Quantifiers

When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain truth value. However, there is another important way, called **quantification**, to create a proposition from a propositional function. Quantification expresses the extent to which a predicate is true over a range of elements. In English, the words *all*, *some*, *many*, *none*, and *few* are used in quantifications. We will focus on two types of quantification here: universal quantification, which tells us that a predicate is true for every element under consideration, and existential quantification, which tells us that there is one or more element under consideration for which the predicate is true. The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

DEFINITION 1 The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

$\forall x\, P(x)$ is true

The notation $\forall x\, P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier.** We read $\forall x\, P(x)$ as "for all $x\, P(x)$" or "for every $x\, P(x)$." An element for which $P(x)$ is false is called a **counterexample** of $\forall x\, P(x)$.

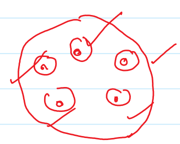
If it is not true for all values of x, then we can find at least one value of x for which this $P(x)$ is false. and if such x exists then $\boxed{'x'}$ is called counter example

$\sim(\forall x\, P(x))$

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\boxed{\forall x\, P(x)}$ ✓	$P(x)$ is true for every x. ✓	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

$\sim(\exists x\, P(x))$



EXAMPLE 8 Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x\, P(x)$, where the domain consists of all real numbers?

Extra Examples

Solution: Because $P(x)$ is true for all real numbers x, the quantification

$$\forall x\, P(x)$$

is true.

$x = 5$

$x+1 = 6 \quad > \quad x = 5$

$x = 4$

$x+1 = 5 \quad > \quad x = 4$

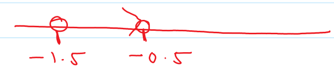
$x = -1.5$

$x+1 = -0.5 \quad > \quad x = -1.5$ ✓

$P(x)$: '$x+1 > x$'

$x+1$ is always greater than x $\quad \forall\, x \in \mathbb{R}$

$\forall x\, P(x)$ is $\boxed{\text{true}}$



EXAMPLE 9 Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x\, Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x\, Q(x)$. Thus

$$\forall x\, Q(x)$$

is false. ◄

Solⁿ $Q(x) : x < 2$

$Q(3) : 3 < 2$ This is false

$\forall x\, Q(x)$ is false.

— X —

$\forall x\, P(x)$ is true

$\therefore$ x_1, x_2, x_3, x_4.

$P(x_1)$ is true.
$P(x_2)$ is true
$P(x_3)$ is true
$P(x_4)$ is true

$P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge P(x_4)$

When all the elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$—it follows that the universal quantification $\forall x\, P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n),$$

because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

$P(x) : x^2 < 10$ for $x \in \{1, 2, 3, 4\}$

Check the truth value of $\forall x\, P(x)$.

$P(1) : 1^2 < 10 \Rightarrow 1 < 10$ (T)

$P(2) : 2^2 < 10 \Rightarrow 4 < 10$ (T)

$P(3) : 3^2 < 10 \Rightarrow 9 < 10$ (T)

$P(1) \wedge P(2) \wedge P(3) \wedge P(4)$ is false
T T T F

$\forall x\, P(x)$ is false.

$P(1)$: $1 < 10$ $\Rightarrow$ $1 < 10$ (T) | ... F

$P(2)$: $2^2 < 10$ $\Rightarrow$ $4 < 10$ (T)

$P(3)$: $3^2 < 10$ $\Rightarrow$ $9 < 10$ (T)

$P(4)$: $4^2 < 10$ $\Rightarrow$ $16 < 10$ (F)

$\forall x\, P(x)$ is false.

— x —

Example 13 shows.

EXAMPLE 13 What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

$\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$

Sol^n $P(x)$: $x^2 \geq x$

① Take set of real no's.

$x = \frac{1}{2}$

$P(\frac{1}{2})$: $(\frac{1}{2})^2 \geq \frac{1}{2}$

$\frac{1}{4} \geq \frac{1}{2}$

$0.25 \geq 0.50$

$\forall x\, P(x)$ is false

② Take set of integer

$\forall x P(x)$ is true.

— x —

DEFINITION 2 The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the *existential quantifier*.

$\exists x\, P(x)$ is true

EXAMPLE 14 Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

Extra Examples

Solution: Because "$x > 3$" is sometimes true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists x P(x)$, is true. ◀

$P(x)$: '$x > 3$'

$P(4)$: '$4 > 3$'

$\exists x P(x)$ is true

This statement is true.

EXAMPLE 15 Let $Q(x)$ denote the statement "$x = x + 1$." What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

Solution: Because $Q(x)$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x Q(x)$, is false. ◀

$Q(x)$: $x = x + 1$

we cannot find any $x \in \mathbb{R}$ s.t the given statement is true.

$\therefore$ $\exists x\, Q(x)$ is false.

— x —

When all elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$— the existential quantification $\exists x P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n),$$

because this disjunction is true if and only if at least one of $P(x_1), P(x_2), \ldots, P(x_n)$ is true.

① $\exists x\, P(x)$ ✓

$P(x_1) \vee P(x_2) \vee \text{----} \vee P(x_n) \longrightarrow (T)$

These are equivalent statement.

EXAMPLE 16 What is the truth value of $\exists x P(x)$, where $P(x)$ is the statement "$x^2 > 10$" and the universe of discourse consists of the positive integers not exceeding 4?

Solution: Because the domain is {1, 2, 3, 4}, the proposition $\exists x P(x)$ is the same as the disjunction

$$P(1) \vee P(2) \vee P(3) \vee P(4).$$

Because $P(4)$, which is the statement "$4^2 > 10$," is true, it follows that $\exists x P(x)$ is true. ◀

$P(x): x^2 > 10$

$P(4): 16 > 10$

$P(x): x^2 > 10$

$P(1): (1)^2 > 10$ (F)

$P(2): 2^2 > 10$ (F)

$P(3): 3^2 > 10$ (F)

$P(4): 4^2 > 10$ (T)

$P(1) \vee P(2) \vee P(3) \vee P(4) \longrightarrow T$

$\exists x\, P(x)$ is true

—x—

1 Direct Proofs

A **direct proof** of a conditional statement $p \rightarrow q$ is constructed when the first step is the assumption that p is true; subsequent steps are constructed using rules of inference, with the final step showing that q must also be true. A direct proof shows that a conditional statement $p \rightarrow q$ is true by showing that if p is true, then q must also be true, so that the combination p true and q false never occurs. In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference, to show that q must also be true. You will find that direct proofs of many results are quite straightforward, with a fairly obvious sequence of steps leading from the hypothesis to the conclusion. However, direct proofs sometimes require particular insights and can be quite tricky. The first direct proofs we present here are quite straightforward; later in the text you will see some that are less obvious.

We will provide examples of several different direct proofs. Before we give the first example, we need to define some terminology.

p. (T) ✓

Q (T) (we prove that φ is also true)

$p \rightarrow \varphi$ T

(T) (T)

p	φ	$p \rightarrow \varphi$
T	T	T

$n = 2k+1$

$n^2 = 2m+1$ ✓

EXAMPLE 1 Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

p: n is odd integer (T)

φ: n^2 is odd integer (T)

$7 = 2(3)+1$

$9 = 2(4)+1$

As n is odd, ✓

$n = 2k+1$ $k \in \mathbb{Z}$

$k \in \mathbb{Z}$ $k^2 \in \mathbb{Z}$

$2k \in \mathbb{Z}$ $2k^2 \in \mathbb{Z}$

$2k + 2k^2 \in \mathbb{Z}$

$m \in \mathbb{Z}$

$$n^2 = (2k+1)^2$$
$$= (2k)^2 + (1)^2 + 2(2k)$$
$$= 4k^2 + 1 + 4k = 4k^2 + 4k + 1$$
$$= 2(2k^2+2k)+1$$
$$= 2m+1 \qquad m = 2k^2+2k \in \mathbb{Z}$$

$p \rightarrow \varphi$: If n is odd integer then n^2 is also odd integer.

EXAMPLE 2 Give a direct proof that if m and n are both perfect squares, then nm is also a perfect square. (An integer a is a **perfect square** if there is an integer b such that $a = b^2$.)

p: m and n are both perfect squares

q: mn is also perfect square.

$36 = 6^2$

$4 = 2^2$

As m and n are perfect squares.

$\therefore \exists$ integers a and b s.t

$m = a^2, \quad n = b^2$

$a \in \mathbb{Z}, b \in \mathbb{Z}$

$ab \in \mathbb{Z}$

$mn = a^2 b^2 = a a b b = (ab)(ab) = (ab)^2$

mn is a perfect square

If m & n are perfect squares then mn is also a perfect square

Proof by Contraposition

cannot easily find a direct proof.

EXAMPLE 3 Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

DEFINITION 2 The real number r is *rational* if there exist integers p and q with $q \neq 0$ such that $r = p/q$. A real number that is not rational is called *irrational*.

EXAMPLE 7 Prove that the sum of two rational numbers is rational. (Note that if we include the implicit quantifiers here, the theorem we want to prove is "For every real number r and every real number s, if r and s are rational numbers, then $r + s$ is rational.)

EXAMPLE 8 Prove that if n is an integer and n^2 is odd, then n is odd.