p, q

$p \to q$

1. p implies q
2. If p then q
3. q whenever p

Conditional Statements

We will discuss several other important ways in which propositions can be combined.

DEFINITION 5 Let p and q be propositions. The *conditional statement* $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

$p \to q$

"If I am elected, then I will lower taxes."

p: I am elected

q: I will lower taxes

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

$p \to q \neq q \to p$

p: you marry me

q: I will bring a car for you.

p	q	p → q	q → p
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

$p \to q \neq q \to p$

EXAMPLE 8 What is the value of the variable x after the statement

$$\textbf{if } 2 + 2 = 4 \textbf{ then } x := x + 1$$

if $x = 0$ before this statement is encountered? (The symbol $:=$ stands for assignment. The statement $x := x + 1$ means the assignment of the value of $x + 1$ to x.)

Solution: Because $2 + 2 = 4$ is true, the assignment statement $x := x + 1$ is executed. Hence, x has the value $0 + 1 = 1$ after this statement is encountered. ◄

CONVERSE, CONTRAPOSITIVE, AND INVERSE We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are three related conditional statements that occur so often that they have special names. The proposition $q \to p$ is called the **converse** of $p \to q$. The **contrapositive** of $p \to q$ is the proposition $\neg q \to \neg p$. The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$. We will see that of these three conditional statements formed from $p \to q$, only the contrapositive always has the same truth value as $p \to q$.

We first show that the contrapositive, $\neg q \to \neg p$, of a conditional statement $p \to q$ always has the same truth value as $p \to q$. To see this, note that the contrapositive is false only when $\neg p$ is false and $\neg q$ is true, that is, only when p is true and q is false. We now show that neither the converse, $q \to p$, nor the inverse, $\neg p \to \neg q$, has the same truth value as $p \to q$ for all possible truth values of p and q. Note that when p is true and q is false, the original conditional statement is false, but the converse and the inverse are both true.

$\boxed{p \to q \equiv \sim q \to \sim p}$

① $p \to q$ (Conditional statement) ✓

$q \to p$ (Converse)

$\sim p \to \sim q$ (Inverse)

$\sim q \to \sim p$ (Contrapositive) ✓

"q whenever p"

EXAMPLE 9 What are the contrapositive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining?"

q whenever p

p: It is raining

q: The home team wins

$p \to q$: If It is raining then the home team wins.

(Converse) $q \to p$: If the home team wins then It is raining

(Inverse) $\sim p \to \sim q$: If It is not raining Then the home team doesn't win

(Contrapositive) $\sim q \to \sim p$: If the home team doesn't win then it is not raining

BICONDITIONALS We now introduce another way to combine propositions that expresses that two propositions have the same truth value.

DEFINITION 6 Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

$5+5 \neq 10$

$2.5 \neq 10$

$p \leftrightarrow q$

$\leftrightarrow$

$5+5 \neq 10$

$2.5 = 10$

$5+5=10$

$2.5=10$

$p : 5+5=10$

$q : 2.5=10$

$2.5.$

p	q	$p \leftrightarrow q$	$q \leftrightarrow p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	T	T

$5+5=10 \rightarrow 2.5=10$

$2.5=10 \rightarrow 5+5=10$

$\boxed{p \leftrightarrow q = q \leftrightarrow p}$

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

EXAMPLE 10 Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement

"You can take the flight if and only if you buy a ticket."

$\leftrightarrow$

p : you can take the flight

q : you buy a ticket

p	q	$p \leftrightarrow q$
T	T	T ✓
T ✓	F ✓	F
F ✓	T ✓	F
F ✓	F ✓	T ✓

① $\wedge$, $\vee$, $\oplus$, $\leftrightarrow$ are Commutative operators

—— X ——

② $\rightarrow$ is not commutative operator

—— X ——

DEFINITION 1 A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

EXAMPLE 1 We can construct examples of tautologies and contradictions using just one propositional variable. Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$, shown in Table 1. Because $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction. ◀

Tautology

p	$\sim p$	$p \vee \sim p$
T ✓	F	T
F ✓	T ✓	T

$p \vee \sim p = T$

$p \vee \sim p$ is tautology

— × —

p: You marry me ✓

q: I will bring a car for you

Contradiction

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

$p \wedge \sim p = F$

$p \wedge \sim p$ is a Contradiction

— × —

p	q	$p \to q$
T	T	T
T	F	F
F	T	
F	F	

EXAMPLE 11 Construct the truth table of the compound proposition

$$(p \vee \neg q) \to (p \wedge q).$$

p	q	$\sim q$	$p \vee \sim q$	$p \wedge q$	$(p \vee \sim q) \to (p \wedge q)$
T	T	F	T	T ✓	T
T	F	T	T	F ✓	F
F	T	F	F	F	T
F	F	T	T	F ✓	F

TABLE 2 De Morgan's Laws.
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

① $\sim(p \wedge q) = \sim p \vee \sim q$

② $\sim(p \vee q) = \sim p \wedge \sim q$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

EXAMPLE 3 Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.

$p \rightarrow q \equiv \sim p \vee q$

V.V.Imp

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

$p \rightarrow q \equiv \sim p \vee q$

TABLE 5 A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T

T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ Distributive law

p	T	$p \wedge T = p$	$p \vee T$
T	T	T	T
F	T	F	T

$\boxed{p \wedge T} = p$ Rem

$\boxed{p \vee T} = T$

Table 1

ntities.

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws

EXAMPLE 6 Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

Sol^n

$\sim(p \Rightarrow q)$

$= \sim(\sim p \vee q)$

$= \sim(\sim p) \wedge \sim q$

$= (p \wedge \sim q)$ Ans

$p \to q = \sim p \vee q$

(De-morgan's Law)

$\sim(\sim p) = p$

EXAMPLE 7 Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: We will use one of the equivalences in Table 6 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$. (*Note:* we could also easily establish this equivalence using a truth table.) We have the following equivalences.

$\neg(p \vee (\neg p \wedge q))$	$\equiv \neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
	$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$	by the first De Morgan law
	$\equiv \neg p \wedge (p \vee \neg q)$	by the double negation law
	$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
	$\equiv \mathbf{F} \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv \mathbf{F}$
	$\equiv (\neg p \wedge \neg q) \vee \mathbf{F}$	by the commutative law for disjunction
	$\equiv \neg p \wedge \neg q$	by the identity law for $\mathbf{F}$

$\boxed{\sim p \wedge \sim q}$ ✓

$\sim(p \vee (\sim p \wedge q))$

$= \sim((p \vee (\sim p)) \wedge (p \vee q))$ (By-distributive Law)

($p \vee \sim p = T$)

$= \sim(T \wedge (p \vee q))$

$\boxed{T \wedge p = p}$

$= \sim(p \vee q)$

$= \sim p \wedge \sim q$ (By De-morgan's law)

EXAMPLE 8 Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to **T**. (*Note:* This could also be done using a truth table.)

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by Example 3} \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and commutative laws for disjunction} \\
&\equiv \mathbf{T} \vee \mathbf{T} && \text{by Example 1 and the commutative law for disjunction} \\
&\equiv \mathbf{T} && \text{by the domination law}
\end{aligned}
$$

Solⁿ $(p \wedge q) \rightarrow (p \vee q)$ $\qquad p \Rightarrow q = \sim p \vee q$

$= \sim(p \wedge q) \vee (p \vee q)$

$= (\sim p \vee \sim q) \vee (p \vee q)$ (By De-morgan's law)

$= (\sim p \vee p) \vee (q \vee \sim q)$ $\qquad \sim p \vee p = T$

$= T \vee T = T$

This is tautological statement

Predicates

Statements involving variables, such as

"$x > 3$," "$x = y + 3$," "$x + y = z$,"

$x > 3$

Subject

is greater than 3

Predicate

this section, we will discuss the ways that propositions can be produced from such statements.

The statement "x is greater than 3" has two parts. The first part, the variable x, is the subject of the statement. The second part—the **predicate**, "is greater than 3"—refers to a property that the subject of the statement can have. We can denote the statement "x is greater than 3" by $P(x)$, where P denotes the predicate "is greater than 3" and x is the variable. The statement $P(x)$ is also said to be the value of the **propositional function** P at x. Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value. Consider Examples 1 and 2.

EXAMPLE 1 Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $\boxed{P(4)}$ and $P(2)$?

Solution: We obtain the statement $P(4)$ by setting $x = 4$ in the statement "$x > 3$." Hence, $P(4)$, which is the statement "$4 > 3$," is true. However, $P(2)$, which is the statement "$2 > 3$," is false. ◀

$P(x)$: $x > 3$

$P(4)$: $4 > 3$ (True)

The truth value of $P(4)$ is $\boxed{\text{True}}$

$P(2)$: $2 > 3$ (False)

The truth value of $P(2)$ is false

EXAMPLE 2 Let $A(x)$ denote the statement "Computer x is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of $A(\text{CS1})$, $A(\text{CS2})$, and $A(\text{MATH1})$?

Solution: We obtain the statement $A(\text{CS1})$ by setting $x = \text{CS1}$ in the statement "Computer x is under attack by an intruder." Because CS1 is not on the list of computers currently under attack, we conclude that $A(\text{CS1})$ is false. Similarly, because CS2 and MATH1 are on the list of computers under attack, we know that $A(\text{CS2})$ and $A(\text{MATH1})$ are true. ◀

$R(x; y, z): \underline{x + y = z}$

EXAMPLE 5 What are the truth values of the propositions $R(1, 2, 3)$ and $R(0, 0, 1)$?

Solution: The proposition $R(1, 2, 3)$ is obtained by setting $x = 1$, $y = 2$, and $z = 3$ in the statement $R(x, y, z)$. We see that $R(1, 2, 3)$ is the statement "$1 + 2 = 3$," which is true. Also note that $R(0, 0, 1)$, which is the statement "$0 + 0 = 1$," is false.

$R(x, y, z)$: '$x + y = z$'

$R(1, 2, 3)$: '$1 + 2 = 3$'   '$3 = 3$'   True.

∴ The truth value of $R(1,2,3)$ is true.

$R(0, 0, 1)$: '$0 + 0 = 1$'

'$0 = 1$'   (False)

The truth value of $R(0, 0, 1)$ is false