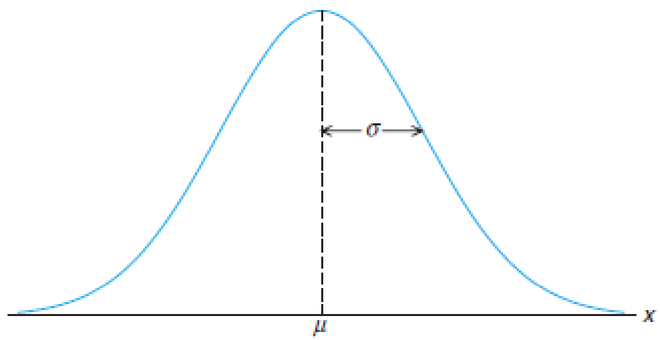
Unit IV

Continuous Probability Distributions

- **Normal Distribution**

A continuous random variable X having the bell shaped distribution is called a normal random variable. The p.d.f. of normal probability curve is given as

$$n(x;\ \mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad , \ -\infty < x < \infty$$



- The curve is bell shaped and symmetrical about the line $x = \mu$.
- Mean, Median and Mode coincide.
- Maximum probability occurs at $x = \mu$ and maximum probability is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}}$.
- The curve has its points of inflection at $x = \mu \pm \sigma$; it is concave downward if $\mu - \sigma < X < \mu + \sigma$ and is concave upwards otherwise.
- The normal curve approaches the horizontal axis asymptotically as we proceed away from mean in either direction.
- $n(x;\ \mu,\ \sigma)$ with mean μ and variance σ^2 and $Z = \frac{X-\mu}{\sigma}$ is standard normal variate with $E(Z) = 0,\ V(Z) = 1$.
- p.d.f. of standard normal variate is $\varphi(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad , \ -\infty < z < \infty$
- Maximum probability occurs at $Z = 0$ for standard normal variate.

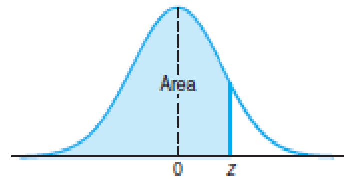


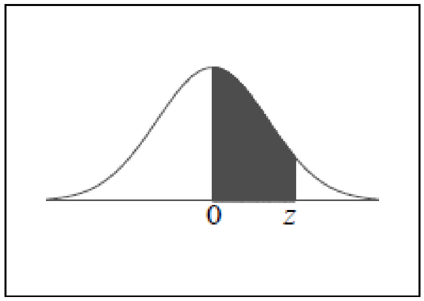
Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Standard Normal Distribution Table



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998

Find

(i) $P(z \leq -1) =$

(ii) $P(z \geq 1) =$

(iii) $P(-1 \leq z \leq 1) =$

(iv) $P(-1.645 \leq z \leq 0) =$

(v) $P(-1.28 \leq Z \leq -0.83) =$

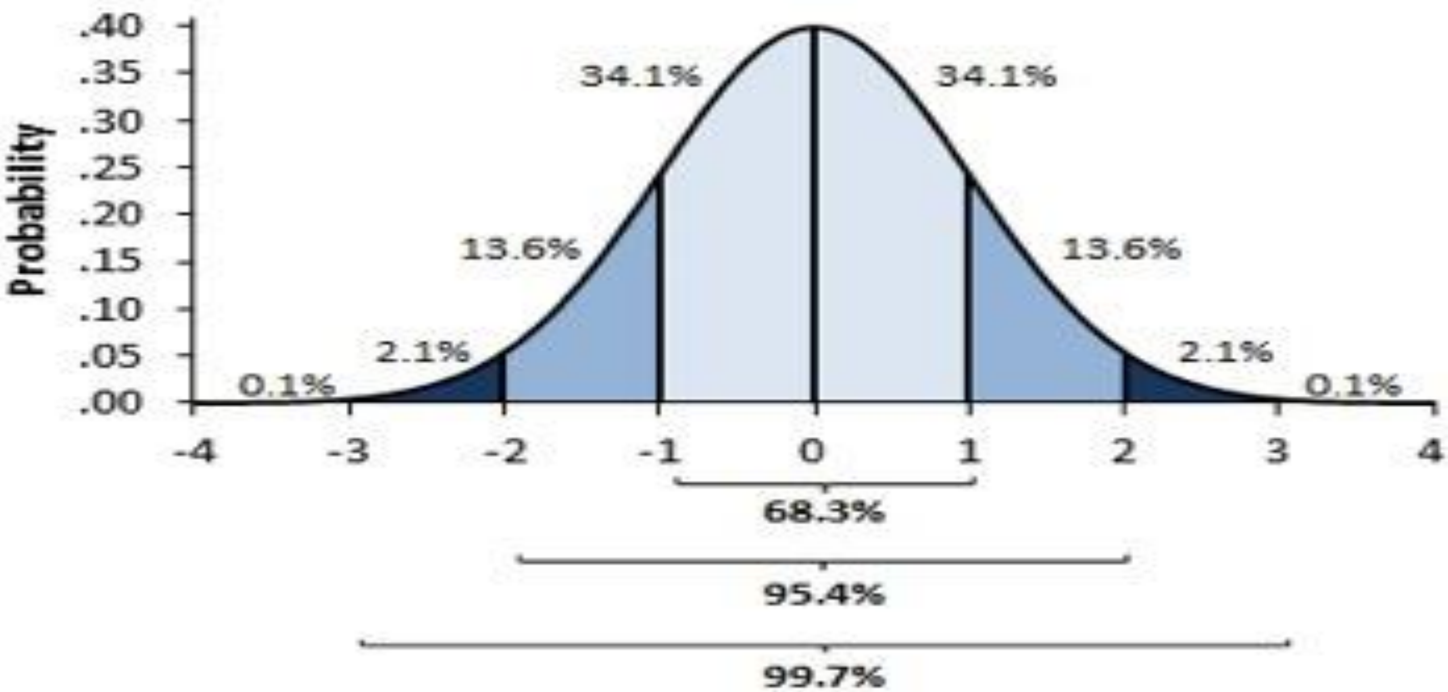
(vi) $P(-1.96 \leq z \leq 1.96) =$

(vii) $P(-2 \leq z \leq 0) =$

(viii) $P(z \leq -2.58) =$

(ix) $P(0.55 \leq z \leq 1.78) =$

(x) $P(-0.67 \leq Z \leq 1.23) =$



$P(-1.645 \leq z \leq 1.645) =$ $2 \times 0.45 = 0.90$

$P(-1.96 \leq z \leq 1.96) =$ $2 \times P(0 \leq Z \leq 1.96) = 0.95$

$P(-2 \leq z \leq 2) =$ 0.9544

) $P(-2.58 \leq z \leq 2.58) =$ 0.99

i) $P(-3 \leq z \leq 3) =$ 0.9973

STEPS TO SOLVE QUESTION ON NORMAL DISTRIBUTION

(i) Convert Normal Variate X to standard normal variate using $Z = \frac{X-\mu}{\sigma}$.

(ii) Convert the probability interval in the provided table form.

Q1.

6.8 Given a normal distribution with $\mu = 30$ and $\sigma = 6$, find

(a) the normal curve area to the right of $x = 17$;

(b) the normal curve area to the left of $x = 22$;

(c) the normal curve area between $x = 32$ and $x = 41$;

(d) the value of x that has 80% of the normal curve area to the left;

(e) the two values of x that contain the middle 75% of the normal curve area.

Given

$$P(Z \le 0.845) = 0.80, P(Z \le -1.15) = 0.125, P(Z \le 1.15) = 0.875,$$
$$P(Z \le -2.167) = 0.0152, P(Z \le -1.33) = 0.0918,$$
$$P(Z \le 1.83) = 0.9664, P(Z \le 0.33) = 0.6293,$$

Q2.

6.11 A soft-drink machine is regulated so that it discharges an average of 200 milliliters per cup. If the amount of drink is normally distributed with a standard deviation equal to 15 milliliters,

(a) what fraction of the cups will contain more than 224 milliliters?

(b) what is the probability that a cup contains between 191 and 209 milliliters?

(c) how many cups will probably overflow if 230-milliliter cups are used for the next 1000 drinks?

(d) below what value do we get the smallest 25% of the drinks?

Given

$$P(0 \le Z \le 0.6) = 0.2257, \qquad P(0 \le Z \le 2) = 0.4772,$$
$$P(0 \le Z \le 1.6) = 0.4452, P(0 \le Z \le 0.675) = 0.25$$

Q3. A normal distribution has mean 25 and variance 25. Find the limits which include 90% of the area under the curve.

Q4. In a university examination of a particular year, 60% of the students failed when mean of the marks was 50% and s.d. 5%. University decided to relax the conditions of passing by lowering the pass marks, to show its result 70%. Find the minimum marks for a student to pass, supposing the marks to be normally distributed and no change in the performance of students take place.

Given: $P(z \leq 0.525) = 0.70, P(z \leq 0.845) = 0.80$

Q5. The height measurements of 600 adult males are arranged in ascending order and it is observed that 180th and 450th entries are 64.2" and 67.8" respectively. Assuming that the sample of heights drawn from a normal population, estimate the mean and s.d. of the distribution.

Given: $P(0 \leq z \leq 0.525) = 0.20, P(0 \leq z \leq 0.675) = 0.25,$
$P(0 \leq z \leq 0.845) = 0.30$

Q6. A certain machine makes electrical resistors having a mean resistance of 40 ohms and s.d. of 2 ohms. Assuming that the resistance follows a normal distribution, find the %age of resistances exceeding 43 ohms if

(i) resistance can be measured to any degree of accuracy.

(ii) resistance is measured to the nearest ohm.

Given: $P(Z < 1.5) = 0.9332\ , P(Z < 1.75) = 0.9599, P(Z < 1.25) = 0.8944$

Q7. The heights of 1000 students are normally distributed with a mean of 174.5 cms and s.d. 6.9 cms. Assuming that the heights are recorded to the nearest half centimeter, how many of these students would be expect to have heights

(a) less than 160.0 cms?

(b) greater than equal to 188.0 cms?

(c) equal to 175.0 cms?

(d) between 171.5 and 182.0 cms inclusive?

Normal approximation to Binomial distribution

If X is a binomial random variable with mean $\mu = np$ and variance $\sigma^2 = npq$, then the limiting form of the distribution of

$$Z = \frac{X - np}{\sqrt{npq}},$$

as $n \to \infty$, is the standard normal distribution $n(z; 0, 1)$.

Normal Approximation to the Binomial Distribution

Let X be a binomial random variable with parameters n and p. For large n, X has approximately a normal distribution with $\mu = np$ and $\sigma^2 = npq = np(1-p)$ and

$$\begin{aligned} P(X \le x) &= \sum_{k=0}^{x} b(k; n, p) \\ &\approx \text{area under normal curve to the left of } x + 0.5 \\ &= P\left(Z \le \frac{x + 0.5 - np}{\sqrt{npq}}\right), \end{aligned}$$

and the approximation will be good if np and $n(1-p)$ are greater than or equal to 5.

Q8.

6.27 The probability that a patient recovers from a delicate heart operation is 0.9. Of the next 100 patients having this operation, what is the probability that

(a) between 84 and 95 inclusive survive?

(b) fewer than 86 survive?

(c) exactly 85 survive?

MGF of Normal distribution

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

For Standard normal variable $M_z(t) = e^{t^2/2}$

$$M_X(t) = E(e^{tX})$$

$$= \int_{-\infty}^{\infty} e^{tX} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $\frac{x-\mu}{\sigma} = u$, $x = \mu + \sigma u$

$dx = \sigma du$

$x \to -\infty$ to ∞

$u \to -\infty$ to ∞

$$M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma u)} \cdot e^{-\frac{u^2}{2}} \cdot \sigma du$$

$$M_X(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2} + \sigma u t} \cdot e^{\mu t} du$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(u^2 - 2\sigma u t)}{2}} du \quad \text{—— } 1$$

Recall gamma function

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} \cdot x^{\alpha-1} dx \quad ; \quad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

i.e., $\int_0^{\infty} e^{-x} x^{\frac{1}{2}-1} dx = \int_0^{\infty} e^{-x} \cdot \frac{1}{\sqrt{x}} dx = \sqrt{\pi}$

Let $x = \frac{u^2}{2}$, $dx = u du$, $x \to 0$ to ∞, $u \to 0$ to ∞

So, $\int_0^{\infty} e^{-u^2/2} \cdot \frac{1}{u/\sqrt{2}} u du = \sqrt{\pi}$

$$\int_0^{\infty} e^{-u^2/2} du \cdot \sqrt{2} = \sqrt{\pi}$$

$$\int_0^{\infty} e^{-u^2/2} du = \sqrt{\frac{\pi}{2}} \quad \text{—— } 2$$

Considering ① again

$$M_X(t) = \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(u^2-2u\sigma t)}{2}} \, du$$

Completing square

$$u^2 - 2u\sigma t = u^2 - 2u\sigma t + \sigma^2 t^2 - \sigma^2 t^2 = (u+\sigma t)^2 - \sigma^2 t^2$$

let $u + \sigma t = v$

$du = dv$

$u \to -\infty$ to ∞, $v \to -\infty$ to ∞

$$M_X(t) = e^{\mu t} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{v^2 - \sigma^2 t^2}{2}\right)} \, dv$$

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot \int_{-\infty}^{\infty} e^{-\frac{v^2}{2}} \, dv$$

Even f^n

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot 2 \int_0^{\infty} e^{-v^2/2} \, dv$$

Using ②

$$= \frac{1}{\sqrt{2\pi}} e^{\mu t + \frac{\sigma^2 t^2}{2}} \cdot 2 \cdot \sqrt{\frac{\pi}{2}}$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

//

Home Exercise: Derive the expression for mean and variance of normal distribution using MGF.

Q9. Practice MCQs

(i).

A normal random variable X has the followir
probability density function

$$f(x) = \frac{1}{\sqrt{8\pi}} e^{-\left\{\frac{(x-1)^2}{8}\right\}}, -\infty < x < \infty$$

Then $\int_{1}^{\infty} f(x)dx =$

A. 0

B. $\frac{1}{2}$

C. $1 - \frac{1}{e}$

D. 1

(ii).

The mean and variance of a normal variate with probability density function

$$f(x) = \frac{1}{\sqrt{32\pi}} \exp\left[-\frac{1}{128}\left(2x - \frac{1}{5}\right)^2\right]$$

are:

(a) Mean = 0.2, variance = 4

(b) Mean = 0.1, variance = 16

(c) Mean = 0.1, variance = 64

(d) Mean = 0.2, variance = 16

(iii)

If X is a normal variate with mean 30 and standard deviation 5,

What is the probability $P(26 \le X \le 34)$?

Given $A(z > 0.8) = 0.2118$.

A. 0.2118

B. 0.4236

C. 0.5764

D. 0.7882

(iv)

A random variable X has normal distribution with mean 100. If $P(100 < X < 120) = \alpha$ then $P(X < 80) =$

A. $1 - 2\alpha$

B. $1 - \alpha$

C. $\frac{1-2\alpha}{2}$

D. $\frac{1-\alpha}{2}$

(v)

If X is a normal random variable with mean μ and variance $\sigma^2 = 9$, and Z is a standardized normal random variable such that $P(X \le 15) = P(Z \le 1)$, then the value of mean μ is:

(a) 7 (b) 12

(c) 7.5 (d) 16

Exponential Distribution

- The time between events in a Poisson process is an exponential distribution.
- Poisson — Exponential

  Events per single unit of time — Time per single event

- The time required for first event to occur is Exponential distribution.
- The mean of the exponential distribution is the parameter β, which is the reciprocal of the parameter in the Poisson distribution.
- β is called mean time between events.

Exponential Distribution The continuous random variable X has an **exponential distribution**, with parameter β, if its density function is given by

$$f(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\beta > 0$.

MGF of Exponential distribution

$$M_X(t) = \frac{1}{1 - t\beta}, \text{provided} \frac{1}{\beta} > t$$

Exponential Distribution

MGF $M_X(t) = E(e^{tx}) = \int_0^\infty \frac{1}{\beta} e^{-x/\beta} e^{tx}\,dx = \frac{1}{\beta}\int_0^\infty e^{-(\frac{1}{\beta}-t)x}\,dx$

$$= \frac{1}{\beta}\left[\frac{e^{-(\frac{1}{\beta}-t)x}}{-(\frac{1}{\beta}-t)}\right]_0^\infty = \frac{1}{\beta}\left[0 - \frac{1}{-(\frac{1}{\beta}-t)}\right] = \frac{1}{1-t\beta}$$

$M_X(t) = \frac{1}{1-t\beta}$, provided $\frac{1}{\beta} > t$ or $t\beta < 1$

The mean and variance of the exponential distribution are

$$\mu = \beta \text{ and } \sigma^2 = \beta^2.$$

$$\text{Mean} = \mu_1' = \left[\frac{d}{dt}(M_x(t))\right]_{t=0} = \left[\frac{-1}{(1-t\beta)^2}\cdot(-\beta)\right]_{t=0} = \beta$$

$$\mu_2' = E(x^2) = \left[\frac{d^2}{dt^2}(M_x(t))\right]_{t=0} = \left[\beta\cdot\frac{(-2)}{(1-t\beta)^3}\cdot(-\beta)\right]_{t=0} = 2\beta^2$$

$$\sigma^2 = 2\beta^2 - (\beta)^2 = \beta^2$$

Q10. If MGF of exponential distribution is given as $M_X(t) = \frac{2}{2-t}$, then find

(a) parameter of exponential distribution.

(b) mean time b/w events.

(c) $P(X < 1)$

Q11.

6.45 The length of time for one individual to be served at a cafeteria is a random variable having an exponential distribution with a mean of 4 minutes. What is the probability that a person is served in less than 3 minutes on at least 4 of the next 6 days?

Q12.

6.46 The life, in years, of a certain type of electrical switch has an exponential distribution with an average life $\beta = 2$. If 100 of these switches are installed in different systems, what is the probability that at most 30 fail during the first year?

Gamma Distribution

The **gamma function** is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx, \qquad \text{for } \alpha > 0.$$

- $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1), \qquad \alpha > 1$
- $\Gamma(n) = (n - 1)!$
- $\Gamma(1) = 1$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Q13. Find value of

(i) $\Gamma(5)$ (ii) $\Gamma(7/2)$ (iii) $\Gamma(11/2)$ (iv) $\int_0^\infty e^{-x}\ x^5\, dx$

Gamma Distribution The continuous random variable X has a **gamma distribution**, with parameters α and β, if its density function is given by

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

- The time until the occurrence of a Poisson event is random variable described by exponential distribution, whereas the time until a specified number of Poisson events occur is a random variable described by gamma distribution.
- The specific number of events is the parameter α in the gamma distribution.
- The special gamma distribution for which $\alpha = 1$ is called the exponential distribution.
- α is called shape parameter and β is called scale parameter.
- If $X_i \sim Exp(\beta)$, then $Y = \sum_{i=1}^{n} X_i \sim$gamma distribution.

MGF of Gamma Distribution

$$M_X(t) = \frac{1}{(1 - t\beta)^\alpha}, \quad provided\ \frac{1}{\beta} > t$$

$M_X(t) =$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-\left(\frac{1}{\beta}-t\right)x} dx$$

$$\left(\frac{1}{\beta}-t\right)x = u$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \left(\frac{u}{\frac{1}{\beta}-t}\right)^{\alpha-1} \cdot e^{-u} \cdot \frac{du}{\left(\frac{1}{\beta}-t\right)} \qquad dx = \frac{du}{\frac{1}{\beta}-t}$$

$x \to 0$ to ∞

$u \to 0$ to ∞

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \cdot \frac{1}{\left(\frac{1}{\beta}-t\right)^\alpha} \int_0^\infty u^{\alpha-1} e^{-u} du$$

$$\frac{1}{\beta^\alpha \Gamma(\alpha)} \frac{\beta^\alpha}{(1-t\beta)^\alpha} \Gamma(\alpha) = \frac{1}{(1-t\beta)^\alpha}$$

The mean and variance of the gamma distribution are

$$\mu = \alpha\beta \text{ and } \sigma^2 = \alpha\beta^2.$$

$$\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \left[-\alpha(1-t\beta)^{-\alpha-1} \cdot (-\beta)\right]_{t=0} = \alpha\beta$$

$$\mu_2' = E(X^2) = \left[\frac{d^2}{dt^2}(M_x(t))\right]_{t=0}$$

$$= \left[\alpha\beta \cdot (-\alpha-1)(1-t\beta)^{-\alpha-2} \cdot (-\beta)\right]_{t=0}$$

$$= \alpha\beta(\alpha+1)\beta$$

$$= \alpha^2\beta^2 + \alpha\beta^2$$

$$\sigma^2 = \alpha^2\beta^2 + \alpha\beta^2 - (\alpha\beta)^2$$

$$\sigma^2 = \alpha\beta^2$$

Q14. In a certain city, the daily consumption of water follows a gamma distribution with $\alpha = 2, \beta = 3$. If the daily capacity of that city is 9 million litres of water, what is the probability that on any given day the water supply is adequate?

Q15.

Suppose that telephone calls arriving at a particular switchboard follow a Poisson process with an average of 5 calls coming per minute. What is the probability that up to a minute will elapse by the time 2 calls have come in to the switchboard?

Q16. On Saturday morning, customers arrive at bakery according to Poisson process at an average rate of 15 per hour.

(a) What is the probability that it takes less than 10 minutes for the first 3 customers to arrive?

(b) What is the average amount of time that will elapse before 3 customers arrive in the bakery?

(c) What is the probability that exactly 15 customers arrive in an hour?

Q17. The mean and variance of gamma distribution are 6 and 12. Find

(i) parameters of Gamma distribution.

(ii) $P(X < 2)$

Gamma distribution of first kind

$$f(x;\alpha) = \begin{cases} \dfrac{e^{-x}x^{\alpha-1}}{\Gamma(\alpha)}, & \alpha > 0, x > 0 \\ 0, & otherwise \end{cases}$$