Unit III: Special Discrete Distributions

- **Binomial Distribution**

1. The experiment consists of repeated trials.
2. Each trial results in two exhaustive and mutually disjoint outcomes, termed as success and failure.
3. The number of trials "n" is finite.
4. The trials are independent of each other.
5. The probability of success is constant for each trial.

X, The **number of successes** in Bernoulli trials is called a **Binomial random variable**. The probability distribution of X is called **Binomial distribution** and is denoted as $b(x; n, p)$

A Bernoulli trial can result in a success with probability p and a failure with probability $q = 1 - p$. Then the probability distribution of the binomial random variable X, the number of successes in n independent trials, is

$$b(x; n, p) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2, \ldots, n.$$

Moment Generating Function (m.g.f.) $M_X(t)$

- $$M_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx} f(x), & \text{for discrete distribution} \\ \int e^{tx} f(x)dx, & \text{for contiuous distribution} \end{cases}$$

- $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r$ **where** $\mu'_r = E(X^r)$ **is rth moment about origin**

$$M_X(t) = E(e^{tX}) = E\left[1 + tX + \frac{t^2X^2}{2!} + \ldots + \frac{t^rX^r}{r!} + \ldots\right]$$
$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \ldots + \frac{t^r}{r!}E(X^r) + \ldots$$

$$= 1 + t\,\mu_1' + \frac{t^2}{2!}\mu_2' + \ldots + \frac{t^r}{r!}\mu_r' + \ldots$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\,\mu_r'$$

- μ_r' =coefficient of $\frac{t^r}{r!}$ in m.g.f.
- $\mu_r' = \left|\frac{d^r}{dt^r}\, M_X(t)\right|_{t=0}$

$$\left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\mu_1' + t\mu_2' + \frac{t^2}{2!}\mu_3' + \frac{t^3}{3!}\mu_4' + \cdots\right]_{t=0} = \mu_1'$$

$$\left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = [\mu_2' + t\mu_3' + \cdots]_{t=0} = \mu_2'$$

MGF of Binomial distribution

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx}\binom{n}{x} p^x q^{n-x} = \sum_{x=0}^{n} (pe^t)^x q^{n-x}\binom{n}{x} = (q + pe^t)^n$$

Mean and Variance of Binomial distribution

$$E(X) = \mu = np, \qquad Var(X) = \sigma^2 = npq$$

Mean and Variance of Binomial distribution using mgf.

$M_X(t) = (pe^t + q)^n$

$E(X) = \mu_1' = \left[\frac{d}{dt}(M_X(t))\right]_{t=0} = \left[\frac{d}{dt}(pe^t+q)^n\right]_{t=0}$

$= \left[n(pe^t+q)^{n-1} \cdot pe^t\right]_{t=0}$

$= n(p+q)^{n-1} \cdot p$ $\quad$ * $p+q=1$

$\boxed{E(X) = np}$

$E(X^2) = \mu_2' = \left[\frac{d^2}{dt^2}(M_X(t))\right]_{t=0} = \left[\frac{d}{dt}\left((pe^t+q)^{n-1} \cdot npe^t\right)\right]_{t=0}$ $\quad$ Using first derivative value

$= np\left[\frac{d}{dt}(pe^t+q)^{n-1} \cdot e^t\right]_{t=0}$ $\quad$ Product Rule

$= np\left[(n-1)(pe^t+q)^{n-2} \cdot pe^t \cdot e^t + (pe^t+q)^{n-1} \cdot e^t\right]_{t=0}$

$= np\left[(n-1)(p+q)^{n-2} \cdot p + (p+q)^{n-1} \cdot 1\right]$

$p+q=1$

$E(X^2) = np[(n-1)p+1]$

$\sigma^2 = E(X^2) - (E(X))^2 = n^2p^2 - np^2 + np - n^2p^2$

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$\sigma^2 = np - np^2 = np(1-p) = npq$.

- **For Binomial distribution, Variance is less than Mean.**

Q1. A and B play a game in which their chances of winning are in the ratio 3:5. Find A's chance of winning the game at least three games out of the five games played.

Q2.

5.2 Twelve people are given two identical speakers, which they are asked to listen to for differences, if any. Suppose that these people answer simply by guessing. Find the probability that three people claim to have heard a difference between the two speakers.

Q3. In 256 sets of 12 tosses of a fair coin, in how many cases may one expect eight heads and four tails?

Q4.

5.16 Suppose that airplane engines operate independently and fail with probability equal to 0.4. Assuming that a plane makes a safe flight if at least one-half of its engines run, determine whether a 4-engine plane or a 2-engine plane has the higher probability for a successful flight.

Q5. Which of the following can be possible values of mean and variance for Binomial distribution?

(A) Mean=5, Variance=7

(B) Mean=3, Variance= -2

(C) Mean=7, Variance=5

(D) Mean=Variance=5

Q6. The mean and variance of Binomial distribution is given as 4 and 4/3 respectively. Find the parameters of distribution.