Wednesday, September 8, 2021 2:30 AM

Recurrence Relation

Application of Recurrence Relation.

Advanced Counting Techniques

In this section we will show that such relations can be used to study and to solve counting problems. For example, suppose that the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present in n hours? To solve this problem,

let a_n be the number of bacteria at the end of n hours. Because the number of bacteria doubles every hour, the relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer. This recurrence relation, together with the initial condition $a_0 = 5$, uniquely determines a_n for all nonnegative integers n. We can find a formula for a_n using the iterative approach followed in Chapter 2, namely that $a_n = 5 \cdot 2^n$ for all nonnegative integers n.

Ans: Let no. of bacteria present in the colony in n hours be $= a_n$ ✓

no of bacteria present in the colony in $n-1$ hours be $= a_{n-1}$ ✓

$$a_n = 2a_{n-1} \longrightarrow (1) \qquad a_0 = 5$$

Changing n to $n-1$

$$a_{n-1} = 2a_{n-2} \longrightarrow (2)$$

using (2) in (1)

$$a_n = 2(2a_{n-2})$$

$$a_n = 2^2 a_{n-2}$$

In general $a_n = 2^n a_{n-n}$

$$= 2^n a_0$$

$$= 5(2)^n$$

no. of bacteria in 5 hours

$$= 5(2)^5$$

$$= 5 \times 32$$

$$= 160$$

no of bacteria in 2 hours $= 5(2)^2$

$$= 5 \times 4$$

$$= 20$$

[Unit - 2]

[Recurrence Relation]

Advanced Counting Techniques

Application of recurrence relation.

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let a_n be the number of bacteria at the end of n hours. Because the number of bacteria doubles every hour, the relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer. This recurrence relation, together with the initial condition $a_0 = 5$, uniquely determines a_n for all nonnegative integers n. We can find a formula for a_n using the iterative approach followed in Chapter 2, namely that $a_n = 5 \cdot 2^n$ for all nonnegative integers n.

Sol^n Suppose that no. of bacteria in n hours $= a_n$

∴ no of bacteria in $n-1$ hours $= a_{n-1}$

$a_n = 2a_{n-1}$ → (1) $a_0 = 5$

Change n to $n-1$ in (1)

$a_{n-1} = 2a_{n-2}$ → (2)

Using (2) in (1)

$a_n = 2(2a_{n-2})$

$a_n = 2^2 a_{n-2}$

$a_n = 2a_{n-1}$

$a_{n-1} = 2a_{n-2}$

$a_{n-2} = 2a_{n-3}$

⋮

Multiply these equations

$a_n \cancel{a_{n-1}} \cancel{a_{n-2}} \cdots \cancel{a_1} = 2^n \cancel{a_{n-1}} \cancel{a_{n-2}} a_{n-3} \cdots \cancel{a_1} a_0$

$\boxed{a_n = 2^n a_0}$

Hly

$a_n = 2^n a_{n-n}$

$= 2^n a_0$

$= 2^n 5$

$a_n = 5 \cdot (2)^n$

Put $n=5$

$a_5 = 5(2)^5$

$= 5 \times 32$

$= 160$ Ans

$a_{n-2} = 2 a_{n-3}$

$\vdots$

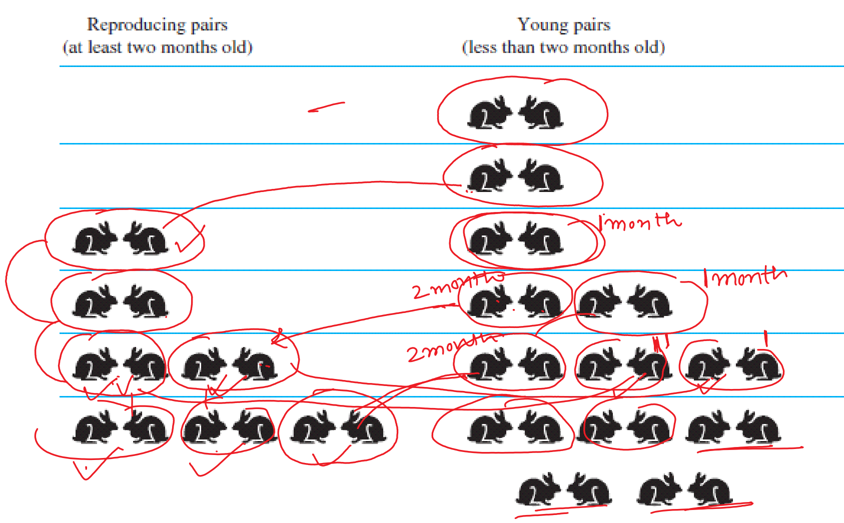
$a_1 = 2 a_0$

$$\boxed{a_n = 2^n a_0}$$

Modelling of recurrence relation.

EXAMPLE 1 **Rabbits and the Fibonacci Numbers** Consider this problem, which was originally posed by Leonardo Pisano, also known as Fibonacci, in the thirteenth century in his book *Liber abaci*. A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month, as shown in Figure 1. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.

Links



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

$$\boxed{f_n = f_{n-1} + f_{n-2} \quad n \geq 2}$$

DEFINITION 1 A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

$\{a_n\} = \{a_1, a_2, a_3, a_{n-k} ----, a_{n-1}, a_n, ------\}$

$a_n = c_1 \underline{a_{n-1}} + c_2 \underline{a_{n-2}} + ----+ c_k \underline{a_{n-k}}$

$a_n - c_1 a_{n-1} - c_2 a_{n-2} --- - c_k \underline{a_{n-k}} = 0$

degree of recurrence relation = largest subscript - Smallest subscript

$= n-(n-k)$

$= \cancel{n}-\cancel{n}+k = k$

$P_n = 1.11 \underline{P_{n-1}}$ degree $= n-(n-1) = \cancel{n}-\cancel{n}+1 = 1$

EXAMPLE 1 The recurrence relation $P_n = (1.11)P_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five.

$f_n = f_{n-1} + f_{n-2}$

$n-(n-2)$
$= \cancel{n} - \cancel{n} + 2 = 2$

— x —

$a_n = a_{n-5}$

Example 2 presents some examples of recurrence relations that are not linear homogeneous recurrence relations with constant coefficients.

EXAMPLE 2 The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear. The recurrence relation $H_n = 2H_{n-1} + 1$ is not homogeneous. The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients.

① $a_n + a_{n-1} + a_{n-2} = 0$ This is called linear-Homogeneous recurrence relation.

② $a_n = a_{n-1} + a_{n-2}^2$

$a_n - a_{n-1} - a_{n-2}^2 = 0$ This is called non-linear, homogeneous recurrence relation.

③ $H_n = 2H_{n-1} + 1$

$H_n - 2H_{n-1} = 1$ This is called linear, non-Homogeneous recurrence relation.

④ $B_n = n B_{n-1}$

$B_n - nB_{n-1} = 0$ This is called linear Homogeneous recurrence relation with variable coefficients.

① $a_{n+2} + 2a_n = 2$ linear, non-Homogeneous recurrence relation.

② $a_n a_{n-2} + a_{n+1}^2 = 0$ non-linear, Homogeneous recurrence relation.

③ $a_n + n^2 a_{n-1} = 0$ linear, Homogeneous recurrence relation with variable coefficients.

——— X ———

How to form characteristic equation and How to write the sol^n.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \dots - c_k a_{n-k} = 0 \longrightarrow (1)$$

$$\left.\begin{array}{l} a_n = 0 \\ a_{n-1} = 0 \\ a_{n-2} = 0 \\ \vdots \\ a_{n-k} = 0 \end{array}\right] \forall n$$

We are interested in finding non-zero sol^n of the recurrence relation.

Set
$$\left[\begin{array}{l} a_n = r^n \quad (r \neq 0) \\ a_{n-1} = r^{n-1} \\ a_{n-2} = r^{n-2} \\ \vdots \\ a_{n-k} = r^{n-k} \end{array}\right.$$

$$r^n - c_1 r^{n-1} - c_2 r^{n-2} - \dots - c_k r^{n-k} = 0$$

Dividing throughout the equation by r^{n-k}.

$$\frac{r^n}{r^{n-k}} - c_1 \frac{r^{n-1}}{r^{n-k}} - c_2 \frac{r^{n-2}}{r^{n-k}} - \dots - c_k \frac{\cancel{r^{n-k}}}{\cancel{r^{n-k}}} = 0$$

$$r^n r^{-n+k} - c_1 r^{n-1} r^{-n+k} - c_2 r^{n-2} r^{-n+k} - \dots - c_k r^{n-k-n+k} = 0$$

$$r^{n-n+k} - c_1 r^{\cancel{n}-1-\cancel{n}+k} - c_2 r^{\cancel{n}-2-\cancel{n}+k} - \dots - c_k r^{n-k-n+k} = 0$$

$$\boxed{r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0}$$

This equation is called characteristic equation.

As this is a polynomial of degree k so it has k roots.

Let $r_1, r_2, \dots, r_k$ be the k roots.

Case (1) $\quad r_1 \neq r_2 \neq \dots \neq r_k$ all roots are distinct

$$a_n = c_1 (r_1)^n + c_2 (r_2)^n + \dots + c_k (r_k)^n.$$

Case (2) $\quad \boxed{r_1 = r_2}$ and other roots are unequal.

$$a_n = (c_1 + c_2 n)(r_1)^n + c_3 (r_3)^n + c_4 (r_4)^n + \dots + c_k (r_k)^n.$$

Case (3) $\quad r_1 = r_2 = \dots = r_k$

Case ③ $r_1 = r_2 = \cdots = r_k$

$$a_n = \left(c_1 + c_2 n + c_3 n^2 + \cdots c_k n^{k-1}\right)(r_1)^n$$

(i) $a_n = (2+n)(2)^n$ ✗

(ii) $a_n = (1+n)(3)^n$ ✓

EXAMPLE 5 What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with initial conditions $\boxed{a_0 = 1}$ and $a_1 = 6$?

Solⁿ: The given recurrence relation is.

$$a_n = 6a_{n-1} - 9a_{n-2}$$

or $$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

Its degree $= n - (n-2)$

$= \cancel{n} - \cancel{n} + 2$

$= 2$

$\therefore$ Its characteristic eqn is

$$\alpha^2 - 6\alpha + 9 = 0$$

$$\alpha^2 - 3\alpha - 3\alpha + 9 = 0$$

$$\alpha(\alpha-3) - 3(\alpha-3) = 0$$

$$(\alpha-3)(\alpha-3) = 0$$

$$\alpha = 3, 3$$

$$\boxed{a_n = (c_1 + c_2 n)(3)^n} \longrightarrow ①$$

Set $\boxed{n=0}$

$$a_0 = (c_1 + c_2(0))(3)^0$$

$$1 = c_1$$

Set $\boxed{n=1}$

$$a_1 = (c_1 + c_2)(3)^1$$

$$\cancel{6} = (c_1 + c_2)\cancel{3}$$

$$2 = c_1 + c_2$$

$$2 = 1 + c_2 \Rightarrow c_2 = 2 - 1 = 1$$

From ①,

$$\boxed{a_n = (1+n)(3)^n}$$

2, 2, 5

$$a_n = (c_1 + c_2 n)(2)^n + c_3(5)^n.$$

2, 2, 2

$$a_n = (c_1 + c_2 n + c_3 n^2)(2)^n$$

Q2 $a_n = a_{n-1} + 2a_{n-2}$ $\boxed{a_0 = 2}$, $a_1 = 7$

Solⁿ The given recurrence relation is.

Q2 $a_n = a_{n-1} + 2a_{n-2}$ $\boxed{a_0 = 2}$, $a_1 = 7$

Sol^n The given recurrence relation is.

$a_n = a_{n-1} + 2a_{n-2}$

$a_n - a_{n-1} - 2a_{n-2} = 0$

Its order $= n-(n-2) = 2$

$\therefore$ Its characteristic eq^n is

$\alpha^2 - \alpha - 2 = 0$

$\alpha^2 - 2\alpha + \alpha - 2 = 0$

$\alpha(\alpha-2) + 1(\alpha-2) = 0$

$(\alpha+1)(\alpha-2) = 0$

$(\alpha+1)(\alpha-2) = 0$

$\alpha = -1, 2$

$a_n = c_1(-1)^n + c_2(2)^n \longrightarrow (1)$

Set $\boxed{n=0}$

$a_0 = c_1(-1)^0 + c_2(2)^0$

$2 = c_1 + c_2$

$c_1 = 2 - c_2 \longrightarrow (3)$

Set $\boxed{n=1}$

$a_1 = c_1(-1)^1 + c_2(2)^1$

$7 = -c_1 + 2c_2$

$7 = -(2-c_2) + 2c_2$

$7 = -2 + c_2 + 2c_2$

$\cancel{9} = \cancel{3}c_2$

$\boxed{c_2 = 3}$, from (3)

$c_1 = 2 - 3 = -1$

$\therefore$ from (1) $a_n = (-1)(-1)^n + 3(2)^n$

— X —

Q1) $a_{n+2} - 7a_{n+1} + 12a_n = 0$ $a_0 = 1$, $a_1 = 2$

$a_n = \cancel{-2(3)^n + 2(4)^n}$

$\boxed{a_n = 2(3)^n - (4)^n.}$ (Correct)

$$\boxed{a_n = 2(3)^n - (4)^n}$$ (Correct)

$4a_n = a_{n-2}$ $a_0 = 1$, $a_1 = 0$, $n \geq 2$

Ans: $a_n = \frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{1}{2}\left(-\frac{1}{2}\right)^n$

Sol^n $4a_n - a_{n-2} = 0$

Its order $= n-(n-2) = \cancel{n}-\cancel{n}+2$
$= 2$

$4\alpha^2 - 1 = 0$

$4\alpha^2 = 1$

$\alpha^2 = \frac{1}{4}$

$\alpha = \pm\sqrt{\frac{1}{4}}$

$\alpha = \pm\frac{1}{2}$

$a_n = c_1\left(\frac{1}{2}\right)^n + c_2\left(-\frac{1}{2}\right)^n \longrightarrow (1)$

Set $\boxed{n=0}$

$a_0 = c_1\left(\frac{1}{2}\right)^0 + c_2\left(-\frac{1}{2}\right)^0$

$1 = c_1 + c_2$

$c_1 = 1 - c_2 \longrightarrow (2)$

Set $n=1$

$a_1 = c_1\left(\frac{1}{2}\right) + c_2\left(-\frac{1}{2}\right)$

$0 = \frac{1}{2}\left[c_1 - c_2\right]$

$0 = c_1 - c_2$

$0 = 1 - c_2 - c_2$

$0 = 1 - 2c_2 \Rightarrow 2c_2 = 1$

$\Rightarrow \boxed{c_2 = 1/2}$

from (2) $c_1 = 1 - \frac{1}{2} = \frac{2-1}{2} = \frac{1}{2}$

$a_n = \frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{1}{2}\left(-\frac{1}{2}\right)^n$

— x —

$a_{n+2} - 10a_{n+1} + 25a_n = 0$ $a_0 = 0$, $a_1 = 2$

$c_1 = 0, \quad c_2 = 2/5$

$a_n = (c_1 + c_2 n)(5)^n$

$= \frac{2}{5} n (5)^n$

$= \underline{2n (5)^{n-1}}$

$f_n = f_{n-1} + f_{n-2} \qquad f_0 = 0, \qquad f_1 = 1$

Solⁿ The given recurrence relation is.

$f_n = f_{n-1} + f_{n-2}$

$f_n - f_{n-1} - f_{n-2} = 0$

Its order $= n - (n-2)$

$= \cancel{n} - \cancel{n} + 2$

$= 2$

Its characteristic eqn is.

$\alpha^2 - \alpha - 1 = 0$

$f_n - f_{n-1} - f_{n-2} = 0 \qquad (2)$

$\alpha^2 - \alpha^1 - 1 = 0$

$f_{n+2} + f_{n+1} + f_n = 0$

$\alpha^2 + \alpha + 1 = 0$

$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4.(1)(-1)}}{2}$

$= \frac{1 \pm \sqrt{1+4}}{2}$

$= \frac{1 \pm \sqrt{5}}{2}$

$f_0 = 0, \qquad f_1 = 1$

$= 1 + \sqrt{5} \qquad 1 - \sqrt{5}$

$= \frac{1 \pm \sqrt{5}}{2}$

$= \frac{1+\sqrt{5}}{2}, \quad \frac{1-\sqrt{5}}{2}$

$f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\quad\right)^n$

Set $n=0$

$f_0 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0$

$0 = c_1 + c_2$

$\boxed{c_1 = -c_2}$

$\boxed{c_2 = -c_1}$

$f_0 = 0, \quad f_1 = 1$

Set $\boxed{n=1}$

$f_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1$

$1 = c_1\left[\frac{1+\sqrt{5}}{2}\right] - c_1\left[\frac{1-\sqrt{5}}{2}\right]$

$1 = \frac{c_1}{2}\left[\cancel{1}+\sqrt{5} - \cancel{1}+\sqrt{5}\right]$

$1 = \frac{c_1}{\cancel{2}}(\cancel{2}\sqrt{5})$

$c_1 = \frac{1}{\sqrt{5}}, \quad c_2 = -\frac{1}{\sqrt{5}}.$

from (1)

$f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$

$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ Ans

—×—

(1) $(a+b)^3 = a^3+b^3+3a^2b+3ab^2 = a^3+3a^2b+3ab^2+b^3$

(2) $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$

$a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 0$

Its order $= \cancel{(n+3)} - \cancel{n} = 3$.

$\therefore$ Its characteristic eqn is.

$\alpha^3 - 3\alpha^2 + 3\alpha - 1^3 = 0$

$(\alpha-1)^3 = 0$

$\alpha = 1, 1, 1$

$\therefore a_n = (c_1 + c_2 n + c_3 n^2)(1)^n$

$\boxed{a_n = (c_1 + c_2 n + c_3 n^2)}$

Q1) $a_{n+3} + 6a_{n+2} + 12a_{n+1} + 8a_n = 0$

Ans: $(c_1 + c_2 n + c_3 n^2)(-2)^n$.

Q2 $a_{n+3} - 9a_{n+2} + 27a_{n+1} - 27a_n = 0$

Ans: $(c_1 + c_2 n + c_3 n^2)(3)^n$.

— X —

$\alpha^3 - 9\alpha^2 + 27\alpha - 27 = 0$

$\alpha^3 - 9\alpha^2 + 27\alpha - (3)^3 = 0$

$-3a^2 b \quad +3ab^2$

$-3a^2(3) \quad 3\alpha(3)^2$

27α

$(\alpha - 3)^3 = 0$

$\alpha = 3, 3, 3$

$a_n = (c_1 + c_2 n + c_3 n^2)(3)^n$ Ans