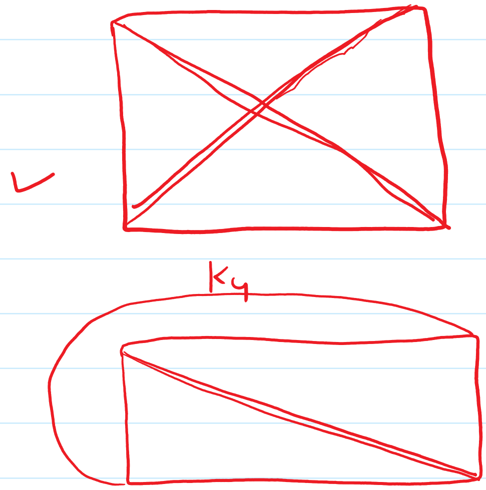
Planar Graphs

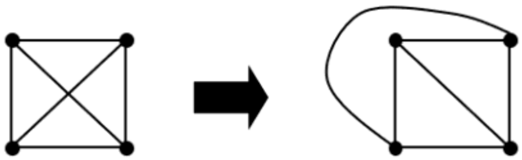
- A graph is called *planar* if it can be drawn in the plane without any edges crossing.
- A crossing of edges is the intersection of the lines or arcs representing them at a point other than their common endpoint.
- Such a drawing is called a *planar representation* of the graph.



K4 is a planar graph.

Example

A graph may be planar even if it is usually drawn with crossings, since it may be possible to draw it in another way without crossings.



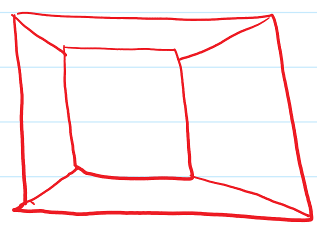
K4 K4

K4 is a planar graph

Example

A graph may be planar even if it represents a 3-dimensional object.

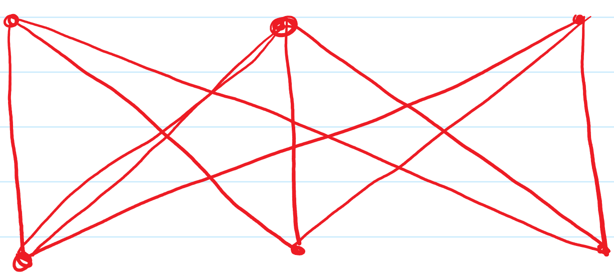
Q_3 ✓



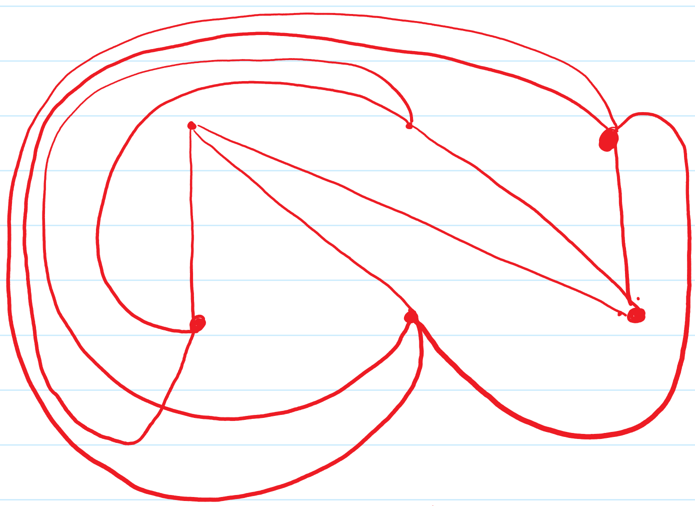
Q_3

Q_3 is a planar graph

Q_3, K_4



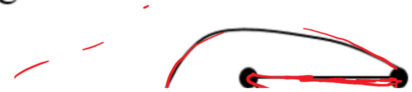
$K_{3,3}$



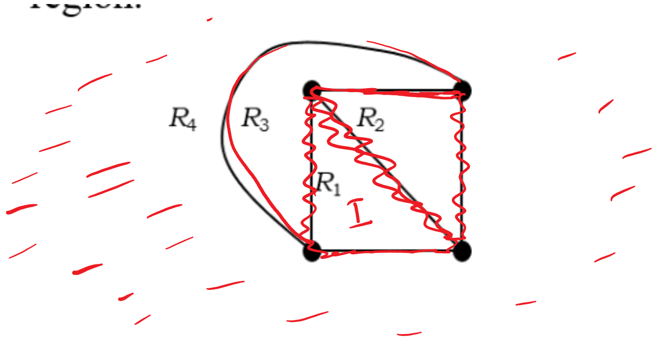
$K_{3,3}$ is not a planargraph

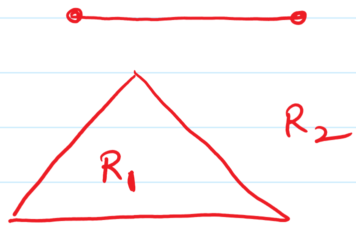
Regions

- Euler showed that all planar representations of a graph split the plane into the same number of *regions*, including an unbounded region.



region.





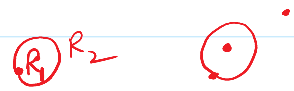
Euler's Formula for a planar graph:

In a connected simple planar graph $\boxed{V - E + R = 2}$

Pf: (1) We shall prove the result with the help of PMI.

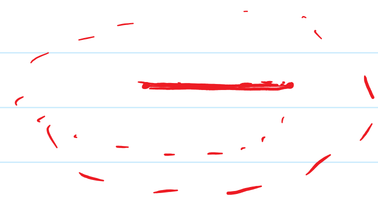
Case 1 $\quad V = 1, \quad E = 1, \quad R = 2$

$V - E + R = 1 - 1 + 2 = 2$



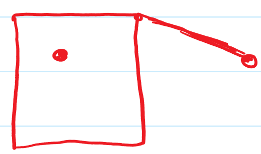
Case 2 $\quad V = 2, \quad E = 1, \quad R = 1$

$V - E + R = 2 - \cancel{1} + \cancel{1} = 2$



(2) Assume that result is true for V_k vertices, E_k edges and R_k regions.

$V - E + R = V_k - E_k + R_k = 2$



(3)

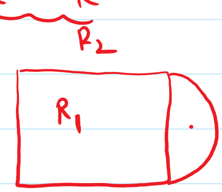
i) increase one vertex, Join this vertex with ~~one~~ one already existing vertex

$v_{k+1} = v_k + 1, \quad E_{k+1} = E_k + 1, \quad R_{k+1} = R_k$

$v_{k+1} - E_{k+1} + R_{k+1} = (v_k + 1) - (E_k + 1) + R_k$

$= v_k + \cancel{1} - E_k - \cancel{1} + R_k = v_k - E_k + R_k = 2$

(ii) We now increase only one edge, Just join two previously existing vertices.



join two previously existing vertices.

$v_{k+1} = v_k$, $\cancel{E}_{k+1} = E_k + 1$, $R_{k+1} = R_k + 1$

$$v_{k+1} - E_{k+1} + R_{k+1} = v_k - (E_k+1) + (R_k+1) = v_k - E_k - \cancel{1} + R_k + \cancel{1}$$
$$= v_k - E_k + R_k = 2$$

$\therefore$ For every positive integer result is true.

$$\boxed{v - E + R = 2}$$

Q If we have 8 vertices of degree 2, then find no. of regions.

Sol^n <u>By Hand Shaking theorem.</u>

$$\sum_{i=1}^{8} \deg(v_i) = 2e$$

$$\deg(v_1) + \deg(v_2) + \cdots + \deg(v_8) = 2e$$

$$2 + 2 + \cdots + 2 = 2e$$

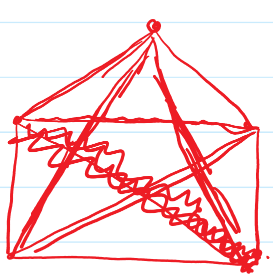
$$\cancel{16} = \cancel{2}e$$

$$\boxed{e = 8}$$

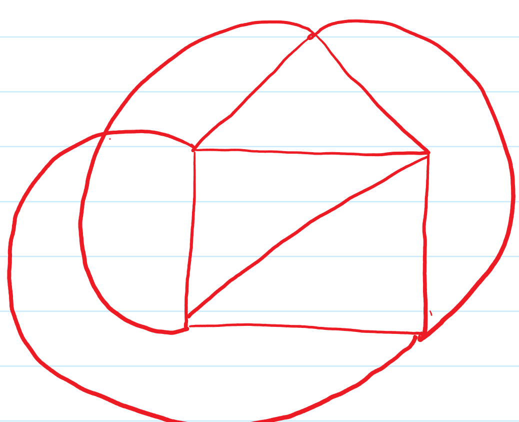
$V - E + R = 2$

$\cancel{8} - \cancel{8} + R = 2$

$\boxed{R = 2}$



K_5



<u>K_5 is a non-planar graph.</u>

Euler's Formula (Cont.)

Euler's Formula (Cont.)

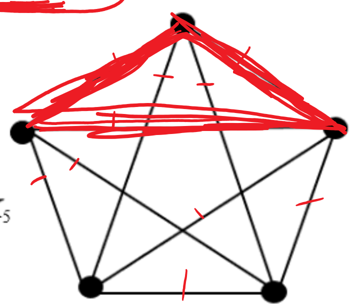
- Corollary 1: If G is a connected planar simple graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v - 6$.
- Is K_5 planar?

(i) $e < v - 6$

(ii) $e < 2v - 6$

(iii) $e \leq 3v - 6$ ✓

(iv) $e \leq 4v - 6$



K_5

We Assume that K_5 is a planar graph.

$v = 5$,

$e \leq 3v - 6$

$10 \leq 3(5) - 6$

$10 \leq 9$

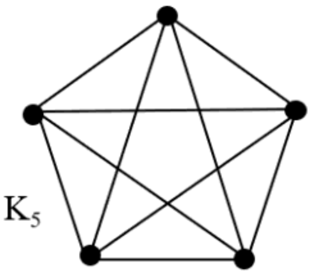
This is not possible

$\Rightarrow$ K_5 is not a planar graph.

—×—

Euler's Formula (Cont.)

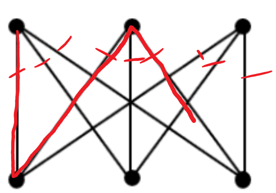
- K_5 has 5 vertices and 10 edges.
- We see that $v \geq 3$.
- So, if K_5 is planar, it must be true that $e \leq 3v - 6$.
- $3v - 6 = 3*5 - 6 = 15 - 6 = 9$.
- So e must be ≤ 9.
- But $e = 10$.
- So, K_5 is nonplanar.



K_5

Euler's Formula (Cont.)

- Corollary 3: If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length 3, then $e \leq 2v - 4$.
- Is $K_{3,3}$ planar?



if possible suppose that $K_{3,3}$ is a planar.
and $K_{3,3}$ doesn't contain a circuit of lenght 3

$e \leq 2v - 4$

$9 \leq 2(6) - 4$

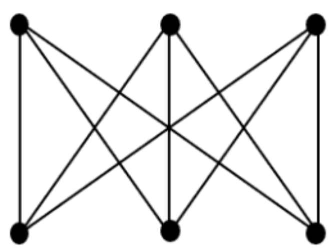
$9 \leq 8$

This is not possible
So this graph $K_{3,3}$ is a

This is not possible
So this graph $K_{3,3}$ is a Non-planar graph.

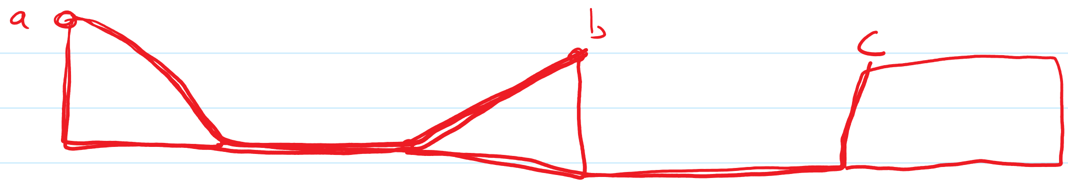
Euler's Formula (Cont.)

- $K_{3,3}$ has 6 vertices and 9 edges.
- Obviously, $v \geq 3$ and there are no circuits of length 3.
- If $K_{3,3}$ were planar, then $e \leq 2v - 4$ would have to be true.
- $2v - 4 = 2*6 - 4 = 8$
- So e must be ≤ 8.
- But $e = 9$.
- So $K_{3,3}$ is nonplanar.

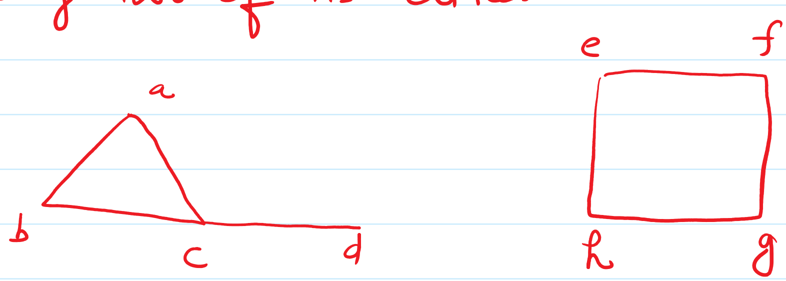


$K_{3,3}$

Connected graph: A graph is called a Connected graph if there exist a path between every pair of vertices.

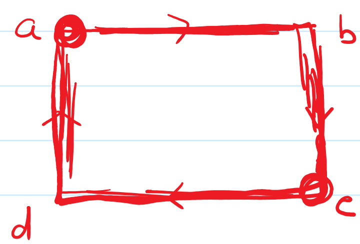


Disconnected graph: A graph is called a disconnected graph if there is no path between any two of its vertices.

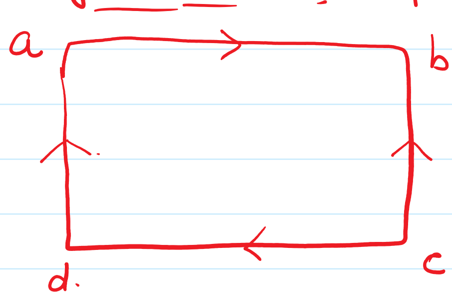


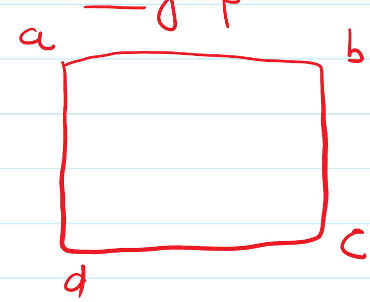
This graph is disconnected graph.

Strongly connected graph: A directed graph is called a strongly connected if there is a directed path from any node u to v and vice-versa.



Weakly connected graph: A directed graph is called a weakly connected if its undirected graph is connected.

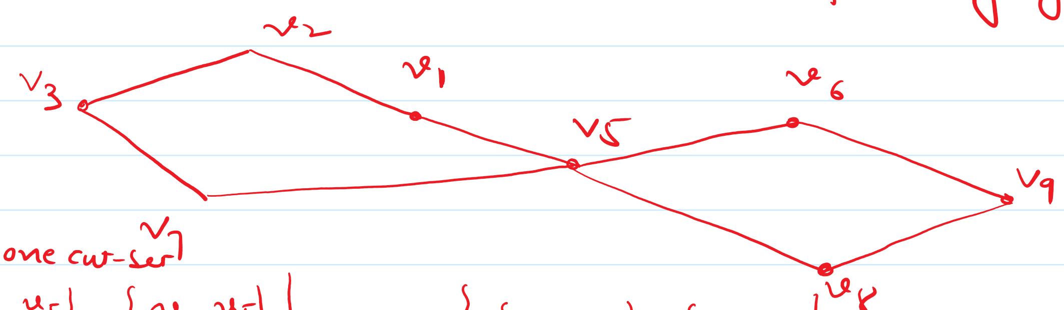




Cut Set :

We mean remove those edges from the graph so that after the removal of these edges graph becomes disconnected but if we incorporate any removed edge in the graph then graph becomes connected.

Q Determine the cut set for the following graph.



This is one cut-set

$\{ \{v_1, v_5\}, \{v_7, v_5\} \}$, $\{ \{v_5, v_6\}, \{v_5, v_8\} \}$