Isomorphism of Graphs

Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a bijection (an one-to-one and onto function) f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if f(a) and f(b) are adjacent in G_2, for all a and b in V_1.

Such a function f is called an **isomorphism**.

In other words, G_1 and G_2 are isomorphic if their vertices can be ordered in such a way that the adjacency matrices M_{G_1} and M_{G_2} are identical.

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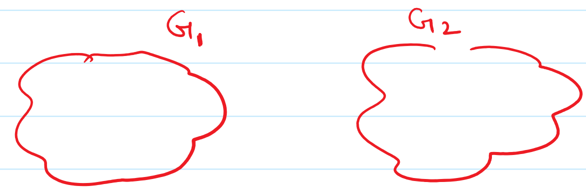
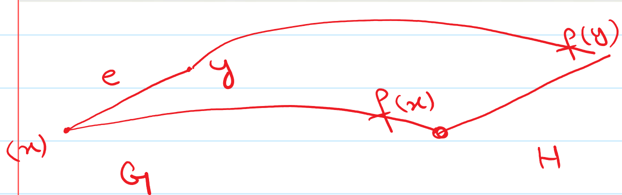


Two Graphs G_1 and G_2 are called isomorphic graphs if the following properties are satisfied.

(i) $\exists$ $f: G_1 \to G_2$ which a bijective map.

(ii) if $(x,y) \in G \Rightarrow (f(x), f(y)) \in H$

(iii) if $(x,y) \notin G \Rightarrow (f(x), f(y)) \notin H$



Isomorphism of Graphs

From a visual standpoint, G_1 and G_2 are isomorphic if they can be arranged in such a way that their **displays are identical** (of course without changing adjacency).

Unfortunately, for two simple graphs, each with n vertices, there are **n! possible isomorphisms** that we have to check in order to show that these graphs are isomorphic.

However, showing that two graphs are **not** isomorphic can be easy.

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Isomorphism of Graphs

For this purpose we can check **invariants**, that is, properties that two isomorphic simple graphs must both have.

For example, they must have

- the same number of vertices,
- the same number of edges, and
- the same degrees of vertices.

Note that two graphs that **differ** in any of these invariants are not isomorphic, but two graphs that **match** in all of them are not necessarily isomorphic.

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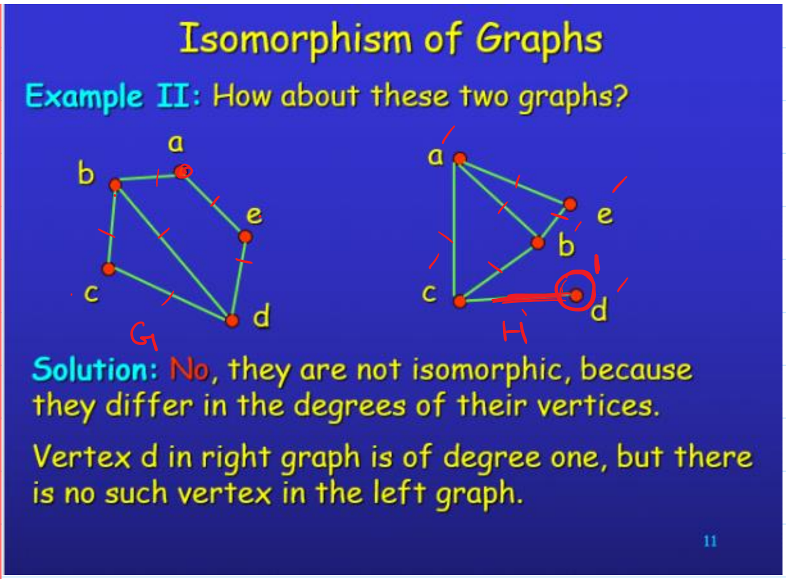
(i) No. of vertices in the both graphs should be same.

(ii) No. of edges in both the graph should be same.

(iii) No of vertices of equal degree should be same.



(i) No of vertices in both the graphs are equal.

(ii) No of Edges in both the Graphs are Same.

(iii) We have one vertex d' of degree 1 in Graph H. but we don't have any vertex of degree 1 in Graph G.

∴ These graphs are not isomorphic

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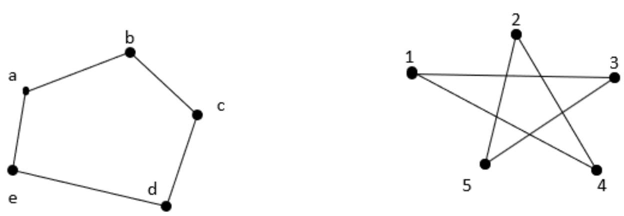
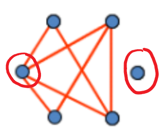
- The two graphs below look different.
- Are from a graph theoretic point of view structurally the 'same'?



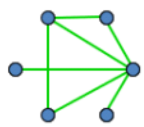
Two non-isomorphic graphs



Vertices: 6

Edges: 7

Vertex sequence: 4, 3, 3, 2, 2, 0.



Vertices: 6

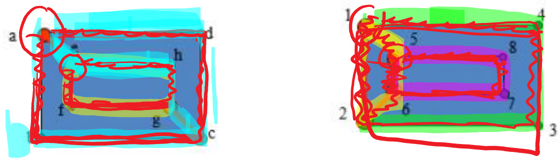
Edges: 7

Vertex sequence: 5, 3, 2, 2, 1, 1.

As in both these graph. degree sequences in both these graphs are not same

∴ Graphs are not isomorphic

Two non-isomorphic graphs



Vertices: 8

Edges: 10

Vertex sequence: 3, 3, 3, 3, 2, 2, 2, 2.

Vertices: 8

Edges: 10

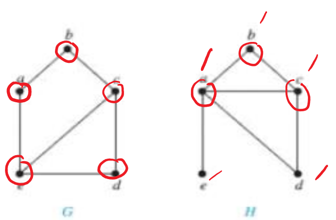
Vertex sequence: 3, 3, 3, 3, 2, 2, 2, 2.

However, induced subgraphs on degree 3 vertices are NOT isomorphic!

These graphs are not isomorphic

EXAMPLE Show that the graphs displayed in Figure are not isomorphic.

Solution: Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one. It follows that G and H are not isomorphic



G: 2, 2, 2, 3, 3

H: 4, 3, 2, 2, 1

So graphs are not isomorphic.

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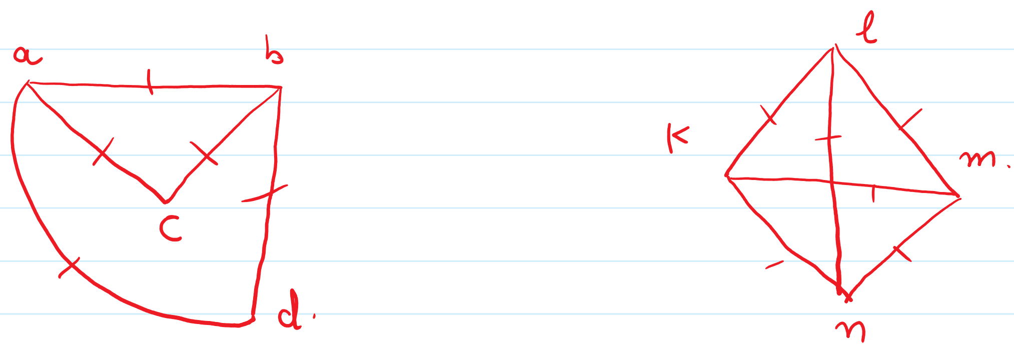
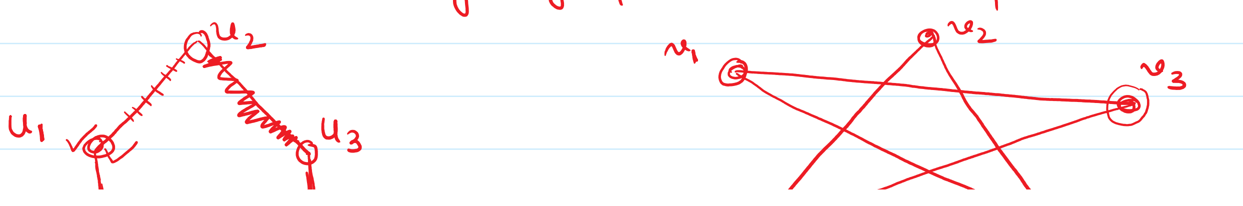
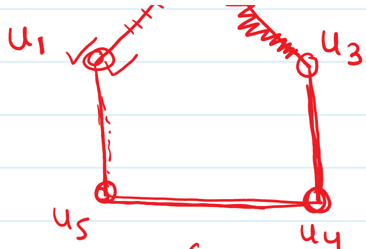


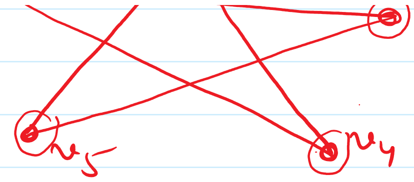
(i) No. of vertices in both the graphs are same
(ii) No of Edges in both the graphs are not same.
$\therefore$ Graphs are not isomorphic.

Q Check the following graphs are isomorphic or not





G



H

(i) no of vertices are same
(ii) no of edges are also same.
(iii) We have same no of vertices of equal degree.

EXAMPLE 2 Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in Figure are isomorphic.

Solution: The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one

EXAMPLE Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in Figure are isomorphic.

Solution: The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between V and W. To see that this correspondence preserves adjacency, note that adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_2 and u_4, and u_3 and u_4, and each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, and $f(u_3) = v_3$ and $f(u_4) = v_2$ consists of two adjacent vertices H.

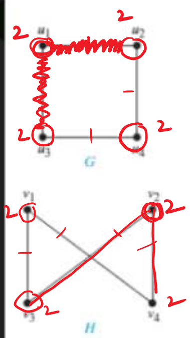


FIGURE The graphs G and H.

$f : G \to H$

$f(u_1) = v_2$

$f(u_3) = v_4$

$f(u_2) = v_3$

$f(u_4) = v_1$

$$A_G = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \checkmark$$

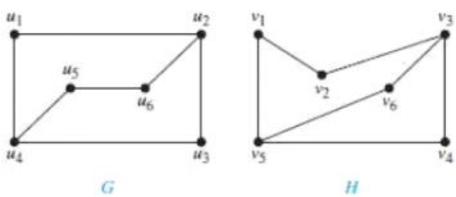
$$B_H = \begin{array}{c} \\ v_2 \\ v_3 \\ v_4 \\ v_1 \end{array} \begin{array}{c} \begin{array}{cccc} v_2 & v_3 & v_4 & v_1 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \checkmark$$

$A_G = B_H$

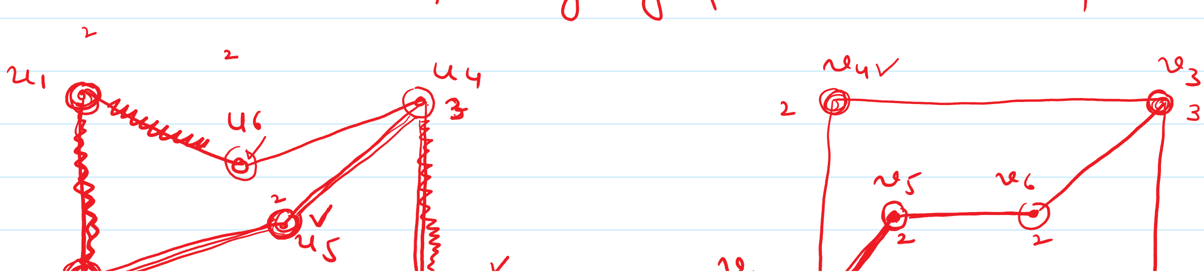
$\therefore$ we can say that graphs are isomorphic.

Determine whether the graphs G and H displayed in Figure 12 are isomorphic.

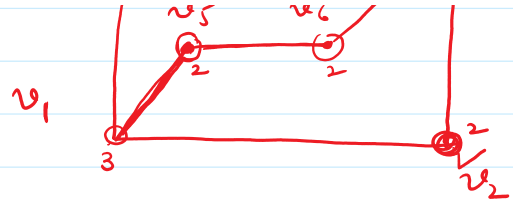
Solution: Both G and H have six vertices and seven edges. Both have four vertices of degree two and two vertices of degree three. It is also easy to see that the subgraphs of G and H consisting of all vertices of degree two and the edges connecting them are isomorphic (as the reader should verify). Because G and H agree with respect to these invariants, it is reasonable to try to find an isomorphism f.



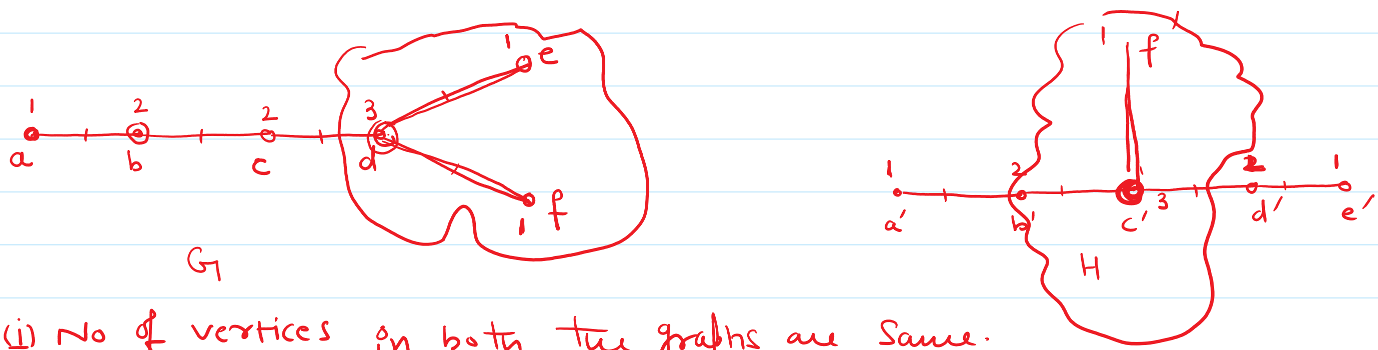
Show that the following graphs are isomorphic.



G_1



Q Check the following graphs are isomorphic or not



(i) No of vertices in both the graphs are same.

(ii) No of Edges in both the graphs are same

(iii) No of vertices of same degree are equal.

we have a vertex d in Graph G of degree 3, which is adjacent to two pendent vertices. but in Graph H. we have a vertex c' of degree 3 which is connected to only one pendent vertex.

∴ As the structure of these graphs are different Graphs are not isomorphic.

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