MTH165

Unit 4
Multivariate functions

L22- Functions of several variables , Limit and Continuity

Unit IV

Multivariate functions : Functions of two variables, Limits and Continuity, Partial derivatives, Total derivative and differentiability, Chain rule, Euler's theorem for Homogeneous functions, Maxima and Minima, Lagrange method of multiplier

Revision

The function $f(x) = 2x^3 - 3x^2 - 12x + 4$ has

(a) no maxima and minima

(b) one maxima and minima

(c) two maxima

(d) two minima

Revision

The maximum value of $\dfrac{\log x}{x}$ is

(a) 1 (b) $\dfrac{2}{e}$ (c) e (d) $\dfrac{1}{e}$

Functions of Two Variables

- Often a dependent variable depends on <u>two</u> or more independent variables:
  - The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point.
    - We can express this by writing $T(x, y)$.
  - The volume V of a circular cylinder depends on its radius r and height h.
    - We write $V(r, h)$.

Examples

- Find the domains of the following functions and evaluate $f(3, 2)$:

(a) $f(x, y) = \dfrac{\sqrt{x + y + 1}}{x - 1}$ (b) $f(x, y) = x \ln(y^2 - x)$

- <u>Solution</u> The expression for (a) makes sense if $x - 1 \neq 0$ and $x + y + 1 \geq 0$, so

$$D = \{(x, y) | x + y + 1 \geq 0, x \neq 1\}$$

Solution (cont'd)

- Also for (a),

$$f(3, 2) = \frac{\sqrt{3 + 2 + 1}}{3 - 1} = \frac{\sqrt{6}}{2}$$

- Here is a sketch of the domain:

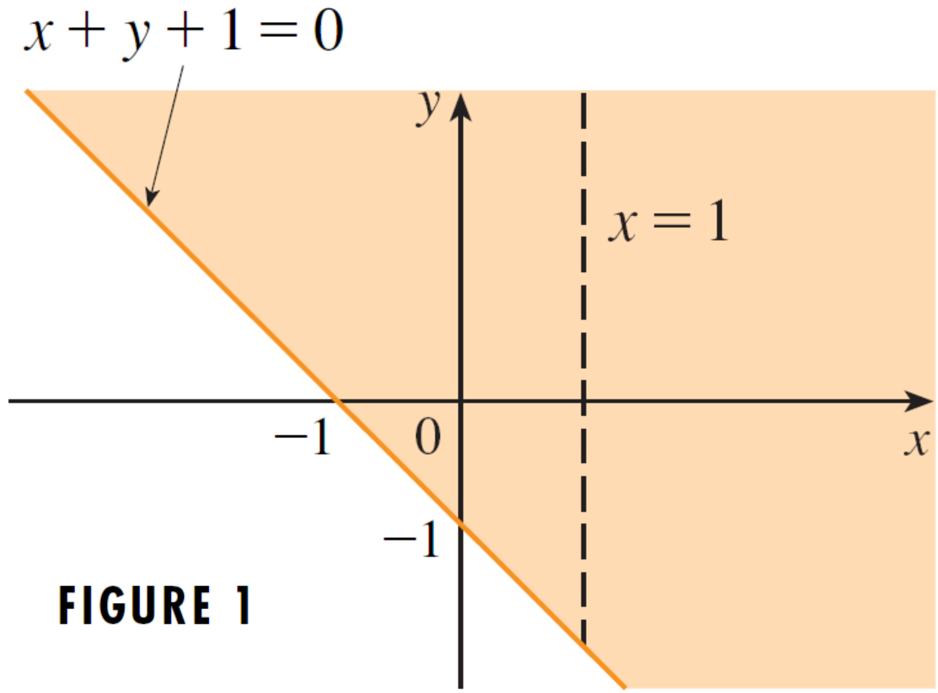


FIGURE 1

Solution (cont'd)

- For part (b),

$$f(3, 2) = 3\ln(2^2 - 3) = 3\ln 1 = 0$$

- Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, the domain of f is

$$D = \{(x, y) | x < y^2\}$$

- This is illustrated on the next slide:

Solution (cont’d)

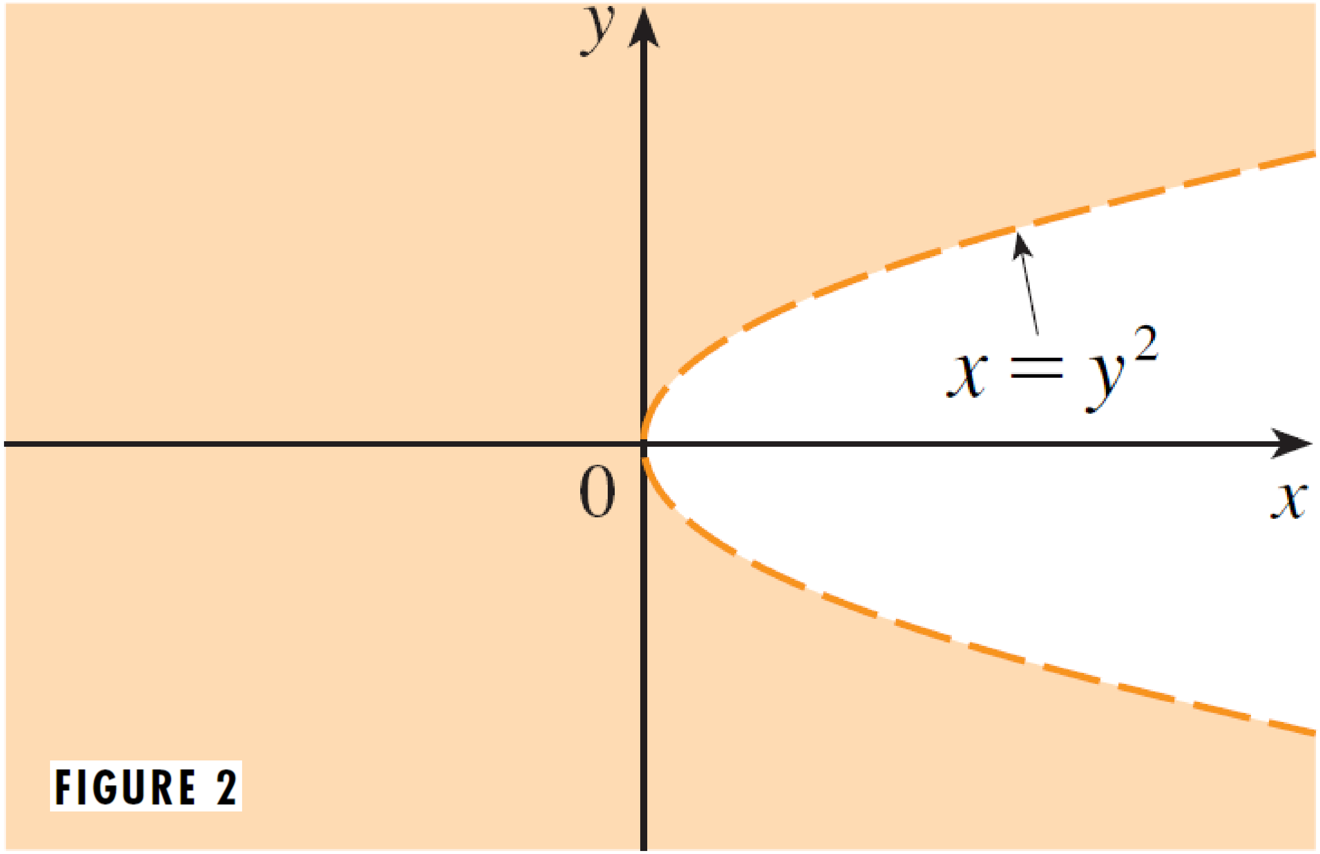
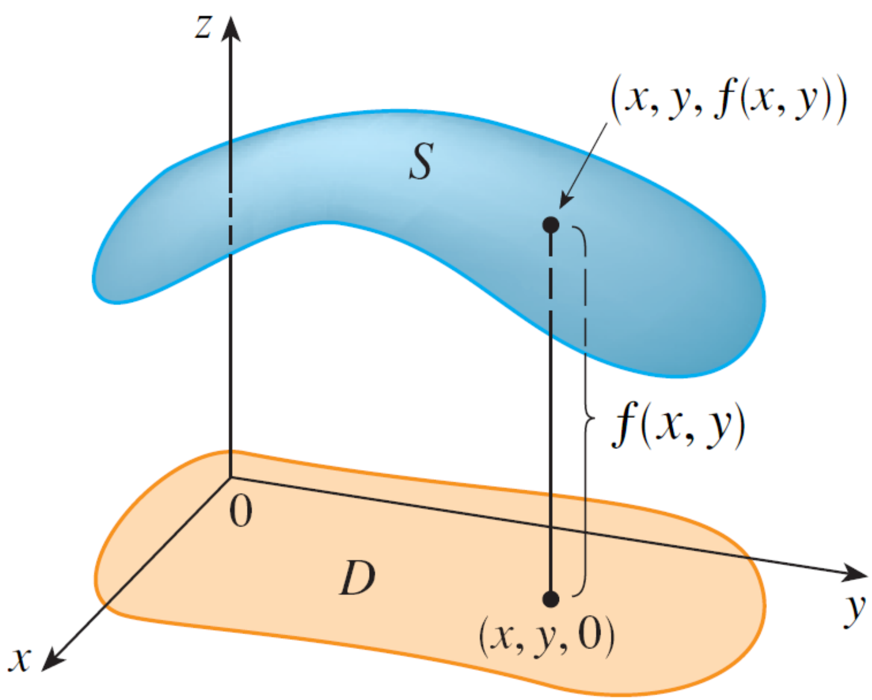


FIGURE 2

- Just as…
  - the graph of a function of one variable is a curve C with equation $y = f(x)$,
- so…
  - the graph of a function of two variables is a surface S with equation $z = f(x, y)$.



MCQ

The type of function which contain only one independent variables is classified as

A. variate function
B. multivariate function
C. univariate function
D. bivariate function

MCQ

The function of two variables in a way that u is dependent variable and v is independent variable is written as

A. $u = f(v)$
B. $f = u(v)$
C. $v = f(u)$
D. $f = v(u)$

- Sketch the graph of the function

$$f(x, y) = 6 - 3x - 2y$$

- The graph of f has the equation

$$z = 6 - 3x - 2y, \quad \text{or} \quad 3x + 2y + z = 6,$$

which represents a plane, let's find the zeros.

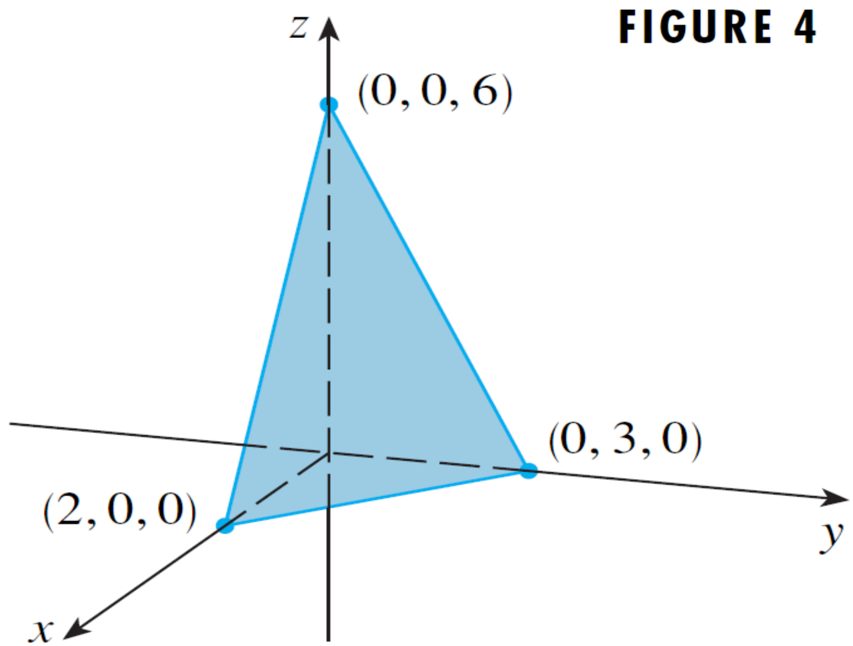
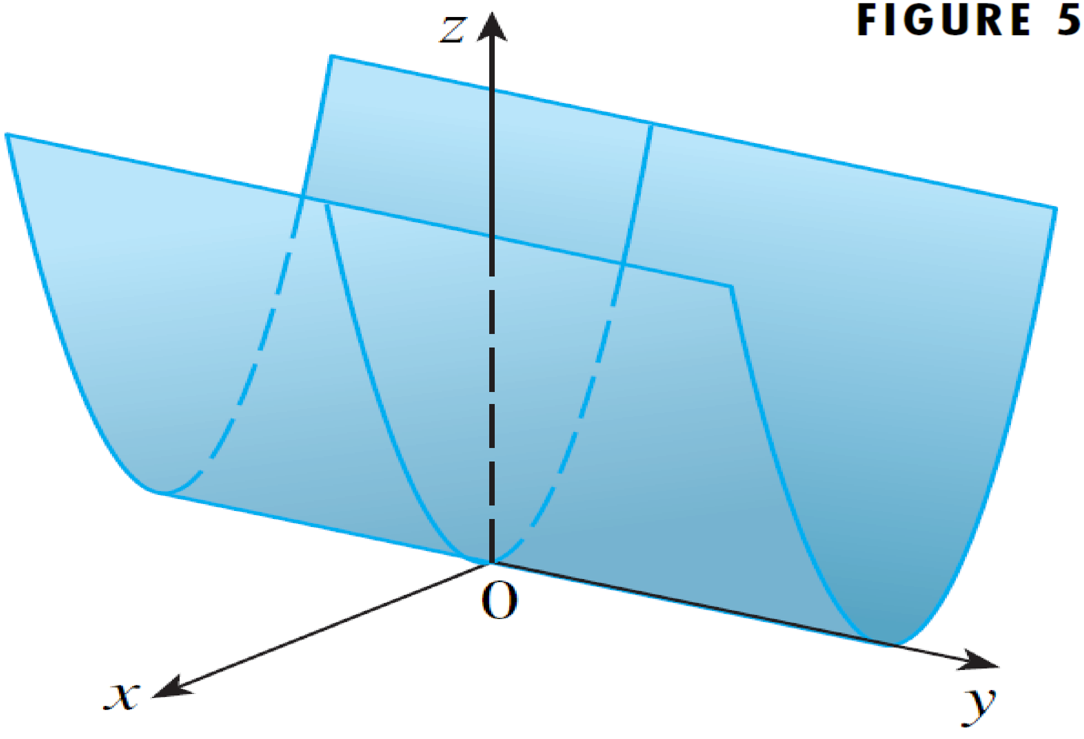


FIGURE 4

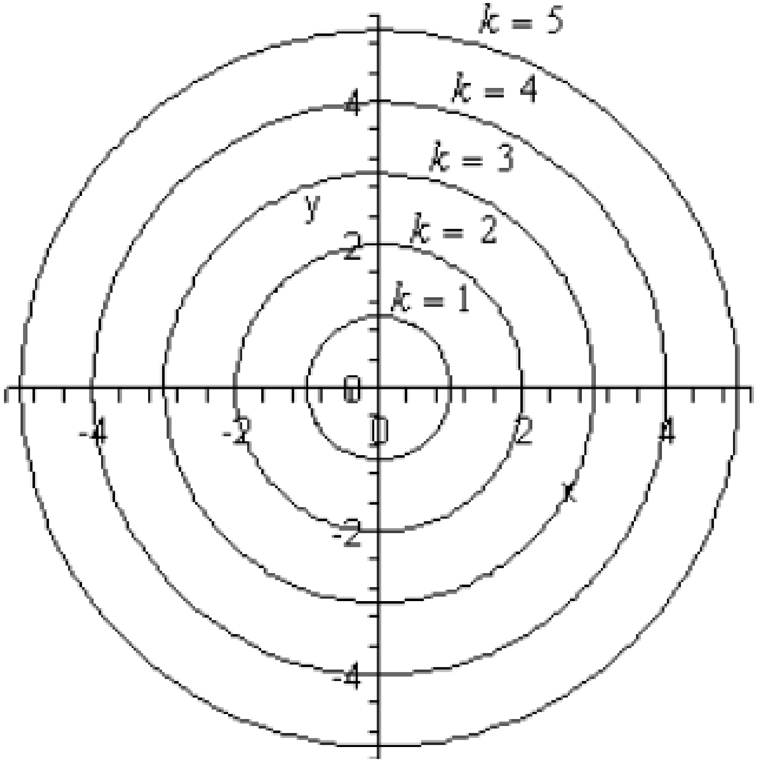
- Sketch the graph of the function $f(x, y) = x^2$.
- Solution The equation of the graph is $z = x^2$, which doesn't involve y.
- Thus any vertical plane $y = k$ intersects the graph in a parabola $z = x^2$.
- The graph is called a *parabolic cylinder*

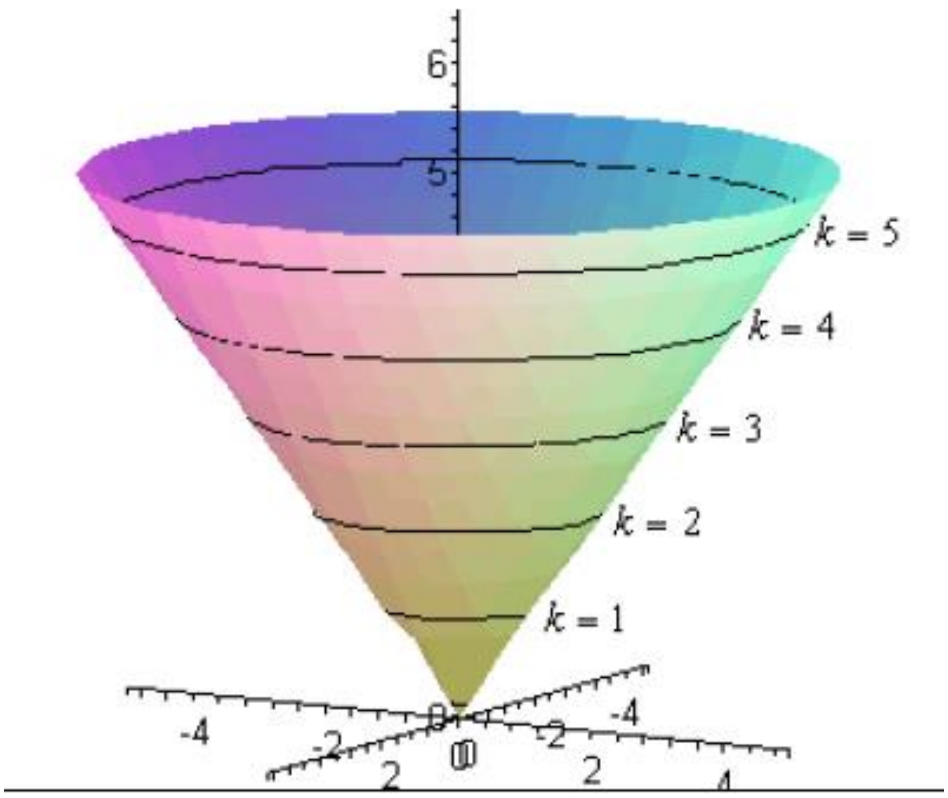
FIGURE 5



--Sketch the function $f(x,y)=\sqrt{x^2+y^2}$

Let's identify what this surface given by $f(x,y)=\sqrt{x^2+y^2}$ and rewrite the function as $z=\sqrt{x^2+y^2}$ or $z^2=x^2+y^2$



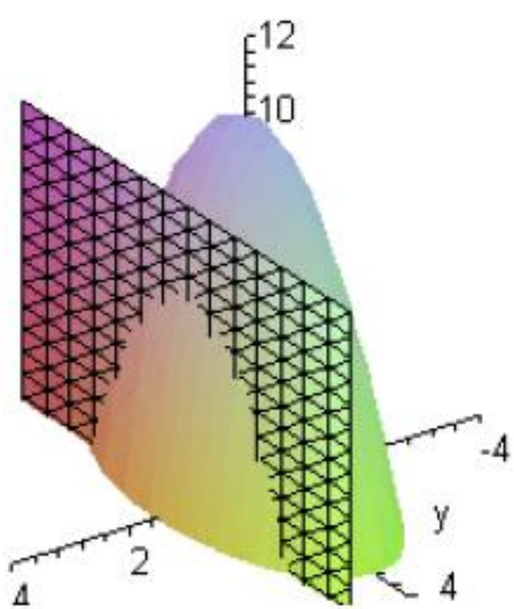


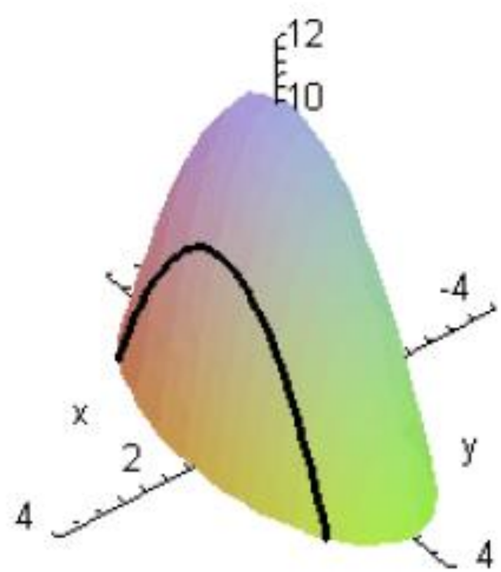
Sketch the traces of $f(x,y) = 10 - 4x^2 - y^2$ for the plane $x = 1$ and $y = 2$.

We'll start with $x = 1$. We can get an equation for the trace by plugging $x = 1$ into the equation. Doing this gives,

$$z = f(1,y) = 10 - 4(1)^2 - y^2 \qquad \Rightarrow \qquad z = 6 - y^2$$

and this will be graphed in the plane given by $x = 1$.

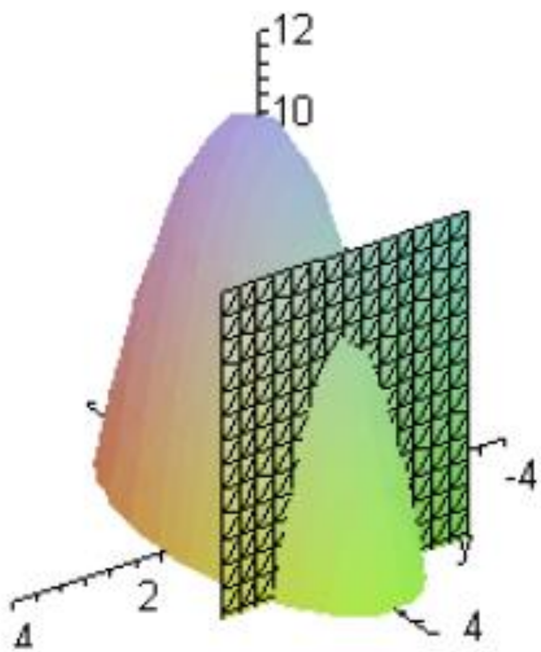


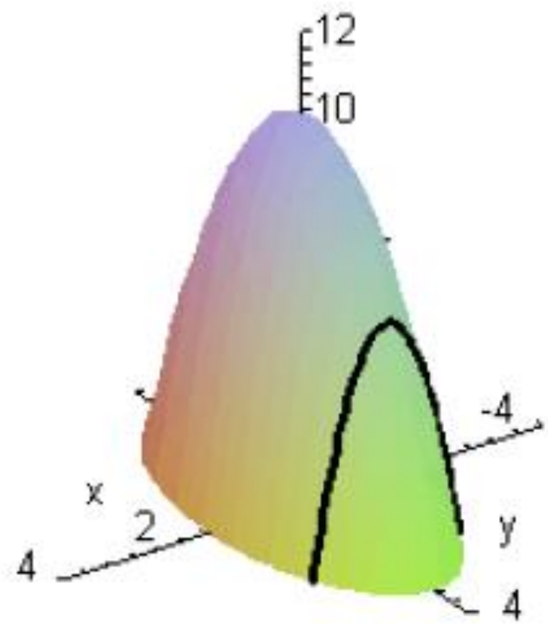


For $y = 2$ we will do pretty much the same thing that we did with the first part. Here is the equation of the trace,

$$z = f(x,2) = 10 - 4x^2 - (2)^2 \qquad \Rightarrow \qquad z = 6 - 4x^2$$

and here are the sketches for this case.

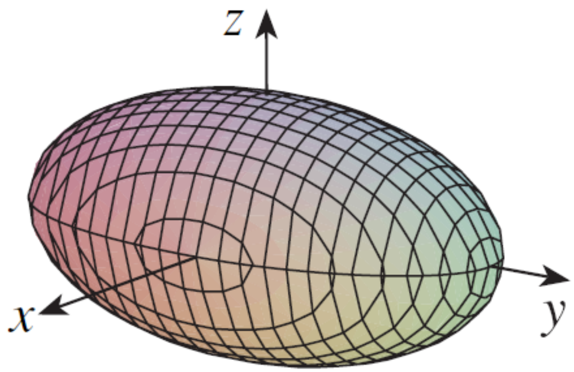




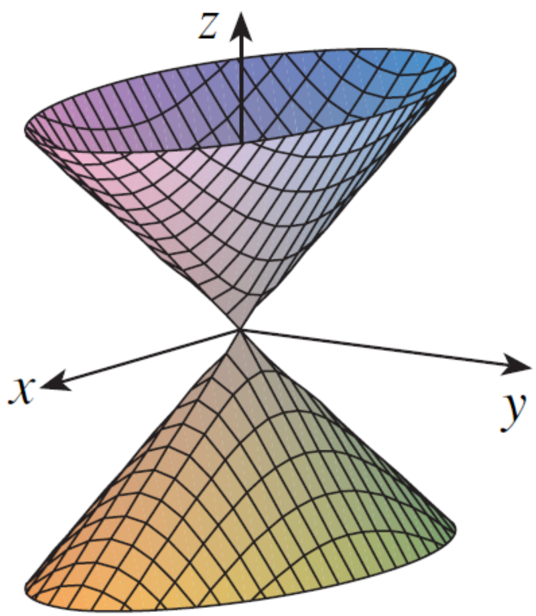
Other Quadric Surfaces

- The following slides show the six basic types of quadric surfaces in standard form.
- All surfaces are symmetric with respect to the z-axis.
- If a quadric surface is symmetric about a different axis, its equation changes accordingly.

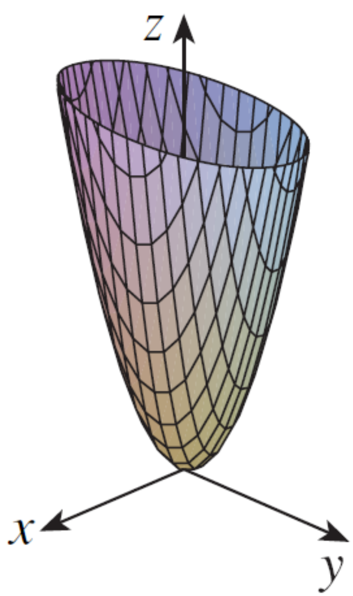
Other Quadric Surfaces (cont'd)

Surface	Equation
Ellipsoid 	$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.

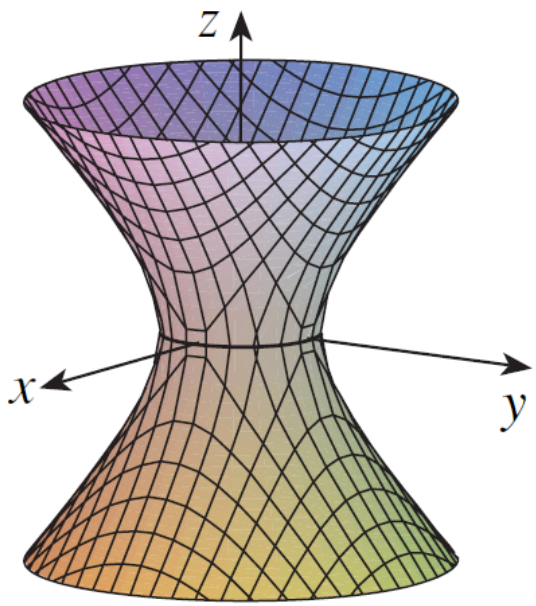
Other Quadric Surfaces (cont'd)

Surface	Equation
Cone 	$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

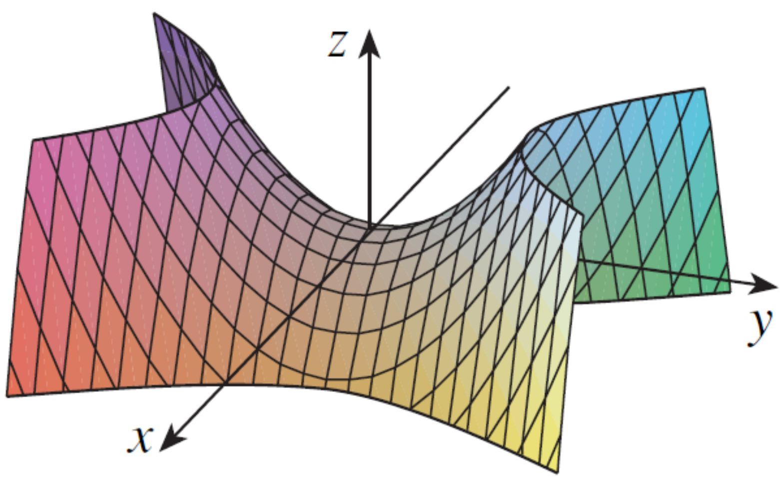
Other Quadric Surfaces (cont'd)

Surface	Equation
Elliptic Paraboloid 	$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.

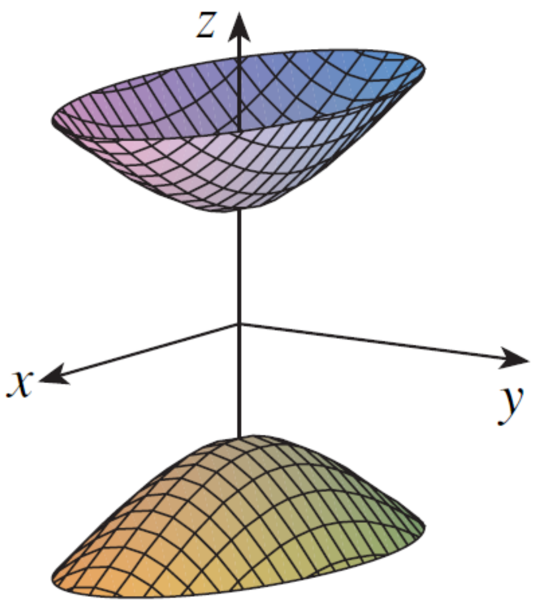
Other Quadric Surfaces (cont'd)

Surface	Equation
Hyperboloid of One Sheet 	$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.

Other Quadric Surfaces (cont'd)

Surface	Equation
Hyperbolic Paraboloid 	$\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.

Other Quadric Surfaces (cont'd)

Surface	Equation
Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

The graph of $f(x, y) = (x^2 + y^2)^{1/2}$ is a paraboloid.

True

False

Limits and Continuity

Limits

DEFINITION: Suppose that *f* is a function of two variables. Let $P_0 = (x_0, y_0)$ be a fixed point in the plane such that every punctured disk $D_*(P_0, r)$ intersects the domain of *f*. We say that the real number ℓ is the *limit* of $f(P)$ as $P = (x, y)$ approaches P_0, and we write

$$\lim_{P \to P_0} f(P) = \ell$$

if, for any $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(P) - \ell| < \varepsilon$ for all points P in the domain of *f* with $0 < d(P, P_0) < \delta$.

EXAMPLE: Define $f(x, y) = x^2 + y^2$. Verify that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

EXAMPLE: Define

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \textit{if} \quad (x,y) \neq (0,0) \\ 0 & \textit{if} \quad (x,y) = (0,0) \end{cases}$$

Discuss the limiting behavior of *f*(x, y) as (x, y) $\rightarrow$ (0, 0).

EXAMPLE: Define $f(x, y) = (x + y + 1) / (x^2 - y^2)$. What is the limiting behavior of f as (x, y) tends to $(1, 2)$?

EXAMPLE: Evaluate the limit

$$\lim_{(x,y)\to(3,-2)} \frac{2x^2+5xy+3y^2}{2x+3y}.$$

MCQ

Find $lt_{(x,y)\to(0,0)} \dfrac{y^6}{x^{10}y^2+x^{15}}$

a) 0

b) 1

c) Does Not exist

d) ∞

MCQ

Find $lt_{(x,y)\to(0,0)} \frac{sec(y).sin(x)}{x}$

a) ∞

b) ½

c) 1

d) ⅓

Continuity

DEFINITION : Suppose that *f* is a function of two variables that is defined at a point $P_0 = (x_0, y_0)$. If $f(x, y)$ has a limit as (x, y) approaches (x_0, y_0), and if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0),$$

then we say that *f* is *continuous* at P_0. If *f* is not continuous at a point in its domain, then we say that *f* is *discontinuous* there.

EXAMPLE: Suppose that

$$f(x,y)=\begin{cases}\dfrac{(x^2-y^2)^2}{x^2+y^2} & \textit{if}\quad (x,y)\neq(0,0)\\ 1 & \textit{if}\quad (x,y)=(0,0).\end{cases}$$

is *f* continuous at (0, 0)?

Rules for Continuity

EXAMPLE: Discuss the continuity of

$$U(x,y) = y^3 \sin(x) - \frac{\cos\left(xy^2\right)}{(2x-y)^2}.$$

at (0,0) .

Functions of Three Variables

EXAMPLE: Show that $V(x, y, z) = z^3 \cos(xy^2)$ is a continuous function.

MTH165

Unit 4
Multivariate functions

L23- Partial Derivatives

Limits and continuity

Method:-

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{y\to 0\\ x\to 0}} f(x,y) = \lim_{\substack{y\to mx\\ x\to 0}} f(x,y) = \lim_{\substack{y\to mx^2\\ x\to 0}} f(x,y)$$

All are same value then, limit exist

If limit exist then it's continuous at given point.

Revision

Find $lt_{(x,y)\to(0,0)} \frac{sin(y)}{x}$

a) 1

b) 0

c) ∞

d) Does Not Exist

Revision

Find $lt_{(x,y)\to(0,0)} \frac{y^7x^{98}-x^{97}y^8+x^{105}}{xy^7+x^8}$

a) Does Not Exist

b) 0

c) 1

d) ∞

WHAT IS PARTIAL DIFFERENTIATION ?

- Let $z=f(x,y)$ be function of two individual variables x & y the derivative with respect to x keeping y constant Is called partial derivative of z with respect to x.
- It is denoted by $\frac{\partial z}{\partial x}$, $\frac{\partial f}{\partial x}$, f_x .
- It is denoted as

$$\frac{\partial z}{\partial x} = \lim_{\partial x \to 0} \frac{f(x+\partial x, y) - f(x,y)}{\partial x}.$$

Partial derivatives of first order

Let $z = f(x,y)$

First order:-

$$\left(\frac{\partial f}{\partial x} or \frac{\partial z}{\partial x} or\ fx\right) or \left(\frac{\partial f}{\partial y} or \frac{\partial z}{\partial y} or\ fy\right)$$

Partial derivatives in higher orders

Second order:-

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \quad or \ \ fxx$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \quad or \ \ fxy$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \quad or \ \ fyx$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \quad or \ \ fyy$$

Third order:-

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = fxxx$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = fyxx$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = fyyx$$

$$\frac{\partial^3 f}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right) = fyyy$$

$$\frac{\partial^3 f}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = fxyx$$

EXAMPLE :

If $a^2x^2 + b^2y^2 = c^2z^2$, evaluate $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$.

EXAMPLE :

If $a^2x^2 + b^2y^2 = c^2z^2$, evaluate $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$.

Solution :

$$a^2x^2 + b^2y^2 = c^2z^2$$

Differentiating partially w.r.t. x,

$$2a^2x = 2c^2z\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{a^2x}{c^2z}$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2z}\left(\frac{1}{} - \frac{x}{z^2}\frac{\partial z}{\partial x}\right)$$

$$= \frac{a^2}{c^2z}\left(1 - \frac{x}{z}\frac{a^2x}{c^2z}\right)$$

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2z}\left(1 - \frac{a^2x^2}{c^2z^2}\right)$$

$$\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2z}\left(1 - \frac{b^2y^2}{c^2z^2}\right)$$

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2z}\left(2 - \frac{a^2x^2 + b^2y^2}{c^2z^2}\right)$$

$$= \frac{1}{c^2z}\left(2 - \frac{c^2z^2}{c^2z^2}\right)$$

$$= \frac{1}{c^2z}$$

$f(x, y) = x^2 + xyz + z$ Find f_x at $(1,1,1)$

a) 0

b) 1

c) 3

d) -1

$f(x, y) = \sin(xy) + x^2 \ln(y)$ Find f_{yx} at $(0, \pi/2)$

a) 33

b) 0

c) 3

d) 1

Show that the function

$$f(x,y)=\begin{cases}(x+y)\sin\left(\dfrac{1}{x+y}\right), & x+y\neq 0\\ 0 & , x+y=0\end{cases}$$

is continuous at (0,0) but its partial derivatives f_x *and* f_y do not exist at (0,0).

Total differential and differentiability

Definition 13.4.1 Total Differential
Let $z = f(x, y)$ be continuous on an open set S. Let dx and dy represent changes in x and y, respectively. Where the partial derivatives f_x and f_y exist, the **total differential of z** is

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy.$$

Example **Finding the total differential**

Let $z = x^4 e^{3y}$. Find dz.

We *can* approximate Δz with dz, but as with all approximations, there is error involved. A good approximation is one in which the error is small. At a given point (x_0, y_0), let E_1 and E_2 be functions of dx and dy such that $E_1 dx + E_2 dy$ describes this error. Then

$$\begin{aligned}\Delta z &= dz + E_1 dx + E_2 dy \\ &= f_x\left(x_0, y_0\right) dx + f_y\left(x_0, y_0\right) dy + E_1 dx + E_2 dy.\end{aligned}$$

If the approximation of Δz by dz is good, then as dx and dy get small, so does $E_1 dx + E_2 dy$. The approximation of Δz by dz is even better if, as dx and dy go to 0, so do E_1 and E_2. This leads us to our definition of differentiability.

Definition **Multivariable Differentiability**

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0) where $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist. Let dz be the total differential of z at (x_0, y_0), let $\Delta z = f(x_0 + dx, y_0 + dy) - f(x_0, y_0)$, and let E_1 and E_2 be functions of dx and dy such that

$$\Delta z = dz + E_1 dx + E_2 dy.$$

1. f is **differentiable at** (x_0, y_0) if

$$\lim_{(dx,dy)\to(0,0)} E_1 = 0 \qquad \text{and} \qquad \lim_{(dx,dy)\to(0,0)} E_2 = 0.$$

2. f is **differentiable on** S if f is differentiable at every point in S. If f is differentiable on $\mathbb{R}^2$, we say that f is **differentiable everywhere**.

Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on an open set S. If f_x and f_y are both continuous on S, then f is differentiable on S.

The existence of first order partial derivatives implies continuity.

a) True

b) False

MTH165

Unit 4
Multivariate functions
L 24-26-Total Derivative , chain rule , Euler's theorem

Compute the partial derivative of the function

$$f(x, y, z) = e^{1-x\cos(y)} + z\, e^{-1/(1+y^2)}$$

with respect to x at the point $(1, 0, \pi)$.

(a) -1

(b) $-1/e$

(c) 0

(d) π/e

If $z = f(x + ay) + g(x - ay)$, then

A. $z_{xx} = z_{yy}$

B. $z_{xx} = a^2 z_{yy}$

C. $z_{yy} = a^2 z_{xx}$

D. $z_{xx} + a^2 z_{yy} = 0$

If $z = cos\left(\frac{x}{y}\right) + \sin\left(\frac{x}{y}\right)$, then $x\, z_x + \ y\, z_y$ is equal to

A. z

B. 2z

C. 0

D. 4z

$f(x, y, z, t) = xy + zt + x^2 yzt$; $x = k^3$; $y = k^2$; $z = k$; $t = \sqrt{k}$

Find ${}^{df}/_{dt}$ at $k = 1$

a) 34

b) 16

c) 32

d) 61

$f(x, y) = x^2 + y^3$; $X = t^2 + t^3$; $y = t^3 + t^9$ Find $^{df}/_{dt}$ at t=1.

a) 0

b) 1

c)-1

d) 164

Homogeneous Function

Consider the function

$$f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots\dots\dots + a_n y^n$$

The degree of each term in x and y is n.

Such functions are called homogenious functions of degree n.

Another def.

A function f(x,y) of two independent variables x and y is said to be homogenious of degree n if f(x,y) can be written in the form $x^n \phi\left(\frac{y}{x}\right)$ where ϕ can be any function

Some examples of homogenious functions

$$(1): F(x,y)=x^n \sin(\frac{y}{x})$$

$$(2): F(x,y)=x^3 - 3xy^2 + y^3$$

$$(3): F(x,y)=\frac{\left(\sqrt{y}-\sqrt{x}\right)}{y-x}$$

Euler's Theorem on Homogeneous Function

If z = F (x,y) be a homogenious function of x,y of degree n

then $x\dfrac{\delta z}{\delta x} + y\dfrac{\delta z}{\delta y} = nz$ for all x,y

Proof: We have

z is a homogenious function of degree n.

so that $z = x^n \phi\left(\dfrac{y}{x}\right)$

$$\Rightarrow \frac{\delta z}{\delta x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$\Rightarrow \frac{\delta z}{\delta x} = nx^{n-1} f\left(\frac{y}{x}\right) + x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$= nx^{n-1} f\left(\frac{y}{x}\right) - yx^{n-2} f'\left(\frac{y}{x}\right)$$

$$\text{Similarly}\ ,\ \frac{\delta z}{\delta y} = x^n f\,'(\frac{y}{x})\left(\frac{1}{x}\right) = x^{n-1} f\,'(\frac{y}{x})$$

Thus ,we have

$$x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = nx^n f(\frac{y}{x}) - yx^{n-1} f\,'(\frac{y}{x}) + yx^{n-1} f\,'(\frac{y}{x})$$

$$\Rightarrow x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = nx^n f(y/x) = nz$$

hence the result.

COROLLARY I:

If $z = f(x, y)$ is a homo. function of x and y of degree n,

then $$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

Example : Verify Euler's theorem for the function

$$z = x^n \log \frac{y}{x}$$

Solution:

z is a homogenious function of x and y of degree n.

$$\therefore\ x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = nz$$

$$Now,\ \frac{\delta z}{\delta x} = nx^{n-1}\log\frac{y}{x} + x^n\left(\frac{x}{y} * \frac{-y}{x^2}\right)$$

$$= nx^{n-1}\log\frac{y}{x} - x^{n-1}$$

$$and \quad \frac{\delta z}{\delta y} = x^n * \frac{x}{y} * \frac{1}{x} = \frac{x^n}{y}$$

Multiply by x and y and add

$$\therefore x\frac{\delta z}{\delta x} + y\frac{\delta z}{\delta y} = x\left(nx^{n-1}\log\frac{y}{x} - x^{n-1} \right) + y * \frac{x^n}{y}$$

$$= n\, x^n \log\frac{y}{x} - x^n + x^n$$

$$= n\, x^n \log\frac{y}{x}$$

$$= \mathrm{n\ z}$$

Example : If u= $\sin^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$,

then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\tan u$

Solution : We have u $= \sin^{-1}\left(\dfrac{x^2+y^2}{x+y}\right)$

Let z $= \dfrac{x^2+y^2}{x+y}$ then sin u = z where $z = \dfrac{x^2 + y^2}{x + y} = x \dfrac{1 + \left(y^2 / x^2 \right)}{1 + \left(y / x \right)}$ *is a* homogenious function of degree one

$\therefore$ By Eular's theorem ,we have

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$$

$$But \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\sin u) = \cos u \frac{\partial u}{\partial x}$$

$$and \quad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\sin u) = \cos u \frac{\partial u}{\partial y}$$

hence $$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z=\cos u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)=\mathrm{sinu}$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$$

Example : If u = $\cot^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{-1}{4}\sin 2u$$

Solution : We have u= $\cot^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

Let z = $\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ then cot u = z

where z = $\dfrac{x+y}{\sqrt{x}+\sqrt{y}} = x^{\frac{1}{2}}\dfrac{1+\dfrac{y}{x}}{1+\dfrac{\sqrt{y}}{\sqrt{x}}}$ *is a* homogenious

function of degree half

$\therefore$ By Eular's theorem ,we have

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\mathrm{n}\ \mathrm{z}=\frac{\mathrm{z}}{2}$$

$$\frac{\partial z}{\partial x}=-\cos ec^{2}x\frac{\partial u}{\partial x}$$

$$and\quad \frac{\partial z}{\partial y}=-\cos ec^{2}x\frac{\partial u}{\partial y}\ \text{ and we have}$$

$$-x\cos ec^{2}x\frac{\partial u}{\partial x}-y\cos ec^{2}x\frac{\partial u}{\partial y}\ =\frac{1}{2}\cot u$$

$$=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\ =\frac{-\cot u}{2\cos ec^{2}x}=\frac{1}{4}\sin 2u$$

Exercise

1 Find the first order partial derivatives of

(a) $\cot^{-1}(x+y)$

(b) $\sin(x^2 y^2)$

(c) $\dfrac{x+y}{x-y}$

2 Find the second order partial derivatives of

(a) $\tan\left(\tan^{-1} x + \tan^{-1} y\right)$

(b) $\dfrac{xy}{\sqrt{1+x^2+y^2}}$

(c) $\log\left(x\tan^{-1} y\right)$

(d) e^{x^y}

3 Varify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

where u is log (ysinx+xsiny)

4 If z= $\sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$\frac{\partial z}{\partial x} = -\frac{y}{x} * \frac{\partial z}{\partial y}$$

5 If z = f(x+ay)+g(x-ay), show that

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

6 If $v=\left(x^2+y^2+z^2\right)^{\frac{3}{2}}$, show that

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0$$

7 If $z(x+y)=\left(x^2+y^2\right)$, show that

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$$

8 If $z=\log\frac{x^2+y^2}{x+y}$, show that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1$$

9 If z = 3xy - $y^3 + \left(y^2 - 2x\right)^{\frac{3}{2}}$, show that

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

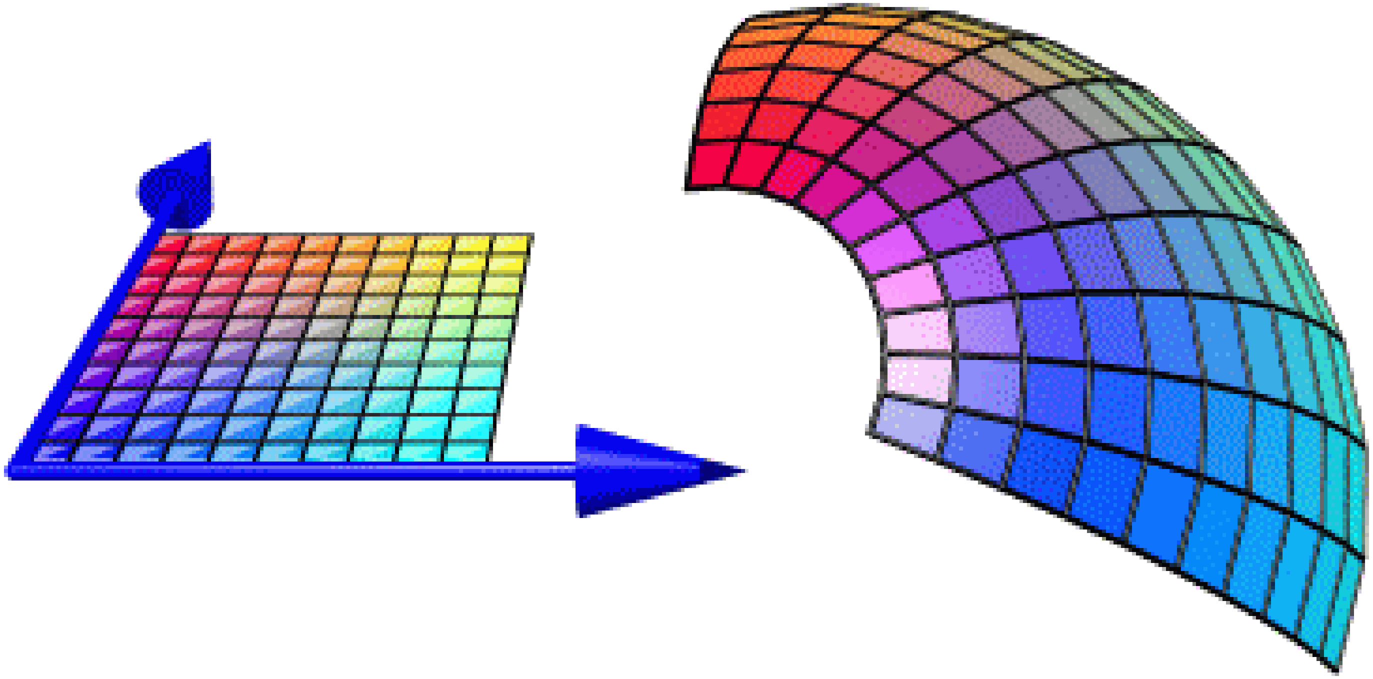
10 If u = $\dfrac{1}{\sqrt{1 - 2xy + y^2}}$ show that

$$\frac{\partial}{\partial x}\left[\left(1 - x^2\right)\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left(y^2\frac{\partial z}{\partial y}\right) = 0$$

11 If z = $\tan^{-1}\dfrac{y}{x}$, then show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Jacobians



Definition of the Jacobian

Definition of the Jacobian

If $x = g(u, v)$ and $y = h(u, v)$, then the **Jacobian** of x and y with respect to u and v, denoted by $\partial(x, y)/\partial(u, v)$, is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Example 1

Find the Jacobian for the change of variables

$x = r\cos\theta$ and

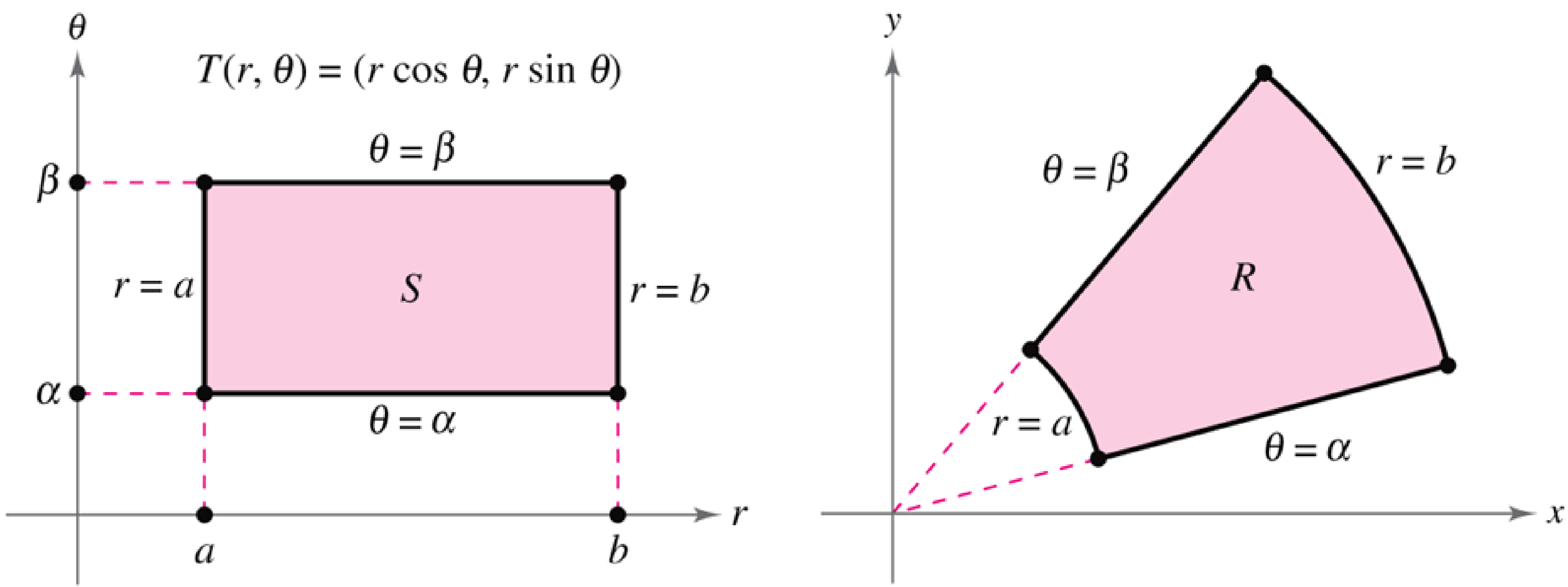
$y = r\sin\theta$

Example 1 Solution

Find the Jacobian for the change of variables
x = r cosϴ and
y = r sinϴ

$$\begin{aligned}
\frac{\partial(x, y)}{\partial(r, \theta)} &= \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= r\cos^2\theta + r\sin^2\theta \\
&= r.
\end{aligned}$$

Why would we change variables?



S is the region in the $r\theta$-plane that corresponds to R in the xy-plane.

If $u = x^2 f\left(\frac{y}{x}\right)$ *then*:

A. $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0.$

B. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$

C. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.$

D. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$

If $x = r\cos\varphi\sin\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\theta$,then the value of $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$ is :

A. 0

B. r

C. $r^2\sin\theta$

D. $r^2\cos\theta$

The Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for the function u = e^x sin y, v = x + log sin y is

A. 1

B. sin x sin y – x y cos x cos y

C. 0

D. $\frac{e^x}{x}$

Properties

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$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

① if $J = \frac{\partial(x,y)}{\partial(u,v)}$ and $J' = \frac{\partial(u,v)}{\partial(x,y)}$ then

$$\boxed{JJ' = 1}$$

For eg. $u = 2xy$, $v = x^2 + y^2$. Find $\frac{\partial(x,y)}{\partial(u,v)}$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2y & 2x \\ 2x & 2y \end{vmatrix} = 4(y^2 - x^2)$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{J'} = \frac{1}{4(y^2 - x^2)}$$ Ans.

② Chain Rule for Jacobians

$z = f(u),\ u = \ldots$

$$\frac{\partial z}{\partial x} = \frac{df}{du} \cdot \frac{\partial u}{\partial x}$$

If u, v are functions of r, s and r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$

if $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$

then find $\dfrac{\partial(u,v)}{\partial(r,\theta)}$

Sol: $\dfrac{\partial(u,v)}{\partial(r,\theta)} = \dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(r,\theta)}$

(a) r^2 (b) $2r^3$ (c) $4r^3$ (d) None of thes

3 Jacobian of Implicit Functions

$$f(x,y) = c$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

if we have

$$f_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$

$$f_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$

$$f_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$ then

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u_1, u_2, u_3)}}$$

For eg:

if $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$.

Find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol: Let $f_1 = u - xyz$, $f_2 = v - x^2 - y^2 - z^2$, $f_3 = w - x - y - z$

then

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{\partial(f_1,f_2,f_3) \,/\, \partial(u,v,w)}{\partial(f_1,f_2,f_3) \,/\, \partial(x,y,z)}$$

(4) <u>Functional Relationship</u> : If u_1, u_2, u_3 be functions of x_1, x_2, x_3 then the necessary and sufficient condition of a functional relationship of the form $f(u_1, u_2, u_3) = 0$ is

$$J = \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = 0$$

if $J \neq 0$ then u, v, w are said to be functionally independent.

Show that $u = x - y + z$, $v = x + y - z$ and $w = x^2 + xz - xy$ are functionally related and find the relationship between them.

Sol

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2x+z-y & -x & x \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 1 & -1 \\ 2x+z-y & -x & x \end{vmatrix} = 0$$

$$\boxed{2w = u(u+v)}$$

MTH165

Unit 4
Multivariate functions

L 27-28-Maxima and minima of function of two variables and Langrange's method of multiplier

MAXIMA AND MINIMA OF FUNCTION OF TWO VARIABLE

- The Function $f(x,y)$ is maximum at (x,y) if for all small positive or negative values of h and k; we have
- $f(x+h\,,\,y+k) - f(x,y) < 0$

- Similarly $f(x,y)$ is minimum at (x,y) if for all small positive or negative values of h and k, we have
- $f(x+h\,,\,y+k) - f(x,y) > 0$

MAXIMA AND MINIMA OF FUNCTION OF TWO VARIABLE

- Thus ,from the defination of maximum of $f(x,y)$ at (x,y) we note that $f(x+h , y+k) - f(x,y)$ preserves the same sign for a maximum it is negative and for a minimum it is positive

- Working rule to find maximum and minimum values of a function $f(x,y)$
- (1) find $\partial f/\partial x$ and $\partial f/\partial y$
- (2) a necessary condition for maximum or minimum value is $\partial f/\partial x=0$, $\partial f/\partial y=0$

MAXIMA AND MINIMA OF FUNCTION OF TWO VARIABLE

- solve simultaneous equations $\partial f/\partial x=0$, $\partial f/\partial y=0$

- Let (a_1,b_1), (a_2,b_2)… be the solutions of these equations.

Find $\partial^2 f/\partial x^2=r$,

$\partial^2 f/\partial x\,\partial y=s$,

$\partial^2 f/\partial y^2=t$

MAXIMA AND MINIMA OF FUNCTION OF TWO VARIABLE

- (3) a sufficient condition for maximum or minimum value is $rt-s^2>0$.

- (4 a) if $r>0$ or $t>0$ at one or more points then those are the points of minima.
- (4 b) if $r<0$ or $t<0$ at one or more points then those points are the points of maxima.

- (5) if $rt-s^2<0$,then there are no maximum or minimum at these points. Such points are called saddle points.

MAXIMA AND MINIMA OF FUNCTION OF TWO VARIABLE

- (6) if $rt-s^2=0$ nothing can be said about the maxima or minima .it requires further investigation.

- (7) if $r=0$ nothing can be said about the maximum or minima . It requires further investigation.

MCQ

What is the saddle point?

a) Point where function has maximum value

b) Point where function has minimum value

c) Point where function has zero value

d) Point where function neither have maximum value nor minimum value

MCQ

For function f(x,y) to have minimum value at (a,b) value is?

a) $rt - s^2 > 0$ and $r < 0$

b) $rt - s^2 > 0$ and $r > 0$

c) $rt - s^2 < 0$ and $r < 0$

d) $rt - s^2 > 0$ and $r > 0$

MAXIMA AND MINIMA OF FUNCTION OF TWO VARIABLE

Example

discuss the maxima and minima of

$$xy+27(1/x + 1/y)$$

Solution

$\partial f/\partial x=y-(27/x^2)$,

$\partial f/\partial y=x-(27/y^2)$

For max. or min ,values we have $\partial f/\partial x=0$, $\partial f/\partial y=0$. $y-(27/x^2)=0$...(1)

$x-(27/y^2)=0$...(2)

Giving $x=y=3$

MAXIMA AND MINIMA OF FUNCTION OF TWO VARIABLE

$\partial^2 f/\partial x^2 = r = 54/x^3$

$\partial^2 f/\partial x\,\partial y = s = 1$,

$\partial^2 f/\partial y^2 = t = 27/y^3$

$r(3,3)=3$

$s(3,3)=1$

$t(3,3)=3$

$rt-s^2=9-1=8>0$, since r,t are both >0

We get minimum value at $x=y=3$ which is 27.

MCQ

Discuss maximum or minimum value of $f(x,y) = y^2 + 4xy + 3x^2 + x^3$.

a) minimum at (0,0)

b) maximum at (0,0)

c) minimum at (2/3, -4/3)

d) maximum at (2/3, -4/3)

MCQ

Divide 120 into three parts so that the sum of their products taken two at a time is maximum. If x, y, z are two parts, find value of x, y and z.

a) x=40, y=40, z=40

b) x=38, y=50, z=32

c) x=50, y=40, z=30

d) x=80, y=30, z=50

Sometimes, we have a *constraint* which restricts us from choosing variables freely:

- Maximize volume subject to limited material costs
- Minimize surface area subject to fixed volume
- Maximize utility subject to limited income

Lagrange's Method of Undetermined Multipliers

To find extreme values of a function we consider a function

i.e. u=f(x, y, z) ....(1)

$\emptyset(x, y, z) = 0$ ....(2)

$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ ....(3)

$\frac{\partial \emptyset}{\partial x}dx = \frac{\partial \emptyset}{\partial x}dx + \frac{\partial \emptyset}{\partial y}dy + \frac{\partial \emptyset}{\partial z}dz$ ....(4)

Application of partial Differentiation

Multiplying the equation (4) by λ and adding with the equation (3) we get

$$du=\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz$$

For u

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

This question will be satisfied if

$$\left.\begin{aligned}\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0\\ \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0\\ \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0\end{aligned}\right] \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The three relations and

$\emptyset(x, y, z) = 0$ are used to determine the values of x, y, z and

Working rule

1. From the equation u=f(x, y, z)+ $\lambda\emptyset$(x, y, z)
2. Consider x, y, z as independent variables and write down the equations.
   $$\frac{\partial u}{\partial x} = 0, \frac{\partial u}{\partial y} = 0, \frac{\partial u}{\partial z} = 0$$
3. Solve these three equations along with $\emptyset(x, y, z) = 0$ and find the value of x,y,z and λ. The values of x, y, z so obtained will give the extreme value of f(x, y, z).

Example

find the maximum value of V (x, y, z)=xyz subjected to the constraint 2x+2y+2z=108

Solution

let $u = xyz + \lambda(2x + 2y + z - 108)$

$$\frac{\partial u}{\partial x} = yz + 2\lambda$$

$$\frac{\partial u}{\partial y} = xz + 2\lambda$$

$$\frac{\partial u}{\partial z} = xy + \lambda$$

Equating these expressions to zero, we get

$yz + 2\lambda = 0$ ....(1)

$xz + 2\lambda = 0$ ....(2)

$xy + \lambda = 0$ ....(3)

Also we have $2x + 2y + z = 108$ ....(4)

$xy + \lambda = 0$ ....(3)

Also we have $2x + 2y + z = 108$ ....(4)

From (1) (2) and (3), we get

$\lambda = \frac{yz}{2} = \frac{xz}{2} = xy$ ....(5)

(5)→ $x = y \; and \; z = 2y$

Substituting for x and z in (4) gives y=18

Thus x=18 and z=36

x=18, y=18, z=36.

V(18, 18, 36) is the only possible of V subject to the constraint.

V(18, 18, 36)=(18) (18) (36) = 11664 is required maximum value.

MCQ

The drawback of Lagrange's Method of Maxima and minima is?

a) Maxima or Minima is not fixed

b) Nature of stationary point is can not be known

c) Accuracy is not good

d) Nature of stationary point is known but can not give maxima or minima

Tutorial

MTH165

$$\underset{x \to 0}{Lt} \frac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)} =$$

(a) $e^{\frac{2}{3}}$ (b) $\frac{2}{3}$ (c) $-\frac{2}{3}$ (d) none of these

Show that the limit $\underset{x\to 0}{Lt}\dfrac{x^2\sin\dfrac{1}{x}}{\sin x}$ exist but they cannot be evaluated by L' Hospital's rule.

(a) 0 (b) 1 (c) e (d) none of these

$$\underset{x\to 0}{Lt}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\frac{1}{x}} =$$

(a) e^3 (b) 2 (c) e^2 (d) none of these

$$\underset{x \to 0}{Lt} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) =$$

(a) 0 (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) none of these

The absolute minimum value of $f(x) = x^2 - 3x$, $0 \le x \le 2$ is

(a) 0 (b) $-\frac{9}{4}$ (c) $\frac{9}{4}$ (d) no minimum value

The extreme value of $f(x) = \log x$ is

(a) 0 (b) -1 (c) 1 (d) no extreme value

$$\underset{(x,y)\to(0,0)}{Lt} \frac{x^3 + y^3}{x^2 + y^2} =$$

(a) 0 (b) 1 (c) ∞ (d) none of these

$$\underset{(x,y)\to(0,1)}{Lt} \tan^{-1}\left(\frac{y}{x}\right) =$$

(a) 0 (b) 1 (c) ∞ (d) does not exist

$$\underset{(x,y)\to(0,0)}{Lt} \frac{x^2 - y^2}{(x+y)^4}$$

(a) 0 (b) 1 (c) ∞ (d) none of these

Tutorial

MTH165

If $u = x^2 + y^2 + z^2$ be such that $x\,u_x + y\,u_y + z\,u_z = \lambda u$, then λ is equal to

A. 1

B. 2

C. 0

D. none of above

If $f(x, y, z) = 0$, then the value of $\frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x}$ is

A. 1

B. -1

C. 0

D. none of these

If $f(x, y) = e^{xy^2}$, the total differential of the function at the point (1, 2) is

A. $e\,(dx + dy)$

B. $e^4\,(dx + dy)$

C. $e^4\,(4dx + dy)$

D. $4e^4\,(dx + dy)$

If $f(x, y)$ is such that $f_x = e^x \cos y$ and $f_y = e^x \sin y$, then which of the following is true?

A. $f(x, y) = e^{x+y} \sin(x + y)$

B. $f(x, y) = e^x \sin(x + y)$

C. $f(x, y)$ does not exist

D. none of above

If $u = x^y$, *the values of* $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are :

A. $xy^{x-1}, y^x \log y$

B. $x^y \log x , yx^{y-1}$

C. $yx^{y-1}, y^x \log y$

D. $yx^{y-1}, x^y \log y$