MTH165

Unit 6
Fourier Series

L 36-37- Introduction and Euler's formulae and change of interval

MCQ

A triangle ABC consists of vertex points A (0,0) B(1,0) and C(0,1). The value of the integral $\int\int 2x \; dxdy$ over the triangle is

A. 1

B. 1/3

C. 1/8

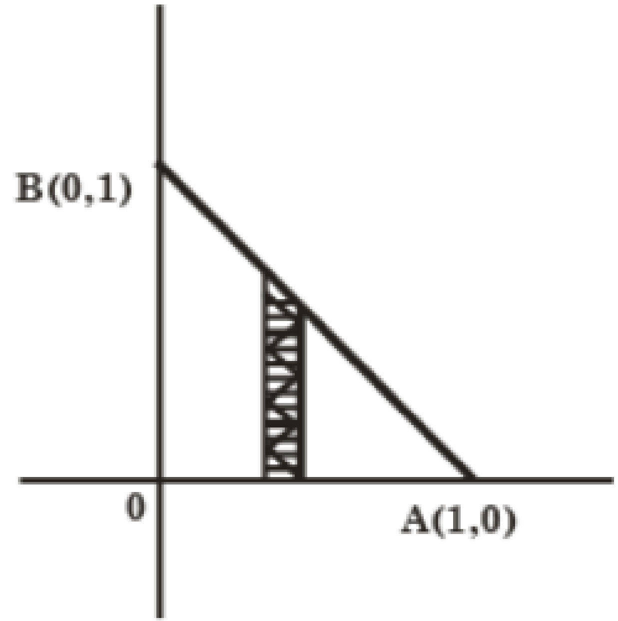
D. 1/9

Solution:

The equation of the line AB is

$$y-0=\frac{1-0}{0-1}(x-1).$$

$$\Rightarrow y+x=1$$



$$\therefore \iint 2x\,dx\,dy = 2\int_{x=0}^{1}\left\{\int_{y=0}^{1-x} x\,dy\right\}dx$$

$$=2\int_{x=0}^{1} x.(1-x)\,dx = 2\int_{0}^{1}\left(x-x^2\right)dx$$

$$=2\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{3}$$

MCQ

The area enclosed between the parabala $y = x^2$ and the straight line $y = x$ is

A. 1/8

B. 1/6

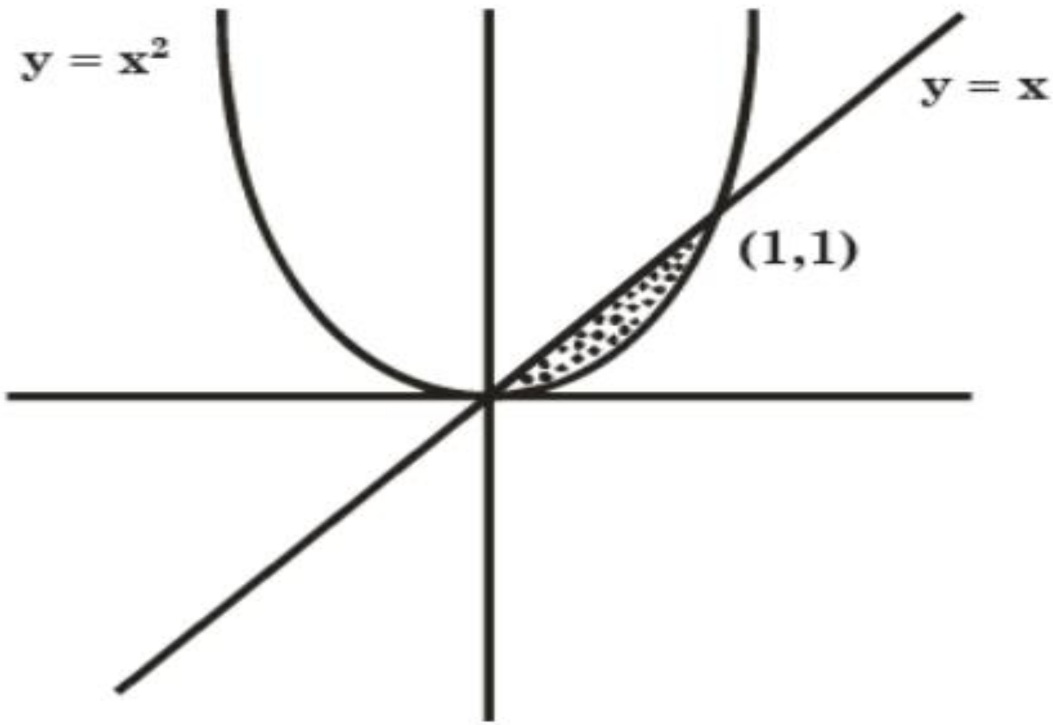
C. 1/3

D. 1/2

Solution:

$$\therefore \text{ Area} = \left|\int_0^1 \left(x^2 - x\right) dx\right|$$

$$= \left|\frac{1}{3} - \frac{1}{2}\right| = \frac{1}{6} \text{ units.}$$



MCQ

Changing the order of the integration in the double integral

$$I = \int_0^8 \int_{x/4}^2 f(x, y)\,dy\,dx \text{ leads to } I$$

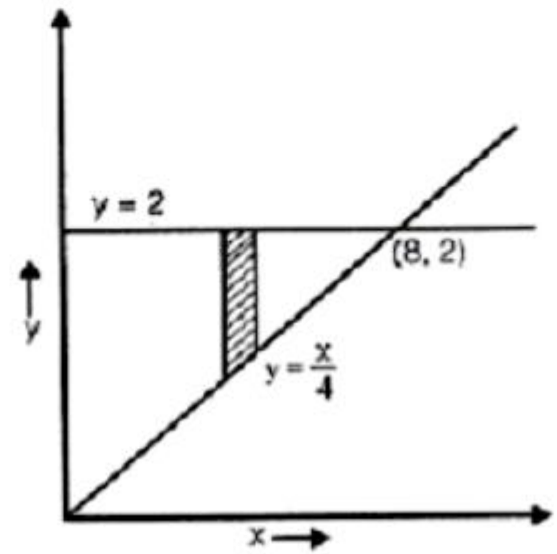
$$= \int_r^s \int_{xp}^q f(x, y)\,dx\,dy. \text{ What is q?}$$

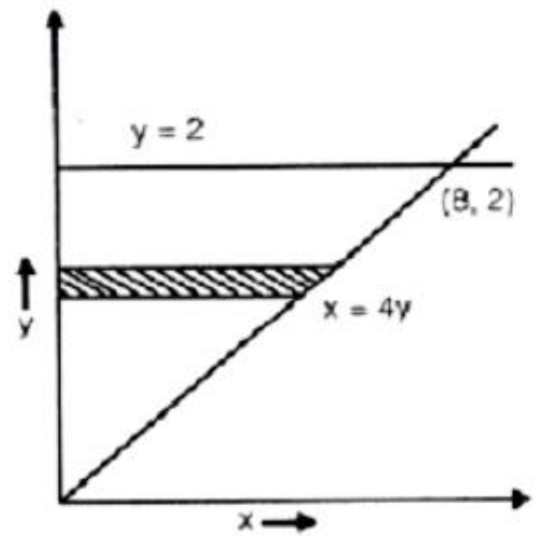
A. 4y

B. $16y^2$

C. x

D. 8





$$I = \int_0^8 \int_{x/4}^2 f(x.y)dx\ dy$$

$$I = \int_0^2 \int_0^{4Y} f(x.y)dx\ dy$$

MCQ

The value of the integral $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dydx}{1+x^2+y^2}$ is:

A. $\frac{\pi}{4}\log\left(\sqrt{2}+1\right)$

B. $\frac{\pi}{4}\log\left(\sqrt{2}-1\right)$

C. $\frac{\pi}{2}\log\left(\sqrt{2}+1\right)$

D. $\frac{\pi}{2}\log\left(\sqrt{2}-1\right)$

MCQ

The value of $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}$ **dydx by changing the order of integration is**

A. zero

B. 3/4

C. 1

D. None of these

MCQ

The area bounded by the curve y = ψ(x), x-axis and the lines x = l , x = m(l <m) is given by

A. $\int_{l}^{m}\int_{0}^{\psi(x)} y\,dx\,dy$

B. $\int_{l}^{m}\int_{0}^{\psi(x)} dy\,dx$

C. $\int_{m}^{l}\int_{0}^{\psi(x)} dx\,dy$

D. None of these

FOURIER SERIES

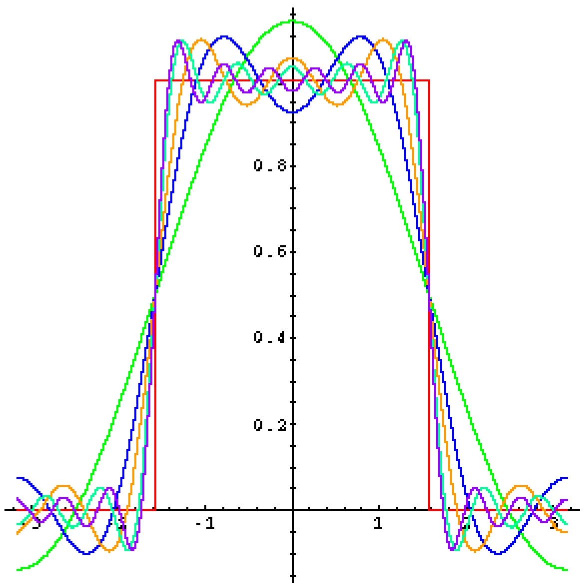
JOSEPH FOURIER

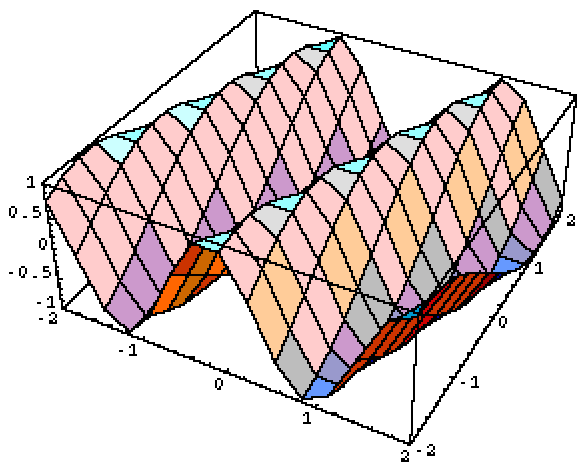
(Founder of Fourier series)

As we know that TAYLOR SERIES representation of functions are valid only for those functions which are continuous and differentiable. But there are many discontinuous periodic function which requires to express in terms of an infinite series containing ‘sine’ and ‘cosine’ terms.

FOURIER SERIES, which is an infinite series representation of such functions in terms of ‘sine’ and ‘cosine’ terms, is useful here. Thus, FOURIER SERIES, are in certain sense, more UNIVERSAL than TAYLOR’s SERIES as it applies to all continuous, periodic functions and also to the functions which are discontinuous in their values and derivatives. FOURIER SERIES a very powerful method to solve ordinary and partial differential equation, particularly with periodic functions appearing as non-homogenous terms.

Fourier waves





FOURIER SERIES can be generally written as,

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

Where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \qquad \text{………(1.1)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\, dx \qquad \text{………(1.2)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\, dx \qquad \text{………(1.3)}$$

Fourier series make use of the orthogonality relationships of the sine and cosine functions.

BASIS FORMULAE OF FOURIER SERIES

The Fourier series of a periodic function $f(x)$ with period 2π is defined as the trigonometric series with the coefficient a0, an and bn, known as *FOURIER COEFFICIENTS*, determined by formulae (1.1), (1.2) and (1.3).

The individual terms in Fourier Series are known as *HARMONICS.*

Every function $f(x)$ of period 2π satisfying following conditions known as *DIRICHLET'S CONDITIONS*, can be expressed in the form of Fourier series.

CONDITIONS :-

1. $f(x)$ is bounded and single value.

   (Afunction $f(x)$ is called single valued if each point in the domain, it has unique value in the range.)

2. $f(x)$ has at most, a finite no. of maxima and minima in the interval.

3. $f(x)$ has at most, a finite no. of discontinuities in the interval.

EXAMPLE:

$\sin^{-1}x$, we can say that the function $\sin^{-1}x$ cant be expressed as Fourier series as it is not a single valued function.

tanx, also in the interval (0,2π) cannot be expressed as a Fourier Series because it is infinite at $x = \pi/2$.

The Fourier series of $f(x)$ in interval $[a, b]$ is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi}{b-a}x + \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi}{b-a}x$$

Where

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$$

$$a_n = \frac{2}{b-a}\int_a^b f(x) \cos\left(\frac{2n\pi}{b-a}x\right)dx$$

$$b_n = \frac{2}{b-a}\int_a^b f(x) \sin\left(\frac{2n\pi x}{b-a}\right)dx$$

MCQ

What is the Fourier series expansion of the function f(x) in the interval (c, c+2π)?

a) $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx)$

b) $a_0 + \sum_{n=1}^{\infty} a_n cos(nx) + \sum_{n=1}^{\infty} b_n sin(nx)$

c) $\frac{a_0}{2} + \sum_{n=0}^{\infty} a_n cos(nx) + \sum_{n=0}^{\infty} b_n sin(nx)$

d) $a_0 + \sum_{n=0}^{\infty} a_n cos(nx) + \sum_{n=0}^{\infty} b_n sin(nx)$

MCQ

What are fourier coefficients?

a) The terms that are present in a fourier series

b) The terms that are obtained through fourier series

c) The terms which consist of the fourier series along with their sine or cosine values

d) The terms which are of resemblance to fourier transform in a fourier series are called fourier series coefficients

MCQ

Find the value of a_0 *for the function* $f(x)=\sqrt{\dfrac{1-\cos x}{2}}$ *in* $(-\pi,\ \pi)$

a) $\frac{4}{\pi}$

b) $\frac{2}{\pi}$

c) $\frac{\pi}{4}$

d) $\frac{\pi}{2}$

Fourier series for EVEN and ODD functions

EVEN FUNCTIONS

If function $f(x)$ is an even periodic function with the period 2L ($-L \le x \le L$), then $f(x)\cos(n\pi x/L)$ is even while $f(x)\sin(n\pi x/L)$ is odd.

Thus the Fourier series expansion of an even periodic function $f(x)$ with period 2L ($-L \le x \le L$) is given by,

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n x}{L}}$$

Where, $$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1,2,\ldots$$

$$b_n = 0$$

ODD FUNCTIONS

If function f(x) is an even periodic function with the period 2L (–L ≤x ≤L), then f(x)cos(nπx/L) is even while f(x)sin(nπx/L) is odd.

Thus the Fourier series expansion of an odd periodic function f(x) with period 2L (–L ≤x ≤L) is given by,

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)}$$

Where,

$$b_n = \frac{2}{L}\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx \qquad n = 1,2,\ldots$$

Examples..

Question.: Find the fourier series of f(x) = x^2+x , -pie $\leq$ x $\leq$pie.

Solution.: The fourier series of f(x) is given by,

$$\boxed{f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),}$$

Using above,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 + x)\,dx$$

$$= \frac{1}{\pi}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi^3}{3}+\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}\right) = \frac{2\pi^3}{3} = a_0$$

Now,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x^2+x)\cos nx dx$$

$$= \frac{1}{\pi}\left[(x^2+x)\left(\frac{\sin nx}{n}\right)-(2x+1)\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2}-(-2\pi+1)\frac{\cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{(-1)^n}{n^2}-(-2\pi+1)\frac{(-1)^n}{n^2}\right]$$

$$= \frac{4(-1)^n}{n^2}$$

Now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x^2 + x)\sin nx dx$$

$$= \frac{1}{\pi}\left[(x^2+x)\left(-\frac{\cos nx}{n}\right) - (2x+1)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(\pi^2+\pi)}{n}(-1)^n + \frac{(\pi^2+\pi)}{n}(-1)^n \right]$$

$$= \frac{(-1)^n}{\pi n}\left[-\pi^2 - \pi + \pi^2 - \pi\right]$$

$$= -\frac{2(-1)^n}{n}$$

Hence fourier series of, f(x) = x^2+x,

$$\boxed{x^2 + x = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos nx - \frac{2(-1)^n}{n}\sin nx\right]}$$

MCQ

If the function f(x) is even, then which of the following is zero?

a) a_n

b) b_n

c) a_0

d) nothing is zero

MTH165

Unit 6
Fourier Series

L 38-39-Even and Odd Function and half range Series

Fourier series for EVEN and ODD functions

EVEN FUNCTIONS

If function f(x) is an even periodic function with the period 2L (–L ≤x ≤L), then f(x)cos(nπx/L) is even while f(x)sin(nπx/L) is odd.

Thus the Fourier series expansion of an even periodic function f(x) with period 2L (–L ≤x ≤L) is given by,

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n x}{L}}$$

Where, $$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1,2,\ldots$$

$$b_n = 0$$

ODD FUNCTIONS

If function f(x) is an odd periodic function with the period 2L (–L ≤x ≤L), then f(x)cos(nπx/L) is odd while f(x)sin(nπx/L) is even.

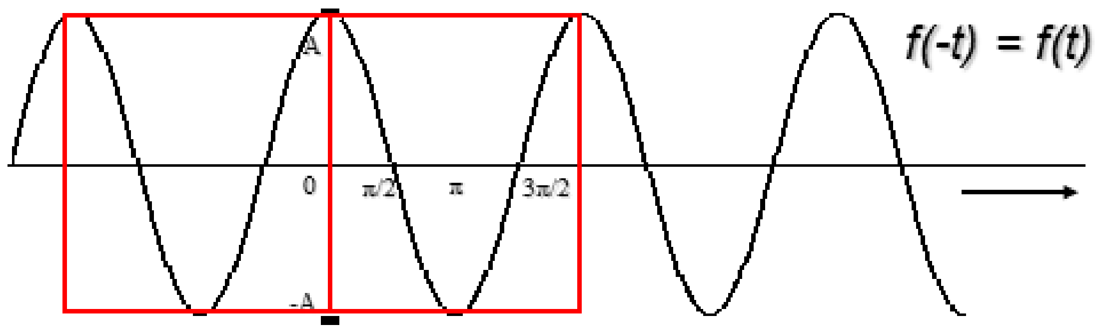
Thus the Fourier series expansion of an odd periodic function f(x) with period 2L (–L ≤x ≤L) is given by,

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)}$$

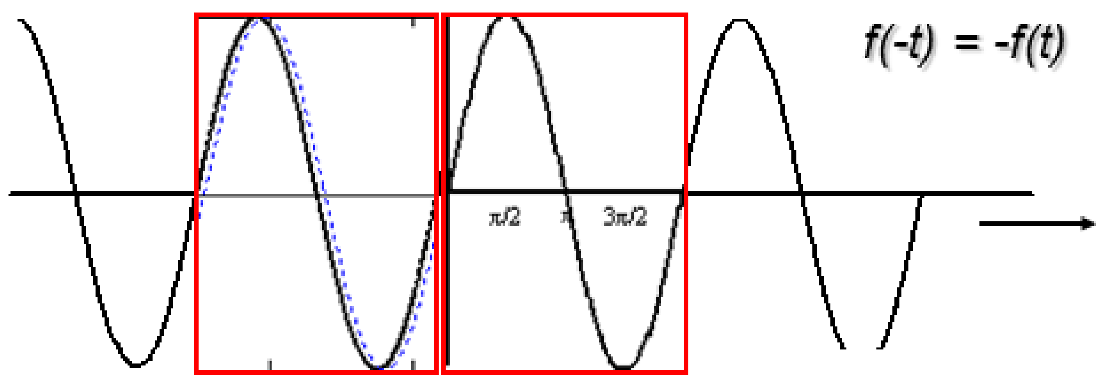
Where,

$$b_n = \frac{2}{L}\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,dx \qquad n = 1,2,\ldots$$

- **Even function**



- **Odd function**



- Note that the integral over a period of an even function is?

If f(t) is even:

Even X Odd = Odd

$$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\sin(n\omega_0 t)\,dt \;\ldots\; n = 1,2,3,\ldots$$

Even X Even = Even

$$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\cos(n\omega_0 t)\,dt \;\ldots\; n = 1,2,3,\ldots$$

Note that the integral over a period of an odd function is zero.

If f(t) is odd:

Odd X Even = Odd

$$a_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\cos(n\omega_0 t)\,dt \;....\; n = 1,2,3,...$$

Odd X Odd = Even

$$b_n = \frac{2}{T_0}\int_{t_1}^{t_1+T_0} f(t)\sin(n\omega_0 t)\,dt \;....\; n = 1,2,3,...$$

- **If the function has:**
  - even symmetry: only the cosine and associated coefficients exist
  - odd symmetry: only the sine and associated coefficients exist
  - even and odd: both terms exist

MCQ

If the function f(x) is even, then which of the following is zero?

a) a_n

b) b_n

c) a_0

d) nothing is zero

MCQ

If $f(x) = x\sin x$ in $(0, 2\pi)$. Then value of a, is

(a) 1 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) None of these

MCQ

If $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

value of b_1 is

(a) 2 (b) 3 (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

MCQ

If $f(t) = \begin{cases} -1 & \text{for } -\pi < t < -\pi/2 \\ 0 & \text{for } -\pi/2 < t < \pi/2 \\ 1 & \text{for } \pi/2 < t < \pi \end{cases}$

then value of a_0 is

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) None of these

HALF RANGE FOURIER SERIES

- Suppose we have a function $f(x)$ defined on $(0, L)$. It can not be periodic (any periodic function, by definition, must be defined for all x).
- Then we can always construct a function $F(x)$ such that:

  - $F(x)$ is periodic with period $p = 2L$, and
  - $F(x) = f(x)$ on $(0, L)$.

Half range Fourier sine series (cont.)

- Expanding the odd-periodic extrapolation F(x) of a function f(x) into a Fourier series,

we find :

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

Where

$$b_n = \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x,$$

Half Range Sine Series in $(0, L)$

It is required to expand $f(x)$ as a Sine series in $0 < x < c$; then we extend the function reflecting it in the origin so that it is an odd function $f(-x) = -f(x)$. So the desired half range Sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi}{b-a} x$$

Here the extended function is odd in $(-c, c)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi x}{2c}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{c} x}$$

where

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin \frac{2n\pi}{b-a} x \, dx$$

$$= \frac{\cancel{2}}{\cancel{2}c} \int_{-c}^{c} \underbrace{\underbrace{f(x)}_{\text{odd}} \cdot \underbrace{\sin \frac{n\pi}{c} x}_{\text{odd}}}_{\text{even}} \, dx$$

$$\boxed{b_n = \frac{2}{c}\int_0^c f(x)\sin\frac{n\pi}{c}x\,dx}$$

Half range Fourier cosine series

- Expanding the even-periodic extrapolation F(x) of a function f(x) into a Fourier series,

We find :

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

With

$$a_0 = \frac{1}{L}\int_0^L F(x)\,\mathrm{d}x, \quad a_n = \frac{2}{L}\int_0^L F(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x,$$

Half range Fourier cosine series (cont.)

- so that the half range Fourier cosine series representation of f(x) is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

with

$$a_0 = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

Half Range Cosine Series in $(0, c)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c}$$

where $$a_0 = \frac{2}{c}\int_0^c f(x)\,dx$$

$$a_n = \frac{2}{c}\int_0^c f(x) \cos \frac{n\pi x}{c}\,dx$$

Example 1

Find the half-range sine series of the function

$$f(x) = \begin{cases} 4, & \text{if } 0 < x < \pi/2 \\ 0, & \text{if } \pi/2 < x < \pi. \end{cases}$$

Solution: $L = \pi$, so that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \;\; b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx.$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx = \\ &= \frac{8}{\pi} \int_0^{\pi/2} \sin(nx)\, dx = \\ &= \frac{8}{n\pi} \int_0^{n\pi/2} \sin u\, du = \\ &= -\frac{8}{n\pi} [\cos u]_0^{n\pi/2} \\ &= \frac{8}{n\pi} [1 - \cos(n\pi/2)]. \end{aligned}$$

Ex 10.5

Q-7 $f(x) = \begin{cases} -x+1 & -\pi \leq x \leq 0 \quad a \\ x+1 & 0 \leq x \leq \pi \quad b \end{cases}$

the value of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

(a) $\frac{\pi}{8}$ (b) $\frac{\pi^2}{4}$ (c) $\frac{\pi^2}{8}$ (d) None of these

$f(-x) = \begin{cases} x+1 \\ -x+1 \end{cases} = f(x) \rightarrow$ even

$a_0 \neq 0, \; a_n \neq 0, \; b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(x+1)\,dx = \frac{2}{\pi}\left[\left(\frac{x^2}{2}+x\right)\right]_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^2}{2}+\pi\right)$$

$$= \cancel{2}\,\frac{(\pi+2)}{\cancel{2}} = \pi+2$$

Find half range Cosine series of $f(x) = (x-1)^2$ in $(0,1)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad (\text{Here } L=1)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = 2\int_0^1 (x-1)^2\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx = 2\int_0^1 f(x)\cos n\pi x\,dx$$

WHY DO WE NEED FOURIER ANALYSIS

In communication we send and recieve information laced signal over a medium, the medium and the hardware corrupts the signal. The receiver has to extract the information from the corrupted signal. The transmitted signal have well defined spectral contents, so if the receiver can do spectral analysis of the received signal then it can extract the information.

FOURIER ANALYSIS

- Fourier Analysis can look at an unknown signal and do an equivalent of a chemical analysis, identifying various frequencies and their relative quantities in the signal.

Fourier Series

Any periodic power signal

Infinite sum of
sines and cosines
at different frequencies

Fourier Series

Tutorial

MTH165

MCQ

1. Which of the following is an “even” function of t?

   (A) t^2
   (B) $t^2 - 4t$
   (C) $\sin(2t) + 3t$
   (D) $t^3 + 6$

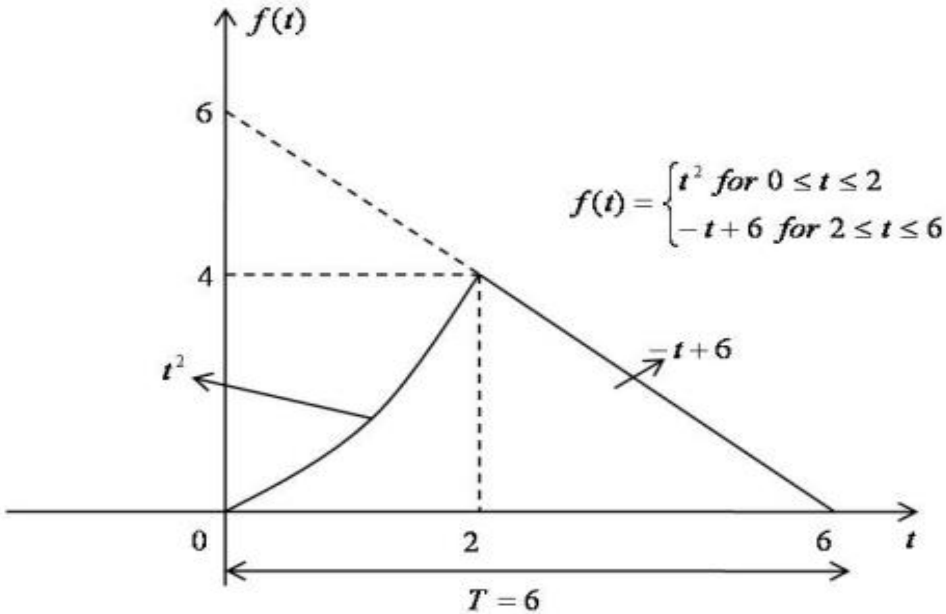
MCQ

A "periodic function" is given by a function which

(A) has a period $T = 2\pi$
(B) satisfies $f(t+T) = f(t)$
(C) satisfies $f(t+T) = -f(t)$
(D) has a period $T = \pi$

MCQ

3. Given the following periodic function, $f(t)$.



The coefficient a_0 of the continuous Fourier series associated with the above given function $f(t)$ can be computed as

(A) $\frac{8}{9}$

(B) $\frac{16}{9}$

(C) $\frac{24}{9}$

(D) $\frac{32}{9}$

MCQ

For the given periodic function $f(t) = \begin{cases} 2t \text{ for } 0 \le t \le 2 \\ 4 \text{ for } 2 \le t \le 6 \,(= T) \end{cases}$. The coefficient b_1 of the continuous Fourier series associated with the given function $f(t)$ can be computed as

(A) -75.6800
(B) -7.5680
(C) -6.8968
(D) -0.7468

MCQ

For the given periodic function $f(t) = \begin{cases} 2t \text{ for } 0 \le t \le 2 \\ 4 \text{ for } 2 \le t \le 6 \end{cases}$ with a period $T = 6$. The Fourier coefficient a_1 can be computed as

(A) -9.2642
(B) -8.1275
(C) -0.9119
(D) -0.5116

Express $f(x)=\frac{1}{2}(\pi - x)$ as a Fourier series with period 2π to be valid in the interval 0 to 2π.

Hence deduce the value of the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Obtain Fourier series for f(x) of period 2l and defined as follows

$f(x) = \begin{cases} l - x, & 0 < x \le l \\ 0, & l \le x < 2l \end{cases}$ Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$ and $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$