$p \to q \longrightarrow p$ implies q
if p then q
q whenever p

Conditional Statements

We will discuss several other important ways in which propositions can be combined.

DEFINITION 5 Let p and q be propositions. The *conditional statement* $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

$p \to q$

"If I am elected, then I will lower taxes."

p: I am elected.

q: I will lower taxes

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

$p \to q \neq q \to p$ Commutative property does not hold.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p: You marry me.

q: I will bring a car for you.

EXAMPLE 8 What is the value of the variable x after the statement

$$\textbf{if } 2 + 2 = 4 \textbf{ then } x := x + 1$$

if $x = 0$ before this statement is encountered? (The symbol $:=$ stands for assignment. The statement $x := x + 1$ means the assignment of the value of $x + 1$ to x.)

Solution: Because $2 + 2 = 4$ is true, the assignment statement $x := x + 1$ is executed. Hence, x has the value $0 + 1 = 1$ after this statement is encountered. ◀

CONVERSE, CONTRAPOSITIVE, AND INVERSE We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are three related conditional statements that occur so often that they have special names. The proposition $q \to p$ is called the **converse** of $p \to q$. The **contrapositive** of $p \to q$ is the proposition $\neg q \to \neg p$. The proposition $\neg p \to \neg q$ is called the **inverse** of $p \to q$. We will see that of these three conditional statements formed from $p \to q$, only the contrapositive always has the same truth value as $p \to q$.

We first show that the contrapositive, $\neg q \to \neg p$, of a conditional statement $p \to q$ always has the same truth value as $p \to q$. To see this, note that the contrapositive is false only when $\neg p$ is false and $\neg q$ is true, that is, only when p is true and q is false. We now show that neither the converse, $q \to p$, nor the inverse, $\neg p \to \neg q$, has the same truth value as $p \to q$ for all possible truth values of p and q. Note that when p is true and q is false, the original conditional statement is false, but the converse and the inverse are both true.

$p \to Q$	Converse	Inverse	Contrapositive
	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$

"q whenever p"

EXAMPLE 9 What are the contrapositive, the converse, and the inverse of the conditional statement

"The home team wins whenever it is raining?"

p: It is raining

Q: The home team wins

$\boxed{p \to q}$: If It is raining, then the home team wins $q \to p \equiv \sim p \to \sim q$

Converse: $q \to p$: If the home team wins, then It is raining

Inverse $\sim p \to \sim q$: If It is not raining, then the home team doesn't win.

Inverse $\sim p \to \sim q$: If it is not raining, then the home team doesn't win.

Contrapositive $\boxed{\sim q \to \sim p}$: If the home team does not win then it is not raining

p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

$\boxed{p \to q \equiv \sim q \to \sim p}$

BICONDITIONALS We now introduce another way to combine propositions that expresses that two propositions have the same truth value.

DEFINITION 6 Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

EXAMPLE 10 Let p be the statement "You can take the flight," and let q be the statement "You buy a ticket." Then $p \leftrightarrow q$ is the statement

"You can take the flight if and only if you buy a ticket."

DEFINITION 1 A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

EXAMPLE 1 We can construct examples of tautologies and contradictions using just one propositional variable. Consider the truth tables of $p \vee \neg p$ and $p \wedge \neg p$, shown in Table 1. Because $p \vee \neg p$ is always true, it is a tautology. Because $p \wedge \neg p$ is always false, it is a contradiction. ◀

EXAMPLE 11 Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q).$$

TABLE 2 De Morgan's Laws.
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

EXAMPLE 3 Show that $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent.