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CLASSTIME	Page No.
Date / /	

DC Assignment 2

Ques 1 For an equiprobable binary baseband data the optimal receiver receives -5mV for 0 and +5mV for 1, corrupted with white noise of PSD 10^{-8} W/Hz. With optimum decision threshold what is the probability of error in reception if data rate is 9600 b/s.

Sol^n (a) We know, probability of error for optimal receiver that uses optimal filter and optimal threshold,

$$P_e = 0.5 \, \mathrm{erfc}\left(\sqrt{\frac{E_s}{\eta}}\right) \text{ where}$$

$$= 10^{-9} \text{ W/Hz}$$

With matched filter based optimal filter,

$$E_s = V^2 T$$

$$= (0.005)^2 (9600)^{-1}$$

$$E_s = 0.26 \times 10^{-8}.$$

$$\therefore P_e = 0.5 \, \mathrm{erfc}\left(\sqrt{\frac{0.26 \times 10^{-8}}{2 \times 10^{-9}}}\right)$$

$$= 0.5 \, \mathrm{erfc}(1.1402)$$

$$\boxed{P_e = 0.0534.}$$

(b) Find percentage increase in error rate if rate is doubled

With doubled rate,

$$E_s = V^2 T$$

$$= (0.005)^2 (2 \times 9600)^{-1}$$

$$E_s = 0.13 \times 10^{-8}$$

Then

$$P_e = 0.5 \, \mathrm{erfc}\left(\sqrt{\frac{0.13 \times 10^{-8}}{2 \times 10^{-9}}}\right)$$

$$\boxed{P_e = 0.1271}$$

CLASSTIME Page No.
Date / /

Percentage increase in error = 138.03 %.

(c) If we want probability of error at increased data rate so what should be input voltage levels?
To restore probability of error input voltage level is to be increased so that E_s is as it was in (a).
Since T is halved, V is to be increased $\sqrt{2}$ times from definition of E_s.
Thus, required voltage levels are
$\pm 5\sqrt{2}$ mV i.e. ± 7.07 mV

Ques2 Find the probability of error of the Match filter.
Ans2 We know that error probability of optimum filter is expressed as:-

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right]$$

Or

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\,df$$

$\therefore$ PSD of white noise $= S_{ni}(f) = \frac{N_0}{2}$

Hence,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{\frac{N_0}{2}}\,df$$

$$= \frac{2}{N_0}\int_{-\infty}^{\infty} |X(f)|^2\,df$$

Also, parseval's power theorem states that

$$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_0^T x^2(t)\,dt$$

or $$\int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} \left(x_1(t) - x_2(t)\right)^2 dt$$

$$\int_{-\infty}^{\infty} |x(f)|^2 df = \int_0^T x_1^2(t)dt + \int_0^T x_2^2(t)dt - 2\int_0^T x_1(t)x_2(t)dt.$$

where $\int_0^T x_1^2(t)\,dt = E_1$ (energy of $x_1(t)$ by standard relations)

and $\int_0^T x_2^2(t)\,dt = E_2$ (energy of $x_2(t)$ " ")

and $\int_0^T x_1(t)\,x_2(t)\,dt = E_{12}$ (energy due to autocorrelation b/w $x_1(t)$ & $x_2(t)$)

If we choose $x_1(t) = -x_2(t)$ then these energies will be equal i.e.

$$E_1 = E_2 = -E_{12} = E$$

$\therefore$

$$\int_{-\infty}^{\infty} |x(f)|^2 df = \left[E + E - 2(-E)\right] = 4E$$

Substitute in eq (2)

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0} \cdot 4E = \frac{8E}{N_0}.$$

Therefore

$$\frac{x_{01}(T) - x_{02}(T)}{\sigma} = 2\sqrt{2}\sqrt{\frac{E}{N_0}}.$$

$\therefore$ Probability of Error of matched filter

$$\boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E}{N_0}}\right)}$$

CLASSTIME Page No.
Date / /

Ques 3 Calculate generalized expression for probability of error when decision threshold is set to optimum.

Soln: Probability of error

$$P_e = P(s_1) \int_{V<\lambda} f(V/s_1)\,dV + P(s_2) \int_{V>\lambda} P(V/s_2)\,dV$$

$$P_e = P(s_1) \int_{V<\lambda} \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[\frac{-(\vartheta - V_1)^2}{2\sigma_n^2}\right] d\vartheta$$

$$+ P(s_2) \int \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[\frac{-(\vartheta - V_2)^2}{2\sigma_n^2}\right] d\vartheta$$

where

$$\lambda = \frac{V_1 + V_2}{2} + \frac{\sigma_n^2 \ln\{P(s_2)/P(s_1)\}}{(V_1 - V_2)}$$

Substite $\frac{\vartheta - V_1}{\sqrt{2}\sigma_n} = y$ in 1st integral & $\frac{\vartheta - V_2}{\sqrt{2}\sigma_n}$ in 2nd Integ. &

$$P_e = \frac{P(s_1)}{\sqrt{\pi}} \int_{V<\lambda_1} \exp[-y]^2\,dy + \frac{P(s_2)}{\sqrt{\pi}} \int \exp[-z^2$$

where $\lambda_1 = \frac{\lambda - V_1}{\sqrt{2}\sigma_n}$ & $\lambda_2 = \frac{\lambda - V_2}{\sqrt{2}\sigma_n}$

$$\lambda_1 = \frac{\frac{V_1 + V_2}{2} + \frac{\sigma_n^2 \ln\{P(s_2)/P(s_1)\}}{V_1 - V_2} - V_1}{\sqrt{2}\sigma_n}.$$

or

$$\lambda_1 = \frac{V_2 - V_1}{2\sqrt{2}\sigma_n} + \frac{\sigma_n \ln\{P(s_2)/P(s_1)\}}{\sqrt{2}(V_1 - V_2)}$$

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and

$$x_2 = \frac{V_1 - V_2}{2\sqrt{2}\,\sigma_n} + \frac{\sigma_n \ln\{P(S_2)/P(S_1)\}}{\sqrt{2}\,(V_1 - V_2)}$$

$$P_e = \frac{P(S_1)}{2}\left[\mathrm{erfc}(-\infty) - \mathrm{erfc}(x_1)\right] + \frac{P(S_2)}{2}\,\mathrm{erfc}(x_2)$$

$$P_e = \frac{P(S_1)}{2}\left[2 - \mathrm{erfc}(x_1)\right] + \frac{P(S_2)}{2}\,\mathrm{erfc}(x_2).$$

$$P_e = \frac{1}{2}\left[2P(S_1) - P(S_1)\,\mathrm{erfc}\left(\frac{V_2 - V_1}{2\sqrt{2}\,\sigma_n} + \frac{\sigma_n \ln\{P(S_2)/P(S_1)\}}{\sqrt{2}\,(V_1 - V_2)}\right) + P(S_2)\,\mathrm{erfc}\left(\frac{V_1 - V_2}{2\sqrt{2}\,\sigma_n} + \frac{\sigma_n \ln\{P(S_2)/P(S_1)\}}{\sqrt{2}\,(V_1 - V_2)}\right)\right]$$

Ques 4 Calculate the probability of error of BFSK & BPSK signal.

Ans 4 Error probability in BFSK.

Two different carrier frequencies are used to transmit two binary levels.

Binary '1' ⇒ $x_1(t) = A\cos(\omega_c + \Omega)t$

Binary '0' ⇒ $x_2(t) = A\cos(\omega_c - \Omega)t$

General Expression for Probability of error (P_e) is -

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left[\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right]$$

Detector which maximises ratio $\gamma = \frac{P_0(T)}{\sigma}$

$= \frac{x_{01}(T) - x_{02}(T)}{\sigma}$ is matched filter.

Output SNR,

$$y^2 = \frac{P_0^2(T)}{\sigma^2} = \int_{-\infty}^{\infty} \frac{|P(t)|^2}{G(t)} df = \frac{2}{N_0} |P(t)|$$

By Parseval's theorem

$$\int_{-\infty}^{\infty} |P(t)|^2 dt = \int_{-\infty}^{\infty} P^2(t)\, dt = \int_0^T P^2(t)\, dt$$

$$\Rightarrow \int_0^T P^2(t)\, dt = A \int_0^T [\cos(w_0 + \pi)t - \cos(w_0 - \pi)t]^2 dt$$

$$= A \int_0^T \Big[\frac{1 + \cos^2 (w_0 + \pi)t}{2} + \frac{1 + \cos^2 (w_0 - \pi)t}{2}$$

$$- 2\cos w_0 t - \cos 2\pi t \Big] dt.$$

Integrating & substituting limits,

$$TA \Big[1 - \frac{\sin 2\pi T}{2\pi T} + \frac{1}{2} \frac{\sin 2(w_0 + \pi)T}{2(w_0 + \pi)T} + \frac{1}{2} \frac{\sin 2(w_0 - \pi)T}{2(w_0 - \pi)T}$$

$$- \frac{\sin 2 w_0 T}{2 w_0 T} \Big]$$

Ratio approaches zero as $w_0 T$ increases

$$\int_0^T P^2(t)\, dt = A^2 T \left[1 - \frac{\sin 2\pi T}{2\pi T} \right]$$

For orthogonal Tone spacing, $\pi T = nT$ and $\frac{\sin 2\pi T}{2\pi T}$

$$\int_0^T P^2(t)\, dt = A^2 T$$

$$\therefore y^2 = \frac{P_0^2(t)}{\sigma^2} = \frac{2A^2 T}{N_0}.$$

Hence $P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{P_0(t)}{2\sqrt{2}\sigma} \right] = \frac{1}{2} \operatorname{erfc}\left[\frac{P_0^2(t)}{8\sigma^2} \right]$

CLASSTIME Page No.
Date / /

$= \frac{1}{2} \operatorname{erfc} \left[\frac{2A^2T}{8N_0}\right]^{1/2} = \frac{1}{2} \operatorname{erfc} \left[\frac{A^2T}{4N_0}\right]^{1/2}$

$P_e = \frac{1}{2} \operatorname{erfc} \left[\frac{A^2T}{2} \cdot \frac{1}{2N_0}\right]^{1/2} = \frac{1}{2} \operatorname{erfc} \left[\frac{E_b}{2N_0}\right]^{1/2}$

$\therefore$ P_e for BFSK

$$\boxed{P_e = \frac{1}{2} \operatorname{erfc} \left[\frac{E_b}{2N_0}\right]^{1/2}}$$

Error probability in BPSK Signal

Binary '1' $\quad x_1(t) = \sqrt{2P} \cos(2\pi f_c t)$

Binary '0' $\quad x_2(t) = -\sqrt{2P} \cos(2\pi f_c t)$

For matched filter detection the presence of white gaussian noise,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0} \int_0^T x^2(t)\, dt$$

we know

$$x(t) = x_1(t) - x_2(t)$$

$$\& \quad x_2(t) = -x_1(t).$$

$$\therefore \quad x(t) = 2x_1(t).$$

$\therefore$

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0} \int_0^T 4x_1^2(t)\, dt$$

$$= \frac{8}{N_0} \int_0^T x_1^2(t)\, dt$$

$$\int x_1^2(t)\, dt = P\left[\int_0^T dt + \int_0^T \cos 4\pi f_c t \, dt\right]$$

$$\int_0^T x_1^2(t)\, dt = P\left[\int_0^T dt\right] = P(t)\Big|_0^T = PT = E.$$

Energy $E = P \times T$.

CLASSTIME Page No.
Date / /

Substituting above in eq (4)

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{8}{N_0} \cdot E.$$

or

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]_{max} = \sqrt{\frac{8E}{N_0}}.$$

$$\therefore \quad P_e = \frac{1}{2}\, \text{erfc}\left\{\frac{1}{2\sqrt{2}}\sqrt{\frac{8E}{N_0}}\right\}$$

$$\boxed{P_e = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E}{N_0}}\right)}$$

Ques Discuss the similarities between MSK & offset QPSK and features that distinguish them.

Ans One of the comparison b/w MSK & OQPSK concern the spectra of two systems.
Maximum phase change in QPSK is $\pm 90°$ or $\pm 180°$ whereas in MSK is $\pm 90°$
MSK is similar to OQPSK, MSK is encoded with bits alternating b/w Quadrature components with Q component delayed by half symbol period. Instead of square pulses as OQPSK uses, MSK encodes each bit as a half sinusoid.
In OQPSK transmitted signal

$$V_{OQPSK}(t) = \sqrt{P_s}\; b_e(t) \cos \omega_0 t + \sqrt{P_s}\, b_o(t) \sin \omega_0 t$$

In MSK transmitted signal.

$$V_{MSK}(t) = \sqrt{2P_0}\left[b_e(t) \sin 2\pi\left(\frac{t}{4T_b}\right)\right] \cos \omega_0 t$$

$$+ \sqrt{2P_0}\left[b_o(t) \cos 2\pi\left(\frac{t}{4T_b}\right)\right] \sin \omega_0 t$$

CLASSTIME Page No.
Date / /

Important Differences b/w OQPSK & MSK

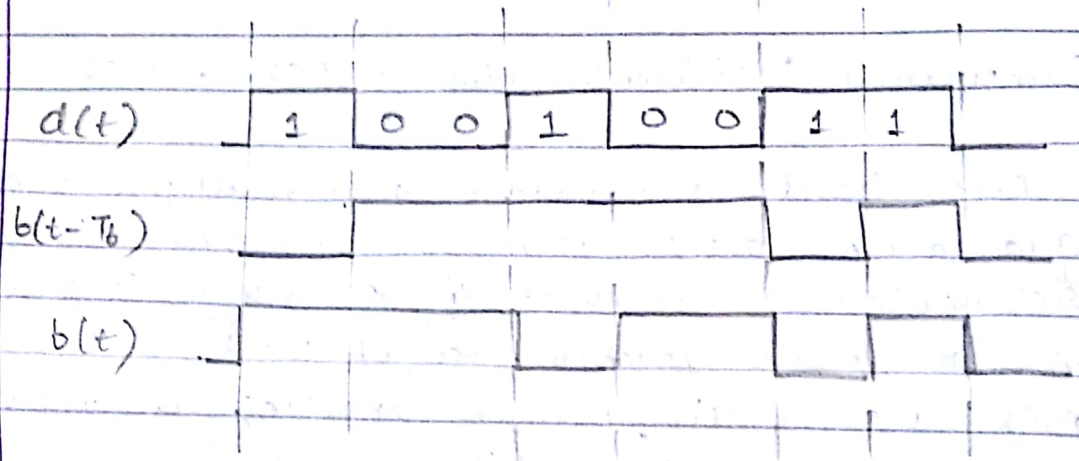
① In MSK baseband waveform, that multiplies the quadrature carrier is much smoother than rectangular waveform of QPSK. While spectrum of MSK has main spectrum lobe which is 1.5 times as main lobe of QPSK, side lobes in MSK are relatively much smaller in comparison to main lobe making filtering much easier.

② The waveform of MSK exhibits phase continuum i.e. there is no abrupt phase changes as in QPSK.

Ques 6 For given data b(t) 10010011 generate DPSK signal m(t). Also from recieved signal m(t) extract original data sequence b(t).

Ans:

Binary data b(t)		1	0	0	1	0	0	1	1
Differentially encoded data d(k)	0^*	0	1	0	0	1	0	0	0
Phase of DPSK	π	π	0	π	π	0	π	π	π
Shifted differentially encoded data (d_{k-1})		0	0	1	0	0	1	0	0
Phase of shifted DPSK		π	π	0	π	π	0	π	π
Phase comparison output		+	−	−	+	−	−	+	+
Detected binary sequence		1	0	0	1	0	0	1	1



Ques: Show that ~~our~~ orthogonal BFSK has minimum separation b/w two data point in comparison to non orthogonal BFSK scheme.

Soln: Geometric representation of orthogonal BFSK

Any signal can be represented as $C_1u_1(t)+C_2u_2(t)$
where

$u_1(t)$ & $u_2(t)$ are orthogonal vectors in space

$$U_1(t)=\sqrt{\frac{2}{T_s}}\cos w_0 t$$

$$U_2(t)=\sqrt{\frac{2}{T_s}}\sin w_0 t$$

Normalised energies associated with $C_1U_1(t)$ & $C_2U_2(t)$ are C_1^2 & C_2^2 & total energy signal is $C_1^2+C_2^2$.

Similarly,

Unit vectors $U_1(t)=\sqrt{\frac{2}{T_b}}\cos 2\pi m f_b t$ (In BFSK)

$$U_2(t)=\sqrt{\frac{2}{T_b}}\cos 2\pi n f_b \{t\} \text{ (In BFSK)}$$

$$f_b=\frac{1}{T_b}$$

U_1 & U_2 are n^{th} harmonics of freq. f_b

$f_H = m f_b$, $f_L = n f_b$.

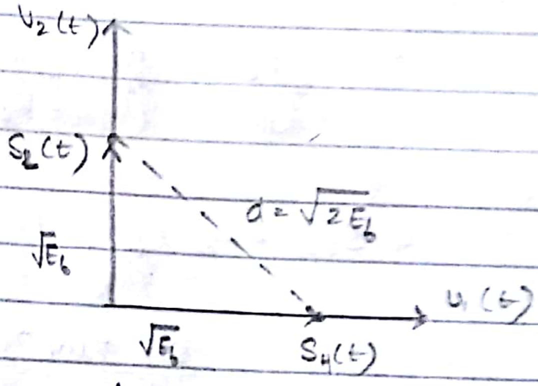
Then corresponding signal vectors are

$S_H(t) = \sqrt{E_b}\ u_1(t)$

$S_L(t) = \sqrt{E_b}\ u_2(t)$.

Distance b/w points is

$d = \sqrt{2E_b}$



Geometrical Representation of Non Orthogonal BFSK.

When two signals are non orthogonal, Gram - Schmidt procedure can be used to represent signals.

$S_H(t) = \sqrt{2P_s}\ \cos\omega_H t$

$= S_{11}\ u_1(t) \qquad 0 \le t \le T_b$

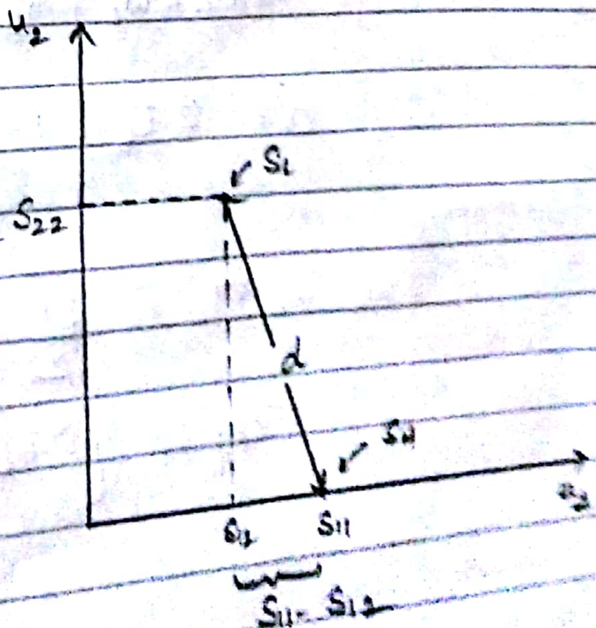
$S_L(t) = \sqrt{2P_s}\ \cos\omega_L t$

$= S_{12}\ u_1(t) + S_{22}\ u_2(t) \qquad 0 \le t \le T_b$

Distance b/w these is

$d^2_{BFSK} = (S_{11} - S_{12})^2 + S_{22}^2$

$d^2_{BFSK} = S_{11}^2 - 2S_{11} S_{12} + S_{12}^2 + S_{22}^2$



$$S_{11}^2 = 2P_s \int_0^{T} \cos^2 \omega_H t \, dt = E_b \left[1 + \frac{\sin 2\omega_H T_b}{2\omega_H T_b} \right]$$

$$S_{12}^2 = \sqrt{2P_s} \int_0^{T} u_1(t) \cos \omega_L t \, dt$$

$$= \frac{E_b}{S_{11}} \left[\frac{\sin(\omega_H - \omega_L) T_b}{(\omega_H - \omega_L) T_b} + \frac{\sin(\omega_H + \omega_L) T_b}{(\omega_H + \omega_L) T_b} \right]$$

$$S_{12}^2 + S_{22}^2 = E_b \left[1 + \frac{\sin 2\omega_L T_b}{2\omega_L T_b} \right]$$

$$d^2 = E_b \left[1 + \frac{\sin 2\omega_H T_b}{2\omega_H T_b} \right] - 2E_b \left[\frac{\sin(\omega_H - \omega_L) T_b}{(\omega_H - \omega_L) T_b} \right.$$

$$\left. + \frac{\sin(\omega_H + \omega_L) T_b}{(\omega_H + \omega_L) T_b} \right] + E_b \left[1 + \frac{\sin 2\omega_L T_b}{2\omega_L T_b} \right]$$

Assume

$$\left| \frac{\sin 2\omega_H T_b}{2\omega_H T_b} \right| << 1 \; ; \; \left| \frac{\sin 2\omega_L T_b}{2\omega_L T_b} \right| << 1$$

$$\left| \frac{\sin(\omega_H + \omega_L) T_b}{(\omega_H + \omega_L) T_b} \right| << \sin \left| \frac{(\omega_H + \omega_L) T_b}{(\omega_H + \omega_L) T_b} \right|$$

$$\therefore \quad d^2 = 2E_b \left[1 - \frac{\sin(\omega_H - \omega_L) T_b}{(\omega_H - \omega_L) T_b} \right]$$

$$(\omega_H - \omega_L) T_b = \frac{3\pi}{2}$$

$$d_{opt} = \left[2E_b \left(1 + \frac{2}{3\pi} \right) \right]^{1/2} = \sqrt{2.4 \, E_b}$$