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Subject :- Statistical Inference

Assignment

Ques-1 : Suppose X is a single observation from a population with probability density function given by

$$f(x) = \theta x^{\theta - 1}$$

for $0 < x < 1$, find the test with the best critical region i.e the most powerful test with significance level $\alpha = 0.05$ for testing the $H_0 : \theta = 3$ v/s simple $H_1 : \theta = 2$

Sol :- Given that

$$f(x) = \theta x^{\theta - 1} \quad 0 < x < 1 \quad \&$$

$H_0 : \theta_0 = 3$, $H_1 : \theta_1 = 2$, $n = 1$, $\alpha = 0.05$

By using NP- Lemma

$$\frac{L_1}{L_0} \geq K$$

Where L_1 & L_0 are the likelihood f^n

$$\frac{\theta_1 x^{\theta_1 - 1}}{\theta_0 x^{\theta_0 - 1}} \geq K$$

$$\left(\frac{\theta_1}{\theta_0}\right)\left(\frac{x^{\theta_1-1}}{x^{\theta_0-1}}\right) \geq K$$

$$x^{\theta_1-\theta_0} \geq K\left(\frac{\theta_0}{\theta_1}\right)$$

$$x^{2-3} \geq K\left(\frac{3}{2}\right)$$

$$x^{-1} \geq K\left(\frac{3}{2}\right)$$

$\because$ By taking reciprocals:-

$$x \leq \left(\frac{2}{3}\right)\frac{1}{K}$$

$$x \leq C, \quad \text{Where } C = \frac{2}{3K}$$

Now,

$$\alpha = P(\text{Reject } H_0 \,|\, H_0 \text{ is true})$$

$$0.05 = P(x \leq c \,|\, H_0 \cdot \theta = 3)$$

$$0.05 = \int_0^c 3x^2 \, dx$$

$$0.05 = \frac{3x^3}{3}\Big]_0^c$$

$$0.05 = c^3$$

$$\boxed{c = 0.3684}$$

The most powerful test for testing $H_0\ \theta = 3$ v/s $H_1 : \theta = 2$ is given by $\omega = \{x \mid x < 0.3684\}$

Now, to find the best test, we need to find the power of the test.

So, the power of the test is given by

$P(\text{Rejecting } H_0 \mid H_1 \text{ is true}) = 1 - \beta$

$P(x < 0.3684 \mid H_1\ \theta = 2) = 1 - \beta$

$$\int_0^{0.3684} \theta x^{\theta - 1}\, dx = 1 - \beta$$

$$\int_0^{0.3684} 2x^{2-1}\, dx = 1 - \beta$$

$$\int_0^{0.3684} 2x\, dx = 1 - \beta$$

$$\frac{2x^2}{2}\Big]_0^{0.3684} = 1 - \beta$$

$$(0.3684)^2 - 0 = 1 - \beta$$

$$\boxed{0.1357 = 1 - \beta}$$

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the power of Critical region is = 0.1357

Hence, this test is said to be the best test with most powerful Critical region.

$$W = \{x \mid x < 0.3684\}$$

Que-2:- If $x_1, x_2, \ldots, x_n$ is a r.s from $N(\mu, 16)$ find the test with best with the best Critical region i.e. find the most powerful test with a Sample of Size $n=16$ on a Significance level $\alpha = 0.05$ to test the Simple $H_0: \mu = 10$ V/S Simple $H_1: \mu = 15$ and a level of Significance.

Sol:- Given that:

$$x_i \sim N(\mu, 16)$$

$$\therefore f(x_i) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2 \left(\frac{x_i - \mu}{\sigma}\right)^2}$$

$-\infty < x < \infty$

$-\infty < \mu < \infty$

$\sigma > 0$

Here $H_0: \mu = 10$ V/S

$H_1: \mu = 15$

$\alpha = 0.05$, $n = 16$

By using NP-Lemma.

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$\cancel{e^{-1/2}[\Sigma}$

$$e^{-\frac{1}{2}\left[\Sigma(x_i-\mu_1)^2 - \Sigma(x_i-\mu_0)^2\right]} \geq K$$

taking $\log_e$ on both Side

$$\frac{-1}{2}\left[\Sigma(x_i-\mu_1)^2 - \Sigma(x_i-\mu_0)^2\right] \geq \log_e K$$

$$\cancel{\frac{1}{2}\left[-2\Sigma x_i(\mu_1-\mu_0) + n \, \mu_1^2\right.}$$

$$\frac{-1}{2}\left[\cancel{\Sigma x_i^2} - 2\mu_1\Sigma x_i + n\mu^2 - \cancel{\Sigma x_i^2} + 2\mu_0\Sigma x_i - n\mu_0^2\right] \geq \log_e K.$$

$$\frac{-1}{2}\left[-2\Sigma x_i(\mu_1-\mu_0) + n(\mu_1^2-\mu_0^2)\right] \geq \log_e K$$

$$\Sigma x_i(\mu_1-\mu_0) - \frac{n}{2}(\mu_1^2-\mu_0^2) \geq \log_e K$$

$$\Sigma x_i(\mu_1-\mu_0) \geq \log_e K + \frac{n}{2}(\mu_1^2-\mu_0^2)$$

$$\Sigma x_i \geq \frac{\log_e K + n/2(\mu_1^2-\mu_0^2)}{(\mu_1-\mu_0)}$$

$\Sigma x_i \geq K$ where

$$K_1 = \frac{\log_e K + n/2(\mu_1^2-\mu_0^2)}{(\mu_1-\mu_0)}$$

$$\bar{x} \geq \frac{K_1}{n}, \quad \boxed{\bar{x} \geq C}$$

$$C = \frac{K_1}{n}$$

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$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ is true})$

$0.05 = P(\bar{x} \geq c \mid H_0 \; \mu = 10)$

We know that
if $x \sim N(\mu, \sigma^2)$ then

$\bar{x} \sim N(\mu, \sigma^2/n)$

$\bar{x} \sim N(\mu = 10, 16/16)$

$\bar{x} \sim$ Standard Normal $(\mu, 1)$

$0.05 = P\left(\frac{\bar{x} - \mu}{\sqrt{\sigma^2/n}} \geq \frac{C - \mu}{\sqrt{\sigma^2/n}}\right)$

$0.05 = P\left(Z \geq \frac{C - 10}{1}\right)$

$0.05 = P(Z \geq C - 10)$

$C - 10 = 1.64$ $\quad \because P(Z \geq 1.64) = 0.05$
$0503)$

$\boxed{C = 11.64}$

The most powerful Critical region for testing $H_0 : \mu = 10$ v/s $H_1 : \mu = 15$ is given by, ω

$\omega = \{x \mid x \geq 11.64\}$

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To find the best test for above critical region we need to find the power of test.

So, the power of test is given

$P(\text{Reject } H_0 \mid H_1 \text{ is true}) = 1-\beta$

$P(\bar{x} \geq 11.64 \mid H_1 : \mu = 15) = 1-\beta$

$P\left(\frac{\bar{x}-15}{1} \geq \frac{11.64-15}{1}\right) = 1-\beta$

$P(Z \geq -3.36) = 1-\beta$

$1 - P(Z \geq 3.36) = 1-\beta$

$1 - 0.0039 = 1-\beta$

$1-\beta = 0.9961$

The power of the test for the Critical region $W = \{x \mid x \geq 11.64\}$ is 0.9961

Hence the above test is said to be the best test with most powerful test. $W = \{x \mid x \geq 11.64\}$