Isomorphism of Graphs

Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a bijection (an one-to-one and onto function) f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if f(a) and f(b) are adjacent in G_2, for all a and b in V_1.

Such a function f is called an **isomorphism**.

In other words, G_1 and G_2 are isomorphic if their vertices can be ordered in such a way that the adjacency matrices M_{G_1} and M_{G_2} are identical.

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$G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$

$G_1 \cong G_2$

if $\exists$ $f: G_1 \to G_2$ where this f is bijection.

(i) $(x, y) \in G_1$ then $(f(x), f(y)) \in G_2$

(ii) $(x, y) \notin G_1$ then $(f(x), f(y)) \notin G_2$

Isomorphism of Graphs

From a visual standpoint, G_1 and G_2 are isomorphic if they can be arranged in such a way that their **displays are identical** (of course without changing adjacency).

Unfortunately, for two simple graphs, each with n vertices, there are **n! possible isomorphisms** that we have to check in order to show that these graphs are isomorphic.

However, showing that two graphs are **not** isomorphic can be easy.

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Isomorphism of Graphs

For this purpose we can check **invariants**, that is, properties that two isomorphic simple graphs must both have.

For example, they must have

- the same number of vertices,
- the same number of edges, and
- the same degrees of vertices.

Note that two graphs that **differ** in any of these invariants are not isomorphic, but two graphs that **match** in all of them are not necessarily isomorphic.

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(i) count no of vertices in both the graph
Check that no. of vertices are equal or not.

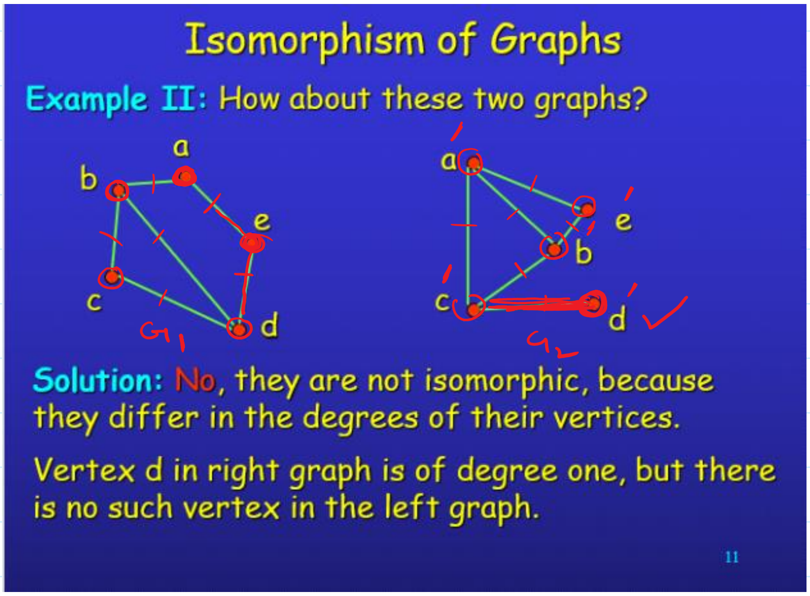
(ii) Count no of edges in both the graphs.

(iii) degree of Each vertex in both the graphs.

no of vertices of Equal degrees in both the graph should be same.

But if all these conditions are satisfied even then we cannot say graphs are isomorphic.

In order to check graphs are isomorphic we construct AdJacency matrices of both the graphs. if their AdJacency matrices are equal then we say that graphs are isomorphic



(i) Total no. vertices in both the graph are Same

(ii) No of Edges in both the graphs are same.

(iii) There is a vertex d' in Graph G_2 of degree 1 but there is no vertex of degree 1 in graph G_1

These two graphs are not isomorphic

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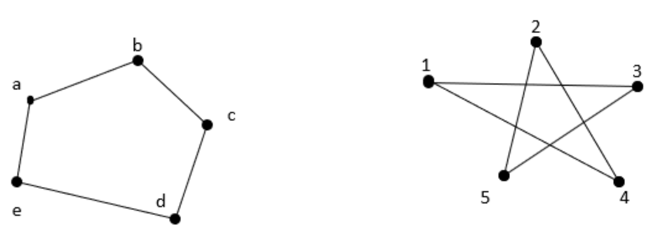
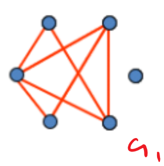
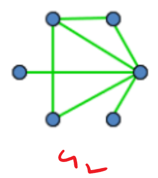
- The two graphs below look different.
- Are from a graph theoretic point of view structurally the 'same'?



Two non-isomorphic graphs





Vertices: 6 ✓
Edges: 7 ✓
Vertex sequence: 4, 3, 3, 2, 2, 0.

Vertices: 6 ✓
Edges: 7 ✓
Vertex sequence: 5, 3, 2, 2, 1, 1.

These graphs are not isomorphic graphs

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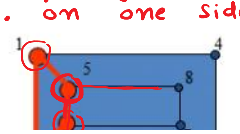
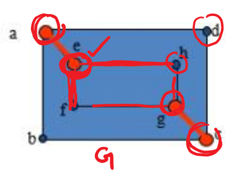
Take any vertex of degree 3 in Graph G. Then this vertex

Two non-isomorphic graphs

is connected to two vertices of degree 2. Take any vertex of degree 3 in Graph H. On one side it is



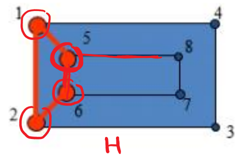
Connected to one vertex of



Vertices: 8

Edges: 10

Vertex sequence: 3, 3, 3, 3, 2, 2, 2, 2.



Vertices: 8

Edges: 10

Vertex sequence: 3, 3, 3, 3, 2, 2, 2, 2.

Connected to one vertex of degree 3 and on other side it is connected to degree 2

∴ Graphs are not isomorphic.

However, induced subgraphs on degree 3 vertices are NOT isomorphic!

EXAMPLE Show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in Figure are isomorphic.

Solution: The function f with $f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$ is a one-to-one correspondence between V and W. To see that this correspondence preserves adjacency, note that adjacent vertices in G are u_1 and u_2, u_1 and u_3, u_2 and u_4, and u_3 and u_4, and each of the pairs $f(u_1) = v_1$ and $f(u_2) = v_4$, $f(u_1) = v_1$ and $f(u_3) = v_3$, $f(u_2) = v_4$ and $f(u_4) = v_2$, and $f(u_3) = v_3$ and $f(u_4) = v_2$ consists of two adjacent vertices in H.

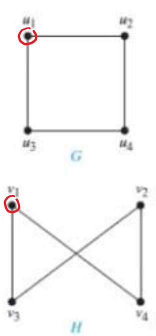


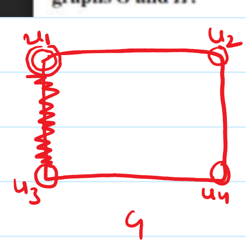
FIGURE **The graphs G and H.**

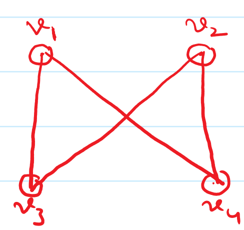
(i) No of vertices are equal.

(ii) No of edges in the graphs are equal.

(iii) In both these graphs we have vertices of degree 2.

$f : G \to H$





$f(u_1) = v_2$

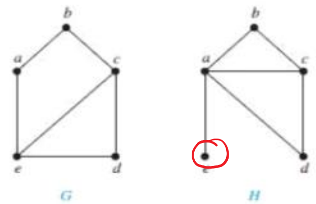
$f(u_2) = v_3$

$f(u_3) = v_4$

$f(u_4) = v_1$

EXAMPLE Show that the graphs displayed in Figure are not isomorphic.

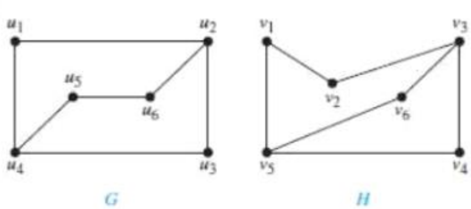
Solution: Both G and H have five vertices and six edges. However, H has a vertex of degree one, namely, e, whereas G has no vertices of degree one. It follows that G and H are not isomorphic



In graph H we have one vertex of degree 1 but in graph G we don't have any vertex of degree 1 these graphs are not isomorphic graphs.

Determine whether the graphs G and H displayed in Figure 12 are isomorphic.

Solution: Both G and H have six vertices and seven edges. Both have four vertices of degree two and two vertices of degree three. It is also easy to see that the subgraphs of G and H consisting of all vertices of degree two and the edges connecting them are isomorphic (as the reader should verify). Because G and H agree with respect to these invariants, it is reasonable to try to find an isomorphism f.



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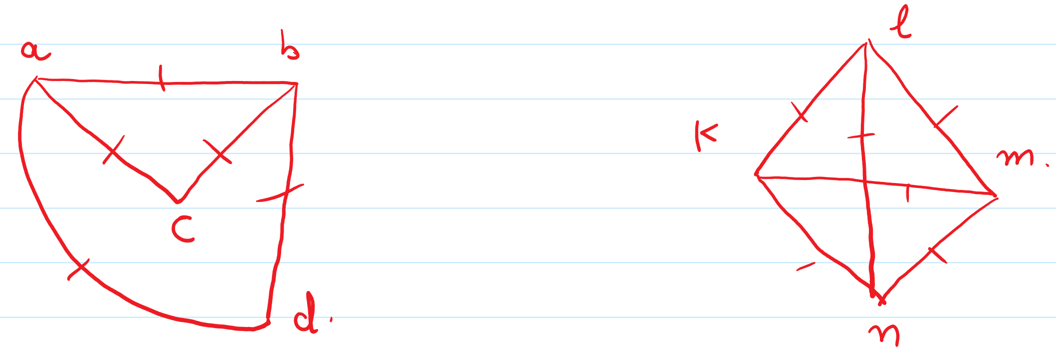

① No of vertices in both the graphs are equal.

② No of Edges in graph G_1 and G_2 are not Equal

∴ Graphs are not isomorphic.

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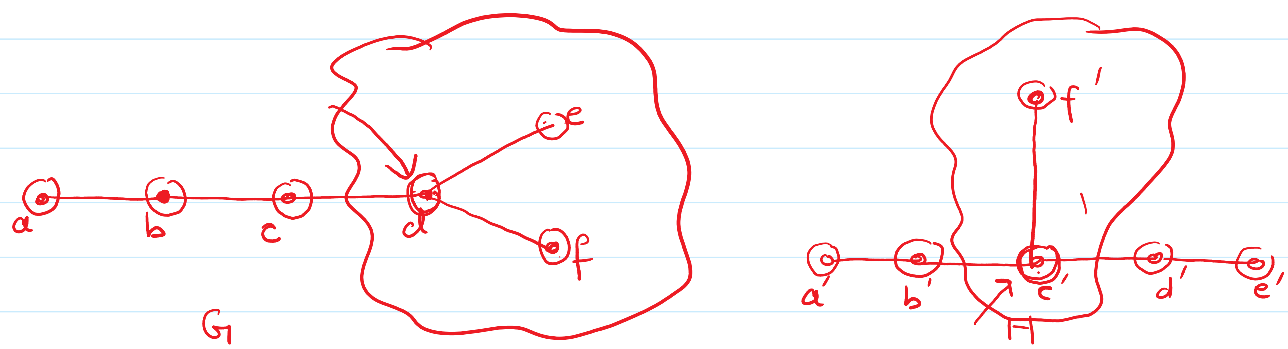

(i) In both the graphs no of vertices are equal.

(ii) No. of Edges in both the graphs are equal.

(iii) In both these graphs we have 1 vertex of degree 3, 2 vertices of degree 2 and 3 vertices of degree 1

Structures of these graphs are not same, Because in graph G vertex of degree 3 is connected to 2 pendent vertices but in graph H vertex of degree 3 is connected to only one pendent vertex.

As these structures are not same, ∴ graphs are not Isomorphic

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