Q. State clearly the basic assumptions that are made in LPP.

3.10. LIMITATIONS OF LINEAR PROGRAMMING

Inspite of wide area of applications, some limitations are associated with linear programming techniques. These are stated below :

1. In some problems objective functions and constraints are not linear. Generally, in real life situations concerning business and industrial problems constraints are not linearly treated to variables.

2. There is no guarantee of getting integer valued solutions, for example, in finding out how many men and machines would be required to perform a particular job, rounding off the solution to the nearest integer will not give an optimal solution. Integer programming deals with such problems.

3. Linear programming model does not take into consideration the effect of time and uncertainty. Thus the model should be defined in such a way that any change due to internal as well as external factors can be incorporated.

4. Sometimes large-scale problems cannot be solved with linear programming techniques even when the computer facility is available. Such difficulty may be removed by decomposing the main problem into several small problems and then solving them separately.

5. Parameters appearing in the model are assumed to be constant. But, in real life situations they are neither constant nor deterministic.

6. Linear programming deals with only single objective, whereas in real life situations problems come across with multiobjectives. *Goal programming* and *multi-objective programming* deal with such problems.

Q. What are the limitations of linear programming technique ?

3.11. APPLICATIONS OF LINEAR PROGRAMMING

In this section, we discuss some important applications of linear programming in our life.

1. Personnel Assignment Problem. Suppose we are given m persons, n-jobs, and the expected productivity c_{ij} of ith person on the jth job. We want to find an assignment of persons $x_{ij} \geq 0$ for all i and j, to n jobs so that the average productivity of person assigned is maximum, subject to the conditions :

$$\sum_{j=1}^{n} x_{ij} \leq a_i \text{ and } \sum_{i=1}^{m} x_{ij} \leq b_j,$$

where a_i is the number of persons in personnel category i and b_j is the number of jobs in personnel category j. For details, refer the chaper of *Assignment Problems.*

2. Transportation Problem. We suppose that m factories (called sources) supply n warehouses (called destinations) with a certain product. Factory F_i ($i = 1\ 2, \ldots, m$) produces a_i units (total or per unit time), and warehouse W_j ($j = 1, 2, 3 \ldots, n$) requires b_j units. Suppose that the cost of shipping from factory F_i to warehouse W_j is directly proportional to the amount shipped; and that the unit cost is c_{ij}. Let the decision variables, x_{ij}, be the amount shipped from factory F_i to warehouse W_j. The objective is to determine the number of units transported from factory F_i to warehouse W_j so that the total transportation cost $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$ is minimized. In the mean time, the supply and demand must be satisfied exactly.

Mathematically, this problem is to find x_{ij} ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$) in order to minimize the total transportation cost $z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}(c_{ij})$, subject to the restrictions of the form

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1, 2, \ldots, m \text{ (factory)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1, 2, \ldots, n \text{ (warehouse)}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j, \text{ and } x_{ij} \geq 0, (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n).$$

For detailed discussion, refer **chapter 10** on ***Transportation Problem.***

3. Efficiencing on Operation of System of Dams. In this problem, we determine variations in water storage of dams which generate power so as to maximize the energy obtained from the entire system. The physical limitations of storage appear as inequalities.

4. Optimum Estimation of Executive Compensation. The objective here is to determine a consistent plan of executive compensation in an industrial concern. Salary, job ranking and the amounts of each factor required on the ranked job level are taken into consideration by the constraints of linear programming.

5. Agricultural Applications. Linear programming can be applied in agricultural planning for allocating the limited resources such as acreage, labour, water, supply and working capital, etc. so as to maximize the net revenue.

6. Military Applications. These applications involve the problem of selecting an air weapon system against gurillas so as to keep them pinned down and simultaneously minimize the amount of aviation gasoline used, a variation of transportation problem that maximizes the total tonnage of bomb dropped on a set of targets, and the problem of community defence against disaster to find the number of defence units that should be used in the attack in order to provide the required level of protection at the lowest possible cost.

7. Production Management. Linear programming can be applied in production management for determining product mix, product smoothing, and assembly time-balancing.

8. Marketing Management. Linear programming helps in analysing the effectiveness of advertising compaign and time based on the available advertising media. It also helps travelling sales-man in finding the shortest route for his tour.

9. Manpower Management. Linear programming allows the personnel manager to analyse personnel policy combinations in terms of their appropriateness for maintaining a steady-state flow of people into through and out of the firm.

10. Physical Distribution. Linear programming determines the most economic and efficient manner of locating manufacturing plants and distribution centres for physical distribution.

Besids above, linear programming involves the applications in the area of administration, education, inventory control, fleet utilization, awarding contract, and capital budgeting etc.

Q. 1. Give a brief account of applications of linear programming problem.

2. Explain the meaning of a Linear Programming Problem stating its uses and give its limitations. **[C.A. (May) 95]**

3. State in brief uses of linear programming Technique.

3.12. ADVANTAGES OF LINEAR PROGARMMING TECHNIQUES

The advantages of linear programming techniques may be out-lined as follows :

1. Linear programming technique helps us in making the optimum utilization of productive resources. It also indicates how a decision maker can employ his productive factors most effectively by choosing and allocating these resources.
2. The quality of decisions may also be improved by linear programming techniques. The user of this technique becomes more objective and less subjective.
3. Linear programming technique provides practically applicable solutions since there might be other constraints operating outside the problem which must also be taken into consideration just because, so many units must be produced does not mean that all those can be sold. So the necessary modification of its mathematical solution is required for the sake of convenience to the decision maker.
4. In production processes, high lighting of bottlenecks is the most significant advantage of this technique. For example, when bottlenecks occur, some machines cannot meet the demand while others remain idle for some tir

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Q. What are the advantages of Linear Programming Technique ?

12

TRANSPORTATION PROBLEMS

12.1. INTRODUCTION

As already defined and discussed earlier, the simplex procedure can be regarded as the most generalized method for linear programming problems. However, there is very interesting class of *'Allocation Methods'* which is applied to a lot of very practical problems generally called *'Transportation Problems'*. Whenever it is possible to place the given linear programming problem in the transportation frame-work, it is far more simple to solve it by *'Transportation Technique'* than by *'Simplex'*.

Let the nature of transportation problem be examined first. If there are more than one centres, called *'origins'*, from where the goods need to be shipped to more than one places called *'destinations'* and the costs of shipping from each of the *origins* to each of the *destinations* being different and known, the problem is to ship the goods from various *origins* to different *destinations* in such a manner that the cost of shipping or transportation is minimum.

Thus, we can formally define the transportation problem as follows :

Definition. ***The Transportation Problem*** *is to transport various amounts of a single homogeneous commodity, that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum.*

For example, a tyre manufacturing concern has m factories located in m different cities. The total supply potential of manufactured product is absorbed by n retail dealers in n different cities of the country. Then, transportation problem is to determine the transportation schedule that minimizes the total cost of transporting tyres from various factory locations to various retail dealers.

The various features of linear programming can be observed in these problems. Here the availability as well as the requirements of the various centres are finite and constitute the limited resources. It is also assumed that the cost of shipping is linear (for example, the costs of shipping of *two* objects will be *twice* that of shipping a *single* object). However, this condition is not often true in practical problems, but will have to be assumed so that the linear programming technique may be applicable to such problems. These problems thus could also be solved by 'Simplex'. Mathematically, the problem may be stated as given in the following section.

12.2. MATHEMATICAL FORMULATION

Let there be m origins, ith origin possessing a_i units of a certain product, whereas there are n destinations (n may or may not be equal to m) with destination j requiring b_j units. Costs of shipping of an item from each of m origins (sources) to each of the n destinations are known either directly or indirectly in terms of mileage, shipping hours, etc. Let c_{ij} be the cost of shipping one unit product from ith origin (source) to jth destination., and 'x_{ij}' be the amount to be shipped from ith origin to jth destination.

It is also assumed that total availabilities Σa_i satisfy the total requirements Σb_j , *i.e.*,

$$\Sigma a_i = \Sigma b_j \quad (i = 1, 2, \ldots, m\,;\, j = 1, 2, \ldots, n) \qquad \ldots(12.1)$$

(***In case,*** $\Sigma a_i \neq \Sigma b_j$ some manipulation is required to make $\Sigma a_i = \Sigma b_j$, which will be shown later).

The problem now is to determine non-negative (≥ 0) values of 'x_{ij}' satisfying both, the availability constraints :

Suppose that this vector is $\mathbf{a}_{kj} = \mathbf{e}_k + \mathbf{e}_{m+j}$. By similar reasoning, we conclude that there must be at least one more vector in $\mathbf{X}$ with the first subscript k; say, $\mathbf{a}_{kl} = \mathbf{e}_k + \mathbf{e}_{m+l}$. By same argument once again, $\mathbf{X}$ must contain at least one vector with the second subscript l. Let it be, say, $\mathbf{a}_{il} = \mathbf{e}_i + \mathbf{e}_{m+l}$.

Thus we have determined four vectors in $\mathbf{X}$, namely $\mathbf{a}_{ij}$, $\mathbf{a}_{kj}$, $\mathbf{a}_{kl}$ and $\mathbf{a}_{il}$ whose corresponding cells, form a loop. Thus the proof is complete.

If the last vector is $\mathbf{a}_{nl} = \mathbf{e}_n + \mathbf{e}_{m+l}$ instead of $\mathbf{a}_{il}$, then as explained just before there must exist at least one more vector with first subscript n. If it is $\mathbf{a}_{nj}$, a loop is complete, if not, let it be $\mathbf{a}_{n0} = \mathbf{e}_n + \mathbf{e}_{m+0}$. $\mathbf{X}$ must contain at least one more vector with second subscript 0. Now two cases will arise :

(1) The first subscript of newly discovered vector is one that has already been identified. In this case a loop has been completed.

(2) The first subscript of the newly discovered vector is also new. In this case, since the number of vectors in $\mathbf{X}$ is finite (by extending the above reasoning), we conclude that eventually a loop must be formed.

Corollary. *A feasible solution to a transportation problem is basic if, and only if, the corresponding cells in the transportation table do not contain a loop.*

Proof : Left as an exercise.

This corollary provides us a method to verify whether the current feasible solution to the transportation problem is basic or not.

Q. 1. A feasible solution to a transportation problem is basic, if and only if, the corresponding cells in the transportation table do not contain.................

2. With reference to a transportation problem define the following terms :
(i) Feasible solution (ii) Basic feasible solution, (iii) Optimal solution, (iv) Non-degenerate basic feasible solution.

3. Define 'loop' in a transportation table. What role do they play ? **[Madurai B.Sc (Math.) 94]**

4. In the classical transportation problem explain as to how many independent equations are there when there are m-origins and n-destinations. What happens and how to handle the solution, when the initial assignment in the problem gives less than this number of occupied cells ?

Initial Basic Feasible Solution

12.8. THE INITIAL BASIC FEASIBLE SOLUTION TO TRANSPORTATION PROBLEM

Methods of finding an *optimal solution* of the transportation problem will consist of two main steps :

(i) ***To find an initial basic feasible solution :***

(ii) ***To obtain an optimal solution by making successive improvements to initial basic feasible solution until no further decrease in the transportation cost is possible.***

There will be fewer improvements to make if initially we start with a better initial basic feasible solution. So first we shall discuss below the methods for obtaining initial basic feasible solution of a T.P.

Remark : Although the transportation problem can be solved using the regular simplex method, its special properties provide a more convenient method for solving this type of problems. This method is based on the same theory of simplex method. It makes use, however, of some shortcuts which provide a less burdensome computational scheme.

12·8·1 Methods for Initial Basic Feasible Solution

Some simple methods are described here to obtain the initial basic feasible solution of the transportation problem. These methods can be easily explained by considering the following numerical example. However, the relative efficiency of these methods is still unanswerable.

Example 1. *Find the initial basic feasible solution of the following transportation problem.*

Table 12.3

Warehouse → Factory ↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Solution.

First Method : ***North-West Corner Rule (Steping Stone Method).*** **[IAS (Maths. 96 Type]**

In this method, first construct an empty 3×4 matrix complete with row and column requirements (*Table 12·4*).

Table 12·4

	W_1	W_2	W_3	W_4	Available
F_1					7
F_2					9
F_3					18
Requirements→	5	8	7	14	

Insert a set of allocations in the cells in such a way that the total in each row and each column is the same as shown against the respective rows and columns. Start with cell (1, 1) at the north-west corner (upper left-hand corner) and allocate as much as possible there. In other words, $x_{11} = 5$, the maximum which can be alloocated to this cell as the total requirement of this column is 5. This allocation ($x_{11} = 5$) leaves the surplus amount of 2 units for row 1 (*Factory* F_1), so allocate $x_{12} = 2$ to cell (1, 2). Now, allocations for first row and first column are complete, but there is a deficiency of 6 units in column 2. Therefore, allocate $x_{22} = 6$ in the cell (2, 2). Column 1 and column 2 requirements are satisfied, leaving a surplus amount of 3 units for row 2. So allocate $x_{23} = 3$ in the cell (2, 3), and column 3 still requires 4 units. Therefore, continuing in this way, from left to right and top to bottom, eventually complete all requirements by an allocation $x_{34} = 14$ in the *south-east* corner. Table 12·5 shows the resulting feasible solution.

Table 12·5

5 (19)	2 (30)			7
	6 (30)	3 (40)		9
		4 (70)	14 (20)	18
5	8	7	14	

On multiplying each individual allocation by its corresponding unit cost in '( )' and adding, the total cost becomes = 5 (19) + 2 (30) + 6 (30) + 3 (40) + 4 (70) + 14 (20) = Rs. 1015 .

Q. Explain the application of North-West Corner Rule with an example.

Second Method : ***The Row Minima Method.***

Step 1. The transportation table of the given problem has 12 cells. Following the ***row minima method,*** since min (19, 30, 50, 10) = 10 , the first allocation is made in the cell (1, 4), the amount of the allocation is given by $x_{14} = \min(7, 14) = 7$. This exhausts the availability from factory F_1 and thus we cross-out the first row from the transportation table (Table 12.6).

Table 12·6

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	x
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(80)	(70)	(20)	18
	5	8	7	7	

Table 12·7

	W_1	W_2	W_3	W_4	
F_1				7	x
F_2	(70)	8 (30)	(40)	(60)	9
F_3	(40)		(70)	(20)	18
	5	8	7	7	

Step 2. In the resulting transportation table (*Table* 12·7) , since min (70, 30, 40, 60) = 30 , the second allocation is made in the cell (2, 2), the amount of allocation being $x_{22} = \min(9, 8) = 8$. This satisfies the requirement of warehouse W_2 and thus we cross-out the second column from the transportation table obtaining new *Table* 12·8 .

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Step 3. In *Table* 12·8 , since min (70, 40, 60) = 40 ,the third allocation is made in the cell (2, 3), the amount being $x_{23} = \min [1, 7] = 1$. This exhausts the availability from factory F_2 ,

Table 12·8

	W_1	W_2	W_3	W_4	
F_1				7 •	x
F_2	(70)	8 • (30)	1 • (40)	(60)	x
F_3	(40)		(70)	(20)	18
	5	x	6	7	

Table 12·9

	W_1	W_2	W_3	W_4	
F_1				7 •	x
F_2	(70)	8 • (30)	1 • (40)	(60)	x
F_3	(40)		(70)	(20)	18
	5	x	6	7	

and thus we cross-out the second row from the table 12.8 getting the *Table 12·9* ,

Step 4. The next allocation is made in the cell (3,4), since min (40, 70, 20) = 20 , the amount of allocation being $x_{34} = \min (7, 18) = 7$. This exhausts the requirement of warehouse W_4 and thus we cross-out the fourth column from the ***Table 12·9*** .

Step 5. The next allocation is made in the cell (3, 1), since min (40, 70) = 40 , the amount of allocation being $x_{31} = \min (5, 11) = 5$ ..This satisfies the requirement of warehouse W_1 and so we cross-out the first column W_1 to get new ***Table 12.11.***

Table 12·10

	W_1	W_2	W_3	W_4	
F_1				7 •	x
F_2		8 •	1 •		x
F_3	(40)		(70)	7 •	11
	5	x	6	x	

Table 12·11

	W_1	W_2	W_3	W_4	
F_1				7 •	x
F_2		8 •	1 •		x
F_3	5 •		6 • 70	7 •	x
	x	x	x	x	

Step 6. The last allocation of amount $x_{33} = 6$ is obviously made in the cell (3, 3). This exhausts the availability from factory F_3 and requirement of warehouse W_3 simultaneously. So we cross-out third row and third column to get the final solution ***Table 12·12*** .

Since the basic cells indicated by (•) do not form a loop, an initial basic feasible solution has been obtained. The solution is displayed in ***Table 12·12*** .

Table 12·12

	W_1	W_2	W_3	W_4	
F_1				7 •	×
F_2		8 •	1 •		×
F_3	5 •		6 •	7 •	×
	×	×	×	×	

The transportation cost is given by

$z = 7 \times 10 + 8 \times 30 + 1 \times 40 + 5 \times 40 + 6 \times 70 + 7 \times 20$
$= \text{Rs. } 1110$.

Third Method : ***The Column Minima Method.***

This method is similar to **row-minima method** except that we apply the concept of minimum cost on columns instead of rows. So, the reader can easily solve the above problem by column minima method also.

Fourth Method : ***Lowest Cost Entry Method*** **(Matrix Minima Method).**

The initial basic feasible solution obtained by this method usually gives a lower beginning cost. In this method, first write the cost and requirements matrix (***Table 12·13***).

Start with the lowest cost entry (8) in the cell (3, 2) and allocate as much as possible, *i.e.*, $x_{32} = 8$. The next lowest cost (10) lies in the cell (1, 4), so allocate $x_{14} = 7$. The next lowest cost (19) lies in the cell (1, 1), so make no allocation, because the amount available from factory F_1 was already used in the cell (1, 4). Next lowest cost entry is (20) in the cell (3, 4) where at the most it is possible to allocate $x_{34} = 7$ in order to complete the requirements of 7 units in column 4.

Further, next lowest cost is (30) in cells (2, 2) and (1, 2) so no allocation is possible, because the requirement of column 2 has already been exhausted. This way, required feasible solution is obtained (*Table 12·13*).

Table 12·13

				Available
• (19)	• (30)	• (50)	7(10)	7
2(70)	• (30)	7(40)	• (60)	9
3(40)	8(8)	• (70)	7(20)	18
Requirements 5	8	7	14	

This feasible solution results in lower transportation cost, *i.e.*,
$2(70) + 3(40) + 8(8) + \overset{7}{8}(40) + 7(10) + 7(20) =$ Rs. 814.
This cost is less by Rs. 201, *i.e.*, Rs. (1015–814) as compared to the cost obtained by *north-west corner rule*.

Q. Explain the application of Matrix-Minimum method with an example.

Fifth Method. ***Vogel's Approximation Method (Unit Cost Penalty Method).*** **[Banasthali (M.Sc.) 93]**

Table 12·14

	W_1	W_2	W_3	W_4	Available
F_1	(19)	(30)	(50)	(10)	7
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(8)	(70)	(20)	18
Requirement	5	8	7	14	

Step 1. In lowest cost entry method, it is not possible to make an allocation to the cell (1, 1) which has the second lowest cost in the matrix. It is trivial that allocation should be made in at least one cell of each row and each column.

Step 2. Next enter the *difference between the lowest and second lowest cost* entries in each column beneath the corresponding column, and put the defference between the lowest and second lowest cost entries of each row to the right of that row. Such individual differences can be thought of a **penalty** for making allocations in second lowest cost entries instead of lowest cost entries in each row or column. For example, allocate one unit in the second lowest cost cell (3, 1) instead of cell (1, 1) with lowest unit cost (19). There will be a loss (penalty) of Rs 21 per unit. In case, the lowest and second lowest costs in a row/ column are equal, the penalty will be taken zero.

Table 12·15

	W_1	W_2	W_3	W_4	Available	Penalties	
F_1	•(19)	•(30)	•(50)	•(10)	7	(9)	β9 − 10 = 9
F_2	•(70)	•(30)	•(40)	•(60)	9	(10)	
F_3	•(40)	8 (8)	•(70)	•(20)	18/10 20 − 8	(12)	+ 8
Requirements :	5	8/0	7	14			
Penalties :	(21)	(22) ↑	(10)	(10)			

Step 3. Select the row or column for which the ***penalty*** is the largest, *i.e.*, (22) (***Table* 12·15**), and allocate the maximum possible amount to the cell (3, 2) with the lowest cost (8) in the particular column (row) making $x_{32} = 8$. If there are more than one largest penalty rows (columns), select one of them arbitrarily.

Step 4. Cross-out that column (row) in which the requirement has been satisfied. In this example, second column has been crossed-out. Then find the corresponding penalties correcting the amount available from factory F_3. Construct the first reduced penalty matrix ***Table* 12·16.**

Table 12·16

	W_1	W_3	W_4	Available	Penalties
F_1	5(19)	•(50)	•(10)	7	(9)
F_2	•(70)	•(40)	•(60)	9	(20)
F_3	•(40)	•(70)	•(20)	10 (Note)	(20)
Requirements	5/0	7	14		
Penalties	(21) ↑	(10)	(10)		

Step 5. Repeat steps 3 and 4 till all allocations have been made. Successive reduced penalty matrices are obtained. Since the largest penalty (21) is now associated with the cell (1, 1), so allocate $x_{11} = 5$. This allocation ($x_{11} = 5$) eliminates the column 1 giving the second reduced matrix (*Table* 12·17).

Table 12·17

	W_3	W_4	Available	Penalty
F_1	•(50)	•(10)	2 (Note)	(40)
F_2	•(40)	•(60)	9	(20)
F_3	•(70)	10(20)	10/0	(50) ←
Requirements :	7	14/4		
Penalties :	(10)	(10)		

The largest penalty (50) is now associated with the cell (3, 4) therefore allocate $x_{34} = 10$. Eliminating the row 3, the third reduced penalty matrix ***Table* 12·18** is obtained.

Table 12·18

	W_3	W_4	Available	Penalties
F_1	•(50)	2(10)	2/0	(40)
F_2	7(40)	2(60)	9/0	(20)
Requirements :	7	4/0 (Note)		
Penalties :	(10)	(50) ↑		

Now, allocate according to the largest penalty (50) as $x_{14} = 2$ and remaining $x_{24} = 2$. Then allocate $x_{23} = 7$.

Step 6. Finally, construct ***Table* 12·19** for the required feasible solution.

The total cost is :
$5(19) + 8(8) + 2(10) + 2(60) + 10(20) + 7(40) = \text{Rs. } 779.$
This cost is Rs. 35 less as compared to the cost obtained by *Lowest Cost Entry Method.*

Table 12·19

	W_1	W_2	W_3	W_4	Available
F_1	5(19)			2(10)	7
F_2			7(40)	2(60)	9
F_3		8(8)		10(20)	18
Requirements :	5	8	7	14	

In order to reduce large number of steps required to obtain the optimal solution, it is advisable to proceed with the initial feasible solution which is close to the optimal solution. Vogel's method often gives the better initial feasible solution to start with. Although Vogel's method takes more time as compared to other two methods, but it reduces the time in reaching the optimal solution.

Short-cut. After a little practice, students may prefer to perform the entire procedure of Vogel's method within the original cost requirement *Table 12·14*. It needs merely to cross-out rows and columns as and when they are completed and to revise requirements, available supplies and penalties as shown below.

Step 3. Select the row or column for which the ***penalty*** is the largest, *i.e.*, (22) (***Table* 12·15**), and allocate the maximum possible amount to the cell (3, 2) with the lowest cost (8) in the particular column (row) making $x_{32} = 8$. If there are more than one largest penalty rows (columns), select one of them arbitrarily.

Step 4. Cross-out that column (row) in which the requirement has been satisfied. In this example, second column has been crossed-out. Then find the corresponding penalties correcting the amount available from factory F_3. Construct the first reduced penalty matrix ***Table* 12·16.**

Table 12·16

	W_1	W_3	W_4	Available	Penalties
F_1	5(19)	•(50)	•(10)	7	(9)
F_2	•(70)	•(40)	•(60)	9	(20)
F_3	•(40)	•(70)	•(20)	10 (Note)	(20)
Requirements	5/0	7	14		
Penalties	(21) ↑	(10)	(10)		

Step 5. Repeat steps 3 and 4 till all allocations have been made. Successive reduced penalty matrices are obtained. Since the largest penalty (21) is now associated with the cell (1, 1), so allocate $x_{11} = 5$. This allocation ($x_{11} = 5$) eliminates the column 1 giving the second reduced matrix (*Table* 12·17).

Table 12·17

	W_3	W_4	Available	Penalty
F_1	•(50)	•(10)	2 (Note)	(40)
F_2	•(40)	•(60)	9	(20)
F_3	/ •(70)	10(20)	10/0	(50) ←
Requirements :	7	14/4		
Penalties :	(10)	(10)		

The largest penalty (50) is now associated with the cell (3, 4) therefore allocate $x_{34} = 10$. Eliminating the row 3, the third reduced penalty matrix ***Table* 12·18** is obtained.

Table 12·18

	W_3	W_4	Available	Penalties
F_1	•(50)	2(10)	2/0	(40)
F_2	7(40)	2(60)	9/0	(20)
Requirements :	7	4/0 (Note)		
Penalties :	(10)	(50) ↑		

Now, allocate according to the largest penalty (50) as $x_{14} = 2$ and remaining $x_{24} = 2$. Then allocate $x_{23} = 7$.

Step 6. Finally, construct ***Table* 12·19** for the required feasible solution.

Table 12·19

	W_1	W_2	W_3	W_4	Available
F_1	5(19)			2(10)	7
F_2			7(40)	2(60)	9
F_3		8(8)		10(20)	18
Requirements :	5	8	7	14	

The total cost is :
5(19) + 8(8) + 2(10) + 2(60) + 10(20) + 7(40) = Rs. 779.
This cost is Rs. 35 less as compared to the cost obtained by *Lowest Cost Entry Method*.

In order to reduce large number of steps required to obtain the optimal solution, it is advisable to proceed with the initial feasible solution which is close to the optimal solution. Vogel's method often gives the better initial feasible solution to start with. Although Vogel's method takes more time as compared to other two methods, but it reduces the time in reaching the optimal solution.

Short-cut. After a little practice, students may prefer to perform the entire procedure of Vogel's method within the original cost requirement *Table 12·14*. It needs merely to cross-out rows and columns as and when they are completed and to revise requirements, available supplies and penalties as shown below.

Step 5. Apply optimality test by examining the sign of each d_{ij} :

(*i*) If all $d_{ij} \geq 0$, the current basic feasible solution is an optimum one.

(*ii*) If at least one $d_{ij} < 0$ (negative), select the variable x_{rs} (having the most negative d_{rs}) to enter the basis.

Step 6. Let the variable x_{rs} enter the basis. Allocate an unknown quantity say θ , to the cell (r , s) . Then construct a loop that starts and ends at the cell (r , s) and connects some of the basic cells. The amount θ is added to and subtracted from the transition cells of the loop in such a manner that the availabilities and requirements remain satisfied.

Step 7. Assign the largest possible value to θ in such a way that the value of at least one basic variable becomes zero and other basic variables remain non-negative (≥ 0) . The basic cell whose allocation has been made zero will leave the basis.

Step 8. Now, return to ***step 3*** and then repeat the process until an optimum basic feasible solution is obtained.

The above iterative procedure determines an optimum solution in a finite number of steps. This method is called *MODI METHOD*. and can be easily remembered with the help of the following FLOW-CHART.

Q. 1. Give an algorithm for solving transportation problem.

2. State the transportation problem. Describe clearly the steps involved in solving the problem.
3. Describe the transportation problem. Give a method of finding an initial feasible solution. Explain what is meant by an optimality test ? Give the method of improving over the initial solution to reach the optimal feasible solution. **[Meerut 94]**
4. Assume that in a transportation problem the demand and supply levels are all positive and integral. Show that there exists an integral optimal solution if the total demand equals total supply levels.
5. Describe the computational procedure of optimality test in a transportation problem.
6. Explain briefly the step-wise description of the computational procedure for solving the transportation problem. **[Delhi B.Sc. (Math.) 91]**
7. Develop methematical model of a balanced transportation problem. Prove that it always has a feasible solution. **[IAS (Maths.) 99]**
8. How do you diagnose that the given transportation problem is having more than one optimal alternate optimal solution. **[AIMS (BE Ind.) Bang. 2002]**

12·10-1. Computational Demonstration of Optimality Test

Example 3. (a) *Obtain an initial basic feasible solution to the transportation problem of* **Example 1.** *Is this solution an optimal solution ? If not, obtain the optimal solution.*

[IGNOU 2001; JNTU (Mach.) 99; Gauhati (MCA) 91]

(b) *If a company is spending Rs. 1000 on transportation of its units to four warehouses from three factories. What can be the maximum saving by optimal scheduling.*

Solution. (a) Computational demonstration for optimality is performed by taking the initial basic feasible solution of ***Example 1*** with $m + n - 1$ allocations in independent positions with transportation cost of Rs 779 obtained (by *Vogel's Method*). This initial basic feasible solution is given in *Table 12·25*.

Table 12·25

	W_1	W_2	W_3	W_4	Available
F_1	5(19)			2(10)	7
F_2			7(40)	2(60)	9
F_3		8(8)		10(20)	18
Required	5	8	7	14	

Step 1. The initial BFS has $m + n - 1$ allocations, that is, $3 + 4 - 1 = 6$ allocations in independent positions. Therefore, condition (1) of optimality test [in sec. *12·9–3*] is satisfied.

Step 2. Since u_i $(i = 1, 2, 3)$ and v_j $(j = 1, 2, 3, 4)$ are to be determined by means of unit cost in the respective occupied cells only, assign a u-value of any particular amount (conveniently *zero*) to any particular row (convenient rule is to select the u_i which has the largest number of allocations in its row). Since all rows contain the same number of allocations, take any of the u_i (say u_3) equal to zero.

$-10+8=-2$

When $u_3 = 0$, $v_4 = 20$ (since $c_{34} = u_3 + v_4$; $c_{34} = 20$). Similarly, $c_{32} = u_3 + v_2$ or $8 = 0 + v_2$ or $v_2 = 8$. Again, $c_{14} = u_1 + v_4$ or $10 = u_1 + 20$ (since $c_{14} = 10$), then $u_1 = -10$. In the same way $60 = 20 + u_2$, which gives $u_2 = 40$; $19 = u_1 + v_1$ or $19 = -10 + v_1$, which gives $v_1 = 29$; $40 = u_2 + v_3$ or $40 = 40 + v_3$, which gives $v_3 = 0$. This completes the set of u_i ($i = 1, 2, 3$) and v_j ($j = 1, 2, 3, 4$) as shown in *Table 12·26*.

Table 12·26

				u_i
• (19)			• (10)	−10
		• (40)	• (60)	40
	• (8)		• (20)	0
v_j 29	8	0	20	

Step 3. To compute the matrix of cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for *empty cells*, it is convenient to write a matrix $[c_{ij}]$ for *empty cells* and the matrix of numbers $[u_i + v_j]$ for *empty cells* only, then subtract the later matrix from the former one.

remaining elements of ... table

Table 12·27 (from Table 12·3)
Matrix $[c_{ij}]$ for empty cells

•	(30)	(50)	•
(70)	(30)	•	•
(40)	•	(70)	•

Table 12·28
Matrix $[u_i + v_j]$ for empty cells

•	−2	−10	•
69	48	•	•
29	•	0	•

Now, subtracting the matrix $[u_i + v_j]$ from the matrix $[c_{ij}]$, *i.e.*, (*Table 12·27 – Table 12·28*), the following matrix $[c_{ij} - (u_i + v_j)]$ of cell evaluations is obtained.

Table 12·29 gives the empty cell evaluations : $d_{12} = 32$, $d_{13} = 60$, $d_{21} = 1$, $d_{22} = -18$, $d_{31} = 11$ and $d_{33} = 70$. The largest negative cell evalua- tion (marked √) is $d_{22} = -18$ which indicates that allocation of one unit to this empty cell (2, 2) will reduce the achieved cost of Rs 779 by Rs. 18. So allocate (say, θ) to cell (2, 2) as much as possible, followed by alternately subtracting and adding the amount of this allocation to other corners of the loop in order to restore feasibility (non-negativity of allocations). For this purpose, the initial basic feasible solution can be read from *Table 12·30*. It is easily seen by the following rule that at the most θ = 2 units can be allocated from cell (2, 4) to cell (2, 2) still satisfying the row and column total and non-negativity restrictions on the allocations.

Table 12·29 d_{ij}

•	32	60	•
1	−18	•	•
11	•	70	•

It is not optimal soln because uncovered element is negative then again solve.

Table 12.30

				Available
5 (19)			2 (10)	7
	+ θ √	7 (40)	2 − θ (60)	9
	8 − θ (8) θ = 2		10 + θ (20)	18
Required 5	8 8 − 2	7	14	

A Rule to Determine θ : Reallocation is done by transfering the maximum possible amount θ in the marked (√) cell. The value of θ, in general, is obtained by equating to zero the minimum of the allocations containing – θ (not + θ) at the corners of the closed loop. That is, in *Table 12·30*, min [8 − θ, 2 − θ] = 0 or 2 − θ = 0 or θ = 2 units. Thus improved basic feasible solution is given in *Table 12·31*.

Table 12·31

				Available
5(19)		–	2(10)	7
	2(30)	7(40)		9
	6(8)		12(20)	18
Required 5	8	7	14	

The cost for this solution becomes

$$= 5\,(19) + 2\,(10) + 2\,(30) + 7\,(40) + 6\,(8) + 12\,(20) = \text{Rs } 743.$$

The cost of Rs 743 is Rs ($2 \times 18 = 36$) less than Rs 779 which was expected also.

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Step 4. Test this improved solution (*Table 12·31*) for optimality by repeating *steps 1, 2* and *3*. In each step, following matrices are obtained.

Table 12·32

Matrix [c_{ij}] for empty cells

•	(30)	(50)	•
(70)	•	•	(60)
(40)	•	(70)	•

Table 12·33

Matrix for u_i and v_j

					u_i
	• (19)			• (10)	−10
		• (30)	• (40)		22
		• (8)		• (20)	0
$v_j \rightarrow$	29	8	18	20	

Table 12·34

Matrix [$u_i + v_j$] for empty cells

•	−2	8	•
51	•	•	42
29	•	18	•

Table 12·35

Matrix [$c_{ij} - (u_i + v_j)$] for empty cells

•	32	42	•
19	•	•	18
11	•	52	•

Since none of the cell evaluations is negative, *i.e.*, $d_{12} = 32$, $d_{13} = 42$, $d_{21} = 19$, $d_{24} = 18$, $d_{31} = 11$ and $d_{33} = 52$, the solution given in ***Table 12·31*** is optimal with minimum cost of Rs. 743.

(b) Maximum saving = Rs. 1000 − Rs. 743 = Rs. 257 . **Ans.**

12·10-2. More Solved Examples

Example 4. *Solve the following transportation problem in which cell entries represent unit costs.*

Table 12·36

	To			Available
	2	7	4	5
	3	3	1	8
From	5	4	7	7
	1	6	2	14
Required	7	9	18	34

Solution. By *Vogel's method,* the following initial basic feasible solution having the transportation cost of Rs. 80 is obtained.
To test the solution for optimality, required tables are given below.

Table 12·37

				Available
	5(2)			5
			8(1)	8
		7(4)		7
	2(1)	**2(6)**	**10(2)**	14
Required	7	9	18	

Table 12·38

[Matrix for set of u_i and v_j]

				u_i
	(2)			1
			(1)	−1
		(4)		−2
	(1)	(6)	(2)	0
v_j	1	6	2	

Table 12·39

[Matrix c_{ij} for empty cells]

•	(7)	(4)
(3)	(3)	•
(5)	•	(7)
•	•	•

Table 12·40

[Matrix $(u_i + v_j)$ for empty cells]

•	7	3
0	5	•
−1	•	0
•	•	•

Table 12·41

Matrix [$c_{ij} - (u_i + v_j)$] for empty cells

•	0	1
3	−2 √	•
6	•	7
•	•	•

From *Table* 12·41, it is observed that the cell evaluation, $d_{22} = -2$, is negative. Therefore, the solution [*Table* 12·37] under test is not optimal.

Corollary. *If* (x_{ij}), $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$ *is an optimal solution for an assignment problem with cost* (c_{ij}), *then it is also optimal for the problem with cost* (c_{ij}') *when*

$$c'_{ij} = c_{ij} \quad \text{for } i, j = 1, 2, \ldots, n \,;\, j \neq k$$
$$c'_{ik} = c_{ik} - A, \text{ where } A \text{ is a constant.}$$

Proof. We have

$$z' = \sum_i \sum_j c'_{ij}\, x_{ij} = \sum_i \Big(\sum_{j \neq k} c'_{ij} + c'_{ik} \Big) x_{ij} = \sum_i \Big(\sum_{j \neq k} c_{ij} + c_{ik} - A \Big) x_{ij} = \Sigma\, \Sigma\, c_{ij}\, x_{ij} - A \sum_i x_{ij}$$
$$= z - A \qquad (\text{since } \sum_i x_{ij} = 1)$$

Thus if (x_{ij}) minimizes z so will it z'.

Theorem 11.2. *In an assignment problem with cost* (c_{ij}), *if all* $c_{ij} \geq 0$ *then a feasible solution* (x_{ij}) *which satisfies* $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} = 0$, *is optimal for the problem.*

Proof. Since all $c_{ij} \geq 0$ and all $x_{ij} \geq 0$, the objective function $z = \Sigma\, \Sigma\, c_{ij}\, x_{ij}$ cannot be negative. The minimum possible value that z can attain is 0. Thus, any feasible solution (x_{ij}) that satisfies $\Sigma\, \Sigma\, c_{ij}\, x_{ij} = 0$ will be optimal.

Theorem 11.3. ***(König Theorem).*** *Let P be the set of 0 elements of a matrix C. Then the maximum number of 0's that can be selected in P such that no row or column of C contains more than one such 0 is equal to the minimum number of lines covering all the elements of P.*

Proof is beyond the scope of the book.

Corollary. *The maximal subset of P provides an optimal assignment when the minimum number of lines to cover all the elements of P is equal to the order of C.*

Proof. Left as an exercise for the reader.

Q. 1. Explain how an assignment problem can be treated as a linear programming problem. Show that the optimal solution to the assignment problem remains the same if a constant is added to or subtracted from any row or column of the cost matrix.

2. If $b_{ij} = c_{ij} - u_i - v_j$ $(i, j = 1, 1, 2, \ldots, n)$ where u_i and v_j are constants, then show that an optimal solution of the assignment problem with cost matrix $B = (b_{ij})$ is also an optimal solution of the assignment problem with cost matrix $C = (c_{ij})$.

[Delhi B.Sc. (Math.) 90]

11.4. HUNGARIAN METHOD FOR ASSIGNMENT PROBLEM

The solution technique of assignment problems can be easily explained by the following practical examples.

Example 1. *A department head has four subordinates, and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours ?* **[JNTU (B. Tech) 2002, 2000; Tamil. (ERODE) 97; IAS (Main) 93; Kerala B.Sc. (Math.) 91; Meerut (Stat.) 90; Kalicut B. Tech 90]**

Table 11·2

Subordinates

		I	II	III	IV
	R1 A	8	26	17	11
Tasks	R2 B	13	28	4	26
	R3 C	38	19	18	15
	R4 D	19	26	24	10

Solution. To understand the problem initially, step by step solution procedure is necessary.

Step 1. Subtracting the smallest element in each row from every element of that row, we get the reduced matrix [*Table* 11·3]

Step 2. Next subtract the smallest element in each column from every element of that column to get the second reduced matrix [*Table* 11·4]

Table 11·3

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Table 11·4

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

Step 3. Now, test whether it is possible to make an assignment using only *zeros*. If it is possible, the assignment must be optimal by *Theorem 11.2* of Section 11·3 . Zero assignment is possible in *Table* 11·4 as follows :

(a) Starting with *row* 1 of the matrix (*Table* 11·4), examine the rows one by one until a row containing exactly *single zero* element is found. Then an experimental assignment (indicated by □) is marked to that cell. Now cross all other zeros in the *column* in which the assignment has been made. This eliminates the possibility of marking further assignments in that column. The illustration of this procedure is shown in *Table* 11·5a.

Table 11·5a

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	0	3	0
D	9	12	14	$\boxed{0}$

Table 11·5b

	I	II	III	IV
A	$\boxed{0}$	14	9	3
B	9	20	$\boxed{0}$	22
C	23	$\boxed{0}$	3	0
D	9	12	14	$\boxed{0}$

(b) When the set of rows has been completely examined, an identical procedure is applied sucessively to columns. Starting with *column* 1, examine all columns until a *column* containing exactly one zero is found. Then make an experimental assignment in that position and cross other zeros in the *row* in which the assignment has been made.

Continue these successive operations on rows and columns until all zeros have been either assigned or crossed-out. At this stage, re-examine rows. It is found that no additional assignments are possible. Thus, the complete '*zero assignment*' is given by A → I , B → III, C → II , D → IV as mentioned in *Table 11·5b*. According to *Theorem* 1, this assignment is also optimal for the original matrix (*Table 11·2*). Now compute the minimum total man-hours as follows :

Optimal assignment	:	A—I	B—III	C—II	D—IV	
Man-hour	:	8	4	19	10	(Total 41 hours.)

Now the question arises : *what would be further steps if the complete optimal assignment after applying Step 3 is not obtained ?* Such difficulty will arise whenever all zeros of any row or column are crossed-out. Following example will make the procedure clear.

Example 2. *A car hire company has one car at each of five depots a ,b , c, d and e. A customer requires a car in each town, namely A, B, C, D, and E. Distance (in kms) between depots (origins) and towns (destinations) are given in the following distance matrix :*

Table 11·6

	a	*b*	*c*	*d*	*e*
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

How should cars be assigned to customers so as to minimize the distance travelled ?

Solution. Applying *Step 1* and *Step 2* as explained in *Example 1* we get the *Table* 11·7 .

Table 11·7

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

Step 3. Row 1 has a single zero in *column* 2. Make an assignment by putting a square '□' around it, and delete other zero (if any) in column 2 by marking '×'.

Table 11·8 a

30	$\boxed{0}$	35	30	15
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$
30	$\cancel{0}$	35	30	20
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5
20	$\cancel{0}$	25	15	15

Now, *column* 1 has a single zero in *row* 4. Make an assignment by putting '□' and cross the other zeros which is not yet crossed. Column 3 has a single zero in *row* 2, make an assignment and delete the other zeros which are uncrossed.

It is observed that there are no remaining zeros ; and *row 3, row 5, column 4,* and *column 5* each has no assignment. Therefore, desired solution cannot be obtained at this stage. we now, proceed to following important steps.

Step 4. Draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. It should, however, be observed that (in all $n \times n$ matrices) less than n lines will cover zeros only when there is no solution among them. Conversely, if minimum number of lines is n , there is a solution.
Following systematic procedure may help us to draw the minimum set of lines :

1. For simplicity, first make the *Table* 11·8*a* again and name it at *Table* 11·8 b.

Table 11·8 b

		L_1				
	30	$\boxed{0}$	35	30	15	✓ ④
L_2	15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	
	30	$\cancel{0}$	35	30	20	✓ ①
L_3	$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
	20	$\cancel{0}$	25	15	15	✓ ②
		③ ✓				

2. Mark (√) *row 3* and *row 5* as they are having no assignments and *column 2* as having zeros in the marked rows 3 and 5.
3. Mark (√) *row 1* because this row contains assignment in the marked *column 2*. No further rows or columns will be required to mark during this procedure.
4. Now start drawing required lines as follows ;
   First draw line (L_1) through *marked* column 2. Then draw lines(L_2 and L_3) through *unmarked* rows (2 and 4) having largest number (2) of uncovered zeros (since no zero is left uncovered, the required lines will be (L_1 , L_2 and L_3) .

Step 5. In this step,

(i) first select the smallest element, say x, among all *uncovered* elements of the *Table* 11·8b [as a result of *step* 4] and

(ii) then subtract this value x from all values in the matrix not covered by lines and add x to all those values that lie at the intersection of any two of the lines L_1 , L_2 and L_3 . (Justification of this rule is given on the next page).

After applying these two rules, we find $x = 15$, and a new matrix is obtained as given in *Table 11·9* .

Table 11·9

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

Step 6. Now re-apply the test of *Step 3* to obtain the desired solution. Therefore, proceeding exactly in the same manner as in *step 3*, obtain the final *Table* 11·10 .

Table 11·10

15	$\boxed{0}$	35	30	15
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$
15	$\cancel{0}$	35	30	20
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5
5	$\cancel{0}$	25	15	15

It is observed that there are no remaining zeros, and every row (column) has an assignment. Since no two assignments are in the same column (they cannot be, if the procedure has been correctly followed), the 'zero assignment' is the required solution.

From original matrix (*Table* 11·6), the minimum distance assignment is given by

Route	A –e	$B-c$	$C-b$	$D-a$	$E-d$	Total Distance Travelled
Distance (*Kms.*)	200	130	110	50	80	570 Kms.

Note. Table 11·10 may be obtained very quickly if we first apply ***Step 2*** and then ***Step I*** in the original Table 11·6 ;

Justification of rules used above in step 5 :

Justification of rules we have used in *Step* 5 is based on the following two facts :

(i) The relative cost of assigning *i*th facility to *j*th job is not changed by the subtraction of a constant either from a column or from a row of the original effectiveness matrix.

(ii) An optimal assignment exists if the total reduced cost of the assignment is zero. This is the case when the minimum number of lines necessary to cover all zeros is equal to the order of the matrix. If however, it is less than *n* further reduction of the effectiveness matrix has to be undertaken.

The underlying logic can be explained with the help of *Table 11·8(b)* in which only 3 ($= n - 2$) lines can be drawn. Here an optimal assignment is not possible. So further reduction is necessary.

Further reduction is made by subtracting the smallest non-zero element 15 from all elements of the matrix *Table 11.8(b)*. This gives the following matrix :

		L_1			
	15	– 15	20	15	0
L_2 ←	0	– 15	15	– 5	– 15
	15	– 15	20	15	0
L_3 ←	15	– 15	5	– 15	– 10
	5	– 15	10	0	0

This matrix contains negative values. Since the objective is to obtain an assignment with reduced cost of zero, the negative numbers must be eliminated. This can be done by adding 15 to only those rows and columns which are covered by three *lines* (L_1, L_2, L_3) as shown above. In doing so the following change is noted.

		L_1			
	15	(–15+15)	20	15	0
L_2 →	(0 + 15)	[(–15+15)+15]	(– 15 + 15)	(– 15 + 15)	(– 15 + 15)
	15	(–15+15)	20	15	5
L_3 →	(– 15 + 15)	[(–15+15)+15]	(5 + 15)	(– 5 + 15)	(– 10 + 15)
	5	(–15+15)	10	0	0

This table is exactly the same as Table 11·9 . In fact, all this is the result of adding the least non-zero element at the intersections; and subtracting from all uncovered elements, and leaving the other elements unchanged.

Q. 1. Show that the procedure of subtracting the minimum elements not covered by any line, from all the uncovered elements and adding the same element to all the elements lying at the intersection of two lines results in a matrix with the same optimal assignments as the original matrix. **[Meerut M.Sc. (Math.) 92, 90; Jodhpur M.Sc. (Math) 92]**

2. State the assignment problem. Describe a method of drawing minimum number of lines in the context of assignment problem. Name the method.

3. Describe any method for solving an assignment problem. **[Delhi B.Sc. (Math.) 93]**

11·4·1. Assignment Algorithm (Hungarian Assignment Method)

Various steps of the computational procedure for obtaining an optimal assignment may be summarized as follows :

Step 1. Subtract the minimum of each row of the effectiveness matrix, from all the elements of the respective rows.

Step 2. Further, modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus obtain the *first modified matrix*.

Step 3. Then, draw the minimum number of horizontal and vertical lines to cover all the zeros in the resulting matrix. Let the minimum number of lines be N. Now there may be two possibilities :

(i) If $N = n$, the number of rows (columns) of given matrix, then an optimal assignment can be made. So make the zero assignment to get the required solution.

(ii) If $N < n$, then proceed to *step 4*.

Step 4. Determine the smallest element in the matrix, not covered by the N lines. Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus, the *second modified matrix* is obtained.

Step 5. Again repeat ***Steps 3 and 4*** until minimum number of lines become equal to the number of rows (columns) of the given matrix *i.e.*, $N = n$.

Step 6. ***(To make zero-assignment)***. Examine the rows successively until a row-wise exactly single zero is found, mark this zero by '☐' to make the assignment. Then, mark a cross (×) over all zeros if lying in the column of the marked '☐' zero, showing that they cannot be considered for future assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for columns also.

Step 7. Repeat the ***Step 6*** successively until one of the following situations arise :

(i) if no unmarked zero is left, then the process ends ; or

(ii) if there lie more than one of the unmarked zeros in any column or row, then mark '☐' one of the unmarked zeros arbitrarily and mark a cross in the cells of remaining zeros in its row and column. Repeat the process until no unmarked zero is left in the matrix.

Step 8. Thus exactly one marked '☐' zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked '☐' *zeros* will give the optimal assignment.

11·4·2 A Rule to Draw Minimum Number of Lines

A very convenient rule of drawing minimum number of lines to cover all the 0's of the reduced matrix is given in the follwing steps :

Step 1. Tick (√) rows that do not have any marked (☐) zero.

Step 2. Tick (√) columns having marked (☐) zeros or otherwise in ticked rows.

Step 3. Tick (√) rows having marked 0's in ticked columns.

Step 4. Repeat *steps 2 and 3* until the chain of ticking is complete.

Step 5. Draw lines through all unticked rows and ticked columns.

This will give us the minimal system of lines.

Q. 1. Give an algorithm to solve an 'Assignment Problem'. **[IGNOU 2001, 99, 97, 96; IAS (Maths) 88]**

2. Write a short note on 'Assignment Problem'.

3. Explain the Humgarian method to solve an assignment problem. **[Meerut (OR) 2003, 02; VTU (BE Mech.) 2002]**

5

LINEAR PROGRAMMING PROBLEM
(SIMPLEX METHOD)

5·1. INTRODUCTION

It has not been possible to obtain the graphical solution to the LP problem of more than two variables. The analytic solution is also not possible because the tools of analysis are not well suited to handle inequalities. In such cases, a simple and most widely used simplex method is adopted which was developed by ***G. Dantzig*** in 1947.

The *simplex method*† provides an ***algorithm*** (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the ***fundamental theorem of linear programming.***

It is clear from Fig. 3.4 (page 78) that feasible solutions may be *infinite* in number (because there are infinite number of points in the feasible region, *OABCD*). So, it is rather impossible to search for the optimum solution amongst all the feasible solutions. But fortunately, the number of basic feasible solutions are finite in number (which are corresponding to extreme points O, A, B, C, D, respectively). Even then, a great labour is required in finding all the basic feasible solutions and to select that one which optimizes the objective function.

The simplex method provides a systematic algorithm which consists of moving from one basic feasible solution (one vertex) to another in a prescribed manner so that the value of the objective function is improved. This procedure of jumping from vertex to vertex is repeated. If the objective function is improved at each jump, then no basis can ever repeat and there is no need to go back to vertex already covered. Since the number of vertices is finite, the process must lead to the optimal vertex in a finite number of steps. The procedure is explained in detail through a numerical example (*see Example 2,* page 118).

The simplex algorithm is an iterative (step-by-step) procedure for solving LP problems. It consists of—

(i) having a trial basic feasible solution to constraint-equations,

(ii) testing whether it is an optimal solution,

(iii) improving the first trial solution by a set of rules, and repeating the process till an optimal solution is obtained.

The computational procedure requires at most m [equal to the number of equations in (3·12)] non-zero variables in the solution at any step. In case of less than m non-zero variables at any stage of computations the degeneracy arises in LP problem. The case of degeneracy has also been discussed in detail in *this chapter*.

Further, it is very interesting to note that a feasible solution at any iteration is related to the feasible solution at the successive iteration in the following way. One of the non-basic variables (which are zero now) at one iteration becomes *basic* (non-zero) at the following iteration, and is called an ***entering variable.*** To compensate, one of the basic variables (which are non-zero now) at one iteration becomes non-basic (zero) at the following iteration, and is called a ***departing variable.*** The other non-basic variables remain zero, and the other basic variables, in general, remain non-zero (though their values may change).

For convenience, re-state the LP problem in standard form :

$$\text{Max. } z = c_1x_1 + c_2x_2 + \dots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \dots + 0x_{n+m} \qquad \text{...(5.1)}$$

subject to the constraints :

$$\left.\begin{array}{llll} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & + x_{n+1} & & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n & & + x_{n+2} & = b_2 \\ \dots \quad \dots & \dots \quad \dots & \dots & \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n & & + x_{n+m} & = b_m \end{array}\right\} \qquad \text{...(5.2)}$$

† For complete development of **'Simplex Method'** please see Appendix-A (**Theory of Simplex Method**) on page 1119.

Hence $\qquad c_{B2}$ = coefficient of x_{B2} = coeff. of $x_1 = c_1 = 1$

$\mathbf{C_B} = (3, 1)$.

Now, using (5.7), the value of the objective function is

$$z = \mathbf{C_B X_B} = (3, 1)\begin{pmatrix} 28/11 \\ 4/11 \end{pmatrix} = \frac{88}{11}.$$

Also, any vector $\mathbf{a}_j = (j = 1, 2, 3, 4, 5)$ can be expressed as linear combination of vectors β_i $(i = 1, 2)$. Therefore, to express $\mathbf{a}_2$ as linear combination of β_1, β_2, we have

$$\mathbf{a}_2 = x_{12}\beta_1 + x_{22}\beta_2 = x_{12}\mathbf{a}_3 + x_{22}\mathbf{a}_1.$$

To compute values of scalars x_{12} and x_{22}, use the result (5.3) to get

$$\mathbf{X}_2 = \mathbf{B}^{-1}\mathbf{a}_2 = -\frac{1}{11}\begin{pmatrix} 1 & -4 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$

Therefore $\qquad x_{12} = 6/11, x_{22} = 4/11$.

Similar treatment can be adopted for expressing other $\mathbf{a}_j$'s as linear combinations of β_1 and β_2.

Now, using (5.6b), the variable z_2 corresponding to vector $\mathbf{a}_2$ can be obtained as

$$z_2 = \mathbf{C_B X}_2 = (3, 1)\begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \left(3 \times \frac{6}{11} + 1 \times \frac{4}{11}\right) = \frac{22}{11}.$$

Similarly z_1, z_3, z_4, z_5 can also be computed.

5·3. COMPUTATIONAL PROCEDURE OF SIMPLEX METHOD

The computational aspect of the simplex procedure is first explained by the following simple example.

Example 2. *Consider the linear programming problem :*

Maximize $z = 3x_1 + 2x_2$, subject to the constraints :

$$x_1 + x_2 \le 4, x_1 - x_2 \le 2, \textit{ and } x_1, x_2 \ge 0.$$

[Kanpur 2000, 96; IAS (Maths.) 92]

Solution. ***Step 1.*** First, observe whether all the right side constants of the constraints are non-negative. If not, it can be changed into positive value on multiplying both sides of the constraints by -1. In this example, all the b_i's (right side constants) are already positive.

Step 2. Next convert the inequality constraints to equations by introducing the non-negative ***slack*** or ***surplus*** variables. The coefficients of slack or surplus variables are always taken zero in the objective function. In this example, all inequality constraints being '$\le$', only slack variables s_1 and s_2 are needed. Therefore, given problem now becomes :

Maximize $\quad z = 3x_1 + 2x_2 + 0s_1 + 0s_2$, subject to the constraints :

$$x_1 + x_2 + s_1 + 0s_2 = 4$$
$$x_1 - x_2 + 0s_1 + s_2 = 2$$
$$x_1, x_2, s_1, s_2 \ge 0.$$

Step 3. Now, present the constraint equations in matrix form :

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Step 4. Construct the starting simplex table using the notations already explained in ***Sec 5·2.***

It should be remembered that the values of non-basic variables are always zero at each iteration. So $x_1 = x_2 = 0$ here. Column $\mathbf{X_B}$ gives the values of basic variables as indicated in the first column. So $s_1 = 4$ and $s_2 = 2$ here. The complete starting basic feasible solution can be immediately read from Table 5·2 as : $s_1 = 4$, $s_2 = 2$, $x_1 = 0$, $x_2 = 0$, and the value of the objective function is *zero.*

Note. *In this step, the variables s_1 and s_2 are corresponding to the columns of basis matrix (identity matrix), so will be called basic variables.* Other variables, x_1 and x_2, are *non-basic variables* which always have the value zero.

Table 5·2 : Starting Simplex Table

initial

		$c_j \rightarrow$	3	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$ (basic feasible soln)	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$ $(\mathbf{S_1})$ (β_1)	$\mathbf{X_4}$ $(\mathbf{S_2})$ (β_2)	MIN. RATIO $\mathbf{X_B/X_k}$ for $\mathbf{X_K} > 0$
					BASIS MATRIX		
s_1	0	4	1	1	1	0	TO BE COMPUTED IN NEXT STEP.
s_2	0	2	1	-1	0	1	
	$z = \mathbf{C_B X_B}$ objective func		$\Delta_1 = -3$ ↑	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	$\Delta_j = z_j - \mathbf{c_j} = \mathbf{C_B X}_j - c_j$

Step 5. Now, proceed to test the basic feasible solution for optimality by the rules given below. This is done by computing the *'net evaluation'* Δ_j for each variable x_j (column vector $\mathbf{X}_j$) by the formula

$$\Delta_j = z_j - \mathbf{c}_j = \mathbf{C_B X}_j - c_j \qquad \text{[from (5·10)]}$$

Thus, we get

$$\begin{aligned} \Delta_1 &= \mathbf{C_B X}_1 - c_1 \\ &= (0\,,0)\,(1\,,1) - 3 \\ &= (0 \times 1 + 0 \times 1) - 3 \\ &= -3 \end{aligned}
\qquad
\begin{aligned} \Delta_2 &= \mathbf{C_B X}_2 - c_2 \\ &= (0\,,0)\,(1\,,-1) - 2 \\ &= (0 \times 1 - 0 \times 1) - 2 \\ &= -2 \end{aligned}
\qquad
\begin{aligned} \Delta_3 &= \mathbf{C_B X}_3 - c_3 \\ &= (0\,,0)\,(1\,,0) - 0 \\ &= (0 \times 1 + 0 \times 0) - 0 \\ &= 0 \end{aligned}
\qquad \Delta_4 = 0$$

Remark. Note that in the starting simplex table Δ_j's are same as $(-c_j)$'s. Also, Δ_j's corresponding to the columns of unit matrix (basis matrix) are always zero. So there is no need to calculate them.

Optimality Test :

(i) If all $\Delta_j\ (= z_j - c_j) \geq 0$, the solution under test will be ***optimal.*** *Alternative optimal solutions* will exist if any non-basic Δ_j is also zero.

(ii) If at least one Δ_j is negative, the solution under test is not optimal, then proceed to improve the solution in the next step.

(iii) If corresponding to any negative Δ_j, all elements of the column $\mathbf{X}_j$ are negative or zero (≤ 0) , then the solution under test will be ***unbounded.***

Applying these rules for testing the optimality of starting basic feasible solution, it is observed that Δ_1 and Δ_2 both are negative. Hence, we have to proceed to improve this solution in ***Step 6.***

Step 6. In order to improve this basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined by the following rules. Such vectors are usually named as ***'incoming vector'*** and ***'outgoing vector'*** respectively.

'Incoming vector'. The incoming vector $\mathbf{X}_k$ is always selected corresponding to the most negative value of Δ_j (say, Δ_k). Here $\Delta_k = \min\ [\Delta_1\,,\Delta_2] = \min\ [-3\,,-2] = -3 = \Delta_1$. Therefore, $k = 1$ and hence column vector $\mathbf{X}_1$ must enter the basis matrix. The column $\mathbf{X}_1$ is marked by an upward arrow (↑).

'Outgoing vector'. The outgoing vector β_r is selected corresponding to the minimum ratio of elements of $\mathbf{x_B}$ by the corresponding *positive* elements of predetermined incoming vector $\mathbf{X}_k$. This rule is called the ***Minimum Ratio Rule.*** In mathematical form, this rule can be written as

$$\frac{x_{Br}}{x_{rk}} = \min_{i} \left[\frac{x_{Br}}{x_{ik}}\,,\, x_{ik} > 0\right]$$

For $k = 1$,

$$\frac{x_{Br}}{x_{r1}} = \min\left[\frac{x_{B1}}{x_{11}}\,,\,\frac{x_{B2}}{x_{21}}\right] = \min\left[\frac{4}{1}\,,\,\frac{2}{1}\right]$$

or

$$\frac{x_{Br}}{x_{r1}} = \frac{2}{1} = \frac{x_{B2}}{x_{21}}\,.$$

Comparing both sides of this equation, we get $r = 2$. So the vector β_2, *i.e.*, $\mathbf{X}_4$ marked with downward arrow (↓) should be removed from the basis matrix. The ***Starting Table*** **5·2** is now modified to ***Table*** **5·3** given below.

Table 5·3

BASIC VARIABLES	Ca	$c_j \rightarrow$ Xa	3 X_1	2 X_2	0 $X_3(S_1)$ (β_1)	0 $X_4(S_2)$ (β_2)	MIN. RATIO (X_B/X_1)
s_1	0	4	1	1	1	0	4/1
s_2	0	2	$\boxed{1}$	-1	0	1	2/1 ← MIN. RATIO
	$z = \mathbf{C_B X_B} = 0$		-3 (min. Δ_j)	-2	0	0	$\leftarrow \Delta_j = z_j - c_j = \mathbf{C_B B}_j - c_j$

key element

↑entering vector ↓leaving vector

Step 7. In order to bring $\beta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in place of incoming vector $\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, unity must occupy in the marked '$\square$' position and zero at all other places of $\mathbf{X}_1$. If the number in the marked '$\square$' position is other than *unity, divide all elements of that row by the 'key element'. (The element at the intersection of minimum ratio arrow* ($\leftarrow$) *and incoming vector arrow* ($\uparrow$) *is called the* ***key element*** *or* ***pivot element****).*

Then, subtract appropriate multiplies of this new row from the other (remaining) rows, so as to obtain zeros in the remaining positions of the column $\mathbf{X}_1$. Thus, the process can be fortified by simple matrix transformation as follows :

The intermediate coefficient matrix is :

	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	
R_1	4	1	1	1	0	
R_2	2	1	-1	0	1	
R_3	$z = 0$	-3	-2	0	0	$\leftarrow \Delta_j$

Apply $R_1 \rightarrow R_1 - R_2$, $R_3 \rightarrow R_3 + 3R_2$ to obtain

$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	
2	0	2	1	-1	
2	1	-1	0	1	
$z = 6$	0	-5	0	3	$\leftarrow \Delta_j$

Now, construct the improved simplex table as follows :

Table 5·4

BASIC VARIABLES	$\mathbf{C_B}$	$c_j \rightarrow$ $\mathbf{X_B}$	3 $\mathbf{X_1}$ (β_2)	2 $\mathbf{X_2}$ ↑	0 $X_3(S_1)$ (β_1)	0 $X_4(S_2)$	MIN-RATIO $(\mathbf{X_B/X_2}, \mathbf{X_2} > 0)$
s_1	0	2	0	$\boxed{2}$	1	-1	$\frac{2}{2}$ ← key row
x_1	3	2	1	-1	0	1	$\cancel{\frac{2}{-1}}$ (negative ratio is not counted)
	$z = \mathbf{C_B X_B} = 6$		0	-5 ↑	0 ↓	3	$\leftarrow \Delta_j$

key column

From this table, the improved basic feasible solution is read as : $x_1 = 2$, $x_2 = 0$, $s_1 = 2$, $s_2 = 0$. The improved value of $z = 6$.

It is of particular interest to note here that Δ_j's are also computed while transforming the table by matrix method. However, the correctness of Δ_j's can be verified by computing them independently by using the formula $\Delta_j = \mathbf{C_B X}_j - c_j$.

Step 8. Now repeat *Steps 5 through* 7 as and when needed until an optimum solution is obtained in Table 5·5.

$$\Delta_k = \text{most negative } \Delta_j = -5 = \Delta_2 .$$

Therefore, $k = 2$ and hence $\mathbf{X_2}$ should be the entering vector (key column). By minimum ratio rule :

Minimum Ratio $\left(\frac{\mathbf{X_B}}{\mathbf{X_2}}, \mathbf{X_2} > \mathbf{0}\right) = \text{Min}\left[\frac{2}{2}, -\right]$ (since negative ratio is not counted, so the second ratio is not considered)

Since *first ratio* is minimum, remove the first vector β_1 from the basis matrix. Hence the key element is 2. Dividing the first row by key element 2, the intermediate coefficient matrix is obtained as :

	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	
R_1	1	0	$\boxed{1}$	1/2	− 1/2	
R_2	2	1	− 1	0	1	
R_3	$z = 6$	0	− 5	0	3	$\leftarrow \Delta_j$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 5R_1$

1	0	1	1/2	− 1/2	
3	1	0	1/2	1/2	
$z = 11$	0	0	5/2	1/2	$\leftarrow \Delta_j$

Now construct the next improved simplex table as follows :

Final Simplex Table 5·5

		$c_j \to$	3	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$ (β_2)	$\mathbf{X_2}$ (β_1)	$\mathbf{S_1}$	$\mathbf{S_2}$	
$\to x_2$	2	1	0	1	1/2	− 1/2	
x_1	3	3	1	0	1/2	1/2	
	$z = \mathbf{C_B}\,\mathbf{X_B} = \mathbf{11}$		0	0	5/2	1/2	$\leftarrow \Delta_j$

The solution as read from this table is : $x_1 = 3$, $x_2 = 1$, $s_1 = 0$, $s_2 = 0$, and max. $z = 11$. Also, using the formula $\Delta_j = \mathbf{C_B}\,\mathbf{X_j} - c_j$ verify that all Δ_j's are non-negative. Hence the optimum solution is

$$x_1 = 3 \;,\; x_2 = 1 \;,\; \max \mathbf{z} = 11 \;.$$

Note. If at the optimal stage, it is desired to bring s_1 in the solution, the total profit will be reduced from 11 (the optimal value) to 5/2 times of 2 units of s_1 in Table 3·4, i.e., $z = 11 - 5/2 \times 2 = 6$. This explains the *economic interpretation* of net-evaluations Δ_j .

5·4. SIMPLE WAY FOR SIMPLEX METHOD COMPUTATIONS

Complete solution with its different computational steps can be more conveniently represented by the following single table (see Table 5.6).

Table 5·6

		$c_j \to$	3	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{S_1}$	$\mathbf{S_2}$	MIN RATIO $(\mathbf{X_B}/\mathbf{X_K})$
s_1	0	4	1	1	1	0	4/1
$\leftarrow s_2$	0	2	$\boxed{1}$	1	0	1	2/1 ← Min
$x_1 = x_2 = 0$	$z = \mathbf{C_B}\mathbf{X_B} = 0$		− 3* ↑	− 2	0	0 ↓	$\leftarrow \Delta_j = z_j - c_j$
$\leftarrow s_1$	0	2	0	$\boxed{2}$	1	−1	2/2 Min ←
$\to x_1$	3	2	1	−1	0	1	—
$x_2 = s_2 = 0$	$z = \mathbf{C_B}\mathbf{X_B} = 6$		0	−5* ↑	0 ↓	3	$\leftarrow \Delta_j$
$\to x_2$	2	1	0	1	1/2	− 1/2	
x_1	3	3	1	0	1/2	1/2	
$s_1 = s_2 = 0$	$z = \mathbf{C_B}\mathbf{X_B} = \mathbf{11}$		0	0	5/2	1/2	$\leftarrow$ All $\Delta_j \geq 0$

Thus, the optimal solution is obtained as : $x_1 = 3$, $x_2 = 1$, max $z = 11$.

$x_1 + 2x_2 + 2x_3 + x_4 = 8$, $3x_1 + 4x_2 + x_3 + x_5 = 7$ and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$, $x_5 \geq 0$.
[**Ans.** One iteration only, $x_1 = x_2 = x_4 = 0$, $x_3 = 4$, $x_5 = 3$, max. $z = 15$]

27. Max. $z = 3x_1 + 2x_2 - 2x_3$
subject to the constraints :
$x_1 + 2x_2 + 2x_3 \leq 10$
$2x_1 + 4x_2 + 3x_3 \leq 15$
$x_1, x_2 < x_3 \geq 0$
[**Ans.** One iteration only.
$x_1 = 15/2$, $x_2 = x_3 = 0$, max. $z = 45/2$]

28 Max. $z = 30x_1 + 23x_2 + 29x_3$
subject to the constraints :
$6x_1 + 5x_2 + 3x_3 \leq 52$
$6x_1 + 2x_2 + 5x_3 \leq 14$.
$x_1, x_2, x_3 \geq 0$.
[**Ans.** Two iterations. $x_1 = 0$, $x_2 = 7$, $x_3 = 0$, max. $z = 161$]

29. Max. $z = 7x_1 + x_2 + 2x_3$,
subject to the constraints :
$x_1 + x_2 - 2x_3 \leq 10$
$4x_1 + x_2 + x_3 \leq 20$
$x_1, x_2, x_3 \geq 0$.
[**Ans.** Two iterations. $x_1 = x_2 = 0$, $x_3 = 20$
max. $z = 40$]

30. Max. $R = 2x - 3y + z$
subject to the constraints :
$3x + 6y + z \leq 6$
$4x + 2y + z \leq 4$
$x - y + z \leq 3$
$x, y, z \geq 0$
[**Ans.** Two iterations. $x = 1/3$,
$y = 0$, $z = 8/3$, max. $R = 10/3$]

31. Max. $R = 2x + 4y + 3z$
subject to the constraints :
$3x + 4y + 2z \leq 60$
$2x + y + 2z \leq 40$
$x + 3y + 2z \leq 80$
$x, y, z \geq 0$.
[**Ans.** Two iterations. $x = 0$, $y = 20/3$,
$z = 50/7$, max. $R = 250/3$.]

32. Max. $z = x_1 - x_2 + x_3 + x_4 + x_5 - x_6$
subject to the constraints :
$x_1 + x_4 + 6x_6 = 9$
$3x_1 + x_2 - 4x_3 + 2x_6 = 2$
$x_1 + 2x_3 + x_5 + 2x_6 = 6$
$x_i \geq 0$, $i = 1, 2, 3, 4, 5, 6$.
[**Ans.** One iteration. $x_1 = 2/3$, $x_2 = x_3 = 0$,
$x_4 = 25/3$, $x_5 = 16/3$, $x_6 = 0$, max $z = 43/3$]

33. A farmer has 1,000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs Rs. 100 for preparation, requires 7 man-days of work and yield a profit of Rs. 30. An acre of wheat cost Rs. 120 to prepare, requires 10 man-days of work and yields a profit of Rs. 40. An acre of soyabeans cost Rs. 70 to prepare, requires 8 man-days of work and yields a profit of Rs. 20. If the farmer has Rs. 1,00,000 for preparation and can count on 8,000 man-days of work, how masny acres should be allocated to each crop to maximize profit ? **[Jammu Univ. (MBA) Feb. 96]**
[**Hint.** Formulation of the problem is :
Max. $z = 30x_1 + 40x_2 + 20x_3$, s.t.
$10x_1 + 12x_2 + 7x_3 \geq 10{,}000$; $7x_1 + 10x_2 + 8x_3 \leq 8{,}000$
$x_1 + x_2 + x_3 \leq 1{,}000$; $x_1, x_2, x_3 \geq 0$.]
[**Ans.** Acreage for carn, wheat and soyabeans are 250, 625 and respectively with max. profit of Rs. 32,500]

5·5. ARTIFICIAL VARIABLE TECHNIQUES

5·5– 1. Two Phase Method **[Garhwal 97; Kanpur (B.Sc.) 90; Rohil. 90]**

Linear programming problems, in which constraints may also have '≥' and '=' signs after ensuring that all b_i are ≥ 0, are considered in this section. In such problems, basis matrix is not obtained as an identity matrix in the starting simplex table, therefore we introduce a new type of variable, called, the ***artificial variable.*** These variables are fictitious and cannot have any physical meaning. The artificial variable technique is merely a device to get the starting basic feasible solution, so that simplex procedure may be adopted as usual until the optimal solution is obtained. Artificial variables can be eliminated from the simplex table as and when they become zero (non-basic). The process of eliminating artificial variables is performed in ***Phase I*** of the solution, and ***Phase II*** is used to get an optimal solution. Since the solution of the LP problem is completed in two phases, it is called ***'Two Phase Simplex Method'*** due to **Dantzig, Orden** and **Wolfe.**

Remarks :

1. **The objective of Phase I is to search for a B.F.S. to the given problem It ends up either giving a B.F.S. or indicating that the given L.P.P. has no feasible solution at all.**
2. **The B.F.S. obtained at the end of Phase 1 provides a starting B.F.S. for the given L.P.P. Phase II is then just the application of simplex method to move towards optimality.**
3. **In Phase II, care must be taken to ensure that an artifical variable is never allowed to become positive, if were present in the basis. Moreover, whenever some artifical variable happens to leave the basis, its column must be deleted from the simplex table altogether.**

Q. 1. Explain the term 'Artificial variable' and its use in linear programming.

2. What do you mean by two phase-method in linear programming problems, why it is used ?

This technique is well explained by the following example.

Example 10. *Solve the problem : Minimize* $z = x_1 + x_2$, *subject to* $2x_1 + x_2 \geq 4$, $x_1 + 7x_2 \geq 7$, *and* $x_1 , x_2 \geq 0$.

[Delhi B.Sc. (Math.) 91, 88; Bharthidasan B.Sc. (Math.) 90; VTU (BE common) Aug. 2002]

Solution. First convert the problem of minimization to maximization by writing the objective function as :

$$\text{Max } (-z) = -x_1 - x_2 \quad \text{or} \quad \text{Max. } z' = -x_1 - x_2 \text{, where } z' = -z.$$

Since all b_i 's (4 and 7) are positive, the 'surplus variables' $x_3 \geq 0$ and $x_4 \geq 0$ are introduced, then constraints become :

$$2x_1 + x_2 - x_3 \quad = 4$$
$$x_1 + 7x_2 \quad - x_4 = 7 .$$

But the basis matrix **B** would not be an identity matrix due to negative coefficients of x_3 and x_4 . Hence the starting basic feasible solution cannot be obtained.

On the other hand, if so-called 'artificial variables' $a_1 \geq 0$ and $a_2 \geq 0$ are introduced, the constraint equations can be written as

$$2x_1 + x_2 - x_3 \quad + a_1 \quad = 4$$
$$x_1 + 7x_2 \quad - x_4 \quad + a_2 = 7 .$$

It should be noted that $a_1 < x_3$, $a_2 < x_4$, otherwise the constraints of the problem will not hold.

Phase I. Construct the first table (*Table 5·14*) where $\mathbf{A}_1$ and $\mathbf{A}_2$ denote the artificial column-vectors corresponding to a_1 and a_2, respectively.

Table 5·14

BASIC VARIABLES	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$
a_1	4	2	1	−1	0	1	0
a_2	7	1	$\boxed{7}$	0	−1	0	1
			↑	×	×	×	↓

Now remove each artificial column vector $\mathbf{A}_1$ and $\mathbf{A}_2$ from the basis matrix. To remove vector $\mathbf{A}_2$ first, select the entering vector either $\mathbf{A}_1$ or $\mathbf{A}_2$, being careful to choose any one that will yield a non-negative (feasible) revised solution. Take the vector $\mathbf{X}_2$ to enter the basis matrix. It can be easily verified that if the vector $\mathbf{A}_2$ is entered in place of $\mathbf{X}_1$, the resulting solution will not be feasible. Thus transformed table (Table 5·15) is obtained.

Table 5·15

BASIC VARIABLES	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$
a_1	3	13/7	0	−1	$\boxed{1/7}$	1	−1/7
x_2	1	1/7	1	0	−1/7	0	1/7
					↑	↓	

(Delete column $\mathbf{A}_2$ for ever at this stage)

This table gives the solution : $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$, $a_1 = 3$, $a_2 = 0$. When the artificial variable a_2 becomes zero (non-basic), we forget about it and never consider the corresponding vector $\mathbf{A}_2$ again for re-entry into the basis matrix.

Similarly, remove $\mathbf{A}_1$ from the basis matrix by introducing it in place of $\mathbf{X}_4$ by the same method. Thus Table 5·16 is obtained.

Table 5·16

BASIC VARAIBLES	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$
x_4	21	13	0	−7	1	7
x_2	4	4	1	−1	0	1

(Delete column $\mathbf{A}_1$ for ever at this stage)

Phase II. Table 5.20

BASIC VARIABLES	C_B	X_B ($c_j \rightarrow$)	X_1 (−1)	X_2 (−1)	X_3 (0)	X_4 (0)	MIN. RATIO (X_B/X_k)
$\leftarrow x_4$	0	21	$\boxed{13}$ $\uparrow$	0	−7	1	21/13 ←
x_2	−1	4	2	1	−1	0	4/2
	$z' = -4$		−1* $\uparrow$	0	1	0 $\downarrow$	$\leftarrow \Delta_j$
$\rightarrow x_1$	−1	21/13	1	0	−7/13	1/13	
x_2	−1	10/13	0	1	1/10	2/13	
	$z' = -31/13$		0	0	6/13	1/13	$\leftarrow \Delta_j \geq 0$

Thus, the desired solution is obtained as : $x_1 = 21/13$, $x_2 = 10/13$, max. $z = 31/13$.

5.5–3. *Alternative Approach* of Two-phase Simplex Method

The two phase simplex method is used to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows :

Phase I. In this phase, the simplex method is applied to a specially constructed ***auxiliary linear programming problem*** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1. Assign a cost − 1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function.

Step 2. Construct the auxiliary linear programming problem in which the new objective function z^* is to be maximized subject to the given set of constraints.

Step 3. Solve the auxiliary problem by simplex method until either of the following three possibilities do arise :

(i) Max $z^* < 0$ and at least one artificial vector appear in the optimum basis at a positive level. In this case given problem does not possess any feasible solution.

(ii) Max $z^* = 0$ and at least one artificial vector appears in the optimum basis at zero level. In this case proceed to *Phase–II.*

(iii) Max $z^* = 0$ and no artificial vector appears in the optimum basis. In this case also proceed to *Phase–II.*

Phase II. Now assign the actual costs to the variables in the objective function and a zero cost to every artificial variable that appears in the basis at the zero level. This new objective function is now maximized by simplex method subject to the given constraints. That is, simplex method is applied to the modified simplex table obtained at the end of *Phase–I*, until an optimum basic feasible solution (if exists) has been attained. The artificial variables which are non-basic at the end of *Phase–I* are removed.

Q. 1. What are artificial variables ? Why do we need them ? Describe briefly the two-phase method of solving a L.P. problem with artificial variables. **[Meerut M.Sc. (Math.) 93]**

2. What do you mean by two phase method for solving a given L.P.P. ? Why is it used ?

3. Explain steps in solving a linear progremming problem by two-phase method.

The following examples will make the *alternative* two-phase method clear.

Example 11. *Use two-phase simplex method to solve the problem : Minimize* $z = x_1 - 2x_2 - 3x_3$, *subject to the constraints* :– $2x_1 + x_2 + 3x_3 = 2$, $2x_1 + 3x_2 + 4x_3 = 1$, *and* $x_1, x_2, x_3 \geq 0$, **[Meerut (Maths) 91]**

Solution. First convert the objective function into maximization form :

$$\text{Max } z' = -x_1 + 2x_2 + 3x_3, \text{ where } z' = -z.$$

Introducing the artificial variables $a_1 \geq 0$ and $a_2 \geq 0$, the constraints of the given problem become,

Dual Problem. Find a column vector $\mathbf{w} \in R^m$, which minimizes $z_w = \mathbf{b}^T\mathbf{w}$, subject to $\mathbf{A}^T\mathbf{w} \geq \mathbf{c}^T$.

Here it is worthnoting that the dual variables are *unrestricted in sign.*

Now the problem is '*what will be the rules and tricks to obtain a dual problem for such linear programming problem which is not given in the standard primal form* (7·1) *considered in* ***subsection 7.2·1.*** ?

The following section is devoted to answer this question.

Q. 1. Define the dual of a linear programming problem.
2. What is dual ? **[Meerut M.Sc. (Math.) 94]**
3. What do you mean by primal and dual problems ? Is the number of constraints in the primal and dual the same ? **[Kanpur 96]**

7.3. GENERAL RULES FOR CONVERTING ANY PRIMAL INTO ITS DUAL

If the system of constraints in a given LPP consists of a mixture of *equations, inequalities* ($\leq$ or $\geq$), *non-negative variables* or *unrestricted variables,* then the dual of the given problem can be obtained by reducing it to standard primal form by adopting the following alogrithm.

Step 1. First convert the objective function to maximization form, if not.

Step 2. If a constraint has inequality sign $\geq$, then multiply both sides by -1 and make the inequality sign $\leq$.

Step 3. If a constraint has an equality sign $(=)$, then it is replaced by two constraints involving the inequalities going in opposite directions, simultaneously.

For example, an equation, $x_1 + 2x_2 = 4$, is replaced by two opposite inequalities ($\leq$ and $\geq$) constraints :

$$x_1 + 2x_2 \leq 4 \quad \text{and} \quad x_1 + 2x_2 \geq 4 .$$

The second inequality with $\geq$ sign, can be further written as $-x_1 - 2x_2 \leq -4$.

Step 4. Every unrestricted variable is replaced by the difference of two non-negative variables.

Step 5. We get the *standard primal form,* of given LPP in which —

(*i*) all the constraints have '$\leq$' sign, where the objective function is of maximization form ; or

(*ii*) all the constraints have '$\geq$' sign, where the objective function is of minimization form.

Step 6. Finally, the dual of the given problem is obtained by :

(*i*) transposing the rows and columns of constraint coefficients;

(*ii*) transposing the coefficients $(c, c_2, \ldots, c_n)$ of the objective function and the right side constants $(b_1, b_2, \ldots, b_m)$;

(*iii*) changing the inequalities from '$\leq$' to '$\geq$' sign; and

(*iv*) minimizing the objective function instead of maximizing it.

[Note. The dual variables that correspond to primal equality constraints must be unrestricted in sign; and those associated with the primal inequalities must be non-negative]

Q. 1. State the general rules for converting any primal LPP into its dual.
2. Set up the dual when its primal is given in canonical form.
3. Write a note on duality in linear programming problem.

Example 1. *Find the dual of the following primal problem :*

Min. $z_x = 2x_2 + 5x_3$, *subject to* $x_1 + x_2 \geq 2,\ 2x_1 + x_2 + 6x_3 \leq 6,\ x_1 - x_2 + 3x_3 = 4,$ *and* $x_1, x_2, x_3 \geq 0$. **[Kanpur 2000, 96]**

Solution. First, convert the problem into *standard primal form,* as follows :

Step 1. *Change the objective function of minimization into maximization one,* that is,

$$\max.\ z'_x = -2x_2 - 5x_3 ,\ \text{where } z'_x = -z_x .$$

Step 2. The inequality $x_1 + x_2 \geq 2$ can be written as $-x_1 - x_2 \leq -2$.

Step 3. The equation $x_1 - x_2 + 3x_3 = 4$ can be expressed as a pair of inequalities :

$$\left\{\begin{array}{l} x_1 - x_2 + 3x_3 \leq 4 \\ x_1 - x_2 + 3x_3 \geq 4 \end{array}\right\} \quad \text{or} \quad \left\{\begin{array}{l} x_1 - x_2 + 3x_3 \leq 4 \\ -x_1 + x_2 - 3x_3 \leq -4 \end{array}\right\}$$

Step 4. Thus, original problem now becomes of the standard primal form :

Max. $z_x = 0x_1 - 2x_2 - 5x_3$, subject to

$$\left.\begin{aligned} -x_1 - x_2 \qquad &\leq -2 \\ 2x_1 + x_2 + 6x_3 &\leq \ \ 6 \\ x_1 - x_2 + 3x_3 &\leq \ \ 4 \\ -x_1 + x_2 - 3x_3 &\leq -4 \\ x_1, x_2, x_3 \qquad &\geq \ \ 0 . \end{aligned}\right\} \qquad \text{...(7·5)}$$

Step 5. Thus, by using rules of *Sec.* 7·3, the required dual is given by :

Min. $z'_w = -2w_1 + 6w_2 + 4w_3 - 4w_4$, subject to

$$\left.\begin{aligned} -w_1 + 2w_2 + w_3 - w_4 &\geq \ \ 0 \\ -w_1 + w_2 - w_3 + w_4 &\geq -2 \\ 6w_2 + 3w_3 - 3w_4 &\geq -5 \\ w_1, w_2, w_3, w_4 &\geq \ \ 0 \end{aligned}\right\} \qquad \text{...(7·6)}$$

Note. It is interesting to note that the primal (7·5) and its dual (7·6) both can be conveniently remembered at the same time by using the following tabular form :

$$\begin{array}{ccccc} & (x_1, & x_2, & x_3) & \text{Min.} \\ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} & & \begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & 6 \\ 1 & -1 & 3 \\ -1 & 1 & -3 \end{bmatrix} & \leq & \begin{bmatrix} -2 \\ 6 \\ 4 \\ -4 \end{bmatrix} . \\ & & \geq & & \\ & & \text{Max. } (0, -2, -5) & & \end{array}$$

Reading horizontally, we have the primal problem (7·5) and reading vertically, we have the corresponding dual problem (7·6) .

7.4. MORE ILLUSTRATIVE EXAMPLES

Example 2. *Write the dual of the following LP problem : Min.* $z = 3x_1 - 2x_2 + 4x_3$, *subject to the constraints*

$$3x_1 + 5x_2 + 4x_3 \geq 7, \quad 6x_1 + x_2 + 3x_3 \geq 4, \quad 7x_1 - 2x_2 - x_3 \leq 10,$$
$$x_1 - 2x_2 + 5x_3 \geq 3, \quad 4x_1 + 7x_2 - 2x_3 \geq 2, \text{ and } x_1, x_2, x_3 \geq 0 .$$

[Madras B.E. (Civil) 91; ICWA (June) 91]

Solution. The given problem can be written in the standard primal form as :

$$\text{Max.} \quad z_x' = -3x_1 + 2x_2 - 4x_3 , \text{ where } z'_x = -z$$

subject to the constraints :

$$\begin{aligned} -3x_1 - 5x_2 - 4x_3 &\leq -7 \\ -6x_1 - x_2 - 3x_3 &\leq -4 \\ 7x_1 - 2x_2 - x_3 &\leq 10 \\ -x_1 + 2x_2 - 5x_3 &\leq -3 \\ -4x_1 - 7x_2 + 2x_3 &\leq -2 \\ x_1, x_2, x_3 &\geq 0 . \end{aligned}$$

Following the rules of *Sec.* 7·3, the dual of this problem becomes :

Min. $z_w' = -7w_1 - 4w_2 + 10w_3 - 3w_4 - 2w_5$, subject to the constraints :

$$\begin{aligned} -3w_1 - 6w_2 + 7w_3 - w_4 - 4w_5 &\geq -3 \\ -5w_1 - w_2 - 2w_3 + 2w_4 - 7w_5 &\geq \ \ 2 \\ -4w_1 - 3w_2 - w_3 - 5w_4 + 2w_5 &\geq -4 \\ w_1, w_2, w_3, w_4, w_5 &\geq \ \ 0 . \end{aligned}$$

Example 3. *Obtain the dual of the following LP problem : Max.* $z = 2x_1 + 3x_2 + x_3$, subject to

$$4x_1 + 3x_2 + x_3 = 6, \ x_1 + 2x_2 + 5x_3 = 4, \quad \textit{and } x_1, x_2, x_3 \geq 0.$$

[Kanpur B.Sc. 95; Meerut 90]

Solution. The given problem is first written in the standard primal form :

Max. $z_x = 2x_1 + 3x_2 + x_3$, subject to the constraints :

$$4x_1 + 3x_2 + x_3 \le 6$$
$$-4x_1 - 3x_2 - x_3 \le -6$$
$$x_1 + 2x_2 + 5x_3 \le 4$$
$$-x_1 - 2x_2 - 5x_3 \le -4$$
$$x_1, x_2, x_3 \ge 0.$$

Following the rules, its dual is obtained as follows :

Minimize $z_w = 6(w_1 - w_2) + 4(w_3 - w_4)$

subject to the constraints :

$$4(w_1 - w_2) + (w_3 - w_4) \ge 2$$
$$3(w_1 - w_2) + 2(w_3 - w_4) \ge 3$$
$$(w_1 - w_2) + 5(w_3 - w_4) \ge 1$$
$$w_1, w_2, w_3, w_4 \ge 0.$$

Again, the dual can also be written as :

Minimize $z_w = 6y_1 + 4y_2$

subject to the constraints :

$$4y_1 + y_2 \ge 2$$
$$3y_1 + 2y_2 \ge 3$$
$$y_1 + 5y_2 \ge 1,$$

y_1, y_2 are unrestricted.

Example 4. *Give the dual of the LP problem : Min.* $z = 2x_1 + 3x_2 + 4x_3$, *subject to the constraints :*

$2x_1 + 3x_2 + 5x_3 \ge 2$, $3x_1 + x_2 + 7x_3 = 3$, $x_1 + 4x_2 + 6x_3 \le 5$, $x_1, x_2 \ge 0$ *and* x_3 *is unrestricted.*

[AIMS (BE Ind.) Bang. 2002]

Solution. Since the variable x_3 is unrestricted in sign, the given LP problem can be transformed into standard primal form by substituting $x_3 = x_3' - x_3''$, where $x_3' \ge 0$, $x_3'' \ge 0$. Therefore, standard primal becomes :

$$\text{Max. } z_x' = -2x_1 - 3x_2 - 4(x_3' - x_3'')$$

subject to the constraints :

$$-2x_1 - 3x_2 - 5(x_3' - x_3'') \le -2$$
$$3x_1 + x_2 + 7(x_3' - x_3'') \le 3$$
$$-3x_1 - x_2 - 7(x_3' - x_3'') \le -3$$
$$x_1 + 4x_2 + 6(x_3' - x_3'') \le 5$$
$$x_1, x_2, x_3', x_3'' \ge 0.$$

unrestricted eq. हो +sign के साथ और -sign के साथ use हो

The dual of the given standard primal is,

Min. $z_w' = -2w_1 + 3(w_2' - w_2'') + 5w_3$

subject to the constraints :

$$-2w_1 + 3(w_2' - w_2'') + w_3 \ge -2$$
$$-3w_1 + (w_2' - w_2'') + 4w_3 \ge -3$$
$$-5w_1 + 7(w_2' - w_2'') + 6w_3 \ge -4$$
$$5w_1 - 7(w_2' + w_2'') - 6w_3 \ge 4$$
$$w_1, w_2', w_2'', w_3 \ge 0$$

Again, we may write

Min. $z_w' = -2w_1 + 3w_2 + 5w_3$,

subject to the constraints :

$$-2w_1 + 3w_2 + w_3 \ge -2$$
$$-3w_1 + w_2 + 4w_3 \ge -3$$
$$5w_1 - 7w_2 - 6w_3 = 4$$

$w_1, w_3 \ge 0$ and w_2 is unrestricted.

Example 5. *Obtain the dual of the LP problem :*

Min. $z = x_1 + x_2 + x_3$. *subject to the constraints :*

$x_1 - 3x_2 + 4x_3 = 5$, $x_1 - 2x_2 \le 3$, $2x_2 - x_3 \ge 4$; $x_1, x_2 \ge 0$ *and* x_3 is *unrestricted.*

[JNTU (B. Tech) 98;Garhwal 97; Meerut M.Sc. (Math.) 94, (TDC) 90; Bharthidasan B.Sc. (Math.) 90]

Solution. Transform the given LP problem into the standard primal form by substituting $x_3 = x_3' - x_3''$, where $x_3' \ge 0$, $x_3'' \ge 0$.

$$\text{Max. } z_x' = -x_1 - x_2 - (x_3' - x_3''),\quad z_x = -z$$

subject to the constraints :

$$x_1 - 3x_2 + 4(x_3' - x_3'') \le 5$$
$$-x_1 + 3x_2 - 4(x_3' - x_3'') \le -5$$
$$x_1 - 2x_2 \le 3$$
$$-2x_2 + (x_3' - x_3'') \le -4$$
$$x_1, x_2, x_3', x_3'' \ge 0.$$

Let w_1', w_1'', w_2, w_3 be the dual variables. The dual problem of above standard primal is obtained as :

Min. $z_w' = 5(w_1' - w_1'') + 3w_2 - 4w_3$,

subject to the constraints :

$$\begin{aligned} (w_1' - w_1'') + w_2 + 0w_3 &\geq -1 \\ -3(w_1' - w_1'') - 2w_2 - 2w_3 &\geq -1 \\ 4(w_1' - w_1'') + 0w_2 + w_3 &\geq -1 \\ -4(w_1' - w_1'') + 0w_2 - w_3 &\geq 1 \\ w_1', w_1'', w_2, w_3 &\geq 0. \end{aligned}$$

This dual can be written in more compact from as : Max. $z_w = -5w_1 - 3w_2 + 4w_3$, subject to the constraints :

$$-w_1 - w_2 \leq 1, \quad 3w_1 + 2w_2 + 2w_3 \leq 1, \quad -4w_1 - w_3 = 1$$

$w_2, w_3 \geq 0$, and w_1 is unrestricted.

Example 6. *Give the dual of the linear programming problem :* Max.$z = 3x_1 - 2x_2$, *subject to*

$x_1 \leq 4, x_2 \leq 6, x_1 + x_2 \leq 5 \; -x_2 \leq -1$, *and* $x_1, x_2 \geq 0$

or $x_1 + x_2 \leq 5, x_1 \leq 4, 1 \leq x_2 \leq 6$; *and* $x_1, x_2 \geq 0$.

Solution. Since the given problem is already present in the standard primal form, we apply rules of ***Section* 7·3** to get the following dual problem : Min.$z = 4w_1 + 6w_2 + 5w_3 - 1w_4$, subject to,

$$\left.\begin{aligned} 1w_1 + 0w_2 + 1w_3 + 0w_4 &\geq 3 \\ 0w_1 + 1w_2 + 1w_3 - 1w_4 &\geq -2 \\ w_1, w_2, w_3, w_4 &\geq 0. \end{aligned}\right] \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} \geq \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Example 7. *Convert the following problem into its dual :*

Min. $z = 2x_1 + 2x_2 + 4x_3$, *subject to*

$2x_1 + 3x_2 + 5x_3 \geq 2, \quad 3x_1 + x_2 + 7x_3 \leq 3, \quad x_1 + 4x_2 + 6x_3 \leq 5; \quad x_1, x_2, x_3 \geq 0.$

Solution. Using the rules of ***Section* 7·3**, we get the standard primal form :

Max. $z' = -2x_1 - 2x_2 - 4x_3$, where $z' = -z$

subject to,

$$\left.\begin{aligned} -2x_1 - 3x_2 - 5x_3 &\leq -2 \\ 3x_1 + x_2 + 7x_3 &\leq 3 \\ x_1 + 4x_2 + 6x_3 &\leq 5 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}\right\} \quad \text{or} \quad \begin{bmatrix} -2 & -3 & -5 \\ 3 & 1 & 7 \\ 1 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}.$$

Applying the usual rules, we get the corresponding dual problem :

Min. $z' = -2w_1 + 3w_2 + 5w_3$

subject to

$$\left.\begin{aligned} -2w_1 + 3w_2 + 1w_3 &\geq -2 \\ -3w_1 + 1w_2 + 4w_3 &\geq -2 \\ -5w_1 + 7w_2 + 6w_3 &\geq -4 \\ w_1, w_2, w_3 &\geq 0. \end{aligned}\right] \quad \text{or} \quad \begin{bmatrix} -2 & 3 & 1 \\ -3 & 1 & 4 \\ -5 & 7 & 6 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \geq \begin{bmatrix} -2 \\ -2 \\ -4 \end{bmatrix}.$$

EXAMINATION PROBLEMS

Obtain the dual of the following linear programming problems :

1. Max. $3x_1 + 4x_2$, subject to $2x_1 + 6x_2 \leq 16$, $5x_1 + 2x_2 \geq 20$; $x_1, x_2 \geq 0$.
   [Ans. Min. $z_w = 16w_1 - 20w_2$, s.t. $2w_1 - 5w_2 \geq 3$, $6w_1 - 2w_2 \geq 4$].
2. Max. $z = x_1 - x_2 + 3x_3$, subject to the constraints :
   $x_1 + x_2 + x_3 \leq 10$, $2x_1 - x_3 \leq 2$, $2x_1 - 2x_2 + 3x_3 \leq 6$; $x_1, x_2, x_3 \geq 0$.
   [Ans. Min. $z_w = 10w_1 + 2w_2 + 6w_2$, s.t. $w_1 + 2w_2 + 2w_3 \geq 1$, $w_1 - 2w_3 \geq -1$, $w_1 - w_2 + 3w_3 \geq 3$; $w_1, w_2, w_3 \geq 0$].
3. Max. $z = 3x_1 + x_2 + 4x_3 + x_4 + 9x_5$, subject to the constraints :
   $4x_1 - 5x_2 - 9x_3 + x_4 - 2x_5 \leq 6$; $2x_1 + 3x_2 + 4x_3 - 5x_4 + x_5 \leq 9$; $x_1 + x_2 - 5x_3 - 7x_4 + 11x_5 \leq 10$, x_1, and $x_2, x_3, x_4, x_5 \geq 0$.
   [Ans. Min. $z_w = 6w_1 + 9w_2 + 10w_3$, s.t. $4w_1 + 2w_2 + w_3 \geq 3$; $-5w_1 + 3w_2 + w_3 \geq 1$,
   $-9w_1 + 4w_2 - 5w_3 \geq 4$, $w_1 - 5w_2 - 7w_3 \geq 1$, $-2w_1 + w_2 + 11w_3 \geq 9$; $w_1, w_2, w_3 \geq 0$]

5.5-4 Big-*M*-Method (Charne's Penalty Method) **[Kanpur (B.Sc.) 92, 91]**

Computational steps of big-*M*-method are as stated below :

Step 1. Express the problem in the standard form.

Step 2. Add non-negative artificial variables to the left side of each of the equations corresponding to constraints of the type ($\geq$) and '='. When artificial variables are added, it causes violation of the corresponding constraints. This difficulty is removed by introducing a condition which ensures that artificial variables will be zero in the final solution (provided the solution of the problem exists). On the other hand, if the problem does not have a solution, at least one of the artificial variables will appear in the final solution with positive value. This is achieved by assigning a very **large price (per unit penalty)** to these variables in the objective function. Such large price will be designated by $-M$ for maximization problems ($+M$ for minimization problems), where $M > 0$.

Step 3. In the last, use the artificial variables for the starting solution and proceed with the usual simplex routine until the optimal solution is obtained.

Q. 1. Explain the use of Big-*M*-method in solving L.P.P. What are its characteristics ?

Example 13. *Solve by using big-M method the following linear programming problem :*

$$\text{Max. } z = -2x_1 - x_2, \text{ subject to } 3x_1 + x_2 = 3,\ 4x_1 + 3x_2 \geq 6,\ x_1 + 2x_2 \leq 4, \text{ and } x_1, x_2 \geq 0.$$

Solution.

Step 1. Introducing slack, surplus and artificial variables, the system of constraint equations become :

$$\begin{aligned} 3x_1 + x_2 \qquad\quad + a_1 \qquad &= 3 \\ 4x_1 + 3x_2 - x_3 \qquad + a_2 &= 6 \\ x_1 + 2x_2 + x_4 \qquad\qquad &= 4 \end{aligned}$$

which can be written in the matrix form as :

$$\begin{array}{cccccc} \mathbf{X_1} & \mathbf{X_2} & \mathbf{X_3} & \mathbf{X_4} & \mathbf{A_1} & \mathbf{A_2} \end{array}$$
$$\begin{bmatrix} 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 4 \end{bmatrix}$$

Step 2. Assigning the large negative price $-M$ to the artificial variables a_1 and a_2, the objective function becomes : Max. $z = -2x_1 - x_2 + 0x_3 + 0x_4 - Ma_1 - Ma_2$.

Step 3. Construct starting simplex table (Table 5·24)

Starting Simplex Table 5·24

		$c_j \rightarrow$	-2	-1	0	0	$-M$	$-M$	
BASIC VARAIBLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{A_1}$	$\mathbf{A_2}$	MIN. RATIO $(\mathbf{X_B}/\mathbf{X_1})$
$\leftarrow a_1$	$-M$	3	$\boxed{3}$	1	0	0	1	0	$3/3 \leftarrow$
a_2	$-M$	6	4	3	-1	0	0	1	$6/4$
x_4	0	4	1	2	0	1	0	0	$4/1$
	$z = -9M$		$(2-7M)$ $\uparrow$	$(1-4M)$	M	0	0 $\downarrow$	0	$\leftarrow \Delta_j$

To apply optimality test, compute

$$\Delta_1 = \mathbf{C_B X_1} - c_1 = (-M, -M, 0)\,(3, 4, 1) - (-2) = 2 + (-3M - 4M + 0) = 2 - 7M$$
$$\Delta_2 = \mathbf{C_B X_2} - c_2 = (-M, -M, 0)\,(1, 3, 2) - (-1) = 1 + (-M - 3M + 0) = 1 - 4M$$
$$\Delta_3 = \mathbf{C_B X_3} - c_3 = (-M, -M, 0)\,(0, -1, 0) + 0 = M$$

$\therefore \quad \Delta_k = \min\,[\Delta_1, \Delta_2, \Delta_3] = \min\,[2 - 7M, 1 - 4M, M] = \Delta_1$. Therefore, $\mathbf{X_1}$ will be entered.

Using minimum ratio rule, find the key element 3 which indicates that $\mathbf{A_1}$ should be removed. Now the transformed table (*Table 5·25*) is obtained in usual manner.

First Improved Table 5·25

	$c_j \rightarrow$		-2	-1	0	0	$-M$	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	MIN RATIO (X_B/X_2)
$\rightarrow x_1$	-2	1	1	1/3	0	0	1/3	0	$1/\frac{1}{3}$
$\leftarrow a_2$	$-M$	2	0	$\boxed{5/3}$	1	0	4/3	1	$2/\frac{5}{3} \leftarrow$
s_4	0	3	0	5/3	0	1	$-1/3$	0	$3/\frac{5}{3}$
	$z = -2 - 2M$		0	$(1-5M)/3$ ↑	M	0	$(-2+7M)/3$	0 ↓	$\leftarrow \Delta_j$

Again compute, $\Delta_2 = \mathbf{C_B X_2} - c_2 = (-2, -M, 0)(1/3, 5/3, 5/3) + 1 = (1-5M)/3$, and similarly, $\Delta_3 = M$, $\Delta_5 = (-2+7M)/3$.

Since *minimum* Δ_j *rule and minimum ratio rule* decide the key element $5/3$, so enter $\mathbf{X_2}$ and remove $\mathbf{A_2}$. Therefore, the second improved table (Table 5.26) is formed.

Table 5·26

	$c_j \rightarrow$		-2	-1	0	0	$-M$	$-M$	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	MIN. RATIO
x_1	-2	3/5	1	0	1/5	0	3/5	$-1/5$	
x_2	-1	6/5	0	1	$-3/5$	0	$-4/5$	3/5	
x_4	0	1	0	0	1	1	1	-1	
	$z = \mathbf{C_B X_B} = -12/5$		0	0	1/5	0	$M - 2/5$	$M - 1/5$	$\leftarrow \Delta_j \geq 0$

To test the solution for optimality, compute

$$\Delta_3 = \mathbf{C_B X_3} - c_3 = (-2, -1, 0)(1/5, -3/5, 1) - 0 = 1/5$$
$$\Delta_5 = \mathbf{C_B A_2} - c_5 = (-2, -1, 0)(3/5, -4/5, 1) + M = M - 2/5$$
$$\Delta_6 = \mathbf{C_B A_2} - c_6 = (-2, -1, 0)(-1/5, -3/5, -1) + M = M - 1/5.$$

Since M is as large as possible, $\Delta_3, \Delta_5, \Delta_6$ are all positive. Consequently, the optimal solution is : $x_1 = 3/5$, $x_2 = 6/5$, max $\mathbf{z} = -12/5$.

Example 14. *Solve the following problem by Big-M-method : Max.* $z = x_1 + 2x_2 + 3x_3 - x_4$, *subject to :* $x_1 + 2x_2 + 3x_3 = 15$, $2x_1 + x_2 + 5x_3 = 20$, $x_1 + 2x_2 + x_3 + x_4 = 10$, *and* $x_1, x_2, x_3, x_4 \geq 0$.

Imp

[IAS (Maths.) 95; Kanpur (B.Sc.) 92; Karala (B.Sc.) 91; Meerut (B.Sc.) 90]

Solution. Since the constraints of the given problem are equations, introduce the artificial variables $a_1 \geq 0$, $a_2 \geq 0$. The problem thus becomes :

Max. $z = x_1 + 2x_2 + 3x_3 - x_4 - Ma_1 - Ma_2$, subject to the constraints :

$$\begin{aligned} x_1 + 2x_2 + 3x_3 \quad + a_1 \quad &= 15 \\ 2x_1 + x_2 + 5x_3 \quad + a_2 &= 20 \\ x_1 + 2x_2 + x_3 + x_4 \quad &= 10 \\ \text{and } x_1, x_2, x_3, x_4, a_1, a_2 &\geq 0. \end{aligned}$$

Now applying the usual simplex method, the solution is obtained as given in the *Table 5·27.*

Table 5·27 (Example 14)

		$c_j \rightarrow$	1	2	3	−1	−M	−M	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	A_1	A_2	MIN RATIO (X_B/X_2)
a_1	$-M$	15	1	2	3	0	1	0	15/3
$\leftarrow a_2$	$-M$	20	2	1	$\boxed{5}$	0	0	1	20/5 ←
x_4	-1	10	1	2	1	1	0	0	10/1
	$z = (-35M - 10)$		$(-3M-2)$	$(-3M-2)$	$(-8M-4)$ ↑	0	0	0 ↓	$\leftarrow \Delta_j$
$\leftarrow a_1$	$-M$	3	$-1/5$	$\boxed{7/5}$	0	0	1	×	$\frac{3}{7/5}$ ←
$\rightarrow x_3$	3	4	2/5	1/5	1	0	0	×	$\frac{4}{1/5}$
x_4	-1	6	3/5	9/5	0	1	0	×	$\frac{6}{9/5}$
	$z = (-3M + 6)$		$(M-2)/5$	$-(7M-16)/5$ ↑	0	0	0 ↓	×	$\leftarrow \Delta_j$
$\rightarrow x_2$	2	15/7	$-1/7$	1	0	0	×	×	—
x_3	3	25/7	3/7	0	1	0	×	×	25/3
$\leftarrow x_4$	-1	15/7	$\boxed{6/7}$	0	0	1	×	×	15/6 ←
	$z = 90/7$		$-6/7$* ↑	0	0	0 ↓	×	×	$\leftarrow \Delta_j$
x_2	2	15/6	0	1	0	1/6	×	×	
x_3	3	15/6	0	0	1	3/6	×	×	
$\rightarrow x_1$	1	15/6	1	0	0	7/6	×	×	
	$z = 15$		0	0	0	75/36	×	×	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution has been obtained as :

$$x_1 = x_2 = x_3 = \frac{15}{6} = \frac{5}{2}, \text{ max } z = 15 .$$

Example 15. *Use penalty (Big-M) method to maximize* : $z = 3x_1 - x_2$ *subject to the constraints :*

$$2x_1 + x_2 \geq 2,\ x_1 + 3x_2 \leq 3,\ x_2 \leq 4, \text{ and } x_1, x_2 \geq 0 .$$

Solution. By introducing the surplus variable $x_3 \geq 0$, artificial variable $a_1 \geq 0$, and slack variables $x_4 \geq 0$, $x_5 \geq 0$, the problem becomes : Max. $z = 3x_1 - x_2 + 0x_3 + 0x_4 + 0x_5 - Ma_1$, subject to the constraints :

$$\begin{aligned} 2x_1 + x_2 - x_3 \; {\scriptstyle s_1} \quad + a_1 &= 2 \\ x_1 + 3x_2 \quad + x_4 \; {\scriptstyle s_2} \quad &= 3 \\ x_2 \quad + x_5 \; {\scriptstyle s_3} &= 4 \\ x_1, x_2, x_3, x_4, x_5, a_1 &\geq 0 . \end{aligned}$$

In matrix form, $\begin{bmatrix} 2 & 1 & -1 & 0 & 0 & 1 \\ 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ a_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Now the solution is obtained as given in Table 5.28

11. A city hospital has the following minimal daily requirements for nurses.

Period	Clock Time (24 hr. day)	Minimal Number of Nurses Required
1	6 A.M. – 10 A.M.	2
2	10 A.M. – 2 P.M.	7
3	2 P.M. – 6 P.M.	15
4	6 P.M. – 10 P.M.	8
5	10 P.M. – 2 A.M.	20
6	2 A.M. – 6 A.M.	6

Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimum number of nurses to be employed so that there will be sufficient number of nurses available for each period. Formulate this as a linear programming problem by setting up appropriate constraints and objective function. Do not solve.

[**Hint.** Let $x_1, x_2, x_3, x_4, x_5, x_6$ be the number of nurses on duty at 6 *A.M.*, 10 *A.M.*, 2 *P.M.size5* , 6 *P.M.*, 10 *P.M.*, and 2 *A.M.*, respectively. Then the required LP formulation will be as follows :

Minimize $\mathbf{z} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

subject to the constraints :

$$\begin{aligned}
x_1 + x_2 &\geq 7\\
x_2 + x_3 &\geq 15\\
x_3 + x_4 &\geq 8\\
x_4 + x_5 &\geq 20\\
x_5 + x_6 &\geq 6\\
x_6 + x_1 &\geq 2\\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0].
\end{aligned}$$

3.3. GRAPHICAL SOLUTION OF TWO VARIABLE PROBLEMS

3.3–1. Graphical Procedure

Simple linear programming problems of two decision variables can be easily solved by *graphical method*. The outlines of graphical procedure are as follows :

Step 1. Consider each inequality–constraint as equation.

Step 2. Plot each equation on the graph, as each one will geometrically represent a straight line.

Step 3. Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality-constraint corresponding to that line is ' ≤ ', then the region *below* the line *lying* in the first quadrant (due to non-negativity of variables) is shaded. For the inequality-constraint with ' ≥ ' sign, the region *above* the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtained is called the ***feasible region.***

Step 4. Choose the convenient value of **z** (say = 0) and plot the objective function line.

Step 5. Pull the objective function line until the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

Step 6. Read the coordinates of the extreme point(s) selceted in *Step 5*, and find the maximum or mininum (as the case may be) value of **z**. The following examples will make the outlined graphical procedure clear.

Q. 1. What is meant by linear programming problem ? Give brief description of the problem with illustrations. How the same can be solved graphically. What are the basic characteristics of a linear programming problem ? **[Meerut (Stat.) 98]**

2. Explain briefly the graphical method of solving linear programming problems. State its advantages and limitations.

3.3-2. Graphical Solution of Properly Behaved LP Problems

Example 26. *Find a geometrical interpretation and solution as well for the following LP problem :*

Maximize $\mathbf{z} = 3x_1 + 5x_2$, *subject to restrictions :*

$x_1 + 2x_2 \leq 2000,\ x_1 + x_2 \leq 1500,\ x_2 \leq 600,$ *and* $x_1 \geq 0, x_2 \geq 0$.

Graphical Solution.

Step 1. ***(To graph the inequality-constraints).*** Consider two mutually perpendicular lines OX_1 and OX_2 as axes of coordinates. Obviously, any point (x_1, x_2) in the positive quadrant will certainly satisfy non-negativity restrictions $x_1 \geq 0, x_2 \geq 0$. To plot the line $x_1 + 2x_2 = 2000$, put $x_2 = 0$, find $x_1 = 2000$ from this equation.

Then mark a point L such that $OL = 2000$ by assuming a suitable scale, say 500 units = 2 cm. Similarly, again put $x_1 = 0$ to find $x_2 = 1000$ and mark another point M such that $OM = 1000$.

Now join the points L and M. This line will represent the equation $x_1 + 2x_2 = 2000$ as shown in the above figure.

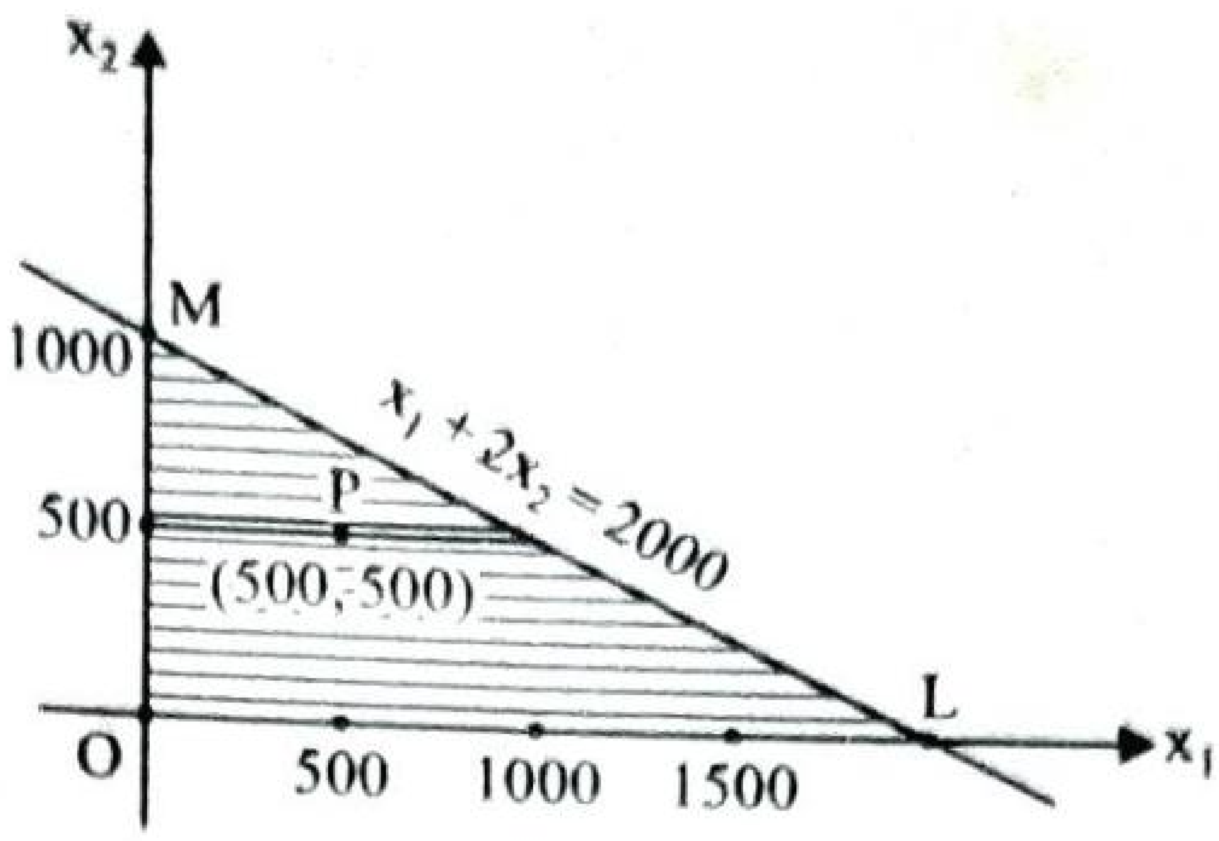

Fig. 3.1

Clearly, any point P lying on or below the line $x_1 + 2x_2 = 2000$ will satisfy the inequality $x_1 + 2x_2 \leq 2000$. (If we take a point (500, 500), *i.e.*, $x_1 = 500$, $x_2 = 500$, then we have $500 + 2 \times 500 < 2000$, which is true).

Similar procedure is now adoped to plot the other two lines : $x_1 + x_2 = 1500$ and $x_2 = 600$ as shown in the Figs. 3.2 and 3.3, respectively. Any point on or below the lines $x_1 + x_2 = 1500$ and $x_2 = 600$ will also satisfy other two inequalities : $x_1 + x_2 \leq 1500$, and $x_2 \leq 600$, respectively.

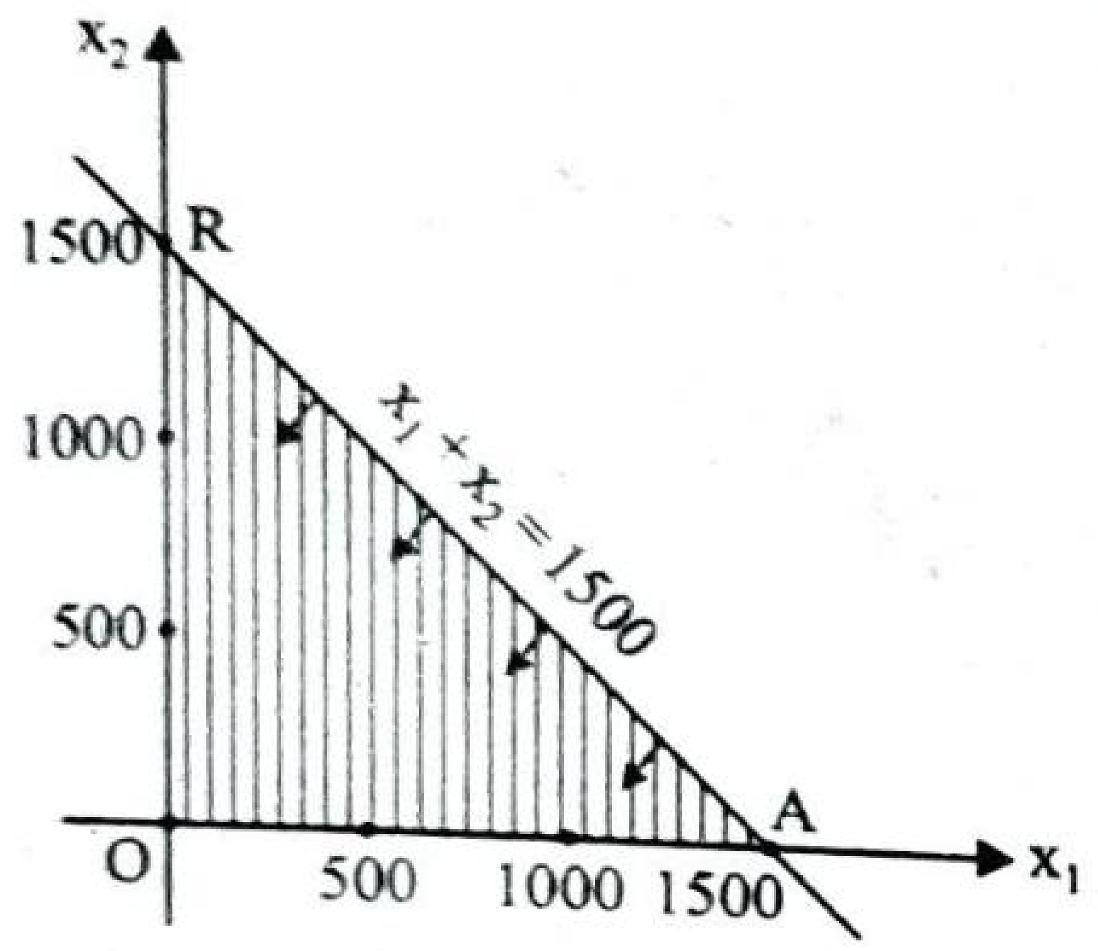

Fig. 3.2

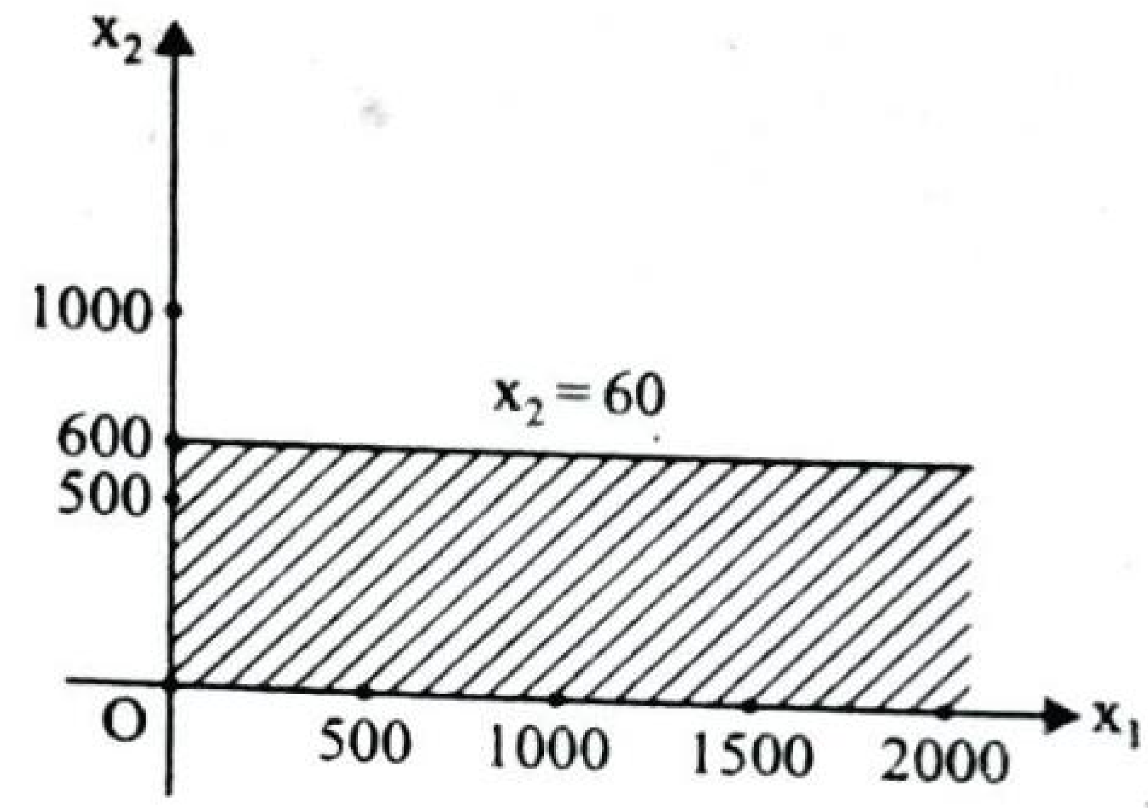

Fig. 3.3

Step 2. Find the *feasible region* or *solution space* by combining the Figs. 3.1, 3.2 and 3.3 together. A common shaded area *OABCD* is obtained (see Fig. 3.4) which is a set of points satisfying the inequality constraints :

$$x_1 + 2x_2 \leq 2000, \; x_1 + x_2 \leq 1500, \; x_2 \leq 600,$$

and non-negativity restrictions as $x_1 \geq 0, x_2 \geq 0$. *Hence any point in the shaded area* (including its boundary) is feasible solution to the given LPP.

Step 3. Find the co-ordinates of the corner points of feasible region O, A, B, C and D.

Step 4. Locate the corner point of optimal solution either by calculating the value of **z** for each corner point O, A, B, C, and D (or by adopting the following procedure).

Here, the problem is to find the point or points in the feasible region (collection of all feasible solutions) which maximize(s) the objective (or profit) function. For some fixed value of $\mathbf{z}$, $\mathbf{z} = 3x_1 + 5x_2$ is a straight line and *any point on it gives the same value of* **z**. Also, it should be noted that the lines corresponding to different values of **z** are parallel, because the gradient $(-3/5)$ of the line $\mathbf{z} = 3x_1 + 5x_2$ remains the same throughout. For $\mathbf{z} = 0$, *i.e.*, $0 = 3x_1 + 5x_2$, means a line which passes through the origin. To draw the line $3x_1 + 5x_2 = 0$, determine the ratio $\dfrac{x_1}{x_2} = \dfrac{-5}{3} = \dfrac{-500}{300}$.

Mark the point E moving 500 units distance from the origin on the negative side of X_1-axis. Then find the points F such that $EF = 300$ units in the positive direction of X_2-axis. Joining the point F and O, draw the line

$3x_1 + 5x_2 = 0$. Now go on drawing the lines parallel to this line until at least a line is found which is farthest from the origin but passes through at least one corner of the feasible region at which the maximum value of **z** is attained. It is also possible that such a line may coincide with one of the edge of feasible region. In that case, every point on that edge gives the maximum value of **z**.

In this example, maximum value of **z** is attained at the corner point B (1000, 500), which is the point of intersection of lines $x_1 + 2x_2 = 2000$ and $x_1 + x_2 = 1500$. Hence, the required solution is $x_1 = 1000$, $x_2 = 500$ and max. value $\mathbf{z} =$ Rs. 5500.

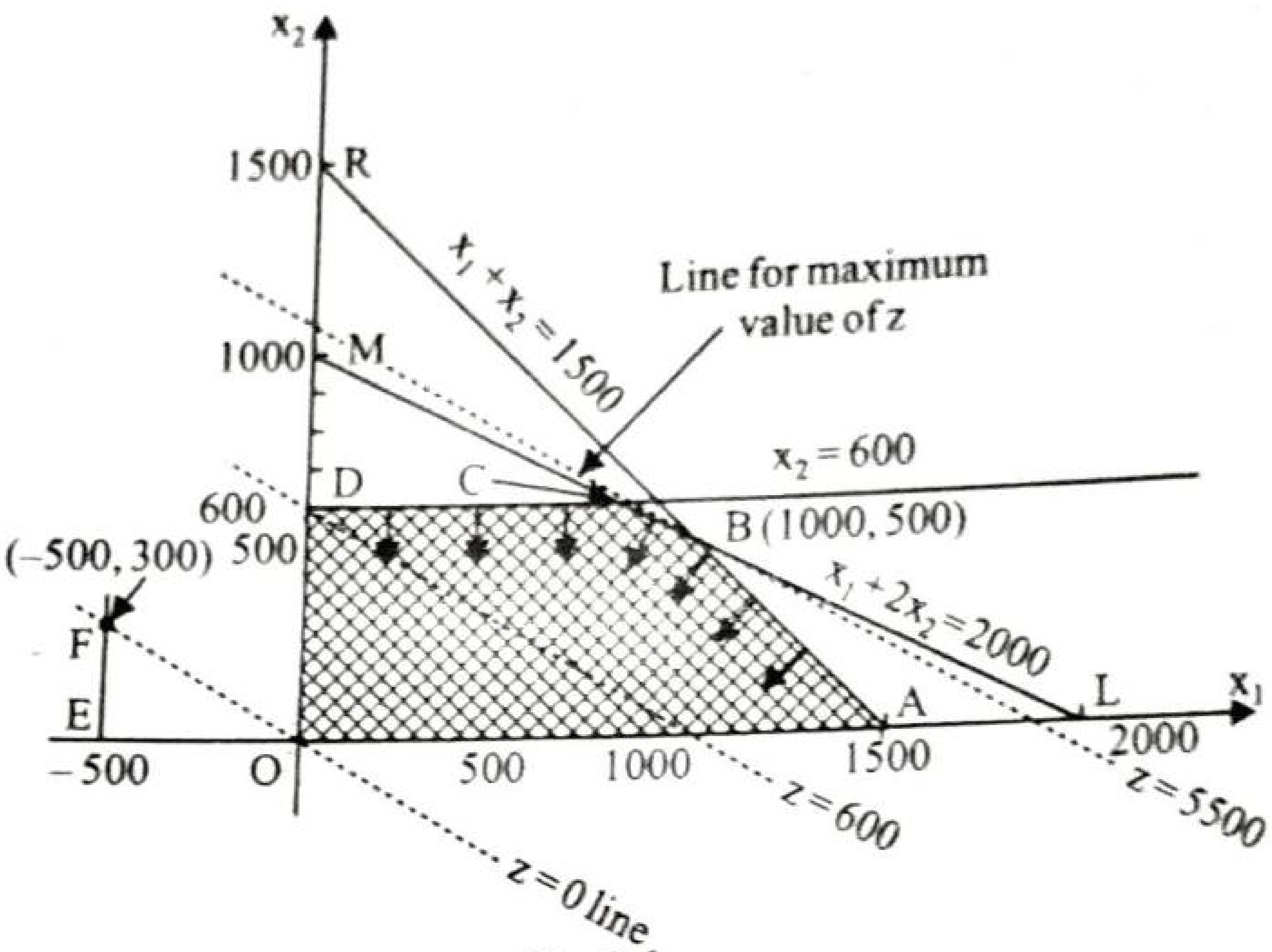

Fig. 3.4.

Note. If the number of vertices of feasible region is small, find the coordinates of vertices. As in above example, $O = (0, 0)$, $A = (1500, 0)$, $B = (1000, 500)$, $C = (800, 600)$, $D = (0, 600)$ are obtained by solving the pair of lines whose intersections are these points, respectively. The value of **z** corresponding to these points will be $z_O = 0$, $z_A = 4500$, $z_B = 5500$, $z_C = 4500$, $z_D = 3000$. Clearly $z_B = 5500$ is maximum for the point B (1000, 500) which gives the required solution.

Example 27. *Consider the problem*

$$\text{Max. } \mathbf{z} = x_1 + x_2, \text{ subject to,}$$

$$x_1 + 2x_2 \le 2000$$

$$x_1 + x_2 \le 1500$$

$$x_2 \le 600$$

and $$x_1, x_2 \ge 0.$$

Graphical Solution. This problem is of the same type as discussed earlier except the objective function is slightly changed. The feasible region will be similar to that of the above problem. Fig. 3.5 shows the objective function lines of the problem for three different values $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3$ of **z**.

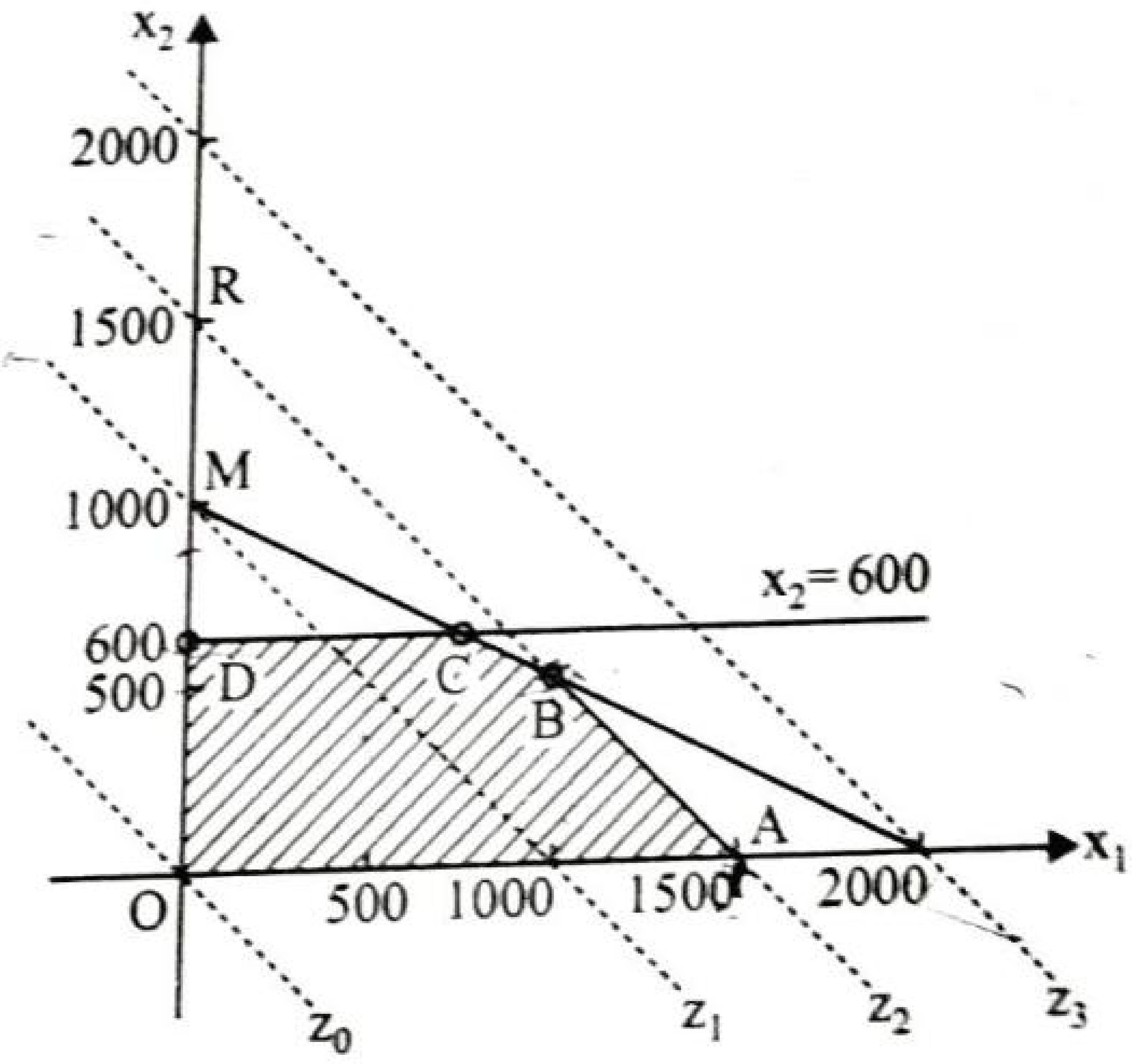

Fig. 3.5

It is clear from Fig. 3.5 that $\mathbf{z}_2$ is the maximum value of **z**. It is quite interesting that the line $\mathbf{z}_2$ representing the objective function lies along the edge AB of the polygon of feasible solutions. This indicates that the values of x_1 and x_2 which maximize **z** are not unique, but any point on the edge AB of $OABCD$ the polygon will give the optimum value of **z**. The maximum value of **z** is always unique, but there will be an infinite number of feasible solutions which give unique value of **z**. Thus, two corners A and B as well as any point on the line AB (segment) give optimal solution of this problem.

It should be noted that if a linear programming problem has more than one optimum solution, there exists alternative optimum solutions. And, one of the optimum solutions will be corresponding to corner point B, *i.e.* $x_1 = 1000$, $x_2 = 500$ with max. profit $\mathbf{z} =$ Rs. 1500.

Example 28. *Solve the following LP problem graphically :*

$$\text{Max. } z = 8000x_1 + 7000x_2, \text{ subject to}$$

$$3x_1 + x_2 \le 66,\ x_1 + x_2 \le 45,\ x_1 \le 20,\ x_2 \le 40 \text{ and } x_1, x_2 \ge 0.$$

Solution. First, plot the lines $3x_1 + x_2 = 66$, $x_1 + x_2 = 45$, $x_1 = 20$ and $x_2 = 40$ and then shade the feasible region as shown in Fig. 3.6.