1. If $f = x^2 + y^2$, $x = r + 3s$, $y = 2r - s$, then $\dfrac{\partial f}{\partial r}$ is

   (i) $4x + 2y$ (ii) $2x + y$ (iii) $2x + 4y$ (iv) $x + 4y$ **Ans.** (iii)

   $$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$
   $$= (2x)(1) + (2y)(2)$$
   $$= 2x + 4y$$

2. If $f = x + 4y$, $x = 2s + t$, $y = s + 2t$, then $\dfrac{\partial f}{\partial t}$ is

   (i) 9 (ii) 8 (iii) 7 (iv) -7 **Ans.** (i)

3. If $z = xy$, $x = e^r \cos\theta$, $y = e^\theta \sin r$, then $\dfrac{\partial z}{\partial r}$ is

   (i) $xy - x e^\theta \cos r$ (ii) $xy + x e^\theta \cos r$
   (iii) $xy + x e^\theta \sin r$ (iv) $xy + y e^\theta \cos r$ **Ans.** (ii)

4. If $z = x^2 + y^2$ and $x = r + t$, $y = r^2 + t^2$, then $\dfrac{\partial z}{\partial t}$ is

   (i) $x + 6yt$ (ii) $2x + 2yt$ (iii) $x + 4yt$ (iv) $2x + 4yt$ **Ans.** (iv)

5. If $z = x + y$, $x = e^{r\cos\theta}$, $y = e^{r\sin\theta}$, then $\dfrac{\partial z}{\partial \theta}$ is

   (i) $r(\cos\theta\, e^{r\cos\theta} - \sin\theta\, e^{r\sin\theta})$ (ii) $r(\cos\theta\, e^{r\sin\theta} - \sin\theta\, e^{r\cos\theta})$
   (iii) $r e^r(\cos\theta - \sin\theta)$ (iv) $r(\cos\theta\, e^{r\sin\theta} + \sin\theta\, e^{r\cos\theta})$ **Ans.** (ii)

6. If $z = x^2 y^2$, and $x = s\log r$, $y = r\log s$ then $\dfrac{\partial z}{\partial r}$ is

   (i) $2xy\left(\dfrac{xs}{r} + y\log s\right)$ (ii) $2xy(ys + x\log s)$
   (iii) $2xy\left(\dfrac{ys}{r} + x\log s\right)$ (iv) $2xy\left(\dfrac{ys}{r} - x\log s\right)$ **Ans.** (iii)

13. If $z = x^2 y^2$, $x = t$ and $y = 2t$ then $\frac{\partial z}{\partial t}$ is equal to

(i) $2xy(2x - y)$
(ii) $xy(2x + y)$
(iii) $2xy(x + 2y)$
(iv) $2xy(2x + y)$

Ans. (iv)

14. If $z = x^3y^3$ then $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$ is equal to

(i) $6xy(x^2 + y^2)$ (ii) $6xy(x + y)$ (iii) $6xy(x - y)$ (iv) $xy(x^2 + y^2)$ Ans. (i)

15. If $z = \sqrt{xy}$ then $\frac{\partial^2 z}{\partial x \partial y}$ is equal to

(i) $4z$ (ii) $\frac{1}{4z}$ (iii) $\frac{z}{4}$ (iv) $\frac{4}{z}$ Ans. (ii)

16. If $u = x^2 + y^2 + z^2$, $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$ and $z = r\cos\theta$ then $\frac{\partial u}{\partial r}$ is equal to

(i) r (ii) $2r$ (iii) r^2 (iv) $2r^2$ Ans. (ii)

17. If $y = e^x + \sin x$, then $\frac{d^2y}{dx^2}$ is equal to .

(i) $e^x + \sin x$ (ii) $e^x - \sin x$ (iii) $e^x - \cos x$ (iv) None of these

Ans. (ii)

18. If $y = \tan x + \sec x$ then $\frac{d^2y}{dx^2}$ is equal to

(i) $\sec x(\tan^2 x + \sec^2 x)$
(ii) $\sec x(\sec x \tan x + \tan^2 x \sec^2 x)$
(iii) $\sec x(2\sec x \tan x + \tan^2 x + \sec^2 x)$
(iv) $2\sec x \tan x + \tan^2 x + \sec^2 x$

Ans. (iii)

20. If $z = f(x, y)$ where $x = \phi(t)$, $y = \psi(t)$, then $\frac{dz}{dt}$ is equal to

(i) $\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$
(ii) $\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} - \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$
(iii) $\frac{\partial z}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial z}{\partial y}$
(iv) $\frac{dx}{dt} + \frac{\partial z}{\partial t}\frac{dx}{dt}$

Ans. (i)

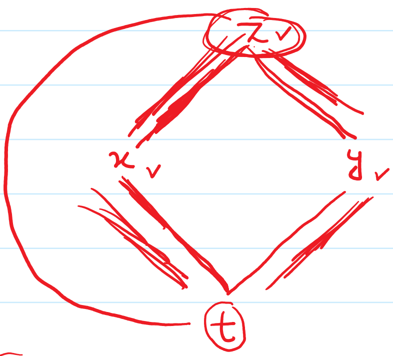
21. If $f(x, y) = 0$, then $\frac{dy}{dx}$ is equal to

(i) $\frac{\frac{\partial y}{\partial f}}{\frac{\partial x}{\partial f}}$ (ii) $-\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}}$ (iii) $-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ (iv) $\frac{\partial y}{\partial x}\cdot\frac{\partial f}{\partial y}$ Ans. (iii)

Differentiation of Composite and Implicit functions

Composite function: Let $z = f(x, y)$ be a function of variables x and y and further $x = g(t)$ and $y = h(t)$.

Then z is called a composite function.

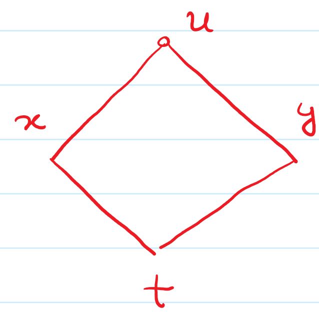


$$\boxed{\frac{dz}{dt}} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Q If $u = x^2 + y^2$, $x = t$, $y = t^2$ find $\frac{du}{dt}$

Sol^n

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

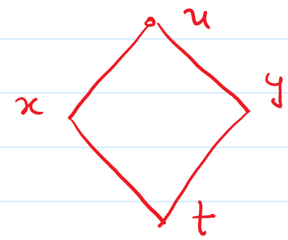


$$= (2x)(1) + (2y)(2t)$$

$$= 2x + 4yt$$

$$= 2t + 4t^2(t) = 2t + 4t^3 \quad \text{Ans.}$$

Q If $u = e^x \sin y$, $x = \log t$, $y = t^2$ find $\frac{du}{dt}$



Sol^n
$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$
$$= \left(e^x \sin y\right)\cdot\frac{1}{t} + e^x \cos y\,(2t)$$
$$= \frac{e^x \sin y}{t} + 2e^x \cos y\, t \quad \text{Ans.}$$

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$Z = f(x, y)$

Partial derivative —

Total derivative —

Total derivative formula.

$$\boxed{dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

Q Find the total derivative of $z = \tan^{-1}\left(\frac{x}{y}\right)$

Sol^n $z = \tan^{-1}\left(\frac{x}{y}\right)$

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2}\,\frac{\partial}{\partial x}\left(\frac{x}{y}\right), \qquad \frac{\partial z}{\partial y} = \frac{1}{1+(x/y)^2}\,\frac{\partial}{\partial y}\left(x y^{-1}\right)$$
$$= \frac{y^{\cancel{2}}}{y^2+x^2}\cdot\left(\frac{1}{y}\right), \qquad = \frac{\cancel{y^2}}{y^2+x^2}\left(-\frac{x}{\cancel{y^2}}\right)$$
$$= \frac{y}{y^2+x^2}, \qquad = -\frac{x}{x^2+y^2}$$

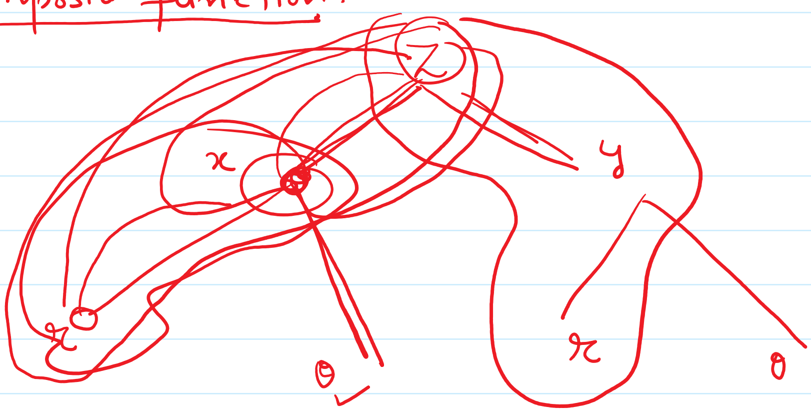
$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy = \frac{y\,dx - x\,dy}{x^2+y^2} \quad \text{Ans}$$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy = \frac{y\,dx - x\,dy}{x^2+y^2}$$ Ans

—— × ——

Composite function.

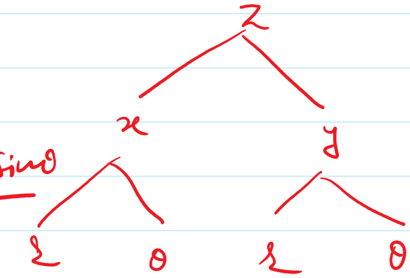
$z = f(x,y)$, $x = f_1(r,\theta)$
$y = f_2(r,\theta)$



$$\frac{\partial z}{\partial r} = \left(\frac{\partial z}{\partial x}\right)\times\left(\frac{\partial x}{\partial r}\right) + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}, \quad \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

Q $z = x^2 + y^2$, $x = r\cos\theta$, $y = r\sin\theta$

find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$



$x = r\cos\theta$, $y = r\sin\theta$

Sol^n $$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= (2x)(\cos\theta) + (2y)(\sin\theta)$$

$$= 2(r\cos\theta)(\cos\theta) + 2(r\sin\theta)(\sin\theta) = 2\{r\cos^2\theta + r\sin^2\theta] = 2r$$

$$= (2x)(\cos\theta) + (2y)(\sin\theta)$$

$$= 2(r\cos\theta)\cos\theta + 2(r\sin\theta)(\sin\theta) = 2\{r\cos^2\theta + r\sin^2\theta\} = 2r$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta} = (2x)(-r\sin\theta) + (2y)(r\cos\theta)$$

$$= 2(r\cos\theta)(-r\sin\theta) + 2(r\sin\theta)(r\cos\theta)$$

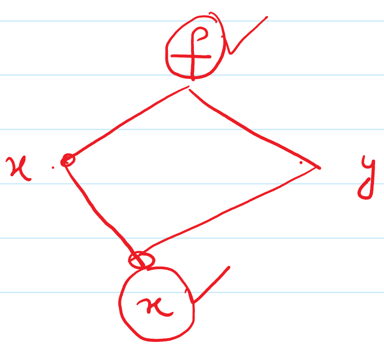
$$= 2r^2\left[-\cos\theta\cancel{\sin\theta} + \sin\theta\cancel{\cos\theta}\right] = 0$$

Implicit function.

A relation of the form $f(x,y)=c$ in which x and y cannot be separated out. is called implicit function.

Relationship b/w derivative and partial derivates.

$f(x,y)=c$ y is treated a function of x.



$$\frac{df}{dx} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$-\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}\,\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{-\,\partial f/\partial x}{\partial f/\partial y}} = -\frac{f_x}{f_y}$$

Q Find $\dfrac{dy}{dx}$ if $ax^2 + 2hxy + by^2 = c$

Solⁿ $f(x,y) = c$

$f = ax^2 + 2hxy + by^2$

$f_x = 2ax + 2hy$

$f_y = 2hx + 2by$

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$= -\frac{(2ax + 2hy)}{(2hx + 2by)} = -\frac{(ax + hy)}{hx + by}$$

Ans.

— × —

Q Find $\dfrac{dy}{dx}$ if $x^3 + 3axy + y^3 = c$

Solⁿ $f = x^3 + 3axy + y^3$

$f_x = 3x^2 + 3ay$

$f_y = 3y^2 + 3ax.$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(3x^2 + 3ay)}{(3y^2 + 3ax)}$$

$$= -\frac{\cancel{3}(x^2 + ay)}{\cancel{3}(y^2 + ax)}$$

$$= -\frac{(x^2 + ay)}{y^2 + ax}$$ Ans

$$\frac{d^2y}{dx^2} = -\left[\frac{f_{xx}(f_y)^2 - 2f_{xy}f_xf_y + f_{yy}(f_x)^2}{(f_y)^3}\right]$$

$f_x = p, \quad f_y = q, \quad f_{xx} = r, \quad f_{xy} = s, \quad f_{yy} = t$

$$\frac{d^2y}{dx^2} = -\left[\frac{rq^2 - 2spq + tp^2}{q^3}\right]$$ Ans.