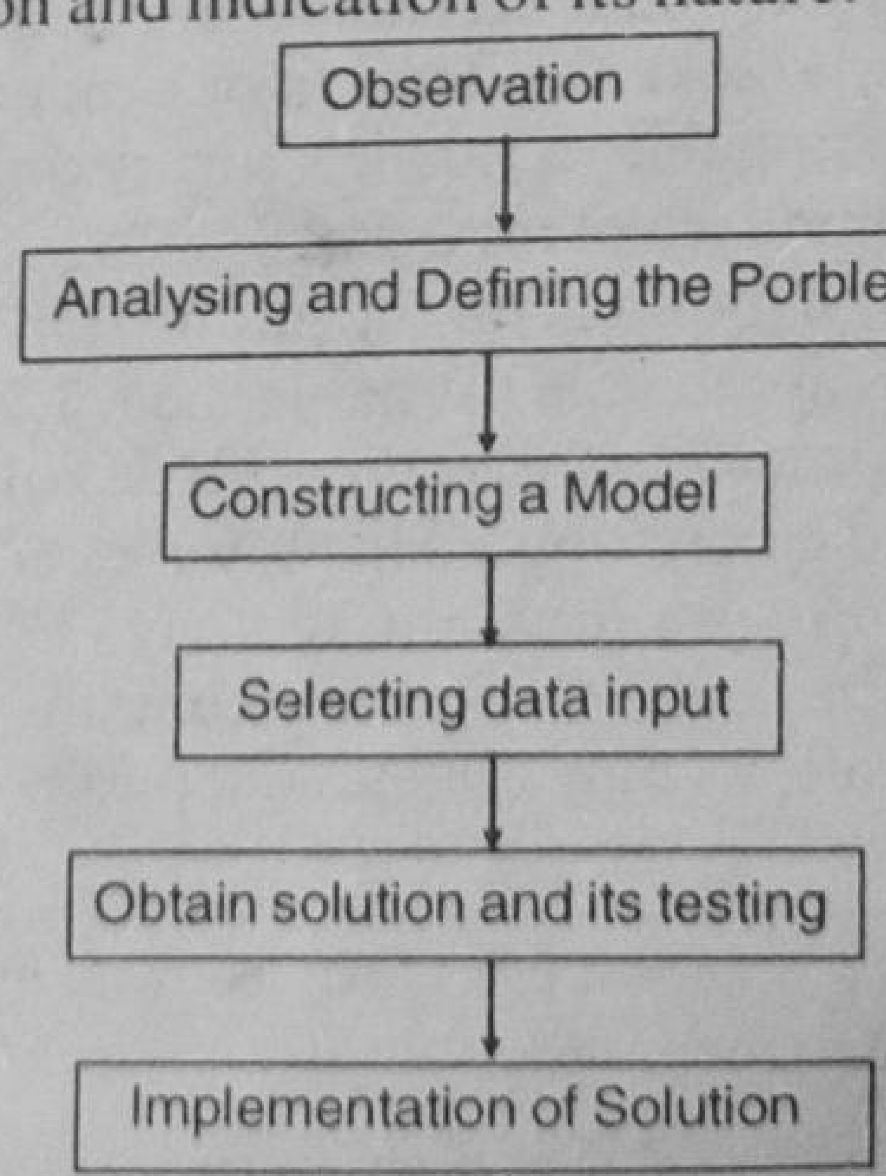
O.R. provides a logical and systematic approach for decision-making. The phases and process study must also be quite logical and systematic. There are six important steps in O.R. study, but in all each and every step does not necessarily follow logical order as below :

***Step I* : Observing the Problem Environment**

The activities in this step are visits, conferences, observations, research etc. With such activities analyst gets sufficient information and support to define the problem.

***Step II* : Analysing and Defining the Problem**

In this step the problem is defined, and objectives and limitations of the study are stated in its context. One thus gets clear grasp of need for a solution and indication of its nature.



***Step III* : Developing a Model**

Step III is to contruct a model. A model is representation of some real or abstract situations. O.R. models are basically mathematical models representing systems, processes or environment in the form of equations, relationships or formulae. The activities in this step are defining interrelationships among variables, formulating equations, using known O.R. models or searching suitable alternate models. The proposed model may be practically tested and modified in order to work under given environmental constraints. A model may also be managerial if not satisfied with the solution it offers.

***Step IV* : Selecting Appropriate Data Input**

No model will work appropriately if data input is not appropriate. Hence using right kind of data is vital in O.R. process. Important activities in this step are analysing internal-external data and facts, collecting openions and using computer data banks. The purpose of the step is to have sufficient input to operate and test the models.

***Step V* : Providing a Solution and Testing its Reasonableness**

Step V is performed to obtain a solution with the help of model and data input. Such a solution is not implemented immediately. First it is tested not behaving properly, updating and modification of the model is considered at this stage. The end result is solution that supports current organization objective.

***Step VI* : Implementing the Solution**

Implementation of the solution is the last step of O.R. process. In O.R., the decision-making is scientific, but implementation of decision involves many behavioural issues. Therefore, the implementing authority has to resolve the behavioural issues. He has to convince not only the workers but also the superiors. The gap between one who provides a solution and the other who wishes to use it has to be eliminated. To achieve this, O.R. analyst as well as management should play a positive role. Needless to say a properly implemented solution obtained through O.R. techniques results an improved working and gets active management support.

Q. 1. What is the importance (role) of Operations Research in decision making. **[Kanpur 96]**

2. Describe in brief the role of quantitative techniques in bussiness management.

3. What are the various phases through which an O.R. team normally has to proceed ?

1.13 BRIEF OUTLINES OF OR-MODELS : QUANTITATIVE TECHNIQUES OF OR

A brief account of some of the important OR models is given below :

1. Distribution (Allocation) Models. Distribution models are concerned with the allotment of available resources so as to minimise cost or maximise profit subject to prescribed restrictions. Methods for solving such type of problems are known as *mathematical programming techniques*. We distinguish between linear and non-linear programming problems on the basis of linearity and non-linearity of the objective function and/or constraints respectively. In linear programming problems, the objective function is linear and constraints are also linear inequalities/equations . Transportation and Assignment models can be viewed as special cases of linear programming. These can be solved by specially devised procedures called *Transportation* and *Assignment Techniques.*

In case the decision variables in a linear programming problem are restricted to either integer or zero-one value, it is known as *Integer* and *Zero-One programming problems*, respectively. The problems having multiple, conflicting and incommensurable objective functions (goals) subject to linear constraints are called *linear goal programming problems*. If the decision variables in a linear programming problem depend on chance, then such problems are called *stochastic linear programming problems*.

2. Production/Inventory Models. Inventory/Production models are concerned with the determination of the optimal (economic) order quantity and ordering (production) intervals considering the factors such as–demand per unit time, cost of placing orders, costs associated with goods held up in the inventory and the cost due to shortage of goods, etc. Such models are also useful in dealing with quantity discounts and multiple products.

3. Waiting Line (or Queueing) Models. In queueing models an attempt is made to predict :

(i) how much average time will be spent by the customer in a queue ?

(ii) what will be an average length of waiting line or queue ?

(iii) what will be the traffic intensity of a queueing system ? etc.

The study of waiting line problems provides us methods to minimize the sum of costs of providing service and cost of obtaining service which are primarily associated with the value of time spent by the customer in a queue.

4. Markovian Models. These models are applicable in such situations where the state of the system can be defined by some descriptive measure of numerical value and where the system moves from one state to another on a probability basis. Brand-swiching problems considered in marketing studies is an example of such models.

5. Competetive Strategy Models (Games Theory). These models are used to determine the behaviour of decision-making under competition or conflict. Methods for solving such models have not been found suitable for industrial applications, mainly because they are referred to an idealistic world neglecting many essential features of reality.

6. Network Models. These models are applicable in large projects involving complexities and inter-dependencies of activities. *Project Evaluation and Review Techniques* (PERT) and *Critical Path Method* (CPM) are used for planning, scheduling and controlling complex project which can be characterised as net-works.

7. Job Sequencing Models. These models involve the selection of such a sequence of performing a series of jobs to be done on service facilities (machines) that optimize the efficiency measure of performance of the system. In other words, sequencing is cencerned with such a problem in which efficiency measure depends upon the order or sequence of performing a series of jobs.

8. Replacement Models. These models deal with the determination of optimum replacement policy in situations that arise when some items or machinery need replacement by a new one. Individual and group replacement policies can be used in the case of such equipments that fail completely and instantaneously.

9. Simulation Models. Simulation is a very powerful technique for solving much complex models which cannot be solved otherwise and thus it is being extensively applied to solve a variety of problems. This technique is more useful when following two types of difficulties may arise :

(i) The number of variables and constraint relationships may be so large that it is not computationally feasible to pursue such analysis.

(ii) Secondly, the model may be much away from the reality that no confidence can be placed on the computational results.

In fact, such models are solved by simulation techniques where no other method is available for its solution.

Operations Research, as its name suggests, gives stress on analysis of operations as a whole. For this purpose it uses any suitable techniques or tools available from the fields of mathematics, statistics, cost analysis or numerical calculations. Some such techniques are listed below :

(1) Linear Programming	(2) Non-linear Programming	(3) Integer Programming
(4) Dynamic Programming	(5) Goal Programming	(6) Games Theory
(7) Inventory Control	(8) PERT-CPM	(9) Simulation

(10) Queueing Theory etc. Here, for example, we describe in brief the *queueing* or *waiting line theory*.

Queues have become an integral part of our daily life. Queues are formed everywhere where a service is fered and the service rate is slower than the arrival rate of customers. People waiting for railway reservations, machines waiting for repairs at a workshop and aeroplanes waiting in the sky to find a place to land at the airport are all examples of queueing.

Costs are associated with the waiting in a line and costs are also associated with adding more service facilities or counters. The purpose of OR study is to decide the optimum number of service facilities so as to minimise the sum of waiting period cost and cost of providing facilities.

Queueing theory works out the expected number of people in the queue, expected waiting time in the queue, expected idle time for the server etc. These calculations then help in deciding the optimum number of service facilities under given constraints.

Q. 1. Write a note on application of various quantitative techniques in different fields of business decision making.
2. Explain various types of O.R. models and indicate their application to production, inventory, and distribution systems.
3. Enumerate six techniques of operations Research and describe one briefly. **[IGNOV 2001 (Jan)]**

1.14 DEVELOPMENT OF OPERATIONS RESEARCH IN INDIA

In 1949, Operations Research came into picture when an OR unit was established at the Regional Research Laboratory, Hyderabad. At the same time, Prof. R.S. Verma (Delhi University) setup an OR team in the Defence Science Laboratory to solve the problems of store, purchase and planning. In 1953, Prof. P.C. Mahalanobis established an OR team in the Indian Statistical Institute, Calcutta, for solving the problem of national planning and survey. In 1957, Operations Research Society of India was formed and this society became a member of the International Federation of Operations Research Societies in 1960. Presently India is publishing a number of research journals, namely, ***'OPSEARCH', 'Industrical Engineering and Management', 'Materials Management Journal of India', 'Defence Science Journal', 'SCIMA', 'Journal of Engineering Production'***, etc.

As far as the OR education in India is concerned University of Delhi was the first to introduce a complete M.Sc. course in OR in 1963. Simultaneously, Institute of Management at Calcutta and Ahemdabad started teaching OR in their MBA courses. Now-a-days, OR has become so popular subject that it has been introduced in almost all Institutes and Universities in various disciplines like, Mathematics, Statistics, Commerce, Economics, Management Science, Medical science, Engineering, etc. Also, realizing the importance of OR in Accounts and Administration, government has introduced this subject for the IAS, CA, ICWA examinations, etc.

Prof. Mahalanobis first applied OR in India by formulating second five-year plan with the help of OR techniques. Planning Commission made the use of OR techniques for planning the optimum size of the Caravelle fleet of Indian air lines. Some of the industries, namely, ***Hindustan Lever Ltd.; Union Carbide, TELCO, Hindustan Steel, Imperial Chemical Industries, Tata Iron & Steel Company, Sarabhai Group, FCI***, etc. have engaged OR teams. ***Kirlosker Company*** is using the assignment technique of OR to maximize profit.

Textile firms like, DCM., Binni's and Calico, etc., are using linear programming techniques. Among other Indian organizations using OR are the **Indian Railways, CSIR, Tata Institute of Fundamental Research, Indian Institute of Science, State Trading Corporation,** etc.

****It is also worthnoting that the present text on* 'OPERATIONS RESEARCH' *is the first book published in India to meet the requirements of various courses on this subject.***

1.15 ROLE OF COMPUTERS IN OPERATIONS RESEARCH

In fact, computers have played a vital role in the development of OR. But OR would not have achieved its present position for the use of computers. The reason is that—in most of the OR techniques computations are so complex and involved that these techniques would be of no practical use without computers. Many large scale applications of OR techniques which require only few minutes on the computer may take weeks, months and sometimes years even to yield the same results manually. So the computer has become as essential and integral part of OR. Now-a-days, OR methodology and computer methodology are growing up simultaneously. It seems that in the near future the line dividing the two methodologies will disappear and the two sciences will combine to form a more general and comprehensive science. It should also be noted that FORTRAN and C-programs are functionally equivalent.

The computor software packages are useful for rapid and effective calculations which is a necessary part of O.R. approach to solve the problems. These are :

(i) QSB+ (Quantitative System for Business Plus), Version 3.0, by Yih-long Chang and Robert S. Sullivan, is a software package that contains problem solving algorithms for OR/MS, as well as modules on basic statistics, non-linear programming and financial analysis.

(ii) QSOM (Quantitative Systems for Operations Management), by Yih-long, is an interactive user-friendly system. It contains problem-solving algorithms for operations management problems and associated information system.

(iii) Value STORM : MS quantitative Modelling for Decision Support, by Hamilton Emmons, A.D. Flowers, Chander Shekhar, M.Khot and Kamlesh Mathur, is a special version of Personal STORM version 3.0 developed for use in OR/MS.

(iv) Excel 97 by Gene Weiss Kopf and distributed by BPB publications, New Delhi, is an easy-to-use task-oriented guide to Excel Spread sheet applications.

(v) LINDO (Linear Interactive Discrete Optimization), developed by Linus Schrage Lindo in his book "An Optimization Modeling System, 4th ed. (Palo Alto, CA : Scientific Press 1991)

SELF-EXAMINATION QUESTIONS

1. (a) What is Operations Research ? A certain wine importer noticed that his sales of wine were not what they should be in comparison to other types of liquor. He hired you as a consultant to look into this problem, with the intention of improving the wine business. What would you do ?

   (b) How does one go about organising for effective Operations Research ? Explain.

10. Find the feasible zone for the constraints $x_1 + x_2 \geq 1$, $x_1 + 2x_2 \geq 6$, $x_1 - x_2 \leq 3$, $x_1 \geq 0$ and state the redundant constraints. If $z = 3x_1 + x_2$, draw two iso-z lines and show the direction of improvement if z is to be minimized. Find z_{min}. Also state what can be the maximum value of z. **[V.T.U. (BE Mech.) 2002]**

11. A firm makes two types of furniture chairs and tables. The contribution for each product as calculated by the accounting department is Rs. 20/- per chair and Rs. 30/- per table. Both products are processed on three machines M_1, M_2, M_3. The time required in hours by each product and total time available in hours per week on each machine are as follows :

Machine	Chairs	Table	Available Time
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

How should the manufacturer schedule his production in order to maximize contribution? Solve graphically. **[VTU (BE Mech.) 2002]**

3.4. GENERAL FORMULATION OF LINEAR PROGRAMMING PROBLEM

The general formulation of the linear programming problem can be stated as follows :

In order to find the values of n decision variables $x_1, x_2, \ldots, x_n$ to maximize or minimize the objective function

$$z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n \quad \ldots(3.7)$$

and also satisfy m-constraints :

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n \ (\leq = \text{or} \geq) \ b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n \ (\leq = \text{or} \geq) \ b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n \ (\leq = \text{or} \geq) \ b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n \ (\leq = \text{or} \geq) \ b_m, \end{array}\right\} \quad \ldots(3.8)$$

where constraints may be in the from of any inequality ($\leq$ or $\geq$) or even in the form of an equation (=), and finally satisfy the non-negativity restrictions

$$x_1 \geq 0, x_2 \geq 0, \ldots, x_j \geq 0, \ldots, x_n \geq 0. \quad \ldots(3.9)$$

However, by convention, the values of right side parameters b_i $(i = 1, 2, 3 \ldots, m)$ are restricted to non-negative values only. It is important to note that any negative b_i can be changed to a positive value on multiplying both sides of the constraint by -1. This will not only change the sign of all left side coefficients and right side parameters but will also change the direction of the inequality sign.

Q. 1. What do you mean by a L.P.P.? What are its limitations ?

2. Define a general linear programming problem. **[Meerut (L.P.) 90]**

3. What is linear programming problem (LPP) ? How can formulate a given problem into LPP ? **[IGNOU 2001, 2000, 98, 97, 96]**

3.5 SLACK AND SURPLUS VARIABLES

[Jaunpur (B.Sc.) 96; Meerut 90]

1. Slack Variables. If a constraint has $\leq$ sign, then in order to make it an equality, we have to add something positive to the left hand side.

The non-negative variable which is added to the left hand side of the constraint to convert it into equation is called the slack variable.

For example, consider the constraints :

$$x_1 + x_2 \leq 2, \ 2x_1 + 4x_2 \leq 5, \ x_1, x_2 \geq 0 \quad \ldots(i)$$

We add the slack variables $x_3 \geq 0$, $x_4 \geq 0$ on the left hand sides of above inequalities respectively to obtain

$$\begin{array}{l} x_1 + x_2 + x_3 \qquad = 2 \\ 2x_1 + 4x_2 \qquad + x_4 = 5 \\ x_1, x_2, x_3, x_4 \geq 0. \end{array}$$

2. Surplus Variables. If a constraint has $\geq$ sign, then in order to make it an equality, we have to subtract something non-negative from its left hand side.

Thus the positive variable which is subtracted from the left hand side of the constraint to convert it into equation is called the surplus variable.

For example, consider the constraints :

$$x_1 + x_2 \geq 2,\ 2x_1 + 4x_2 \geq 5, \text{ and } x_1, x_2 \geq 0. \qquad \text{...(ii)}$$

We subtract the surplus variables $x_3 \geq 0$, $x_4 \geq 0$ from the left hand sides of above inequalities respectively to obtain

$$\begin{aligned} x_1 + x_2 - x_3 \quad &= 2 \\ 2x_1 + 4x_2 \quad - x_4 &= 5 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

3.6. STANDARD FORM OF LINEAR PROGRAMMING PROBLEM

The standard form of the linear programming problem is used to develop the procedure for solving general linear programming problem. The characteristics of the standard form are explained in the following steps :

Step 1. ***All the constraints should be converted to equations except for the non-negativity restrictions which remain as inequalities*** (≥ 0). Constraints of the inequality type can be changed to equations by augmenting (adding or subtracting) the left side of each such constraint by non-negative variables. These new variables are called ***slack variables*** and are added if the constraints are $(\leq)$ or subtracted if the constraints are $(\geq)$. Since in the case of $\geq$ constraint, the subtraced variable represents the surplus of the left side over the right side, it is common to refer to it as surplus variable. For convenience, however, the name 'slack' variable will also be used to represent this type of variable. In this respect, a surplus is regarded as a negative slack.

For example, consider the constraints : $3x_1 - 4x_2 \geq 7$, $x_1 + 2x_2 \leq 3$.

These constraints can be changed to equations by introducing slack variables x_3 and x_4 respectively.

Thus, we get

$$3x_1 - 4x_2 - x_3 = 7,\ x_1 + 2x_2 + x_4 = 3, \text{ and } x_3 \geq 0, x_4 \geq 0.$$

Step 2. ***The right side element of each constraint should be made non-negative (if not).*** The right side can always be made positive on multiplying both sides of the resulting equation by (-1) whenever it is necessary.

For example, consider the constraint as $3x_1 - 4x_2 \geq -4$

which can be written in the form of the equation $3x_1 - 4x_2 - x_3 = -4$

by introducing the surplus variable $x_3 \geq 0$.

Again, multiplying both sides by (-1), we get $-3x_1 + 4x_2 + x_3 = 4$ which is the constraint equation in standard form.

Step 3. ***All variables must have non-negative values.***

A variable which is *unrestricted in sign* (that is, positive, negative or zero) is equivalent to the difference between two non-negative variables. Thus, if x is unconstrained in sign, it can be replaced by $(x' - x'')$, where x' and x'' are both non-negative, that is, $x' \geq 0$ and $x'' \geq 0$.

Step 4. ***The objective function should be of maximization form.***

The minimization of a function $f(x)$ is equivalent to the maximization of the negative expression of this function, $f(x)$, that is,

$$\text{Min. } f(x) = -\text{Max } [-f(x)]$$

For example, the linear objective function

$$\text{Min. } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \qquad \text{...(3.10)}$$

is equivalent to Max $(-z)$, *i.e.* Max $z' = -c_1x_1 - c_2x_2 - \ldots - c_nx_n$ with $z = -z'$.

Consequently, in any L.P problem, the objective function can be put in the maximization form.

Standard Form of General LPP with '$\leq$' Constraints :

Now, applying above steps systematically to general form of L.P. problem with all $(\leq)$ constraints, the following standard form is obtained. Of course, no difficulty will arise to convert the general LPP with mixed constraints $(\leq = \geq)$.

$$\text{Max. } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0x_{n+1} + \ldots + 0x_{n+m} \qquad \text{...(3.11)}$$

subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + x_{n+1} \qquad\qquad = b_1 \\ a_{21}x_2 + a_{22}x_2 + \ldots + a_{2n}x_n \qquad + x_{n+2} \qquad = b_2 \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \qquad\qquad + x_{n+m} = b_m \end{array}\right\} \quad \ldots(3.12)$$

where $x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0, x_{n+1} \geq 0, \ldots, x_{n+m} \geq 0.$ ...(3.13)

Note.

1. It should be remembered that the coefficient of slack variables $x_{n+1}, x_{n+2}, \ldots x_{n+m}$ in the objective function are assumed to be zero, so that the conversion of constraints to a system of simultaneous linear equations does not change the function to be optimized.
2. Since in the case of (≥ 0) constraints, the subtracted variable represents the surplus variable. However, the name slack variable may also represent this type. In this respect, a surplus is regarded as a negative slack.

Q. Define slack and surplus variables as involved in the L.P.P, How are these variables useful in solving a L.P.P.?

[AIMS (Bang.) MBA 2002]

Example 33. *Express the following L.P. problem in standard form.*

Min. $z = x_1 - 2x_2 + x_3$, *subject to* :

$2x_1 + 3x_2 + 4x_3 \geq -4$, $3x_1 + 5x_2 + 2x_3 \geq 7$, $x_1 \geq 0$, $x_2 \geq 0$ *and* x_3 *is unrestricted in sign.*

Solution. Proceeding according to above rules, the standard LP form becomes :

Max $(z') = -x_1 + 2x_2 - (x_3' - x_3'')$, where $z' = -z$, subject to

$$-2x_1 - 3x_2 - 4(x_3' - x_3'') + x_4 \qquad = 4$$

$$3x_1 + 5x_2 + 2(x_3' - x_3'') \qquad - x_5 = 7$$

$$x_1 \geq 0, x_2 \geq 0, x_3' \geq 0, x_3'' \geq 0, x_4 \geq 0, x_5 \geq 0.$$

Of course, the number of variables will now increase to *six*.

3.7. MATRIX FORM OF LP PROBLEM

The linear programming problem in standard form [(3.11), (3.12), (3.13)] can be expressed in matrix form as follows :

Maximize $z = \mathbf{CX}^{\mathbf{T}}$ ***(objective function)***

subject to $\mathbf{AX} = \mathbf{b}, \mathbf{b} \geq \mathbf{0}$ ***(constraint equation)***

$\mathbf{X} \geq \mathbf{0}.$ ***(non-negativity restriction)***

where $\mathbf{X} = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$,

$\mathbf{C} = (c_1, c_2, \ldots, c_n, 0, 0, \ldots, 0)$, and $\mathbf{b} = (b_1, b_2, \ldots, b_m)$.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} & 1 & 0 & \ldots & 0 \\ a_{21} & a_{22} & \ldots & a_{2n} & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & 0 & 0 & \ldots & 1 \end{bmatrix}$$

Similar treatment can be adopted in the case of mixed constraints ($\leq, =, \geq$). Following example will make this point clear.

The vector $\mathbf{x}$ is assumed to include all decision variables, (i.e. original, slack and surplus). For convenience, $\mathbf{x}$ is used to represent all types of variables. The vector $\mathbf{c}$ gives the corresponding coefficients in the objective function. For example, if the variable is slack, its corresponding coefficient will be zero.

Example 42. *Express the following LP problem in the matrix form.*

Max. $z = 2x_1 + 3x_2 + 4x_3$, *subject to*

$x_1 + x_2 + x_3 \geq 5$, $x_1 + 2x_2 = 7$, $5x_1 - 2x_2 + 3x_3 \leq 9$, and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

Solution. This problem can be written in standard form as

Max. $z = 2x_1 + 3x_2 + 4x_3 + 0x_4 + 0x_5$ or Max. $z = (2, 3, 4, 0, 0)\,(x_1 x_2 x_3 x_4 x_5)^T$

subject to $x_1 + x_2 + x_3 - x_4 = 5$, $x_1 + 2x_2 = 7$, $5x_1 - 2x_2 + 3x_3 + x_5 = 9$

$$c_{B2} = \text{coefficient of } x_{B2} = \text{coeff. of } x_1 = c_1 = 1$$

Hence $\quad \mathbf{C_B} = (3, 1)$.

Now, using (5.7), the value of the objective function is

$$z = \mathbf{C_B X_B} = (3, 1)\begin{pmatrix} 28/11 \\ 4/11 \end{pmatrix} = \frac{88}{11}.$$

Also, any vector $\mathbf{a_j} = (j = 1, 2, 3, 4, 5)$ can be expressed as linear combination of vectors β_i $(i = 1, 2)$. Therefore, to express $\mathbf{a}_2$ as linear combination of β_1, β_2, we have

$$\mathbf{a}_2 = x_{12}\beta_1 + x_{22}\beta_2 = x_{12}\mathbf{a}_3 + x_{22}\mathbf{a}_1.$$

To compute values of scalars x_{12} and x_{22}, use the result (5.3) to get

$$\mathbf{X}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = -\frac{1}{11}\begin{pmatrix} 1 & -4 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$

Therefore $\quad x_{12} = 6/11, x_{22} = 4/11.$

Similar treatment can be adopted for expressing other $\mathbf{a}_j$'s as linear combinations of β_1 and β_2.

Now, using (5.6b), the variable z_2 corresponding to vector $\mathbf{a}_2$ can be obtained as

$$z_2 = \mathbf{C_B X_2} = (3, 1)\begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \left(3 \times \frac{6}{11} + 1 \times \frac{4}{11}\right) = \frac{22}{11}.$$

Similarly z_1, z_3, z_4, z_5 can also be computed.

5·3. COMPUTATIONAL PROCEDURE OF SIMPLEX METHOD

The computational aspect of the simplex procedure is first explained by the following simple example.

Example 2. *Consider the linear programming problem :*

Maximize $z = 3x_1 + 2x_2$, subject to the constraints :

$$x_1 + x_2 \le 4, x_1 - x_2 \le 2, \text{ and } x_1 , x_2 \ge 0 .$$

[Kanpur 2000, 96; IAS (Maths.) 92]

Solution. *Step 1.* First, observe whether all the right side constants of the constraints are non-negative. If not, it can be changed into positive value on multiplying both sides of the constraints by -1. In this example, all the b_i 's (right side constants) are already positive.

Step 2. Next convert the inequality constraints to equations by introducing the non-negative ***slack*** or ***surplus*** variables. The coefficients of slack or surplus variables are always taken zero in the objective function. In this example, all inequality constraints being '≤', only slack variables s_1 and s_2 are needed. Therefore, given problem now becomes :

Maximize $\quad z = 3x_1 + 2x_2 + 0s_1 + 0s_2$, subject to the constraints :

$$x_1 + x_2 + s_1 + 0s_2 = 4$$
$$x_1 - x_2 + 0s_1 + s_2 = 2$$
$$x_1 , x_2 , s_1 , s_2 \ge 0 .$$

Step 3. Now, present the constraint equations in matrix form :

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Step 4. Construct the starting simplex table using the notations already explained in ***Sec 5·2.***

It should be remembered that the values of non-basic variables are always zero at each iteration. So $x_1 = x_2 = 0$ here. Column $\mathbf{X_B}$ gives the values of basic variables as indicated in the first column. So $s_1 = 4$ and $s_2 = 2$ here. The complete starting basic feasible solution can be immediately read from Table 5·2 as : $s_1 = 4$, $s_2 = 2$, $x_1 = 0$, $x_2 = 0$, and the value of the objective function is *zero*.

Note. In this step, the variables s_1 and s_2 are corresponding to the columns of basis matrix (identity matrix), so will be called *basic variables*. Other variables, x_1 and x_2, are *non-basic variables* which always have the value zero.

Table 5·2 : Starting Simplex Table

initial

		$c_j \rightarrow$	3	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$ (basic feasible soln)	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}\,(\mathbf{S_1})$ (β_1)	$\mathbf{X_4}\,(\mathbf{S_2})$ (β_2)	MIN. RATIO $\mathbf{X_B/X_k}$ for $\mathbf{X_K > 0}$
					BASIS MATRIX		
s_1	0	4	1	1	1	0	TO BE COMPUTED IN NEXT STEP.
s_2	0	2	1	−1	0	1	
	$z = \mathbf{C_B X_B}$ objective func		$\Delta_1 = -3$ ↑	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	$\Delta_j = z_j - \mathbf{c_j} = \mathbf{C_B X}_j - c_j$

Step 5. Now, proceed to test the basic feasible solution for optimality by the rules given below. This is done by computing the *'net evaluation'* Δ_j for each variable x_j (column vector $\mathbf{X}_j$) by the formula

$$\Delta_j = z_j - \mathbf{c}_j = \mathbf{C_B X}_j - c_j \qquad \text{[from (5·10)]}$$

Thus, we get

$$\begin{aligned} \Delta_1 &= \mathbf{C_B X}_1 - c_1 \\ &= (0\,,0)\,(1\,,1) - 3 \\ &= (0 \times 1 + 0 \times 1) - 3 \\ &= -3 \end{aligned} \quad \Bigg| \quad \begin{aligned} \Delta_2 &= \mathbf{C_B X}_2 - c_2 \\ &= (0\,,0)\,(1\,,-1) - 2 \\ &= (0 \times 1 - 0 \times 1) - 2 \\ &= -2 \end{aligned} \quad \Bigg| \quad \begin{aligned} \Delta_3 &= \mathbf{C_B X}_3 - c_3 \\ &= (0\,,0)\,(1\,,0) - 0 \\ &= (0 \times 1 + 0 \times 0) - 0 \\ &= 0 \end{aligned} \quad \Bigg| \quad \Delta_4 = 0$$

Remark. Note that in the starting simplex table Δ_j's are same as $(-c_j)$'s. Also, Δ_j's corresponding to the columns of unit matrix (basis matrix) are always zero. So there is no need to calculate them.

Optimality Test :

(i) If all $\Delta_j\ (= z_j - c_j) \geq 0$, the solution under test will be ***optimal.*** *Alternative optimal solutions* will exist if any non-basic Δ_j is also zero.

(ii) If at least one Δ_j is negative, the solution under test is not optimal, then proceed to improve the solution in the next step.

(iii) If corresponding to any negative Δ_j, all elements of the column $\mathbf{X}_j$ are negative or zero (≤ 0) , then the solution under test will be ***unbounded.***

Applying these rules for testing the optimality of starting basic feasible solution, it is observed that Δ_1 and Δ_2 both are negative. Hence, we have to proceed to improve this solution in ***Step 6.***

Step 6. In order to improve this basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined by the following rules. Such vectors are usually named as ***'incoming vector'*** and ***'outgoing vector'*** respectively.

'Incoming vector'. The incoming vector $\mathbf{X}_k$ is always selected corresponding to the most negative value of Δ_j (say, Δ_k). Here $\Delta_k = \min\ [\Delta_1\ ,\ \Delta_2] = \min\ [-3\ ,-2] = -3 = \Delta_1$. Therefore, $k = 1$ and hence column vector $\mathbf{X}_1$ must enter the basis matrix. The column $\mathbf{X}_1$ is marked by an upward arrow (↑).

'Outgoing vector'. The outgoing vector β_r is selected corresponding to the minimum ratio of elements of $\mathbf{x_B}$ by the corresponding *positive* elements of predetermined incoming vector $\mathbf{x}_k$. This rule is called the ***Minimum Ratio Rule.*** In mathematical form, this rule can be written as

$$\frac{x_{\mathrm{Br}}}{x_{rk}} = \min_{i} \left[\frac{x_{\mathrm{B}r}}{x_{ik}}\ ,\ x_{ik} > 0 \right]$$

For $k = 1$,

$$\frac{x_{\mathrm{B}r}}{x_{r1}} = \min \left[\frac{x_{\mathrm{B}1}}{x_{11}}\ ,\ \frac{x_{\mathrm{B}2}}{x_{21}} \right] = \min \left[\frac{4}{1}\ ,\ \frac{2}{1} \right]$$

or

$$\frac{x_{\mathrm{B}r}}{x_{r1}} = \frac{2}{1} = \frac{x_{\mathrm{B}2}}{x_{21}}\ .$$

Comparing both sides of this equation, we get $r = 2$. So the vector β_2, *i.e.*, $\mathbf{X}_4$ marked with downward arrow (↓) should be removed from the basis matrix. The ***Starting Table*** **5·2** is now modified to ***Table*** **5·3** given below.

Table 5·3

		$c_j \rightarrow$	3	2	0	0	
BASIC VARIABLES	Ca	Xa	X_1	X_2	$X_3(S_1)$ (β_1)	$X_4(S_2)$ (β_2)	MIN. RATIO (X_B/X_1)
s_1	0	4	1	1	1	0	4/1
s_2	0	2	$\boxed{1}$	1	0	1	2/1 ← MIN. RATIO
	$z = C_BX_B = 0$		− 3 (min. Δ_j)	− 2	0	0	$\leftarrow \Delta_j = z_j - c_j = C_BB_j - c_j$

key element

↑entering vector ↓leaving vector

Step 7. In order to bring $\beta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in place of incoming vector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, unity must occupy in the marked '$\square$' position and zero at all other places of $\mathbf{x}_1$. If the number in the marked '$\square$' position is other than *unity, divide all elements of that row by the 'key element'. (The element at the intersection of minimum ratio arrow* (←) *and incoming vector arrow* (↑) *is called the* ***key element or pivot element****).*

Then, subtract appropriate multiplies of this new row from the other (remaining) rows, so as to obtain zeros in the remaining positions of the column $\mathbf{x}_1$. Thus, the process can be fortified by simple matrix transformation as follows :

The intermediate coefficient matrix is :

	X_B	X_1	X_2	X_3	X_4	
R_1	4	1	1	1	0	
R_2	2	1	− 1	0	1	
R_3	$z = 0$	− 3	− 2	0	0	$\leftarrow \Delta_j$

Apply $R_1 \rightarrow R_1 - R_2$, $R_3 \rightarrow R_3 + 3R_2$ to obtain

X_B	X_1	X_2	X_3	X_4	
2	0	2	1	−1	
2	1	− 1	0	1	
$z = 6$	0	− 5	0	3	$\leftarrow \Delta_j$

Now, construct the improved simplex table as follows :

Table 5·4

		$c_j \rightarrow$	3	2	0	0	
BASIC VARIABLES	C_B	X_B	X_1 (β_2)	X_2	$X_3(S_1)$ (β_1)	$X_4(S_2)$	MIN-RATIO $(X_B/X_2, X_2 > 0)$
s_1	0	2	0	$\boxed{2}$	1	−1	$\frac{2}{2}$ ← key row
x_1	3	2	1	− 1	0	1	$\cancel{\frac{2}{-1}}$ (negative ratio is not counted)
	$z = C_BX_B = 6$		0	− 5 ↑	0 ↓	3	$\leftarrow \Delta_j$

key column

From this table, the improved basic feasible solution is read as : $x_1 = 2$, $x_2 = 0$, $s_1 = 2$, $s_2 = 0$. The improved value of $z = 6$.

It is of particular interest to note here that Δ_j's are also computed while transforming the table by matrix method. However, the correctness of Δ_j's can be verified by computing them independently by using the formula $\Delta_j = C_BX_j - c_j$.

Step 8. Now repeat *Steps* 5 *through* 7 as and when needed until an optimum solution is obtained in Table 5·5.

$$\Delta_k = \text{most negative } \Delta_j = -5 = \Delta_2 .$$

Therefore, $k = 2$ and hence $\mathbf{X}_2$ should be the entering vector (key column). By minimum ratio rule :

$$\text{Minimum Ratio}\left(\frac{\mathbf{X_B}}{\mathbf{X_2}}, \mathbf{X}_2 > \mathbf{0}\right) = \text{Min}\left[\frac{2}{2}, -\right]$$ (since negative ratio is not counted, so the second ratio is not considered)

Since *first ratio* is minimum, remove the first vector β_1 from the basis matrix. Hence the key element is 2. Dividing the first row by key element 2, the intermediate coefficient matrix is obtained as :

	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	
R_1	1	0	$\boxed{1}$	1/2	− 1/2	
R_2	2	1	− 1	0	1	
R_3	$z = 6$	0	− 5	0	3	$\leftarrow \Delta_j$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 5R_1$

1	0	1	1/2	− 1/2	
3	1	0	1/2	1/2	
$z = 11$	0	0	5/2	1/2	$\leftarrow \Delta_j$

Now construct the next improved simplex table as follows :

Final Simplex Table 5·5

		$c_j \rightarrow$	3	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$ (β_2)	$\mathbf{X_2}$ (β_1)	$\mathbf{S_1}$	$\mathbf{S_2}$	
$\rightarrow x_2$	2	1	0	1	1/2	− 1/2	
x_1	3	3	1	0	1/2	1/2	
	$z = \mathbf{C_B X_B} = 11$		0	0	5/2	1/2	$\leftarrow \Delta_j$

The solution as read from this table is : $x_1 = 3$, $x_2 = 1$, $s_1 = 0$, $s_2 = 0$, and max. $z = 11$. Also, using the formula $\Delta_j = \mathbf{C_B X_j} - c_j$ verify that all Δ_j's are non-negative. Hence the optimum solution is

$$x_1 = 3 , x_2 = 1 , \text{max } z = 11 .$$

Note. If at the optimal stage, it is desired to bring s_1 in the solution, the total profit will be reduced from 11 (the optimal value) to 5/2 times of 2 units of s_1 in Table 3·4, i.e., $z = 11 - 5/2 \times 2 = 6$. This explains the *economic interpretation* of net-evaluations Δ_j .

5·4. SIMPLE WAY FOR SIMPLEX METHOD COMPUTATIONS

Complete solution with its different computational steps can be more conveniently represented by the following single table (see Table 5.6).

Table 5·6

		$c_j \rightarrow$	3	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{S_1}$	$\mathbf{S_2}$	MIN RATIO ($\mathbf{X_B/X_K}$)
s_1	0	4	1	1	1	0	4/1
$\leftarrow s_2$	0	2	$\boxed{1}$	1	0	1	2/1 ← Min
$x_1 = x_2 = 0$	$z = \mathbf{C_B X_B} = 0$		− 3* ↑	− 2	0	0 ↓	$\leftarrow \Delta_j = z_j - c_j$
$\leftarrow s_1$	0	2	0	$\boxed{2}$	1	−1	2/2 Min ←
$\rightarrow x_1$	3	2	1	−1	0	1	—
$x_2 = s_2 = 0$	$z = \mathbf{C_B X_B} = 6$		0	−5* ↑	0 ↓	3	$\leftarrow \Delta_j$
$\rightarrow x_2$	2	1	0	1	1/2	− 1/2	
x_1	3	3	1	0	1/2	1/2	
$s_1 = s_2 = 0$	$z = \mathbf{C_B X_B} = 11$		0	0	5/2	1/2	← All $\Delta_j \geq 0$

Thus, the optimal solution is obtained as : $x_1 = 3$, $x_2 = 1$, max $z = 11$.

25

PROJECT MANAGEMENT BY PERT–CPM

25.1. INTRODUCTION

A project defines a combination of interrelated activities which must be executed in a certain order before the entire task can be completed. The activities are interrelated in a logical sequence in such a way that some activities cannot start until some others are completed. An activity in a project is usually viewed as job requiring time and resources for its completion. Until recently, planning was seldom used in the design phase. As the technological development took place at a very rapid speed and the designs become more complex with more inter-departmental dependence and interaction, the need for planning in the development phase become inevitable.

Until five decades ago, the best known 'planning tool' was the so called *Gantt bar chart* which specifies the start and finish times for each activity on a horizontal time-scale, but the disadvantage is the interdependency between different activities (that mainly controls the progress of the project) which cannot be determined from the bar chart. Growing complexities of modern projects have demanded more systematic and effective planning techniques with the objective of optimizing the efficiency of executing the project. Efficiency implies effecting the utmost reduction in the time required to complete the project while accounting for economic feasibility of using available resources. Project management has evolved as a new field with the development of *two* 'analytic' techniques for planning, scheduling and controlling of projects. These are the *Critical Path Method (CPM)* and the *Project Evaluation and Review Technique (PERT)*.

25.2. HISTORICAL DEVELOPMENT OF CPM / PERT TECHNIQUES

In 1956-58, above two techniques were developed by two different groups almost simultaneously. CPM was developed by Walker from E.L. du pont de Nemours Company to solve project scheduling problems and was later extended to a more advanced status by Mauchly Associates. During the same time, PERT was developed by the team of engineers working on the polar[illegible] Missile programme of US Navy. This was a large project involving many departments and there were ma[illegible] activities about which they had a very little information about the duration of the project. Under such conditions, the project was to be completed within a specified time. To coordinate activities of various departments, this group used PERT and devised the technique independent of CPM.

The methods are essentially network-oriented techniques using the same principle. PERT and CPM are basically time-oriented methods in the sense that they both lead to the determination of a time schedule for the project. The significant difference between two approaches is that the time estimates for the different activities in CPM were assumed to be deterministic while in PERT these were described probabilistically. Now a days, PERT and CPM actually comprise one technique and the differences, if any are only historical. Therefore, these techniques are referred to as '*project scheduling*' techniques.

[VTU (BE Mech.) 2002]

Q. Distinguish between PERT and CPM techniques.

25.3. APPLICATIONS OF PERT / CPM TECHNIQUES

These methods have been applied to a wide variety of problems in industries and have found acceptance even in government organizations.

These include :

(*i*) construction of a dam or canal system in a region, (*ii*) construction of a building or highway,
(*iii*) maintenance or overhaul of aeroplanes or oil refinery, (*iv*) space flight,
(*v*) cost control of a project using PERT/COST, (*vi*) designing a prototype of a machine,
(*vii*) development of supersonic planes.

25.4. BASIC STEPS IN PERT/CPM TECHNIQUES

Project scheduling by PERT/CPM consists of *four* main steps :

1. Planning. The planning phase is started by splitting the total project into small projects. These smaller projects, in turn, are divided into activities and are analysed by the department or a section. The relationship of each activity with respect to other activities are defined and established, and the corresponding responsibilities and the authority are also stated. Thus, the possibility of overlooking any task necessary for the completion of the project is reduced substantially.

2. Scheduling. The ultimate objective of the scheduling phase is to prepare a time chart showing the start and finish times for each activity as well as its relationship to other activities of the project. Moreover, the schedule must pinpoint the critical path (in view of time) activities which require special attention if the project is to be completed in time. For non-critical activities, the schedule must show the amount of slack or float times (defined later) which can be used advantageously when such activities are delayed or when limited resources are to be utilized effectively. In this phase, it is possible to resource requirements such as time, manpower, money machines, etc.

3. Allocation of Resources. Allocation of resources is performed to achieve the desired objective. A resource is a physical variable such as labour, finance,equipment and space which will impose a limitation on time for the project. When resources are limited and conflicting, demands are made for the same type of resources a systematic method for allocation of resources become essential. Resource allocation usually incurs a compromise, and the choice of this compromise depends on the judgement of managers.

4. Controlling. The final phase in project management is *controlling*. Critical path methods facilitate the application of the priciple of management by expectation to identifying areas that are critical to the completion of the project. By having progress reports from time to time and updating the network continuously, a better financial as well as technical control over the project is exercised. Arrow diagrams and time charts are used for making periodic progress reports. If necessary, new course of action is determined for the remaining portion of project.

25.5 NETWORK DIAGRAM REPRESENTATION

In project scheduling, the first step is to sketch an arrow diagram which shows inter-dependencies and the precedence relationship among activities (as defined below) of the project. In a network representation of a project, certain basic definitions are used.

1. Activity. Any individual operation, which utilises resources and has an end and a beginning, is called *activity*. An arrow is commonly used to represent an activity with its head indicating the direction of progress in the project. These are usually classified into following *four* categories :

(*i*) ***Predecessor activity.*** Activities that must be completed immediately prior to the start of another activity are called *predecessor activities.*

(*ii*) ***Successor activity***. Activities that cannot be started until one of more of other activities are completed, but immediately succeed them are called *successor activities.*

(*iii*) ***Concurrent activity.*** Activities which can be accomplished concurrently are known as concurrent activities. It may be noted that an activity can be a predecessor or a successor to an event or it may be concurrent with one or more of the other activities.

(*iv*) ***Dummy activity.*** An activity which does not consume any kind of resource but merely depicts the technological dependence is called a *dummy activity.*

It may be noted that the dummy activity is inserted in the network to clarify the activity pattern in the following *two* situations :

(*i*) to make activities with common starting and finishing points distinguishable, and

(*ii*) to identify and maintain the proper precedence relationship between activities that are not connected by events.

For example, consider a situation where A and B are concurrent activities, C is dependent on A, and D is dependent on A and B both. Such a situation can be handled by using a dummy activity as follows (*Fig.* 25.1.):

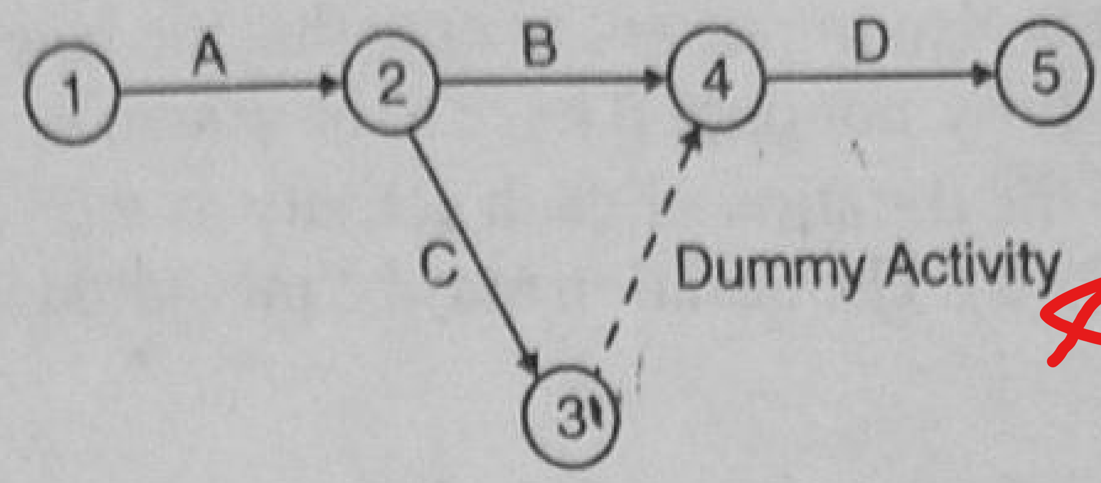


Fig. 25.1.

In another situation, consider the following diagram where job A and C have the same job reference and they can be started independently on completion of A. But, D could be started only after completion of B and C. This relationship is shown by the dotted line (*Fig.* 25.2.).

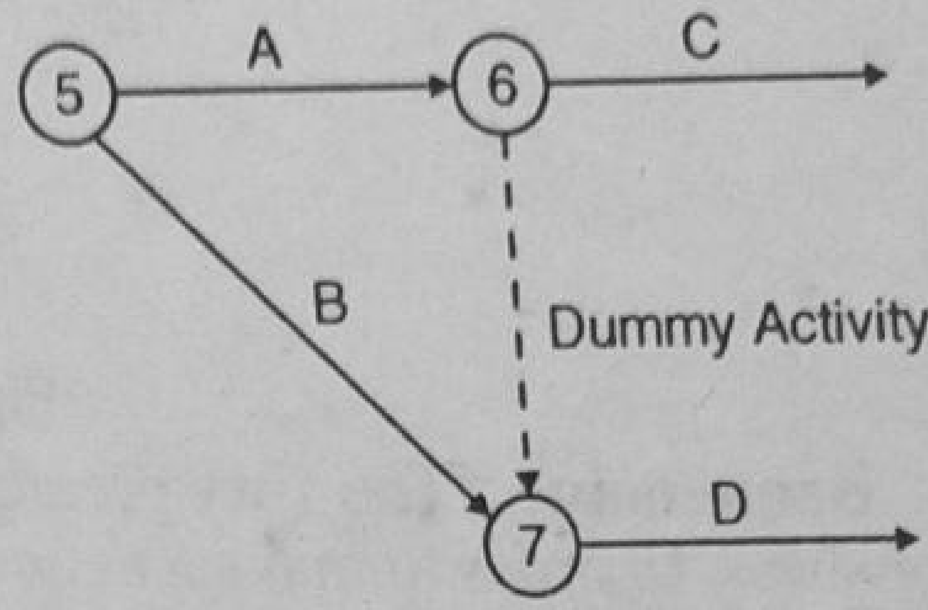


Fig. 25.2.

2. Event. An event represents a point in time signifying the completion of some activities and the beginning of new ones. This is usually represented by a circle 'O' in a network which is also called a *node* or *connector*.

The events can be further classified into following three categories (as shown below in the figure):

(*i*) **Merge event.** When more than one activity comes and joins an event, such event is known as *merge event*.

(*ii*) **Burst event.** When more than one activity leaves an event, such event is known as a *burst event*.

(*iii*) **Merge and burst event.** An activity may be a merge and burst event at the same time as with respect to some activities it can be a merge event and with respect to some other activities it may be a burst event.

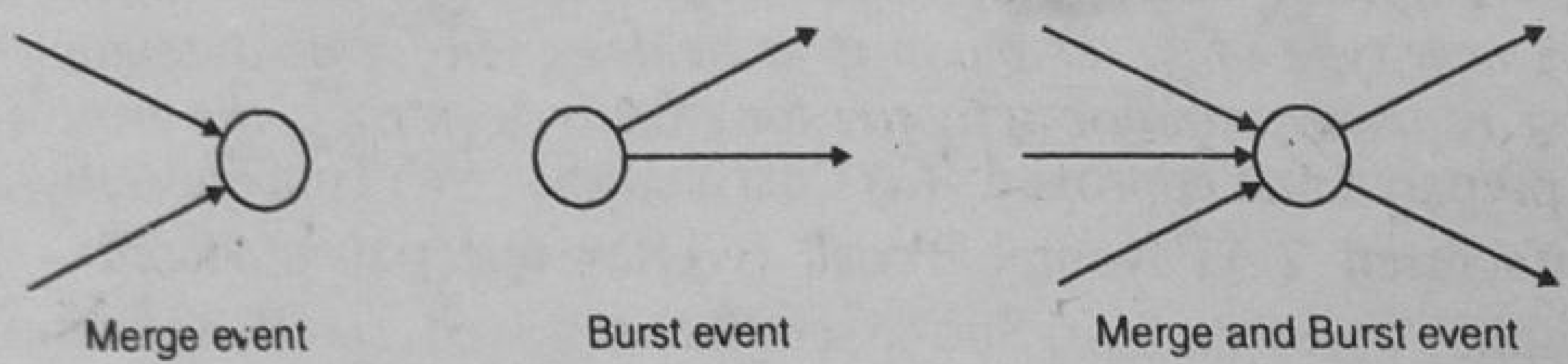


Fig. 25.3.

Remarks :

1. An event is that particular instant of time at which some *specific part* of a project has been or is to be achieved. While, an activity is actual performance of a task. An activity requires time and resources for its completion.

   Examples of events : design completed, pipe line laid, etc.

   Examples of activities : assembly of parts, mixing of concrete, preparing budget, etc.

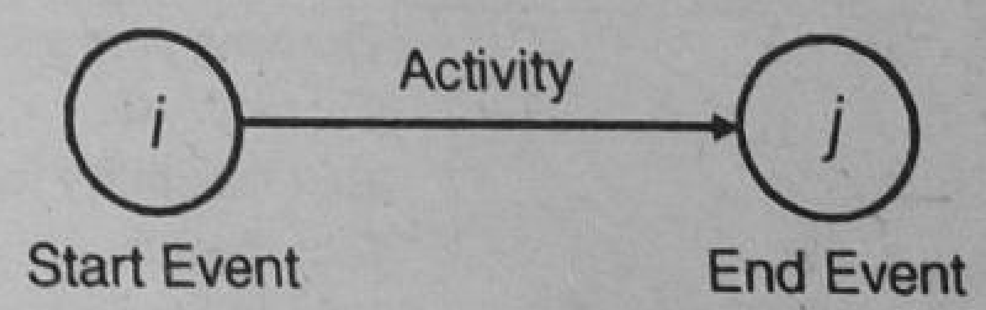


Fig. 25.4.

2. Events are described by such words as : complete, start, issue, approve, tested. etc.

   While, the word like : design, procure, test, develop, prepare etc. shows that work is being accomplished and thus represent activities.

3. While drawing networks, it is assumed that (*i*) time flows from left to right, and (*ii*) head events always have number higher than that of tail event. Thus activity ($i - j$) always means that the job which begins at event '*i*' is completed at event '*j*'.

4. Network representation is based on the following *two* axioms :

   (*i*) An event is not said to be complete until all the activities flowing into it are completed.

   (*ii*) No subsequent activity can begin until its tail event is reached or completed.

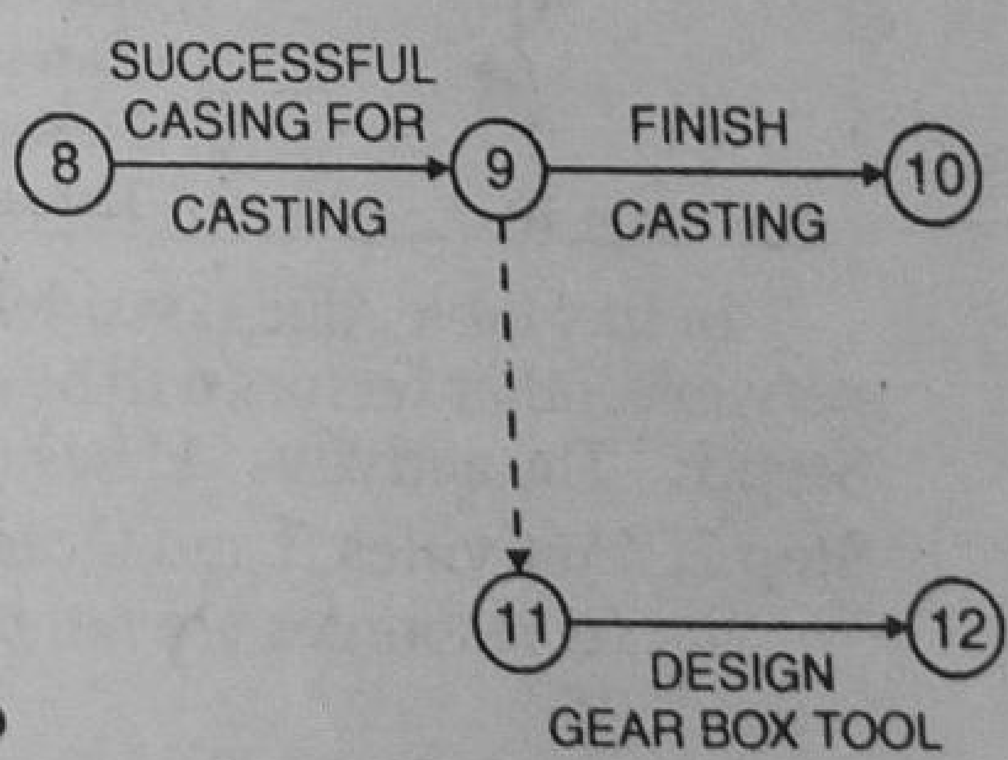


Fig. 25.5

Illustration. Designing tools for a gear box is an *activity*. A decision to start designing tools may depend on having a successful casing for gear-box casting.

In terms of technological sequence, casting as such has little bearing on the tooling of the gear box, but the management would prefer to have a successful casing before gear-box tool is designed. Thus the dependence of gear-box tools on successful casing is shown as a dummy activity (*Fig.* 25.5.).

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It is also important to note that the length of the arc (or arrow) need not be proportional to the durati the activity nor does it have to be drawn as a straightline.

If the duration of each activity as well as their logical sequence is known, it can be shown in a netw (Fig. 25.6). The duration may be measured in days.

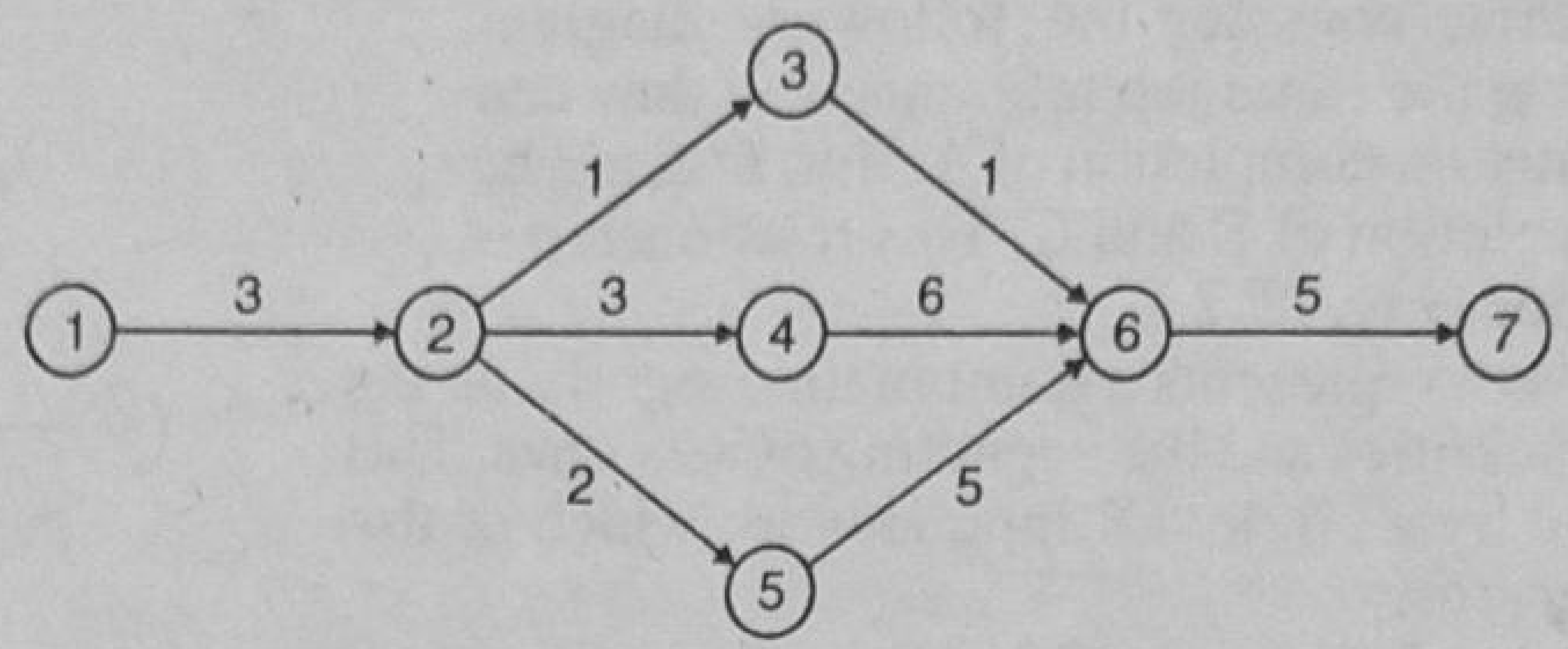


Fig. 25.6. Activity duration on the network

3. Sequencing. The first prerequisite in the development of a network is to maintain the preceden relationships. In order to make a network, following points should be taken into consideration :

(*i*) What job or jobs precede it ? (*ii*) What job or jobs could run concurrently ?

(*iii*) What job or jobs follow it ? (*iv*) What controls the start and finish of a job ?

Since all further calculations are based on the network, it is necessary that a network be drawn with fu care. There are many ways to draw a network. In this text, the method will be used which follows th precedence table. It is generally agreed that dummy activities be used as liberally as needed in the in attempt, while revising the same network, every attempt should be made to minimize them.

The following example of water pump shows basic steps required in drawing a network.

Illustration. *A new type of water pump is to be designed for an automobile. Major specifications ar given. Following list represents major activities for effective control of the project :*

(*i*) Drawings prepared and approved (*ii*) Cost analysis (*iii*) Tool feasibility (economics)

(*iv*) Tool manufactured (*v*) Favourable cost (*vi*) Raw materials procured

(*vii*) Sub-assemblies ordered (*viii*) Sub-assemblies received (*ix*) Parts manufactured

(*x*) Final assembly (*xi*) Testing and shipment.

Based on this available information, a precedence table may be formed (Table 25.1).

Precedence Table 25.1.

Activity	Description	Preceding Activity
A	Drawing prepared and approved	–
B	Cost analysis	A
C	Tool feasibility (economics)	A
D	Tool manufactured	C
E	Favourable cost	B, C
F	Raw materials procured	D, E
G	Sub-assemblies ordered	E
H	Sub-assemblies received	G
I	Parts manufactured	D, F
J	Final assembly	I, H
K	Testing and shipment	J

In this table, due consideration has been given to precedings of an activity/activities. While drawing the network, other factors will be considered.

Step 1. The activity 'A' has no preceding activity and is represented by an arrowed line (*Fig* 25.7).

Step 2. Activities B and C are preceded by an activity 'A' and activities B and C could be done concurrently (if resources are not binding). No other activity can be scheduled at this stage. This is shown in *Fig.* 25.8.

Step 3. The activity 'D' can be sequenced easily. Cost favourable activity 'E' cannot be scheduled unless activities B and C are scheduled. Further, it is observed that dependence of 'cost favourable activity' on the *'economy of tooling'* is from a *'technical view-point'* and does not consume any resource, and

The basic scheduling computations can be put under the following three groups :

25.8-2. Forward Pass Computations (For Earliest Event Time)

[Meerut (OR) 2003]

Before starting computations, the occurrence time of *initial* network *event* is fixed. Then, the forward pass computation yields the *earliest start* and *earliest finish* time for each activity (i, j), and indirectly the earliest expected occurrence time for each event. This is mainly done in three steps.

Step 1. The computations begin from the '*start*' node and move towards the '*end*' node. For easiness, the forward pass computations start by assuming the earliest occurrence time of *zero* for the initial project event.

Step 2. (*i*) *Earliest starting time* of activity (i, j) is the earliest event time of the tail end event *i.e.*, $(Es)_{ij} = E_i$.

(*ii*) *Earliest finish time* of activity (i, j) is the earliest starting time + the activity time *i.e.*,

$$(Ef)_{ij} = (Es)_{ij} + D_{ij} \quad \text{or} \quad (Ef)_{ij} = E_i + D_{ij}$$

(*iii*) *Earliest event time* for event *j* is the *maximum* of the earliest finish times of all activities ending into that event. That is,

$$E_j = \max_i \,[(Ef)_{ij} \text{ for all immediate predecessor of}(i, j)] \quad \text{or} \quad E_j = \max_i \,[E_i + D_{ij}]$$

The computed 'E' values are put over the respective circles representing each event.

25.8-3. Backward Pass Computations (For Latest Allowable Time)

[Meerut (OR) 2003]

The latest event times (L) indicates the time by which all activities entering into that event must be completed without delaying the completion of the project. These can be computed by reversing the method of calculation used for earliest event times. This is done in the following steps :

Step 1. For ending event assume $E = L$. Remember that all E's have been computed by *forward pass computations.*

Step 2. *Latest finish time* for activity (i, j) is equal to the *latest event time* of event *j*, i.e., $(Lf)_{ij} = L_j$.

Step 3. *Latest starting time* of activity (i, j) = the *latest completion time* of (i, j) – *the activity time*.

or $\quad (Ls)_{ij} = (Lf)_{ij} - D_{ij} \quad$ or $\quad (Ls)_{ij} = L_j - D_{ij}$.

Step 4. *Latest event time for event i is* the *minimum* of the *latest start time of* all activities originating from that event, *i.e.*,

$$L_i = \min_j \,[(Ls)_{ij} \text{ for all immediate successors of } (i, j)] = \min_j \,[(Lf)_{ij} - D_{ij}] = \min_j \,[L_j - D_{ij}]$$

The computed 'L' values are put over the respective circles representing each event.

25.8-4. Determination of Floats and Slack Times

When the network diagram is completely drawn, properly labelled, and earliest (E) and latest (L) event times are computed as discussed so far, the next object is to determine the *floats* and *slack times* defined as follows :

There are mainly *three* kinds of floats as given below :

(1) Total float. *The amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time without affecting the overall project duration time.* **[VTU (BE Mech.) 2003]**

Mathematically, the total float of an activity $(i - j)$ is the difference between the *latest start time* and *earliest start time* of that activity. Hence the *total float* for an activity $(i - j)$, denoted by $(Tf)_{ij}$, can be calculated by the formula :

$$(Tf)_{ij} = (\textit{Latest start} - \textit{Earliest start}) \text{ for activity}(i - j)$$

or $$(Tf)_{ij} = (Ls)_{ij} - (Es)_{ij} \quad \text{or} \quad (Tf)_{ij} = (L_j - D_{ij}) - E_i$$

where E_j, L_j and D_{ij} are defined in **sec.25.8-1.** This is the most important type of float because of concerning with the overall project duration.

(2) Free float. *The time by which the completion of an activity can be delayed beyond the earliest finish time without affecting the earliest start of a subsequent(succeeding) acitivity.* **[VTU (BE Mech.) 2003]**

Mathematically, the free float for activity (i, j), denoted by $(Ff)_{ij}$, can be calculated by the formula :

Conclusions drawn from total float values :

The value of total float for any activity is useful for drawing the following conclusions :

Total Float Value

- **Negative**
  - Resources are not adequate and activity may not finish in time
  - Induct extra resources or say critical path needs crashing in order to reduce the negative float.
- **Zero**
  - Resources are just sufficient to complete the activity or say the activity cannot be delayed.
- **Positive**
  - Resources are extra
  - One has the freedom to reallocate the resources or the activity can be delayed by so much period.

Q. 1. Briefly explain the four types of floats used in Network Analysis. **[CA. (May) 92]**

2. Explain in brief the following terms commonly used in network of PERT/CPM.
(i) Activity (ii) Event (iii) Dummy activity (iv) Path (v) Critical path. **[Meerut (OR) 2003; VTU (BE Mech.) 2002]**

25·8-5. Determination of Critical Path

Before defining *critical path*, let us first discuss about the meaning of *critical event* and *critical activity*.

Critical event. *Since the slack of an event is the difference between the latest and earliest event times, i.e., slack* $(i) = L_i - E_i$, *the events with zero slack times are called critical events.*

In other words, *the event (i) is said to be critical if* $E_i = L_i$.

Critical activity. *Since the difference between the latest start time and earliest start time of an activity is usually called as* ***total float****, the activities with zero total float are known as* ***critical activities.***

In other words, *an activity is said to be* ***critical*** *if a delay in its start will cause a further delay in the completion date of the entire project.*

Obviously, a non-critical activity is such that the time between its earliest start and its latest completion dates (as allowed by the project) is longer than its actual duration. In this case, non-critical activity is said to have a *slack* or *float* time.

Critical Path. *The sequence of critical activities in a network is called the* ***critical path.***

The critical path is the *longest path* in the network from the starting event to ending event and defines the minimum time required to complete the project.

By the term '*path*' we mean a sequence of activities such that it begins at the starting event and end at the final event. The length of a path is the sum of the individual times of the activities lying on the path.

If the activities on critical path are delayed by a day, the project would also be delayed by a day unless the times of the future critical activities are reduced by a day by different means. The critical path is denoted by *double* or *darker* lines to make distinction from the other non-critical paths.

Main features of critical path. The critical path has two main features :

(*i*) If the project has to be shortened, then some of the activities on that path must also be shortened. The application of additional resources on other activities will not give the desired result unless that critical path is shortened first.

(*ii*) The variation in actual performance from the expected activity duration time will be completely reflected in one-to-one fashion in the anticipated completion of the whole project.

The *critical path* identifies all critical activities of the project. The method of determining such a path is explained by the following numerical example.

Example 1. *Consider the following network where nodes have been numbered according to the Fulkerson's rule. Numbers along various activities represent the normal time* (D_{ij}) *required to finish that activity, e.g. activity* (3)—(6) *will take 5 days (months, weeks, hours depending on the time units). For this project, we are interested to find out the time it will take to complete this project.*

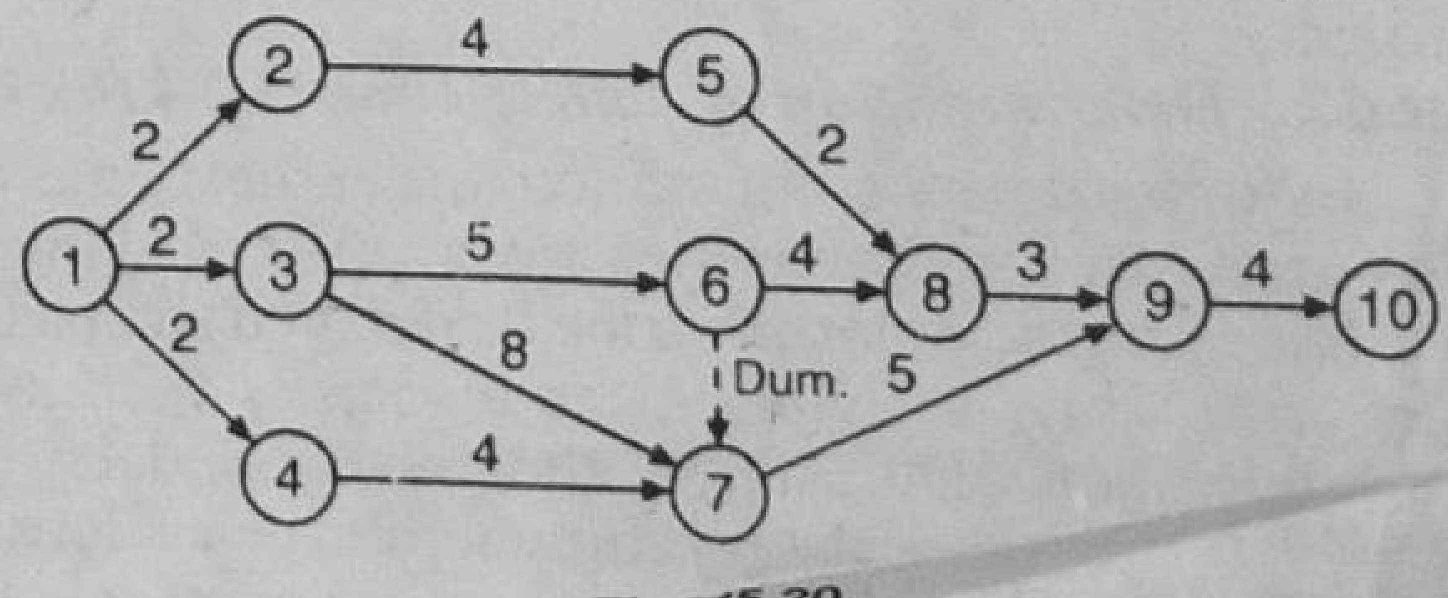


Fig. 25.20.

What jobs are critical to the completion of the project in time, etc ?

Solution. For this, it is necessary to find out the earliest and the latest completion time for each activity in the network. The earliest and the latest times are re-calculated by using *'forward pass'* and *'backward pass'* computations, respectively.

To understand the procedure, we define :

E_i = the earliest expected occurrence time of event i,

L_j = the latest allowable event occurrence time for event j.
This is the latest time by which the event j must be started without increasing the project duration.

D_{ij} = the expected duration to complete the activity i–j.

The solution now starts by the forward pass computation.

Step 1. ***Determination of Earliest Time (E_j) : Forward Pass Computation***

The purpose of the forward pass computation is to find out earliest start times for all the activities. For this, it is necessary to assign some initial value to the starting node 1. Usually this value is taken to be zero so that the subsequent earliest time could be interpreted as the project duration up to that point in question.

Rules for the computation are as follows :

***Rule* 1.** Initial event is supposed to occur at time equal to zero, that is, $E_1 = 0$.

***Rule* 2.** Any activity can start immediately when all preceding activities are completed.
The earliest time E_j for node j is given by $E_j = \max\limits_i [E_i + D_{ij}]$,

where i is the collection of nodes which precede node j.

***Rule* 3.** Repeat **step 2** for the next eligible activity until the end node is reached.

Numerical Calculation :

Consider the network (*Fig.* 25.20.) by assumption $E_1 = 0$ and $E_2 = \max\limits_i [E_i + D_{i2}]$.

For node 2, node 1 is the only predecessor and hence $i = 1$ contains only one element. Therefore,

$$E_2 = E_1 + D_{12} = 0 + 2 = 2.$$

Likewise, values of E_3, E_4, E_5 and E_6 can be computed as :

$E_3 = E_1 + D_{13} = 0 + 2 = 2$, $E_4 = E_1 + D_{14} = 0 + 2 = 2$, $E_5 = E_2 + D_{25} = 2 + 4 = 6$, $E_6 = E_3 + D_{36} = 2 + 5 = 7$.

Consider node 7, where there are three emerging activities, *i.e.* $E_7 = \max\limits_i (E_i + D_{i7})$,

The collection i consists of nodes 3,4 and 6 that are preceding node 7. Therefore,

$$E_7 = \max [\ E_3 + D_{37} = 2 + 8 = 10,\ E_4 + D_{47} = 2 + 4 = 6,\ E_6 + D_{67} = 7 + 0 = 7\ ] = 10$$

$$E_8 = \max [E_5 + D_{58} = 6 + 2 = 8,\ E_6 + D_{68} = 7 + 4 = 11] = 11$$

$$E_9 = \max [E_8 + D_{89} = 11 + 3 = 14,\ E_7 + D_{79} = 10 + 5 = \mathbf{15^*}] = 15$$

and $$E_{10} = E_9 + D_{9,10} = 15 + 4 = 19.$$

From this computation, it can be inferred that this job will take 19 days to finish as this is the longest path of the network. Activities along this longest path are : 1—3, 3—7, 7—9 and 9—10. This longest path is called the critical path. In any network, it is not possible that there can be only one critical path. For example, if in the above network, let $D_{36} = 6$ days, then two critical paths exist having the same duration for completion of project.

Step 2. ***Determination of Latest Time (L_i) : Backward Pass Computation***

In forward pass computation, the earliest time when a particular activity will be completed is known. It is also seen that some activities are not critical to the completion of the job. The question, a manager would like to ask is : Can their starting time be delayed so that the total completion time is still the same ? Such a question may arise while scheduling the resources such as : manpower, equipment, finance and so on. If delay is allowable, then what can be the maximum delay ? For, this is the latest time for various activities which is desired. The backward pass computation procedure is used to calculate the latest time for various activities. In the forward pass computation assignment of $E_1 = 0$ was arbitrary, likewise for the backward pass computation, it is possible to assign the project terminal event the date on which the project should be over. If no such date is prescribed, then the covention is of setting latest allowable time determined in forward pass computation.

Rules of the backward pass computation are as follows :

Rule 1. Set $L_i = E_i$ or T_S

where T_s is the scheduled date for completion and E_i is the earliest terminal time.

Rule 2. $L_i = \min_j \{L_j - D_{ij}\}$, *i.e.* the latest time for activities is the minimum of the latest time of all succeeding activities reducing their activity time.

Rule 3. Repeat *rule* 2 until starting activity is reached.

Latest times for activities of the network are calculated below :

By *rule 1*, set $L_{10} = 19$. Applying *rule 2*, it is to determine L_9, L_8 and L_7,

$$L_9 = \min_j \{L_j - D_{9,j}\} = 19 - 4 = 15 \text{ for } j = 10.$$

$$L_8 = \min_j \{L_j - D_{8,j}\} = L_9 - D_{8,9} = 15 - 3 = 12 \qquad (j \text{ contains only one node } 9)$$

$$L_7 = \min_j \{L_j - D_{7,j}\} = L_9 - D_{7,9} = 15 - 5 = 10 \qquad (j \text{ contains node } 9).$$

Now consider node 6. for this node, there are two succeeding activities, namely 6—8 and 6—7.

Hence, $$L_6 = \min_{j=(7,8)} [L_j - D_{6,j}] = \min \begin{bmatrix} L_7 - D_{6,7} \\ L_8 - D_{6,8} \end{bmatrix} = \min \begin{bmatrix} 10 - 0 = 10 \\ 12 - 4 = 8^* \end{bmatrix} = 8$$

Similarly, $$L_5 = L_8 - D_{5,8} = 12 - 2 = 10, \qquad L_4 = L_7 - D_{4,7} = 10 - 4 = 6$$

$$L_3 = \min \begin{bmatrix} L_6 - D_{3,6} \\ L_7 - D_{3,7} \end{bmatrix} = \min \begin{bmatrix} 8 - 5 = 3 \\ 10 - 8 = 2^* \end{bmatrix} = 2, \; L_2 = L_5 - D_{2,5} = 10 - 4 = 6$$

$$L_1 = \min_{j=(2,3,4)} [L_j - D_{1,j}] = \min \begin{bmatrix} 6 - 2 = 4 \\ 2 - 2 = 0^* \\ 6 - 2 = 4 \end{bmatrix} = 0.$$

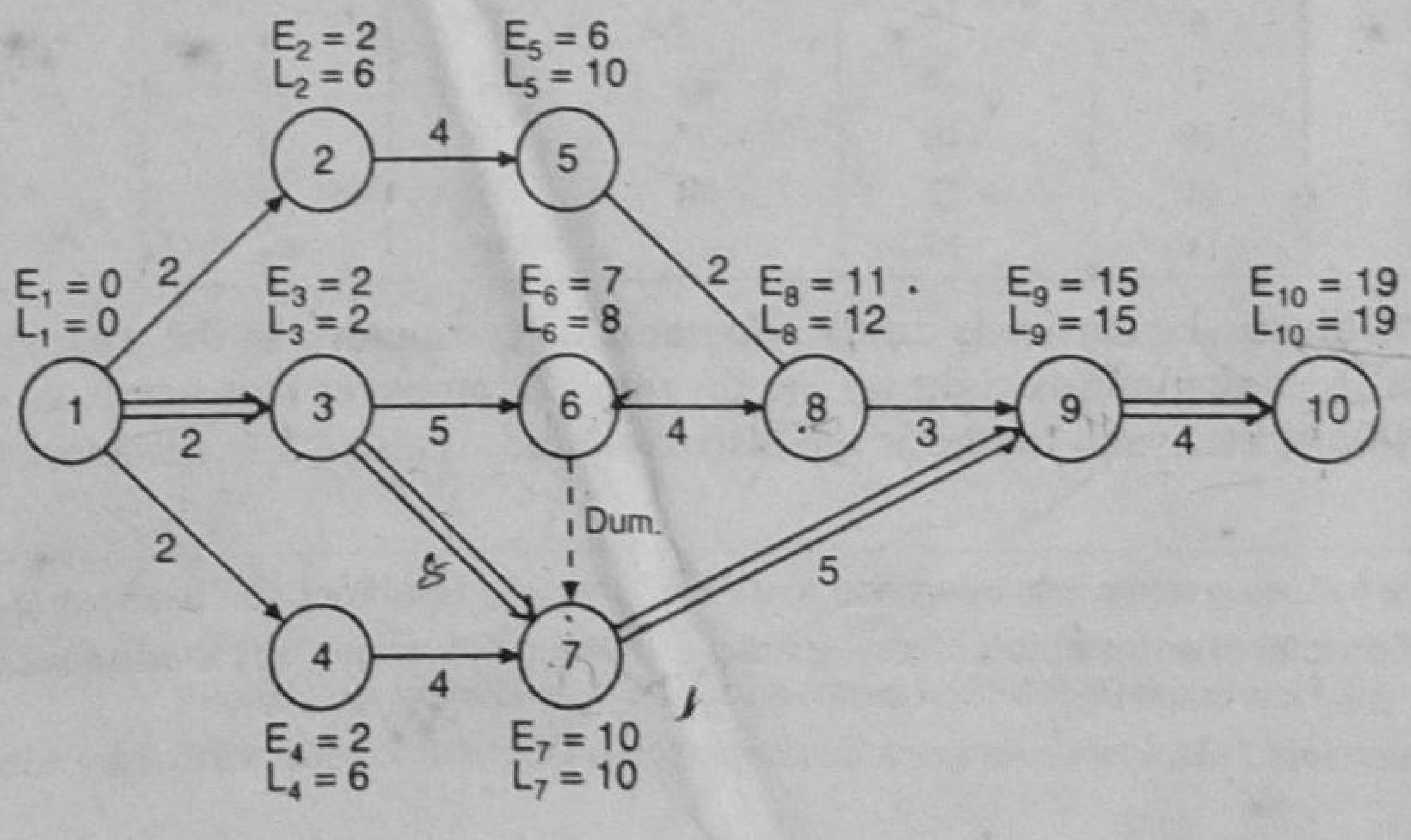


Fig. 25.21

(1,2) The minimum value of $L_1 = 0$ is no surprising result. Since, started with $L_i = E_i$, it is always possible to (1,3) get $L_1 = 0$. If this is not so, it means that some error is made in calculations of forward pass or backward pass (1,4) (2,5) Thus *Fig.* 25.21 shows earliest and latest times of each event.

(3,4) Recall that path 1—3—7—9—10 was defined as the critical path of this network. Along this path, it is (3,7) observed that the latest and earliest times are the same implying that any activity along this path cannot be (4,5) delayed without affecting the duration of the projcet.

Step 3. *Computation of Floats.*

By definition, for the activity 8—9, the float is one day $(L_8 - E_8 = 12 - 11 = 1)$. This float represents the amount by which this particular activity can be delayed without influencing the duration of the project.

Also, by definition, free float, if any, will exist only on the activities merge points. To illustrate the concept of free float, consider path 1—2—5—8—9, total float on activity 8—9 is one day and since this is the last activity prior to merging two activities, this float is *free float* also. Similarly, consider the activity 5—8 which has a total float of 4 days but has only 3 days of free float because 1 day of free float is due to the activity 8—9. If the activity 5—8 is delayed up to three days, the early start time of no activity in the network will be affected. Therefore, the concept of free float clearly states that the use of free float time will not influence any succeeding activity float time.

If free float of any activity comes out to be negative, it is taken zero.

For example, independent float of $(1, 2)$ = free float of $(1, 2) - (L_1 - E_1) = 0 - (0 - 0) = 0$.

Step 4. *To Identify Critical Path*

The earlier calculation shows that the path or paths which have *zero float* are called the *critical* ones. If this logic is extended little further, it would provide a guide rule to determine the next most critical path and so on. Such an information will be useful for managers in the control of projects. In this example, path 1—3—8—9—10 happens to be next to critical path; because it has float of one day on many of its activities.

Table 25.2.

Activity $(i-j)$	Duration (D_{ij})	E_i Start L_i		E_j Finish L_j		Float		
		Earliest	Latest	Earliest	Latest	Total	Free	Independent
(1)	(2)	(3) E_i	(4) = (6) − (2)	(5) = (3) + (2)	(6) L_j	(7) = (4) − (3)	(8) = $E_j - E_i - D_{ij}$	(9) = (8) − $(L_i - E_i)$
1–2	2	0	4	2	6	4	0	0
1–3	2	0	0	2	2	0	0	0
1–4	2	0	4	2	6	4	0	0
2–5	4	2	6	6	10	4	0	0
3–6	5	2	3	7	8	1	0	0
3–7	8	2	2	10	10	0	0	0
4–7	4	2	6	6	10	4	4	0
5–8	2	6	10	8	12	4	3	0
6–8	4	7	8	11	12	1	0	0
7–9	5	10	10	15	15	0	0	0
8–9	3	11	12	14	15	1	1	0
9–10	4	15	15	19	19	0	0	0

The method discussed earlier is easily adoptable on computers. In the case of small networks, we can perform most of the calculations right on the diagram. In an event that a person would like to use tableau format to find floats, etc, such methods are also available. Table 25.2 summarizes float times and other information.

Q. 1. Define the following terms with reference to a PERT chart : (i) Total float, (ii) Free float, (iii) Independent float.

2. The local chapter of an institute is planning a dinner meeting with a nationally known speaker and you are responsible for organising it. How could PERT/CPM methodology be useful for this simulation ? [CA. (Nov) 92]

3. What is a project ? Give two examples. List the important four district features that are common to all projects.

4. Define a dummy arrow used in a network. State two purposes for which it is used. Mention four conventions that are used in drowings the network. [CA (May) 93]

[C

Example 2. *A project consists of a series or tasks labelled A, B,..., H, I with the relationships ($W < X, Y$, means X and Y cannot start until W is completed; $X, Y < W$ means W cannot start until both X and Y are completed). With this notation, construct the network diagram having the constraints :*

$$A < D, E ; B, D < F ; C < G ; B < H; F, G < I.$$

have

valu

obse

find also the optimum time of completion of the

as follows :

Task	:	A	B	C
Time	:	23	8	20

Solution. The resulting network is shown in Fig.

establish the correct precedence relationship.

ding order shows the direction of progress in th

o determine the critical path (*optimum time of*

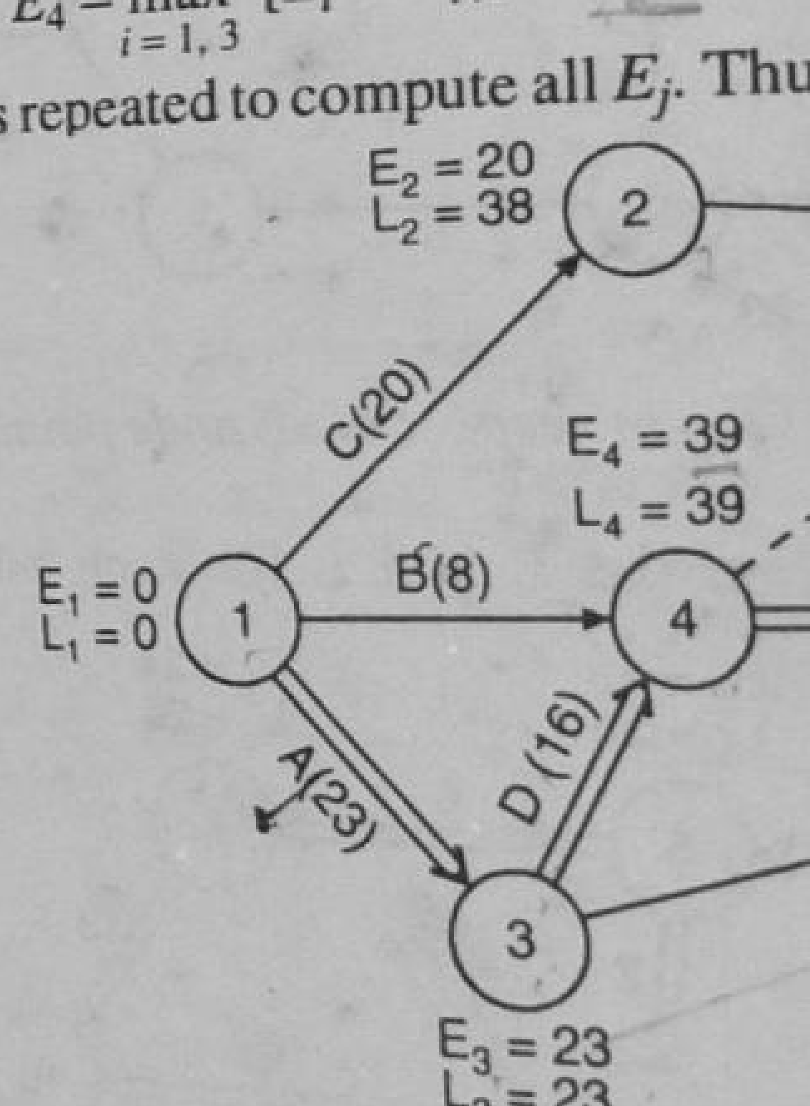
atest finish L_j for each task (i, j). The calculatio

$$E_1 = 0,\ E_2 = E_1 + D_{12} = 0 + 20$$

o determine the value of E_4, since there are two

$$E_4 = \max_{i=1,3}\ [E_i + D_{i4}] = \max\ [E$$

This procedure is repeated to compute all E_j. Thu



F

$$E_5 = \max_{i=2,4}\ [E_i + D_{i5}] = \max\ [39 + 0,\ 20 + 1$$

$$E_7 = \max_{i=3,5,6}\ [E_i + D_{i7}] = \max\ [23 + 24,\ 39 +$$

The value of L_i are calculated proceeding back

$$L_7 = E_7 = 67,\ L_6 = L_7 - D_{67} = 67 - 10 = 5$$

$$L_5 = \min_{j=6,7}\ [L_j - D_{5j}] = \min\ [57 - 0,\ 67 - 4]$$

$$L_4 = \min_{j=5,6}\ [L_j - D_{4j}] = \min\ [57 - 0,\ 57 - 1$$

$$L_3 = \min_{j=4,7}\ [L_j - d_{3j}] = \min\ [39 - 16,\ 67 - 2$$

$$L_2 = L_5 - D_{25} = 57 - 19 = 38$$

$$L_1 = \min_{j=2,3,4}\ [L_j - D_{1j}] = \min\ [38 - 20,\ 23 -$$

Table 25.3. N

Task (i, j)	Normal Time (D_{ij})	Earliest Ti	
		Start (E_i)	Fi
(1, 2)	20	0	
(1, 3)	23	0	
(1, 4)	8		
(2, 5)	19		
(3, 4)	16		
(3, 7)			
(4, 5)			

Step 3. *Computation of Floats.*

By definition, for the activity 8—9, the float is one day $(L_8 - E_8 = 12 - 11 = 1)$. This float represents the amount by which this particular activity can be delayed without influencing the duration of the project.

Also, by definition, free float, if any, will exist only on the activities merge points. To illustrate the concept of free float, consider path 1—2—5—8—9, total float on activity 8—9 is one day and since this is the last activity prior to merging two activities, this float is *free float* also. Similarly, consider the activity 5—8 which has a total float of 4 days but has only 3 days of free float because 1 day of free float is due to the activity 8—9. If the activity 5—8 is delayed up to three days, the early start time of no activity in the network will be affected. Therefore, the concept of free float clearly states that the use of free float time will not influence any succeeding activity float time.

If free float of any activity comes out to be negative, it is taken zero.

For example, independent float of $(1, 2)$ = free float of $(1, 2) - (L_1 - E_1) = 0 - (0 - 0) = 0$.

Step 4. *To Identify Critical Path*

The earlier calculation shows that the path or paths which have *zero float* are called the *critical* ones. If this logic is extended little further, it would provide a guide rule to determine the next most critical path and so on. Such an information will be useful for managers in the control of projects. In this example, path 1—3—8—9—10 happens to be next to critical path; because it has float of one day on many of its activities.

Table 25.2.

Activity $(i\text{–}j)$	Duration (D_{ij})	E_i Start Earliest	L_i Latest	E_j Finish Earliest	L_j Latest	Float Total	Free	Independent
(1)	(2)	(3) E_i	(4) = (6) − (2)	(5) = (3) + (2)	(6) L_j	(7) = (4) − (3)	(8) = $E_j - E_i - D_{ij}$	(9) = (8) − $(L_i - E_i)$
1–2	2	0	4	2	6	4	0	0
1–3	2	0	0	2	2	0	0	0
1–4	2	0	4	2	6	4	0	0
2–5	4	2	6	6	10	4	0	0
3–6	5	2	3	7	8	1	0	0
3–7	8	2	2	10	10	0	0	0
4–7	4	2	6	6	10	4	4	0
5–8	2	6	10	8	12	4	3	0
6–8	4	7	8	11	12	1	0	0
7–9	5	10	10	15	15	0	0	0
8–9	3	11	12	14	15	1	1	0
9–10	4	15	15	19	19	0	0	0

The method discussed earlier is easily adoptable on computers. In the case of small networks, we can perform most of the calculations right on the diagram. In an event that a person would like to use tableau format to find floats, etc, such methods are also available. Table 25.2 summarizes float times and other information.

Q. 1. Define the following terms with reference to a PERT chart : (i) Total float, (ii) Free float, (iii) Independent float.

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3. What is a project ? Give two examples. List the important four district features that are common to all projects. **[CA (May) 93]**

4. Define a dummy arrow used in a network. State two purposes for which it is used. Mention four conventions that are used in drowings the network. **[C**

Example 2. *A project consists of a series or tasks labelled A, B,..., H, I with the relationships (W < X, Y, means X and Y cannot start until W is completed; X, Y < W means W cannot start until both X and Y are completed). With this notation, construct the network diagram having the constraints :*

$$A < D, E ; B, D < F ; C < G ; B < H ; F, G < I.$$

The activity (Job) dependencies are as below :

(*i*) *A, B, C* are starting activities.
(*ii*) Activities *D, E* and *F* can start when once *A* is completed.
(*iii*) Activity *G* can start after *B* and *D* are completed.
(*iv*) Activity *H* can start after *C* and *E* are completed.
(*v*) Activities *G, F* and *H* are the final activities.
(1) Draw the network and indicate the critical path.
(2) What is the total time required to complete the project ? (based on normal times).
(3) If the project is to be completed in 8 days, what is the minimum cost to be incurred ? Indicate this cheapest cost schedule.

13. Explain network problems. Describe the method of drawing network diagrams.

14. A project has the following details. The indirect cost of the project per week is Rs. 1000/-

Activity	Normal time (weeks)	Crash time (weeks)	Direct cost slop (Rs./week)
1–2	6	4	100
1–3	10	6	300
1–4	15	7	600
2–4	4	3	700
3–5	15	10	500
4–5	15	8	800

(i) Draw the network and find the normal duration of the project with normal total cost.
(ii) Determine the optimal duration and its corresponding cost.
(iii) If all the activities are crashed to their maximum values, determine the duration and total cost of project.

[AIMS (Bangl.) MBA 2002]

25.10 PROJECT EVALUATION AND REVIEW TECHNIQUE (PERT)

In the network analysis discussed so far, it is implicitly assumed that the time values are deterministic, variations in time are insignificant. This assumption is valid in regular jobs such as maintenance of a machine etc., construction of a building or road, planning for production, as these are done from time to time and various activities could be timed very well. However, in research projects or design of a gear box of a new machine, various activities are based on judgement. A reliable time estimate is difficult to get because the technology is changing repidly. Time values are subject to chance variations.

The main objective in the analysis through PERT is to find out the completion for a particular event within specified date. If yes, what are the chances of completing the job ? The PERT approach takes into account the uncertainties. In this approach, three time values are associated with each activity : the *optimistic value*, the *pessimistic value*, and the *most likely value*. These three time values provide a measure of uncertainty associated with that activity.

Def. 1. The ***optimistic time*** *is the shortest possible time in which the activity can be finished. It assumes that every thing goes very well. This is denoted by* t_0.

Def. 2. *The* ***most likely time*** *is the estimate of the normal time the activity would take. This assumes normal delays. If a graph is plotted in the time of completion and the frequencey of completion in that time period, then the most likely time will represent the highest frequency of occurrence. This is denoted by* t_m.

Def. 3. *The* ***pessimistic time*** *represents the longest time the activity could take if everything goes wrong. As in optimistic estimate, this value may be such that only one in hundred or one in twenty will take time longer than this value. This is denoted by* t_p.

These three time values are shown in *Fig*. 25.46.

In order to obtain these values, one could use time values available for similar jobs, but most of the time the estimator may not be so fortunate to have this data. Secondly, values are the functions of manpower, machines and supporting facility. A better approach would be to seek opinion of 'experts in the field' keeping in view the resources available.

This estimate does not take into account such natural catastrophes as fire, etc.

In PERT calculation, all values are used to obtain the per cent expected value.

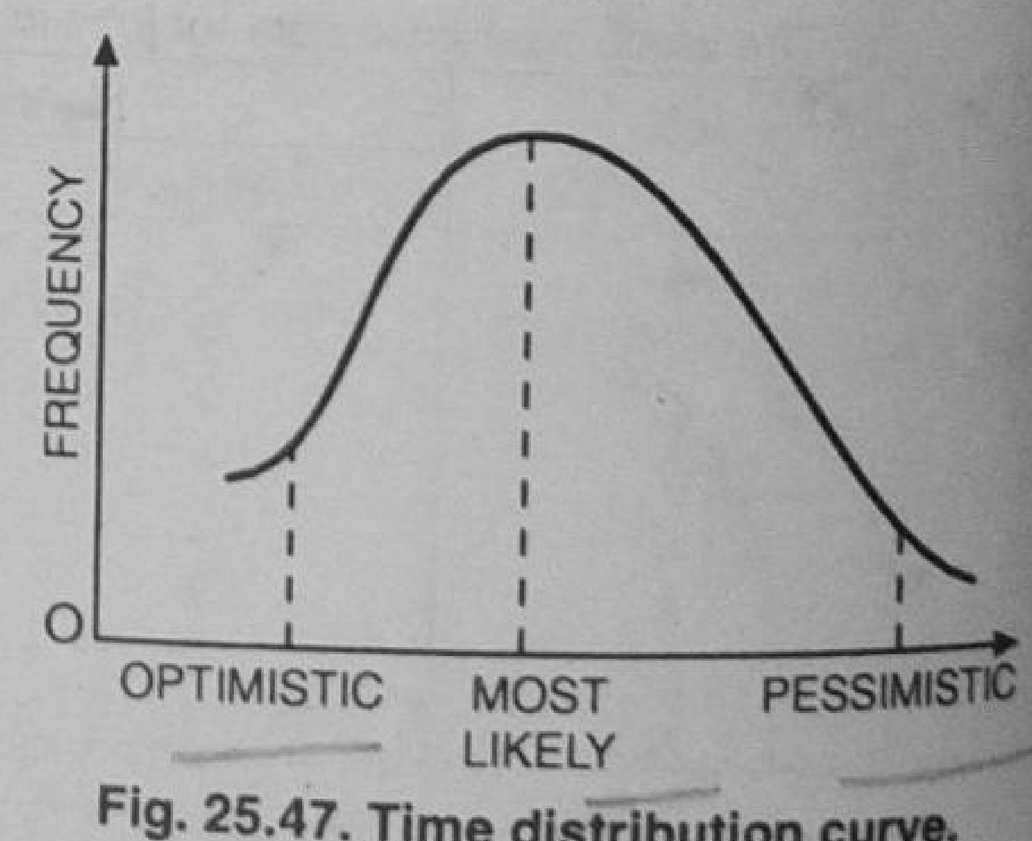

Fig. 25.47. Time distribution curve.

Def. 4. ***Expected time*** *is the average time an activity will take if it were to be repeated on large number of times and is based on the assumption that the activity time follows Beta distribution*. This is given by the formula :*

$$t_e = (t_o + 4t_m + t_p)/6.$$

Def. 5. *The* ***variance*** *for the activity is given by the formula :*

$$\sigma^2 = [(t_p - t_o)/6]^2,$$

where t_o is the *optimistic* time, t_p is the *pessimistic* time, t_m is the *most-likely* time, t_e is the *expected* time and σ^2 is the *variance.*

PERT computations are essentially the same as used earlier.

Q. 1. Explain the following terms in PERT :
(*i*) Optimistic time, (*ii*) Normal time, (*iii*) Pessimistic time, (*iv*) Expected time, (*v*) Variance in relation to activities.

2. What are the requirements for the application of PERT ? Give an algorithm for PERT and state the limitations of this technique. **[Meerut (OR) 2003]**

The main difference is that instead of activity duration, expected time t_e for the activity is considered. With each node, variance is associated. Thus, the duration of the project is the mean expected time with variance.

Consider the network of *Fig* 25.20 again. *Table* 25.6 give three time estimates for each activity, the expected value and the variance also.

Table 25.6

Activity	t_o	t_m	t_p	$t = (t_o + 4t_m + t_p)/6$	$\sigma^2 = [(t_p - t_o)/6]^2$
(1–2)	1.0	2.00	3.0	2	4/36
(1–3)	1.5	2.00	2.5	2	1/36
(1–4)	1.5	2.75	3.5	3	4/36
(2–5)	3.0	3.00	7.0	4	16/36
(3–6)	4.0	4.50	8.0	5	16/36
(3–7)	6.0	8.25	9.0	8	9/36
(4–7)	3.0	3.50	7.0	4	16/36
(5–8)	2.0	2.00	2.0	2	0
(6–8)	2.0	4.00	6.0	4	16/36
(7–9)	2.0	4.50	8.0	5	36/36
(8–9)	2.0	3.00	4.0	3	4/36
(9–10)	2.5	4.25	4.5	4	4/36

Once, expected values have been calculated, these are used in finding the critical path. In this particular example, three estimates are so chosen that mean values are same as before and hence critical path calculations are same as before. However, the interpretation of the critical path is now different. In this case, the expected duration of job taken time less than 9 days or more than 19 days too. Then, meaning of the expected duration is that—if the same job is performed again under similar conditions, the average duration will be 19 days. If the job takes 19 days, then there is probability value associated with it which can be calculated under some assumptions. Since duration of each activity is a random variable, the duration of a path which consists of a set of activities will also be a random variable. To calculate the exact distribution of the duration of a path will be difficult and for management decisions it is enough to know the mean and the variance. Mean value has been calculated using the method discussed earlier. The same approach is used to find the variance.

Rules for finding variance of events.

(*i*) Variance for the initial event is zero. Set $V_1 = 0$.

---

* The Beta distribution was chosen possibily because it is a
(1) Unimodal distribution
(2) Finite non-negative end points
(3) Non-symmetric or symmetric Beta density is given by

$$f(x, \alpha, \beta) = \frac{1}{B(\alpha+1, \beta+1)} x^{\alpha}(1-x)^{\beta}, \begin{cases} 0 < x < 1 \\ \alpha > -1, \beta > -1 \end{cases}$$

If x takes on values between limits *a* and *b*, a new variable $y = (x - a)$[illegible]$- a)$ can be defined and this takes values between zero and one. Beta function is defined by

$$B(\alpha, \beta) = \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)}$$, where $\Gamma\alpha$ is "gamma[illegible]

Example 15. *In Example 14, also find the critical path of the network :*

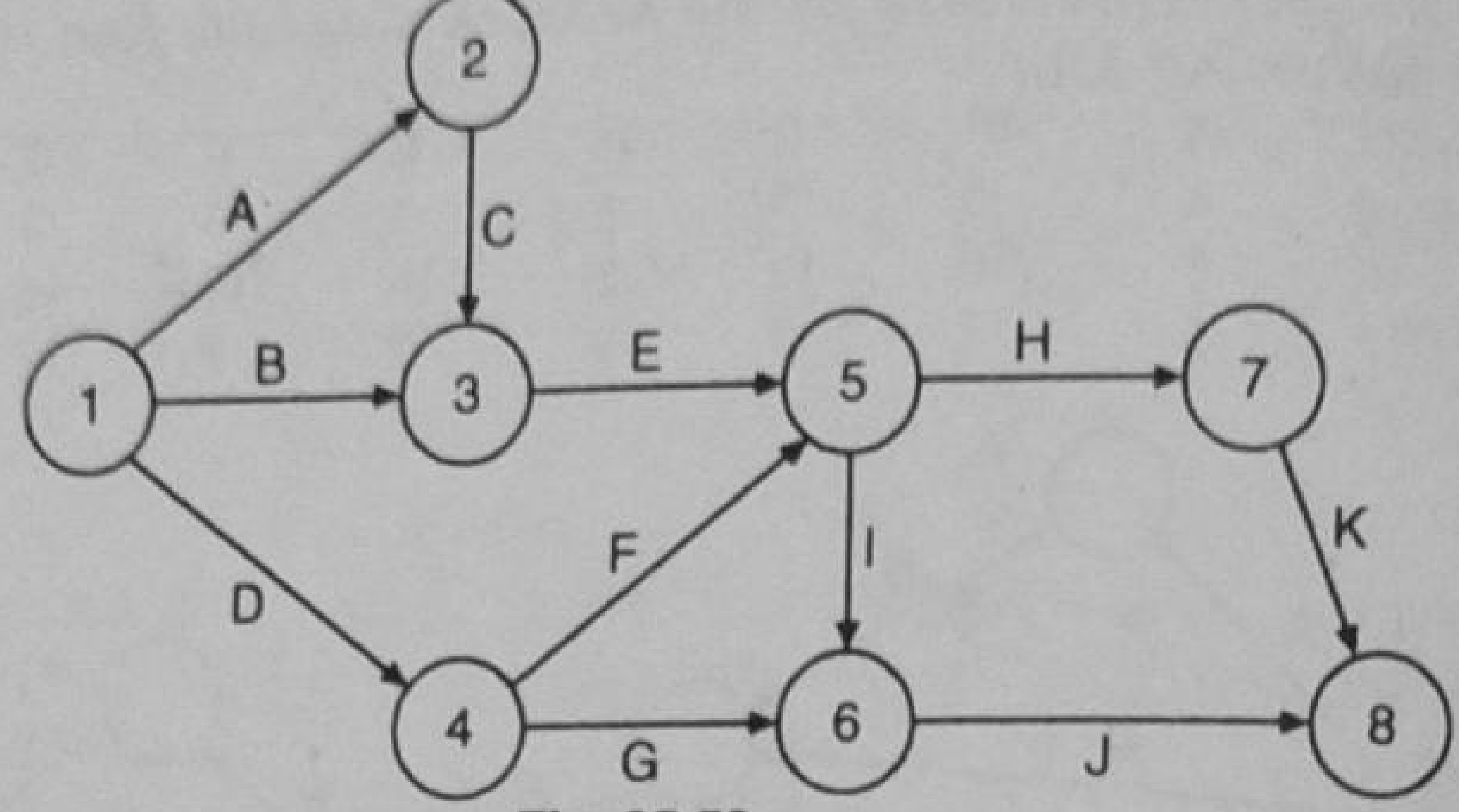

Fig. 25.50

Solution.

Expected Time Computations

Task	Last time a	Greatest time b	Most likely time m	Expected time $(a + b + 4m)/6$
A	4	8	5	5⅓
B	5	10	7	7⅙
C	8	12	11	10⅔
D	2	7	3	3½
E	4	10	7	7
F	6	15	9	9½
G	8	16	12	12
H	5	9	6	6⅓
I	3	7	5	5
J	5	11	8	8
K	6	13	9	9⅙

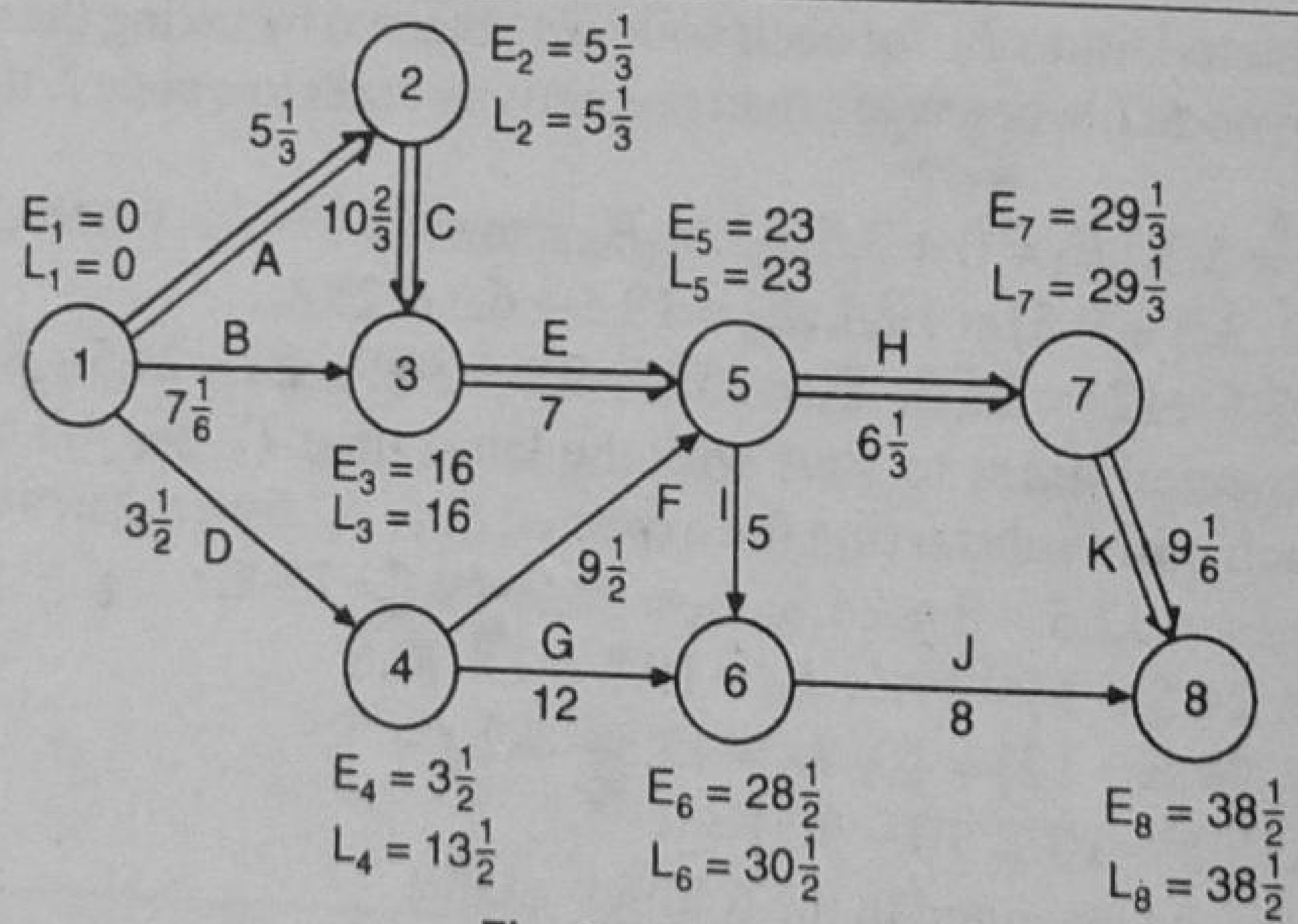

Fig. 25.51

Task	Expected time (t_e)	Start		Finish		Total float
		Earliest	Latest	Earliest	Latest	
A	5⅓	0	0	5⅓	5⅓	0
B	7⅙	0	8⅚	7⅙	16	8⅚
C	10⅔	5⅓	5⅓	16	16	0
D	3½	0	10	3½	13½	10
E	7	16	16	23	23	0
F	9½	3½	13½	13	23	10
G	12	3½	18½	15½	30½	15
H	6⅓	23	23	29⅓	29⅓	0
I	5	23	25½	28	30½	2½
J	8	28	30½	36	38½	2½
K	9⅙	29⅓	29⅓	31½	38½	0

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11

ASSIGNMENT PROBLEMS

11.1. INTRODUCTION

As already discussed earlier, linear programming relates to the problems concerning distributions of various resourses (such as *money, machines, time* etc.), satisfying some constraints which can be algebraically represented as linear *equations/inequalities* so as to *maximize profit* or *minimize cost.* This chapter deals with a very interesting method called the *'Assignment Technique'* which is applicable to a class of very practical problems generally called *'Assignment problems'*.

The name *'Assignment Problem'* originates from the classical problems where the objective is to assign a number of origins (jobs) to the equal number of destinations (persons) at a minimum cost (or maximum profit). To examine the nature of assignment problem, ***suppose there are n jobs to be performed and n persons are available for doing these jobs. Assume that each person can do each job at a time, though with varying degree of efficiency. Let c_{ij} be the cost (payment) if the ith person is assigned the jth job, the problem is to find an assignment (which job should be assigned to which person) so that the total cost for performing all jobs is minimum.*** Problems of this kind are known as *assignment problems.*

Table 11·1

Jobs

		1	2	...	j	...	n
	1	c_{11}	c_{12}	...	c_{1j}	...	c_{1n}
	2	c_{21}	c_{22}	...	c_{2j}	...	c_{2n}
Persons	:	:	:		:		:
	i	c_{i1}	c_{i2}	...	c_{ij}	...	c_{in}
	:	:	:		:		:
	n	c_{n1}	c_{n2}	...	c_{nj}	...	c_{nn}

Further, such types of problems may consist of assigning men to offices, classes to rooms, drivers to trucks, trucks to delivery routes, or problems to research teams, etc. The assignment problem can be stated in the form of $n \times n$ cost-matrix $[c_{ij}]$ of real number as given in Table 11·1 .

Q. 1. Define Assignment Problem.
2. What is an assignment problem ? **[IGNOU 2001, 99, 97, 96]**

11.2. MATHEMATICAL FORMULATION OF ASSIGNMENT PROBLEM

Mathematically, the assignment problem can be stated as :

$$\textit{Minimize the total cost}: z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}, \; i = 1, 2, \ldots, n \;;\; j = 1, 2, \ldots, n \qquad \ldots(11\cdot1)$$

subject to restrictions of the form :

$$x_{ij} = \begin{cases} 1 & \text{if } ith \text{ person is assigned } jth \text{ job} \\ 0 & \text{if } not \end{cases} \qquad \ldots(11\cdot2)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \qquad (\textit{one job is done by the ith person}, \; i = 1, 2, \ldots, n) \qquad \ldots(11\cdot3)$$

Corollary. *If* (x_{ij}), $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$ *is an optimal solution for an assignment problem with cost* (c_{ij}), *then it is also optimal for the problem with cost* (c_{ij}') *when*

$$c'_{ij} = c_{ij} \quad \text{for } i, j = 1, 2, \ldots, n \,;\, j \neq k$$

$$c'_{ik} = c_{ik} - A \text{, where } A \text{ is a constant.}$$

Proof. We have

$$z' = \sum_i \sum_j c'_{ij} x_{ij} = \sum_i \Big(\sum_{j \neq k} c'_{ij} + c'_{ik} \Big) x_{ij} = \sum_i \Big(\sum_{j \neq k} c_{ij} + c_{ik} - A \Big) x_{ij} = \Sigma \Sigma\, c_{ij} x_{ij} - A \sum_i x_{ij}$$

$$= z - A \qquad \Big(\text{since } \sum_i x_{ij} = 1\Big)$$

Thus if (x_{ij}) minimizes z so will it z'.

Theorem 11.2. *In an assignment problem with cost* (c_{ij}), *if all* $c_{ij} \geq 0$ *then a feasible solution* (x_{ij}) *which satisfies* $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} = 0$, *is optimal for the problem.*

Proof. Since all $c_{ij} \geq 0$ and all $x_{ij} \geq 0$, the objective function $z = \Sigma \Sigma\, c_{ij} x_{ij}$ cannot be negative. The minimum possible value that z can attain is 0. Thus, any feasible solution (x_{ij}) that satisfies $\Sigma \Sigma\, c_{ij} x_{ij} = 0$ will be optimal.

Theorem 11.3. ***(König Theorem).*** *Let P be the set of 0 elements of a matrix C. Then the maximum number of 0's that can be selected in P such that no row or column of C contains more than one such 0 is equal to the minimum number of lines covering all the elements of P.*

Proof is beyond the scope of the book.

Corollary. *The maximal subset of P provides an optimal assignment when the minimum number of lines to cover all the elements of P is equal to the order of C.*

Proof. Left as an exercise for the reader.

Q. 1. Explain how an assignment problem can be treated as a linear programming problem. Show that the optimal solution to the assignment problem remains the same if a constant is added to or subtracted from any row or column of the cost matrix.

2. If $b_{ij} = c_{ij} - u_i - v_j$ $(i, j = 1, 1, 2, \ldots, n)$ where u_i and v_j are constants, then show that an optimal solution of the assignment problem with cost matrix $B = (b_{ij})$ is also an optimal solution of the assignment problem with cost matrix $C = (c_{ij})$.

[Delhi B.Sc. (Math.) 90]

11.4. HUNGARIAN METHOD FOR ASSIGNMENT PROBLEM

The solution technique of assignment problems can be easily explained by the following practical examples.

Example 1. *A department head has four subordinates, and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours ?* **[JNTU (B. Tech) 2002, 2000; Tamil. (ERODE) 97; IAS (Main) 93; Kerala B.Sc. (Math.) 91; Meerut (Stat.) 90; Kalicut B. Tech 90]**

Table 11·2

		Subordinates			
		I	II	III	IV
Tasks	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Solution. To understand the problem initially, step by step solution procedure is necessary.

Step 1. Subtracting the smallest element in each row from every element of that row, we get the reduced matrix [*Table* 11·3]

Step 2. Next subtract the smallest element in each column from every element of that column to get the second reduced matrix [*Table* 11·4]

Table 11·3

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Table 11·4

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Step 3. Now, test whether it is possible to make an assignment using only *zeros*. If it is possible, the assignment must be optimal by *Theorem 11.2* of Section 11·3 . Zero assignment is possible in *Table* 11·4 as follows :

(a) Starting with *row* 1 of the matrix (*Table* 11·4), examine the rows one by one until a row containing exactly *single zero* element is found. Then an experimental assignment (indicated by □) is marked to that cell. Now cross all other zeros in the *column* in which the assignment has been made. This eliminates the possibility of marking further assignments in that column. The illustration of this procedure is shown in *Table* 11·5a.

Table 11·5a

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	0	3	0
D	9	12	14	$\boxed{0}$

Table 11·5b

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	$\boxed{0}$	3	0
D	9	12	14	$\boxed{0}$

(b) When the set of rows has been completely examined, an identical procedure is applied sucessively to columns. Starting with *column* 1, examine all columns until a *column* containing exactly one zero is found. Then make an experimental assignment in that position and cross other zeros in the *row* in which the assignment has been made.

Continue these successive operations on rows and columns until all zeros have been either assigned or crossed-out. At this stage, re-examine rows. It is found that no additional assignments are possible. Thus, the complete '*zero assignment*' is given by A → I , B → III, C → II , D → IV as mentioned in *Table 11·5b*. According to *Theorem* 1, this assignment is also optimal for the original matrix (*Table 11·2*). Now compute the minimum total man-hours as follows :

Optimal assignment	:	A—I	B—III	C—II	D—IV	
Man-hour	:	8	4	19	10	(Total 41 hours.)

Now the question arises : *what would be further steps if the complete optimal assignment after applying Step 3 is not obtained ?* Such difficulty will arise whenever all zeros of any row or column are crossed-out. Following example will make the procedure clear.

Example 2. *A car hire company has one car at each of five depots a ,b , c, d and e. A customer requires a car in each town, namely A, B, C, D, and E. Distance (in kms) between depots (origins) and towns (destinations) are given in the following distance matrix :*

Table 11·6

	a	*b*	*c*	*d*	*e*
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

How should cars be assigned to customers so as to minimize the distance travelled ?

Solution. Applying ***Step 1*** and ***Step 2*** as explained in *Example 1* we get the *Table* 11·7 .

Table 11·7

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15