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A 6-DOF ROCKET MODEL FOR CONTROL ANALYSIS

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***Abstract*— A "flat earth" model of a thrust vectoring rocket dynamics was developed. Some of the model features are variable mass, moments of inertia, CG and center of pressure positions. Also, aerodynamic fluctuations are implemented, so as nozzle vibrations and thrust instabilities. Simple simulation results without any control input are presented and discussed, leading to the conclusion that the mode is well suited for the sake of control analysis, although future enhancements may provide yet better insights.**

Keywords— **rocket model, simulation, control**

I. INTRODUCTION

What follows is a partial result of a broader research where the analysis of quantitative and qualitative aspects of different control laws for rocket launch maneuvering is at stake.

Here, a 6-DoF mathematical model of a rocket's dynamics will be presented (section II). Although many good models are readily available from the literature (see, e. g., Refs. [1], [2] and [3]), the one presented here was developed from scratch based of fundamental physical principles (Refs. [8], [9] and [10]), primarily for stability and control analysis. It is simplified in many aspects, of which the main one is the adoption of a "flat earth" reference frame instead of an inertial frame in the center of the planet. This leads to considerable error in term of rocket position, but has little effect on control analysis. The model is limited to first stage burn, and is suited for rockets that rely only in thrust vectoring for maneuvering control.

Aerodynamic fluctuations, nozzle vibrations and thrust instabilities are represented. The model has been implemented in Simulink® and tested under two different scenarios, yet without any control. Section II presents some of the results obtained, and discusses the validity of the model.

II. ROCKETS'S MATHEMATICAL MODEL

A. Reference Frames, Angles, Forces and Moments

Fig. 1 shows the relationship between the two main reference frames adopted: frame $OXYZ$ is the earth reference frame, fixed to the launch point, and $oxyz$ is the body reference frame, fixed to the rocket's body, with origin at its the center of gravity (CG). Between these, one can see a rotation defined by the 3 angles Ψ, Θ and Φ.[i] In the figure, earth frame has been translated to the rocket's CG for clarity.

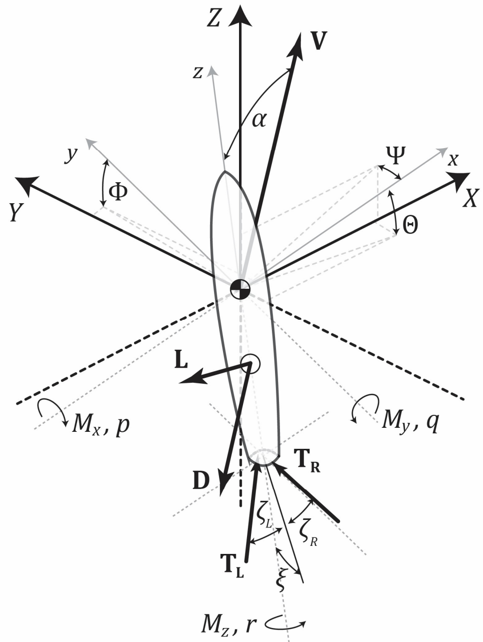

Fig. 1 Free Body Diagram of the 6-DoF dynamics and kinematics of the rocket.

Except for Φ, the angles are represented in their negative direction. The figure also shows the velocity vector (**V**) in earth frame, applied to the origin of the body frame, forming with axis oz an angle α. Aerodynamic forces lift (**L**) and drag (**D**) are also represented, both in a plane defined by vectors **V** and the unity "body vector" $\mathbf{v}_c' = [0 \quad 0 \quad 1]^T$ in body frame, and applied at the rocket's center of pressure (CoP), distant d_{CoP} from the CG. Two thrust vectors, $\mathbf{T}_L$ and $\mathbf{T}_R$, representing two nozzles are shown, distant $d_T/2$ from the axis defined by the intersection of plane xz and the rocket's base plane.

They form with the body plane yz angles ζ_L and ζ_R, and with plane xz the single angle ξ. These angles represent thrust

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vectoring, which accounts for the rocket's maneuver control. The two thrust vectors are applied at the rocket's base, distant z_{CG} from the CG. These forces produce moments M_x, M_y and M_z about the CG, and around the body axis, resulting in angular accelerations and velocities, the later represented by p, q and r. In the figure, except for M_y and q, moments and angular velocities are represented in their positive direction.

Once it is paramount to constantly move from one reference frame to the other, it is convenient to define a coordinate transformation function and it´s inverse. If $\mathbf{v}$ is a tridimensional vector in earth frame and $\mathbf{v}'$ is the same vector in the body frame, then

$$\varphi(\Phi,\Theta,\Psi) = \mathbf{v}' = \mathbf{R}\mathbf{v} \tag{1}$$

$$\varphi^{-1}(\Phi,\Theta,\Psi) = \mathbf{v} = \mathbf{R}^T\mathbf{v}' \tag{2}$$

Where (using c for cos and s for sin, Ref. [7])

$$\mathbf{R} = \begin{bmatrix} c\Psi c\Theta & c\Theta s\Psi & -s\Theta \\ c\Psi s\Phi s\Theta & c\Phi c\Psi + s\Phi s\Psi s\Theta & c\Theta s\Phi \\ s\Phi s\Psi + c\Phi c\Psi s\Theta & c\Phi s\Psi s\Theta - c\Psi s\Phi & c\Phi c\Theta \end{bmatrix} \tag{3}$$

B. Accelerations

From basic dynamics, the rocket's linear acceleration in earth coordinate frame, can be written as:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = (m\mathbf{I})^{-1}\mathbf{F} = \frac{1}{m}\begin{bmatrix} T_X + D_X + L_X \\ T_Y + D_Y + L_Y \\ T_Z + D_Z + L_Z + mg \end{bmatrix} \tag{4}$$

Where m is the instantaneous mass, $\mathbf{I}$ is the identity matrix, $\mathbf{F} = \mathbf{T} + \mathbf{D} + \mathbf{L} + m\mathbf{g}$ is the sum of forces on the rocket, with components in the OX, OY and OZ direction.

Since the rocket's axial symmetry allows us to consider its longitudinal axis as a principal axis, and since, by the same symmetry $J_s = J_{xx} = J_{yy}$ is the moment of inertia around body's ox and oy axis, then the inertia tensor is given by:

$$\mathbf{J} = \begin{bmatrix} J_s & 0 & 0 \\ 0 & J_s & 0 \\ 0 & 0 & J_{zz} \end{bmatrix} \tag{5}$$

And the resultant moment vector is:

$$\mathbf{M} = \mathbf{J}\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times (\mathbf{J}\boldsymbol{\omega}) \tag{6}$$

Where $\boldsymbol{\omega} = [p \quad q \quad r]^T = [\omega_x \quad \omega_y \quad \omega_z]^T$ is the angular velocity vector. From the last equation, we find the angular acceleration as:

$$\dot{\boldsymbol{\omega}} = \begin{bmatrix} 1/J_s & 0 & 0 \\ 0 & 1/J_s & 0 \\ 0 & 0 & 1/J_{zz} \end{bmatrix}\begin{bmatrix} M_x - \omega_y\omega_z(J_{zz} - J_s) \\ M_y - \omega_x\omega_z(J_s - J_{zz}) \\ M_z \end{bmatrix} \tag{7}$$

The task now is to find expressions for the terms in Equations (4) and (7). Starting with the aerodynamic terms.

C. Aerodynamic Forces and Moments

Aerodynamic forces are drag ($\mathbf{D}$) and lift ($\mathbf{L}$). We can proceed by first determining these forces directions, then their magnitudes, and then the position of the CoP, where they are applied, from which one can compute the aerodynamic moment (M_{aero}).

By definition, drag is parallel and opposite to velocity, and so, if D is drag's magnitude, then

$$\mathbf{D} = -D\frac{\mathbf{V}}{\|\mathbf{V}\|} \tag{8}$$

Lift, by definition, is perpendicular to velocity. By inspection of Fig. 1, one can see that lift is in the same plane defined as the plane that contains $\mathbf{V}$ and $\mathbf{v}_c = \mathbf{R}^T\mathbf{v}_c'$. From the right hand rule for the cross product of two vectors, $\mathbf{V} \times \mathbf{v}_c$ gives a vector perpendicular to both, that points in the direction of one's thumb when her finger's close from $\mathbf{V}$ to $\mathbf{v}_c$. If now one makes the cross product of this resulting vector again with $\mathbf{V}$, then the direction of lift in earth frame is found. So, if L is the magnitude of lift, then

$$\mathbf{L} = L\frac{(\mathbf{V} \times \mathbf{v}_c) \times \mathbf{V}}{\|(\mathbf{V} \times \mathbf{v}_c) \times \mathbf{V}\|} \tag{9}$$

From Equations (1), vectors $\mathbf{L}' = \begin{bmatrix} L_x' & L_y' & L_z' \end{bmatrix}^T$ and $\mathbf{D}' = \begin{bmatrix} D_x' & D_y' & D_z' \end{bmatrix}^T$, can be found, which are lift and drag in the body frame. It should be noticed that forces in the yz plane in the oy direction cause moments around ox, and that forces in the xz plane in the direction ox cause moments around oy. So it is convenient to define vectors:

$$\mathbf{L}^* = \begin{bmatrix} L_y' & -L_x' & 0 \end{bmatrix}^T \tag{10}$$

$$\mathbf{D}^* = \begin{bmatrix} D_y' & -D_x' & 0 \end{bmatrix}^T \tag{11}$$

And so, being d_{CoP} the distance from the CoP to the CG, the aerodynamic moment is

$$\mathbf{M}_{aero} = d_{CoP}\begin{bmatrix} L_y' + D_y' \\ -L_x' - D_x' \\ 0 \end{bmatrix} \tag{12}$$

For the aerodynamic forces magnitudes, one can start with the well-known equations:

$$L = \frac{1}{2}\rho V^2 S_{ref} C_L \tag{13}$$

$$D = \frac{1}{2}\rho V^2 S_{ref} C_D \tag{14}$$

Where the aerodynamic coefficients must be defined. There is no simple way for doing it. Specially because the flight of a rocket is subjected to many states where mostly Mach number plays an important role. So, in the present modeling, it was decided to use interpolating functions over the range of a set of data. This set of data can be obtained through experiment, CFD, or whatever source is appropriate enough for a given rocket. These data may then be fed to functions of the form:

$$C_L' = C_{L_\alpha}(Ma)\alpha \tag{15}$$
$$C_D' = C_{D_{\alpha=0}}(Ma) + k_1\alpha^{k_2} \tag{16}$$
$$d_{CoP}' = a(Ma)\alpha^{b(Ma)} \tag{17}$$

In Equations (15), (16) and (17), $C_{L_\alpha}(Ma)$ is an interpolating function of a set of data that gives the slope of the C_L curve as a function of Mach, $C_{D_{\alpha=0}}(Ma)$ is an interpolation function over a set of data for the drag coefficient at zero angle of attack α, and $a(Ma)$ and $b(Ma)$ are interpolation functions over sets of data for the CoP position modeled as a power function. Coefficients k_1, k_2 should be adjusted to fit drag coefficient as α varies.

The primes in the coefficients indicate that they are nominal values. As already mentioned, we want to introduce fluctuations in the aerodynamic coefficients, representing all sorts of buffeting and other interferences, especially those related to shock waves. So we want these fluctuations to be proportional to the coefficients themselves, but also maximum when $Ma \longrightarrow 1$.

For this, one can define a function that returns a random number with Gaussian distribution around 0 and maximum standard deviation when $Ma = 1$. The returned value of the function is then multiplied by a fraction of the coefficient, defining the maximum fluctuation value in general.

As for the standard deviation value, a Gaussian function of the form:

$$\sigma_a = e^{-\left(\frac{Ma-1}{\delta}\right)^2} \tag{18}$$

will make it maximum ($\sigma_a = 1$) when $Ma = 1$. The term δ controls the spread of the bell shaped curve. A larger value implies a fatter curve. The value of this standard deviation shall be then fed into a random number generator function with Gaussian distribution around 0. Obviously, there is no such a thing as a random number function in mathematics, but computer environments such as Matlab/Simulink® can provide such functions.

Now, let us define an implicit function that takes as argument the standard deviation and returns a random number as discussed above. Let C_{fr} be a generic coefficient ratio and $f_g(\sigma_a)$ be the function. Then the fluctuations are simply $C_{fr}f_g(\sigma_a)$. This means essentially that 95% of the random numbers generated will be between $100C_{fr}C_{fr}f_g(\sigma_a)$ % of the coefficient, and it will be the most when Ma 1. So:

$$C_L = C_L' + C_{L_{fr}}f_g(\sigma_a) \tag{19}$$
$$C_D = C_D' + C_{D_{fr}}f_g(\sigma_a) \tag{20}$$
$$d_{CoP} = d_{CoP}' + C_{CoP_{fr}}f_g(\sigma_a) \tag{21}$$

Equations (19)-(21) give the actual values of the aerodynamic coefficients. It a computer implementation, it may be useful to conditionally use these fluctuations, so that they may be shut off and on, depending on the needs.

The last term we need is α, which is not exactly the angle of attack:[ii]

$$\alpha = \cos^{-1}\frac{\mathbf{V}\cdot\mathbf{v}_c}{\|\mathbf{V}\|\|\mathbf{v}_c\|} \tag{22}$$

D. Thrust Forces and Moments

From inspection of Fig. 1, thrust vector in body frame is given by:

$$\mathbf{T}' = \begin{bmatrix} T_L\sin\zeta_L + T_R\sin\zeta_R \\ (T_L + T_R)\sin\xi \\ (T_L\cos\zeta_L + T_R\cos\zeta_R)\cos\xi \end{bmatrix} = \begin{bmatrix} T_x' \\ T_y' \\ T_z' \end{bmatrix} \tag{23}$$

To have moments about the oz axis, a differential thrust must also be defined as:

$$T_{diff} = T_R\sin\zeta_R - T_L\sin\zeta_L \tag{24}$$

And so, the moment vector of the thrust forces, in body frame, is:

$$M_T = \begin{bmatrix} z_{CG}T_y' \\ -z_{CG}T_x' \\ \frac{d_T}{2}T_{diff} \end{bmatrix} \tag{25}$$

Usually, the thrust of rocket engines are given as values for sea level (SL) and vacuum (vac). If so, a simple linear function for thrust magnitude can be written as:

$$T = T_{vac} + \frac{T_{SL} - T_{vac}}{\rho_{SL}}\rho \tag{26}$$

As with aerodynamics, we may introduce thrust magnitude fluctuations, but in this case they may be simpler, once they have no dependency on Mach number. Defining nominal values for each nozzle's thrust as:

$$T_L' = T_R' = \frac{1}{2}T \tag{27}$$

Now, being σ_T the standard deviation for thrust, and defining a fraction as T_{fr}, then the actual values are:

$$T_L = T_L' + T_{fr}f_g(\sigma_T) \tag{28}$$
$$T_R = T_R' + T_{fr}f_g(\sigma_T) \tag{29}$$

E. Mass and Inertia

Considering a liquid fuel rocket with fuel and oxidizer tanks, and considering as structural mass (m_s) all mass that is not propellant, then if $m_p = m_f + m_o$ is the mass of propellant (fuel plus oxidizer), then the instantaneous mass of the rocket is:

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$$m = \begin{cases} m_s + m_{p_{ini}} - \dot{m}_p t & for\, m_p > 0 \\ m_s & for\, m_p \leq 0 \end{cases} \tag{30}$$

Subscripts ini and fin account for initial and final. If α_f is oxidizer-fuel ratio, then:

$$m_p = m_f(1 + \alpha_f) \tag{31}$$

From where the mass of fuel and oxidizer can be computed. Considering now the mass of fuel and oxidizer in the tanks, then one can indicate their initial and final CG's positions as $z_{CG_{f,ini}}$, $z_{CG_{f,fin}}$, $z_{CG_{o,ini}}$ and $z_{CG_{o,fin}}$, and so the instantaneous CG position of the mass of fuel and oxidizer are:

$$z_{CG_f} = \left(1 - \frac{m_f}{m_{f_{ini}}}\right) z_{CG_{f,fin}} + \frac{m_f}{m_{f_{ini}}} z_{CG_{f,ini}} \tag{32}$$

$$z_{CG_o} = \left(1 - \frac{m_o}{m_{o_{ini}}}\right) z_{CG_{o,fin}} + \frac{m_o}{m_{o_{ini}}} z_{CG_{o,ini}} \tag{33}$$

And the overall instantaneous CG position is given by:

$$z_{CG} = \frac{z_{CG_s} m_s + z_{CG_f} m_f + z_{CG_o} m_o}{m_s + m_f + m_o} \tag{34}$$

Where z_{CG_s} is the CG's position of the unvarying mass. For the moment of inertia, it is acceptable to consider the rocket and the fuel and oxidizer masses as solid cylinders with the same diameter D_J. Being so, then the moment of inertia about the oz axis is given by:

$$J_{zz} = \frac{1}{8}(m_s + m_f + m_o){D_J}^2 \tag{35}$$

For the moment of inertia J_s about ox and oy axis, the lengths of the masses and the distances from their CG's and the overall CG must me computed. The length of the rocket mass is the length of the rocket, a constant geometric data. The lengths of the fuel and oxidizer masses depend on how much propellant remains in the tanks, so:

$$L_f = 2\left(z_{CG_f} - z_{CG_{f,fin}}\right) \tag{36}$$

$$L_o = 2\left(z_{CG_o} - z_{CG_{o,fin}}\right) \tag{37}$$

The distances are:

$$d_s = z_{CG} - \frac{L_s}{2} \tag{38}$$

$$d_f = z_{CG} - z_{CG_f} \tag{39}$$

$$d_o = z_{CG} - z_{CG_o} \tag{40}$$

And so:

$$J_s = \frac{1}{12}\left(m_s {L_s}^2 + m_f {L_f}^2 + m_o {L_o}^2\right) + m_s {d_s}^2 + m_f {d_f}^2 + m_o {d_o}^2 \tag{41}$$

F. Atmosphere

Atmospheric models are well known and one is free to use the one that suits better. Nonetheless, care must be taken that the model used gives good values for density (ρ), gravitational acceleration (g), and speed of sound (V_s) up to the altitude where the rocket is supposed to go.

In the present case, with respect to gravitational acceleration, a good approximation is given by:

$$g = g_{SL}\left(\frac{R_e}{R_e + h}\right)^2 \tag{42}$$

With respect to the other parameters, the COESA 1976 model was used (Reference [4]). Once the computations are to be implemented in a computer environment, the pre-defined Matlab/Simulink® function may suffice. Note should be taken, though, that this model is good enough only up to about 85 km, over which density was set to zero and speed of sound was set to constant and equal to its value at 85 km.

G. Thrust Vectoring Actuators

From Fig. 1, rocket vectoring is defined by angles ζ_L, ζ_R and ξ. A very simple actuation model was implemented, where a desired (subscripted as dsr) angular position of the nozzles results from the controllers commands (subscripted as cmd). As for angular position ξ, $\xi_{dsr} = \xi_{cmd}$, but this is not the case for ζ angles, where a differential (subscripted as $diff$) command is also present. In this case, we have:

$$\zeta_{L_{dsr}} = \zeta_{cmd} + \frac{1}{2}\zeta_{diff} \tag{43}$$

$$\zeta_{R_{dsr}} = \zeta_{cmd} - \frac{1}{2}\zeta_{diff} \tag{44}$$

In each case, the position is the result of the angular movement of the nozzle from its actual (subscripted as act) angular position to the desired one, at a constant angular speed λ. Once there will be an error e_n from the actual to the desired position, a very simple controller should move the nozzle according to:

$$\xi' = \xi_{act} + \int \lambda \operatorname{sign}(e_n)\, dt \tag{45}$$

Analogous equations are immediately written for ζ_L and ζ_R. Care must be taken to not allow movement beyond the possible range of the nozzles, using with some sort of function saturation.

Also here, to represent nozzle vibrations, a nominal angular position is defined as in Equation (45), and a "real" positions are determined after some Gaussian random perturbation of the nominal position. Using the same notation as was done for the previous cases, we have:

$$\xi = \xi' + \xi_{fr} f_g(\sigma_n) \tag{46}$$

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$$\zeta_L = \zeta_L' + \zeta_{fr} f_g(\sigma_n) \quad (47)$$
$$\zeta_R = \zeta_R' + \zeta_{fr} f_g(\sigma_n) \quad (48)$$

H. **Attitude**

Attitude is that which is to be controlled, after all. For representing it, the most common and logical choice is the use of quaternions. For the adopted convention, if a quaternion is a four-dimensional vector defined as $\mathbf{Q} = [q_0 \quad q_1 \quad q_2 \quad q_3]^T$, where q_0 is the scalar part, then it may represent Euler angles as:

$$\mathbf{Q} = \begin{bmatrix} \cos\frac{\Phi}{2}\cos\frac{\Theta}{2}\cos\frac{\Psi}{2} + \sin\frac{\Phi}{2}\sin\frac{\Theta}{2}\sin\frac{\Psi}{2} \\ -\cos\frac{\Phi}{2}\sin\frac{\Theta}{2}\sin\frac{\Psi}{2} + \cos\frac{\Theta}{2}\cos\frac{\Psi}{2}\sin\frac{\Psi}{2} \\ \cos\frac{\Phi}{2}\cos\frac{\Psi}{2}\sin\frac{\Theta}{2} + \sin\frac{\Phi}{2}\cos\frac{\Theta}{2}\sin\frac{\Psi}{2} \\ \cos\frac{\Phi}{2}\cos\frac{\Theta}{2}\sin\frac{\Psi}{2} - \sin\frac{\Phi}{2}\cos\frac{\Psi}{2}\sin\frac{\Theta}{2} \end{bmatrix} \quad (49)$$

Inversely, one can extract Euler angles from a quaternion through:

$$\begin{bmatrix} \Phi \\ \Theta \\ \Psi \end{bmatrix} = \begin{bmatrix} \tan^{-1}\dfrac{2q_2q_1 + 2q_0q_1}{q_3{}^2 - q_2{}^2 - q_1{}^2 + q_0{}^2} \\ -\sin^{-1}(2q_1q_3 - 2q_0q_2) \\ \tan^{-1}\dfrac{2q_2q_1 + 2q_0q_3}{q_1{}^2 + q_0{}^2 - q_3{}^2 - q_2{}^2} \end{bmatrix} \quad (50)$$

One should remember to normalize the quaternion before applying Equation 44. Defining vector $\boldsymbol{\omega}^* = [0 \quad \boldsymbol{\omega}]^T$, then the derivative of quaternion $\mathbf{Q}$ containing the representation of the actual attitude is:

$$\dot{\mathbf{Q}} = \frac{1}{2}\mathbf{Q} \otimes \boldsymbol{\omega}' \quad (51)$$

Being $\otimes$ the quaternion product. Integrated in time, the next attitude is found from Equation (50).

I. **State Equations**

The rocket dynamics has been modeled. One can see that it is a highly non-linear second order system. The choice of state variables should reflect the objectives one must accomplish. In the case of control analysis, angular and linear positions and velocities, plus angular nozzle positions are a good choice. It´s possible then to define state variables as vectors:

$$\mathbf{X} = [X \quad Y \quad Z]^T \quad (52)$$
$$\mathbf{V} = [V_X \quad V_Y \quad V_Z]^T \quad (53)$$
$$\boldsymbol{\Theta} = [\Psi \quad \Theta \quad \Phi]^T \quad (54)$$
$$\boldsymbol{\omega} = [\omega_x \quad \omega_y \quad \omega_z]^T \quad (55)$$
$$\boldsymbol{\Xi} = [\xi \quad \zeta_L \quad \zeta_R]^T \quad (56)$$

These can be put together in a state vector $\mathbf{S}$ as:

$$\mathbf{S} = [\mathbf{X} \quad \mathbf{V} \quad \mathbf{Q} \quad \boldsymbol{\omega} \quad \boldsymbol{\Xi}]^T \quad (57)$$

Where $\mathbf{Q} = \mathbf{f}(\boldsymbol{\Theta})$. Deriving Equation (57), we have:

$$\dot{\mathbf{S}} = [\mathbf{V} \quad \mathbf{a} \quad \dot{\mathbf{Q}} \quad \dot{\boldsymbol{\omega}} \quad \dot{\boldsymbol{\Xi}}]^T \quad (58)$$

Being the input vector $\mathbf{u} = [\xi_{cmd} \quad \zeta_{cmd} \quad \zeta_{diff}]^T$, then we have:

$$\dot{\mathbf{S}} = \begin{bmatrix} \mathbf{V} \\ \mathbf{a} \\ \dot{\mathbf{Q}} \\ \dot{\boldsymbol{\omega}} \\ \dot{\boldsymbol{\Xi}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1(a) \\ \mathbf{f}_2(\mathbf{T}, \mathbf{F}_{aero}, \mathbf{g}, m) \\ \mathbf{f}_3(\mathbf{Q}, \boldsymbol{\omega}) \\ \mathbf{f}_4(\mathbf{M}, \boldsymbol{\omega}, \mathbf{J}) \\ \mathbf{f}_5(\mathbf{u}, t) \end{bmatrix} \quad (59)$$

After examining equations written so far, the Equation 53 can be reduced to:

$$\dot{\mathbf{S}} = \begin{bmatrix} \mathbf{V} \\ \mathbf{a} \\ \dot{\mathbf{Q}} \\ \dot{\boldsymbol{\omega}} \\ \dot{\boldsymbol{\Xi}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1(\mathbf{a}) \\ \mathbf{f}_2(\mathbf{X}, \mathbf{V}, \mathbf{Q}, \Xi, t) \\ \mathbf{f}_3(\mathbf{Q}, \boldsymbol{\omega}) \\ \mathbf{f}_4(\mathbf{X}, \mathbf{V}, \Xi, \mathbf{J}) \\ \mathbf{f}_5(\mathbf{u}, t) \end{bmatrix} \quad (60)$$

Explicit form for function $\mathbf{f}_1, \dots, \mathbf{f}_5$ are too complex to be written here, although they are what has been developed so far. This system has one vector (three scalars) inputs and outputs, so that the output function $\mathbf{g}_{out}$ is defined as:

$$\boldsymbol{\Theta} = \mathbf{g}_{out}(\mathbf{X}, \mathbf{V}, \Xi, t) \quad (61)$$

II. SIMULATION ANALYSIS AND DISCUSSION

The model has been implemented in Simulink® and fed with data based on the Russian Angara 1.2 rocket, after some minor adaptations. Table 1 summarizes these data.

Table 1 – Data used in simulations

m_s	39812 kg
$m_{o_{ini}}$	92738 kg
$m_{o_{ini}}$	35262 kg
$\dot{m}_p$	533.33 kg/s
z_{CG_s}	24.73 m
$z_{CG_{f,ini}}$	7.45 m
$z_{CG_{f,fin}}$	5.1 m
$z_{CG_{o,ini}}$	18.5 m
$z_{CG_{o,fin}}$	10.0 m
α_f	2.63
T_{SL}	1922 kN
T_{vac}	2085 kN
λ	0.5585 rad/s
$(\xi, \zeta_L, \zeta_R)_{max}$	± 0.1396 rad
S_{ref}	10.75 m²
L_{ref}	41 m
D_{ref}	3 m

d_T	2 m

Aerodynamic data was produced with a simplified CFD analysis, corrected by general literature data (References [5] and [6]). The graphs in Fig. 2 shows the resulting data for the coefficients in Equations (15) - (17). Note that where $k_1 = 1.4$ and $k_2 = 2$ where chosen, and Interpolation was made using

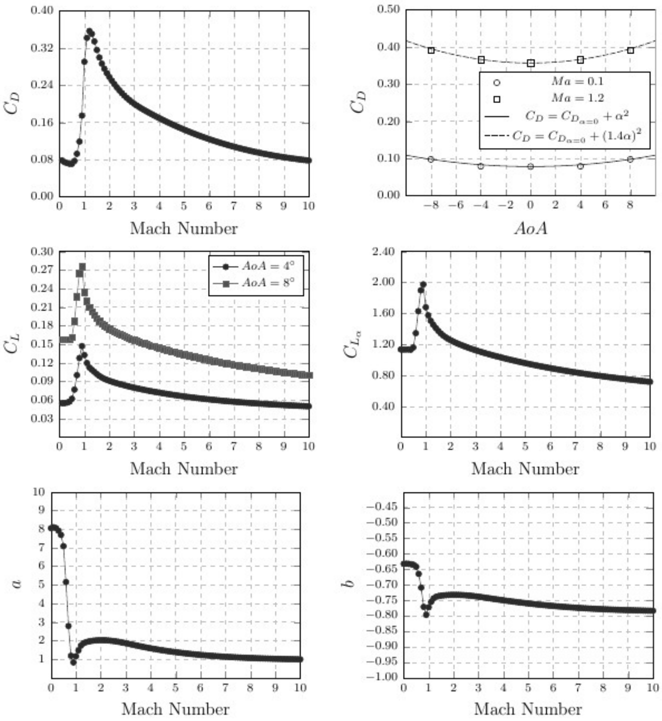

Fig. 3 – Aerodynamic data set.

Matlab®'s spline function.

With these data, some simulations were attempted, basically to understand the model's behavior and validate it. Fig. 2 shows that in this case, an altitude of about 800 km was reached, which is 150 km more than estimated with Tsiokolvsky's equation (using $V_e = 3757$ m/s). The other graphs in the figure show some other aspects of the rocket's behavior. The drag rise around $Ma = 1$ can be noticed, and the engine shutdown at $t = 240$ s, when the propellant is over. At the same time, the increasing speed starts to diminish, as expected.

As for mass and inertia parameters, the graphs in Fig. 4 show their behavior. Instantaneous mass and moment of inertia about oz axis diminish linearly to a minimum, when there is no more mass variation, as expected from equations (31) and (35). Moment of inertia about ox and oy axes cannot be linear, since they depend on the distances of varying masses to a varying CG position. The same is obviously true for the instantaneous CG position.

One should notice that the parameters shown in the graphs of Fig. 4 depend only on time, and will vary identically in any case where the same set of configuration data is used. These

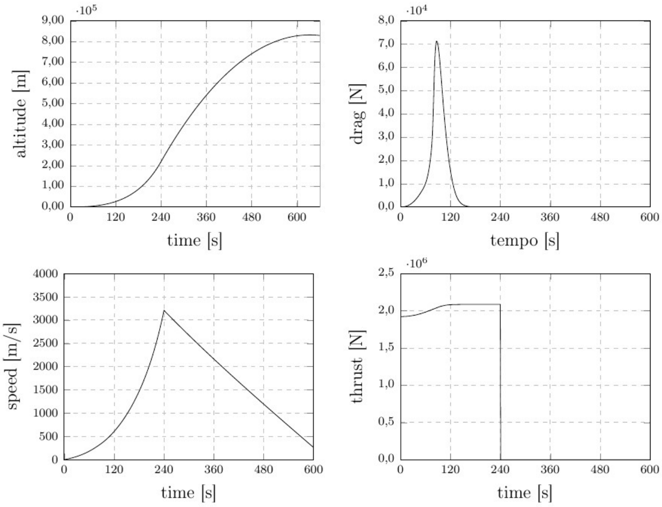

Fig. 2 – Simulation 1: rocket's apogee, drag, speed and thrust. Disturbances are off, and there is no controlling input.

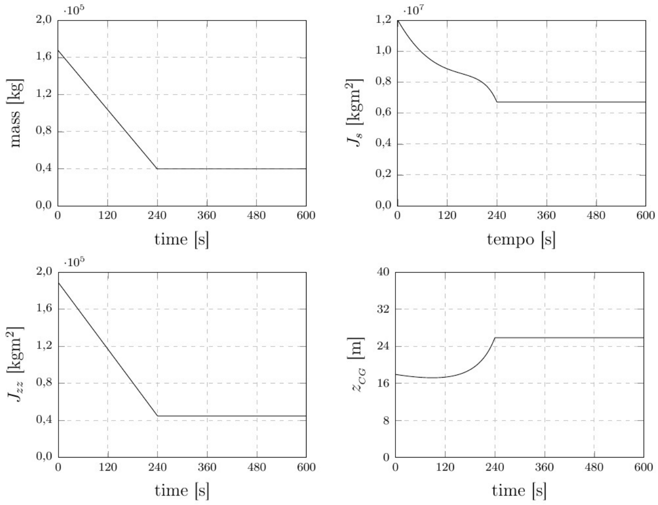

Fig. 4 – Simulation 1: mass, inertia properties and CG position.

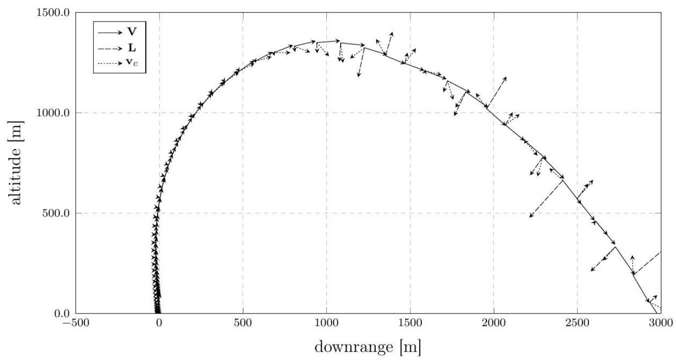

Fig. 5 – Simulation 2: rocket's trajectory from non-vertical initial position. Velocity, lift and body vectors relations.

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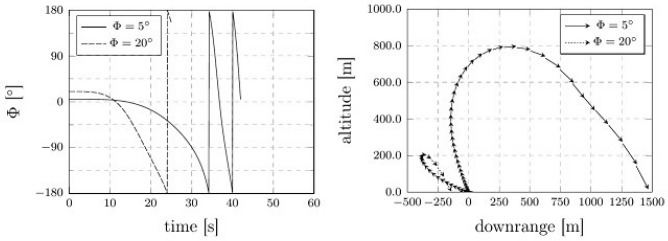

Fig. 6 – Simulation 2: Phi angle (degrees) variation and trajectory for two different initial conditions.

graphics show that the rocket's parameters are behaving as expected, thus validating the model so far.

Another simulation was made with an initial $\Phi = 1°$. In this case, since the CoP position is ahead of the CG position, a divergent behavior is expected. In fact, any initial attitude different than the perfect vertical alignment leads to instability of the rocket.

In this case, a plot (Fig. 5) was made relating the rocket's trajectory with velocity, lift and body vectors, so that the correct direction of lift force could be verified.

One can notice that after about 500 m downrange, the body vector starts to spin around, while lift force, always perpendicular to **V** changes direction as the body vector rotates, accordingly to Equation (9). There are no forces or movement in the X direction.

Attitude variation and trajectory are given for other 2 different Φ values for initial conditions in Fig. 6. One may notice some similitude between the graph in Fig. 5 and Fig. 6 (right) with the phase portraits, which also suggests that the rocket is unstable for all but the perfect vertical initial condition.

Now turning on all disturbances while keeping the controllers off, the perturbations introduced dramatically

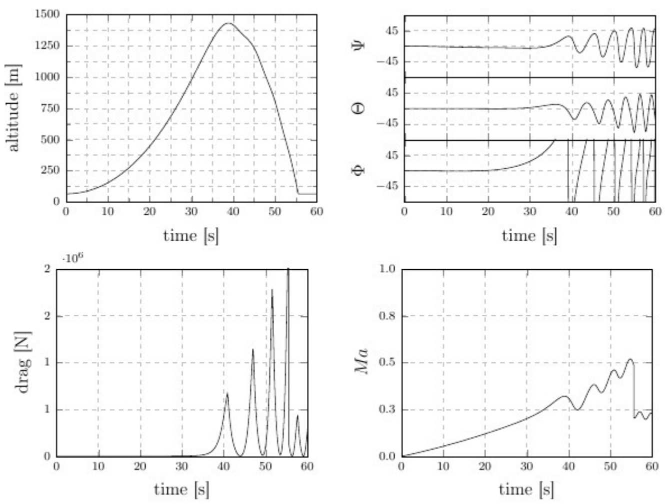

Fig. 7 – Simulation 3: Apogee, attitude, drag and Mach number. Disturbances are on, and controls are off.

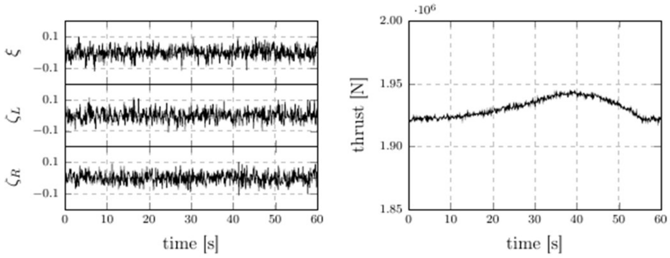

Fig. 8 – Simulation 3: nozzle positions and thrust magnitude with instabilities on.

change the rocket's behavior, as can be seen from the graphs in Fig. 7.

The initial attitude is the same as in Simulation 1 (C.f. Fig. 2), i.e. perfect vertical. One may see that the perturbations introduced break out the stability of the rocket even in this case. The effects of these instabilities in nozzle position and thrust magnitude can be seen in the graphs of Fig. 8.

The trajectory, together with velocity, body and lift vectors can be seen in Fig. 9.

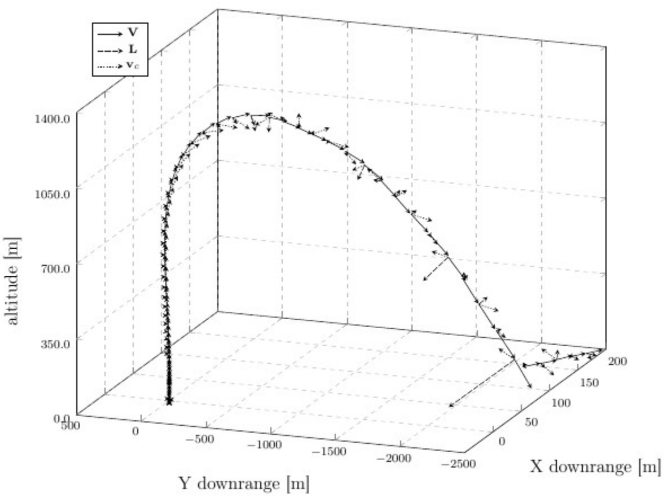

Fig. 9 – Simulation 3: trajectory and velocity, body and lift vectors.

Inspection of Fig. 9 should convince the reader of the appropriate vector's directions.

IV.CONCLUSION

From the data and results presented so far, it seems reasonable to conclude that the model here proposed is good enough in the stablished limits. Nonetheless, many improvements can yet be made. To cite a few:

A better and complete reference framework, one that includes all kinds of accelerations related to the planet's center would lead to better accuracy as to velocity and position, giving the model uses other than only control analysis.

Also, Equations (15) - (17) are inaccurate when α is large. In general, this is not a problem since one does not expect a rocket to operate in this condition. But it would give a better understanding of its behavior when uncontrolled, as in the case of its free fall, for instance.

Other sources of perturbations should be introduced. Atmospheric winds, gusts and turbulence would be a good addition, allowing to understand and develop better control laws. The same is true for attitude sensor noise.

The next step in this research is to implement some control laws and analyze and compare their differences and effectiveness in stabilizing and controlling the rocket in some maneuvers.

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[i] Note that this representation of Euler angles is not identical to the one usually adopted in aerospace context, where the rocket body would be longitudinally aligned with the ox axis, and the oz and oy axis would be in the opposite directions. The reason for this choice resides in the mathematical simplicity and the convenience of avoiding the problematic representation of angles close to 90° (see Reference [7]).

[ii] Angle of attack is, by definition, the angle between velocity and an aircraft’s horizontal plane. In the case of a rocket, there is no practical need to distinguish between this and an angle β, between velocity and the body’s lateral plane.