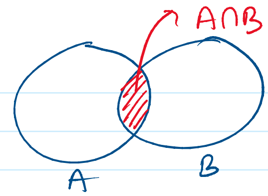


$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

5.7 THE INCLUSION–EXCLUSION PRINCIPLE

Let A and B be any finite sets. Recall Theorem 1.9 which tells us:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

In other words, to find the number $n(A \cup B)$ of elements in the union of A and B, we add $n(A)$ and $n(B)$ and then we subtract $n(A \cap B)$; that is, we "include" $n(A)$ and $n(B)$, and we "exclude" $n(A \cap B)$. This follows from the fact that, when we add $n(A)$ and $n(B)$, we have counted the elements of $(A \cap B)$ twice.

The above principle holds for any number of sets. We first state it for three sets.

Theorem 5.8: For any finite sets A, B, C we have

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

That is, we "include" $n(A), n(B), n(C)$, we "exclude" $n(A \cap B), n(A \cap C), n(B \cap C)$, and finally "include" $n(A \cap B \cap C)$.

Note

four Sets -

$|A| = n(A)$

Odd Number Include

Even - Exclude

$$\begin{aligned}|A_1 \cup A_2 \cup A_3 \cup A_4| = {} & \underbrace{|A_1| + |A_2| + |A_3| + |A_4|}_{\text{odd (Include)}} \\ & \underbrace{- |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4|}_{\text{Even (Exclude)}} \\ & - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| \\ & + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|.\end{aligned}$$

THE PRINCIPLE OF INCLUSION–EXCLUSION Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$\begin{aligned}|A_1 \cup A_2 \cup \cdots \cup A_n| = {} & \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| \\ & + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.\end{aligned}$$

$(-1)^{n+1}$

$n(A \cup B \cup C) =$

EXAMPLE 5.11 Find the number of mathematics students at a college taking at least one of the languages French, German, and Russian, given the following data:

65 study French,	20 study French and German,	
45 study German,	25 study French and Russian,	8 study all three languages.
42 study Russian,	15 study German and Russian,	

INCLUSION–EXCLUSION PRINCIPLE

5.61. Suppose 32 students are in an art class A and 24 students are in a biology class B, and suppose 10 students are in both classes. Find the number of students who are:

(a) in class A or in class B; (b) only in class A; (c) only in class B.

5.62. A survey of 80 car owners shows that 24 own a foreign-made car and 60 own a domestic-made car. Find the number of them who own:

(a) both a foreign made car and a domestic made car;

(b) only a foreign made car;

(c) only a domestic made car.

5.63. Consider all integers from 1 up to and including 100. Find the number of them that are:
(a) odd or the square of an integer; (b) even or the cube of an integer.

5.64. In a class of 30 students, 10 got A on the first test, 9 got A on a second test, and 15 did not get an A on either test. Find: the number of students who got:

(a) an A on both tests;

(b) an A on the first test but not the second;

(c) an A on the second test but not the first.

5.65. Consider all integers from 1 up to and including 300. Find the number of them that are divisible by:

(b) an A on the first test but not the second;

(c) an A on the second test but not the first.

5.65. Consider all integers from 1 up to and including 300. Find the number of them that are divisible by:

(a) at least one of 3, 5, 7; (c) by 5, but by neither 3 nor 7;
(b) 3 and 5 but not by 7; (d) by none of the numbers 3, 5, 7.

5.66. In a certain school, French (F), Spanish (S), and German (G) are the only foreign languages taught. Among 80 students:

(i) 20 study F, 25 study S, 15 study G.

(ii) 8 study F and S, 6 study S and G, 5 study F and G.

(iii) 2 study all three languages.

Find the number of the 80 students who are studying:

(a) none of the languages; (c) only one language; (e) exactly two of the languages.
(b) only French; (d) only Spanish and German;

5.67. Find the number m of elements in the union of sets A, B, C, D where:

(i) A, B, C, D have 50, 60, 70, 80 elements, respectively.

(ii) Each pair of sets has 20 elements in common.

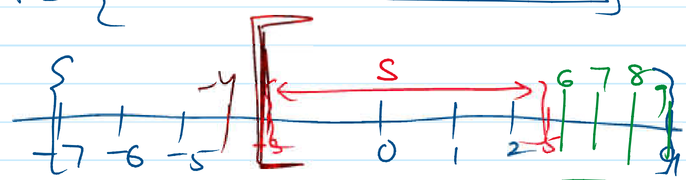
(iii) Each three of the sets has 10 elements in common.

(iv) All four of the sets have 5 elements in common.

$S \subseteq A$

$A = \{-7 \quad 9\}$

$S = \{-3 - 5\}$



$a \in A$ for S

$\forall x \in S \quad x \le a$

Upper Bound of S = 5, 6, 7, 8, 9

l.u.b = Least Smallest = 5

Lower Bound $a \in A$ is L.B of S if $\forall x \in S \quad x \ge a$

Lower bound of S = -3, -4, -5, -6, -7

glb = greatest (Maxi) = -3

Unique

5.6 THE PIGEONHOLE PRINCIPLE

Many results in combinational theory come from the following almost obvious statement.

Pigeonhole Principle: If n pigeonholes are occupied by $n + 1$ or more pigeons, then at least one pigeonhole is occupied by more than one pigeon.

This principle can be applied to many problems where we want to show that a given situation can occur.

Generalized Pigeonhole Principle: If n pigeonholes are occupied by $kn + 1$ or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by $k + 1$ or more pigeons.

EXAMPLE 5.10 Find the minimum number of students in a class to be sure that three of them are born in the same month.

JAN Feb - - - - Dec

$n = 12$

$k+1 = 3$

JAN Feb — — — — Dec

$n = 12$

$K+1 = 3$

$\boxed{K=2}$

In One Month there will be three born Studen

Mm: Total No. of Student (pigeon) $= kn+1 = 2\times 12+1 = 24+1 = 25$

PIGEONHOLE PRINCIPLE

5.19. Find the minimum number n of integers to be selected from $S = \{1, 2, \ldots, 9\}$ so that: (a) The sum of two of the n integers is even. (b) The difference of two of the n integers is 5.

(a) The sum of two even integers or of two odd integers is even. Consider the subsets $\{1, 3, 5, 7, 9\}$ and $\{2, 4, 6, 8\}$ of S as pigeonholes. Hence $n = 3$.

(b) Consider the five subsets $\{1, 6\}$, $\{2, 7\}$, $\{3, 8\}$, $\{4, 9\}$, $\{5\}$ of S as pigeonholes. Then $n = 6$ will guarantee that two integers will belong to one of the subsets and their difference will be 5.

5.20. Find the minimum number of students needed to guarantee that five of them belong to the same class (Freshman, Sophomore, Junior, Senior).

Here the $n = 4$ classes are the pigeonholes and $k + 1 = 5$ so $k = 4$. Thus among any $kn + 1 = 17$ students (pigeons), five of them belong to the same class.

5.21. Let L be a list (not necessarily in alphabetical order) of the 26 letters in the English alphabet (which consists of 5 vowels, A, E, I, O, U, and 21 consonants).

(a) Show that L has a sublist consisting of four or more consecutive consonants.

(b) Assuming L begins with a vowel, say A, show that L has a sublist consisting of five or more consecutive consonants.

(a) The five letters partition L into $n = 6$ sublists (pigeonholes) of consecutive consonants. Here $k + 1 = 4$ and so $k = 3$. Hence $nk + 1 = 6(3) + 1 = 19 < 21$. Hence some sublist has at least four consecutive consonants.

(b) Since L begins with a vowel, the remainder of the vowels partition L into $n = 5$ sublists. Here $k + 1 = 5$ and so $k = 4$. Hence $kn + 1 = 21$. Thus some sublist has at least five consecutive consonants.

PIGEONHOLE PRINCIPLE

5.68. Find the minimum number of students needed to guarantee that 4 of them were born: (a) on the same day of the week; (b) in the same month.

5.69. Find the minimum number of students needed to guarantee that 3 of them:

(a) have last names which begin with the same first letter;

(b) were born on the same day of a month (with 31 days).

5.70. Consider a tournament with n players where each player plays against every other player. Suppose each player wins at least once. Show that at least 2 of the players have the same number of wins.

5.71. Suppose 5 points are chosen at random in the interior of an equilateral triangle T where each side has length two inches. Show that the distance between two of the points must be less than one inch.

5.72. Consider any set $X = \{x_1, x_2, \ldots, x_7\}$ of seven distinct integers. Show that there exist $x, y \in X$ such that $x + y$ or $x - y$ is divisible by 10.