Vector Spaces (Linear Spaces)

A set V along with the set of properties, is called a vector space.

Set of properties (All the elements in the vector space are called vectors).

Properties: (5 properties for addition & 5 more for scalar multiplication).

1. Closure: $\forall\ x, y \in V$, $\exists$ a unique $z \in V$ such that $z = x + y$.

2. Commutative property: $\forall\ x, y \in V$, $x + y = y + x$.

3. Associative (under addition): $\forall\ x, y, z \in V$, $(x + y) + z = x + (y + z)$.

4. Additive Identity: $\exists$ a unique $\theta \in V$, such that $\forall\ x \in V$, $x + \theta = \theta + x = x$.

5. Additive Inverse: $\forall\ x \in V$, $\exists$ a unique $(-x) \in V$, such that $x + (-x) = (-x) + x = \theta$.

6. For any scalar α, $\alpha x \in V$, $\forall\ x \in V$.

7. Associative (under scalar multiplication): For any scalars α and β, $\alpha(\beta x) = (\alpha\beta)x$, $\forall\ x \in V$.

8. Identity scalar: $\exists$ a unique scalar 1 (need not be number - just a notation), such that $1 \cdot x = x \cdot 1 = x$, $\forall\ x \in V$.

9. Distributive w.r.t vector Addition: For any scalar α, $\alpha(x + y) = \alpha x + \alpha y$, $\forall\ x, y \in V$.

10. Distributive w.r.t scalar Addition: For any scalars α, β; $(\alpha + \beta)x = \alpha x + \beta x$, $\forall\ x \in V$.

→ Set of all real numbers can form a vector space.

→ Set of integers cannot form a vector form because 6th property (if αx is real but not integer) is not followed.

→ Set of all points in 2-D form a vector space.

→ Set of " " " 3-D " "

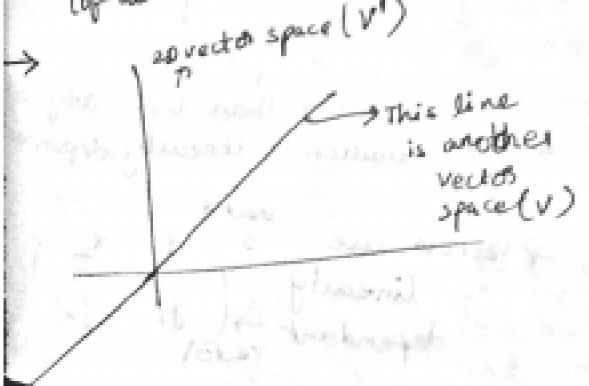
→ Set of all points in an n-Dimensional space always form a vector space.

In n-D space; each point is a vector.

→ Set of all 2-dimensional matrix with real entities is a vector space.

(of all n-dimensional matrices with real entities) ⇒

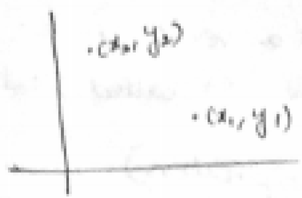
→



V is subset of V'. ⇒ $V \subset V'$.

V can be created by considering only one point (all other elements can be created by scalar multiplication of point with α)

P8) →



Q⇒ whether any x and y (on the 2D space) can be created by using two arbitrary points (x_1, y_1) & (x_2, y_2)

$$\begin{pmatrix} x \\ y \end{pmatrix} = \alpha_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Now whether unique α_1 & α_2 is enough to create 2D space

⇕ when

Rank of $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ is equal to order.

→ $\begin{array}{l} \text{vector 1} \rightarrow \\ \rightarrow \\ \text{vector 2} \end{array} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ If a vector is obtained by applying some operations on another vector (operations mean multiplying with some scalar etc then, these are linearly dependant.

i.e.,

if Rank of the matrix is less than the order, then ~~one~~ vectors are linearly dependant on each other.

* (one vector is " " " other) (det = 0).

That means.,

Any x and y is space can be created by 2 arbitrary points if scalars are unique i.e., rank is equal to order.

i.e., scalars aren't unique for linearly dependant vectors.

→ A set of vectors are linearly dependant if there exist non-zero scalars such that linear combination of them is zero.

$$n\text{-Dimension} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \alpha_1 \bar{x_1} + \alpha_2 \bar{x_2} + \alpha_3 x_3 + \dots\dots + \alpha_n \bar{x_n}$$

If $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$ is the only solution then they are linearly independant.

Other than this any other ~~solution~~ solution - linearly dependant.

For 2 Dimension

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$ ⇒ vectors are linearly dependant $\begin{array}{c} \text{vector} \rightarrow \\ \\ \rightarrow \text{vector} \end{array} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$

Vector $\Rightarrow \bar{x}_1$

Vector spaces

Linear combination of vectors

Express a given vector by a linear combination of a set of vectors

Linear dependance.

V_n = n-dimensional vector space.

<u>Theorem-1</u>: In a vector space of n-dimensions, every basis contains n-elements.

$\Rightarrow$ A set of vectors $\bar{y}_1, \bar{y}_2, \cdots \bar{y}_n$ is said to form a basis; if any vector of the same dimension / vector space can be generated by the given set of vectors.

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \alpha_1 \bar{y}_1 + \alpha_2 \bar{y}_2 + \cdots + \alpha_n \bar{y}_n \qquad \bar{y}_1 = \begin{pmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1n} \end{pmatrix}$$

Ex: $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \alpha_1 \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix}}_{\text{form a Basis}}$$

$\rightarrow$ simplest basis for any n-dimensional vector

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \cdots\cdots \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

{ Basis cannot contain a null vector }

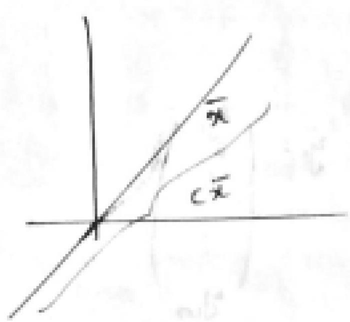
$\boxed{\text{Theorem 1}}$ In a vector space of n-dimensions, every basis contains n-elements.

$\boxed{\text{Theorem 2}}$

Any linearly independent set of vectors $\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n$ in V_n, form a basis in V_n.

Span of a vector :

Span of a vector $\bar{x}$ is defined as $c \cdot \bar{x}$, where c is any scalar.



Span ⇒ set of any vectors obtained to multiplying a scalar to a vector.

$c = 0 \Rightarrow$ Null vector.

span of a set of vectors:

Span of a set of vectors $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n$ is set of all possible vectors formed by linear combination of given vectors, multiplied by a scalar each.

$\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$

$L = \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 + \cdots + \alpha_n \bar{x}_n$.

(If (2 points) $\bar{x}_1$ & $\bar{x}_2$ are linearly independent; then any point in 2-D space can be generated

If they are linearly dependent: only line joining them can be formed

(1,2), (2,4)

* Inner product of two vectors $\bar{x}\begin{pmatrix}x_1\\x_2\\x_3\\\vdots\\x_n\end{pmatrix}$ and $\bar{y}\begin{pmatrix}y_1\\y_2\\y_3\\\vdots\\y_n\end{pmatrix}$ is $(\bar{x}^T\bar{y})$

→ We can convert any vector into a unit vector. This is called normalization.

Vector subspace

~~A Subspac~~

A subset of a vector space V_n which is again a vectorspace is called a vector subspace.

$V_n \Rightarrow$ vector space

$W \subset V_n$

and W also follows all propeties of vector space.

$\Rightarrow W =$ vector subspace

Ex: A 2-D vector space has a vector subspace which is a line passing through the origin.

Theorem

If $S \subseteq V_n$, then S is a subspace of V_n iff

1. $\bar{x}+\bar{y} \in S \quad \forall \bar{x}, \bar{y} \in S$
2. $c\bar{x} \in S \quad \forall \bar{x} \in S$ and any scalar c.

(No need to check all the ten propeties)

$c\bar{x}+d\bar{y} \in S$.

Theorem

Let $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_r$ be r vectors in V_n; $r \le n$. Then the set S of all the linear combinations of these vectors also forms a vector space.

→ Normal form of a matrix $A_{m\times n} \sim \underset{B}{\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}}$ $r \le \min(m,n)$.

unit matrix.

[First Quadrant always should be unit square matrix]

Given a matrix A; we can get equivalent normal form $\underline{B}$ of A by multiple P to left of A & Q to right of A.

$R_2 \leftarrow R_2 - 4R_1$

$R_3 \leftarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 3/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 3/2 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/2 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftarrow \frac{2}{3} R_2$

$$\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/3 & 0 & 2/3 \\ -2 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \leftarrow C_2 - \frac{1}{2} C_1$

$C_3 \leftarrow C_3 - \frac{1}{2} C_1$

[Column operations → Q should be changed]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ -1/3 & 0 & 2/3 \\ -2 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \leftarrow C_3 - C_2$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Normalised form } (B)} = \underbrace{\begin{bmatrix} 1/2 & 0 & 0 \\ -1/3 & 0 & 2/3 \\ -2 & 1 & 0 \end{bmatrix}}_{P} A \underbrace{\begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{Q}$$

$B = PA$

B obtained only by row operations

$B = AQ$

B obtained only by column operations

→ To find basis for a given set of vectors.

→ If three vectors of a vector space are given and they are linearly independant,

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ forms a basis.

→ ~~But~~ Many vectors are given and if some of them are linearly

~~indep~~ dependant; then we can get the vector space from a lower dimensional basis.

Pb)
To find basis for given set of vectors.

$[1\ 2\ 1]^T \quad [2\ 4\ 2]^T \quad [1\ 2\ 2]^T \quad [2\ 4\ 0]^T$

given matrix

$$\underline{4\times 3} \quad \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & 4 \\ 1 & 2 & 2 & 0 \end{bmatrix}$$

[no. of elements in basis = dimension of vector = 3.
⇒ <u>atmost</u> 3 vectors in basis]

Columns here are more. So, if we can normalise them so that every vector in it is linearly independent (by applying column operations)

$$\underset{3\times4}{\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 2 & 4 \\ 1 & 2 & 2 & 0 \end{bmatrix}} = \underset{3\times4}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}} \underset{4\times4}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

$C_2 \leftarrow C_2 - 2C_1$

$C_3 \leftarrow C_3 - C_1$

$C_4 \leftarrow C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & \underbrace{1 \quad -2}_{\text{linearly dependant}} \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \leftarrow C_4 + 2C_3$

$$\underset{B}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}} = \begin{bmatrix} \cdot & \cdot & \cdot \end{bmatrix} \underset{Q}{\begin{bmatrix} 1 & -2 & -1 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

[Q is here upper triangular matrix]

It cannot be reduced any further.

so, the non-zero columns [vectors] in the equivalent normal form represent the basis. (like rank) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form the basis.

[2 dimensional basis]

→ If we are given with two different sets of vectors.

Q) Are they from the same vector space?

→ Theorem: Two sets of vectors forming the columns of matrix A and column C, will span the same vector space if

1. Rank(A) = Rank(C) [If ranks are same ⇒ no. of elements in basis ⇒ same ⇒ same dimension for both sets]

2. $\underbrace{[A : C]}_{\substack{\text{This denotes} \\ \text{contatenation of columns} \\ \text{of A and C}}} \sim [B : 0]$ [vectors in A and C should be linearly dependant ⇒ i.e., basis of A should also generate vector set of C ⇒ rank of matrix formed by combination of A and C = rank (A) or rank (C) [Rank(A) = rank(C)]]

0 ⇒ zero

B ⇒ ~~reduced~~ transformed form of A

[By applying operations on A, ~~B is reduced~~ B is obtained and all other columns become zero]

Ex. Pb) $\{[3\ 1\ 0]^T, [0\ 2\ 3]^T\ [1\ 1\ 1]^T\}$

$\{[-1\ -3\ -4]^T, [7\ 11\ 3]^T\ [6\ 8\ 9]^T\ [3\ -11\ -18]^T\}$

?? $A_{m\times n}$

$A\bar{x} = \theta$ θ ⇒ Null vector

Solution of it forms a vector space ⇒ null space.

→ There will be no case of no sol^n.

because rank(A) = rank($A\bar{x}$)

[dimension-1 ⇒ straight line
dimension-2 ⇒ plane]

→ One solution case

↳ null space

$\begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$ dimension of null space = nullity.

rank (A) + nullity (A) = min (m, n)

rank(A) + nullity (A) = dimension of (A).

→ If rank = order ⇒ nullity = 0 [only one solution ⇒ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$]

Linear
Transformation: $\longrightarrow$ (Doing translation, rotation etc.)

$\rightarrow$ A mapping T from V_n to another vector space W_m

$$T : V_n \longrightarrow W_m$$

1. $T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v}) \quad \forall\ \bar{u}, \bar{v} \in V_n$

2. $T(\alpha \bar{u}) = \alpha T(\bar{u}) \quad \forall\ \bar{u} \in V_n$ and any scalar α.

[The transformation is called linear transformation if properties 1 & 2 are satisfied].

<u>Ex</u>: $T : V_3 \longrightarrow V_2$ $\quad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$T(\bar{x}) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$$

$\bar{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \bar{v} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

1) $\rightarrow T(\bar{u} + \bar{v}) = T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right) = T\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix} = T\begin{pmatrix} x_1 + x_2 + y_1 + y_2 \\ x_2 + x_3 + y_2 + y_3 \end{pmatrix}$

$\left[\because T(\bar{x}) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}\right.$ $\quad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$T(\bar{u}) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$$

$$T(\bar{v}) = \begin{pmatrix} y_1 + y_2 \\ y_2 + y_3 \end{pmatrix}$$

$\rightarrow T(\bar{u}) + T(\bar{v}) = \begin{pmatrix} x_1 + x_2 + y_1 + y_2 \\ x_2 + x_3 + y_2 + y_3 \end{pmatrix}$

$$\therefore T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v})$$

2) $T(\alpha\bar{u}) = T\left(\alpha\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right)$

$$= \begin{pmatrix} \alpha x_1 + \alpha x_2 \\ \alpha x_2 + \alpha x_3 \end{pmatrix}$$

$$\alpha \cdot T(\bar{u}) = \alpha \cdot \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha x_1 + \alpha x_2 \\ \alpha x_2 + \alpha x_3 \end{pmatrix}$$

$\therefore T(\alpha\bar{u}) = \alpha \cdot T(\bar{u})$

Both 1) and 2) are satisfied. Hence ; $T : V_3 \to V_2$; $T(\bar{x}) = \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \end{pmatrix}$ is a linear transformation.

Ex2: $T : V_3 \to V_2$ $\quad T(\bar{x}) = \begin{pmatrix} x_1^2 + x_2 \\ x_2^2 + x_3 \end{pmatrix}$

$\Rightarrow$ Non linear transformation.

$\to$ Linear tranformation is called affine transformation.

Theorem: If a transformation satisfies the following criterion, then it is a linear transformation and viceversa.

$$T(\alpha\bar{u} + \beta\bar{v}) = \alpha T(\bar{u}) + \beta T(\bar{v}) \qquad \forall \bar{u}, \bar{v} \in V,$$
α, β are any scalars.

Proof: ~~viceversa~~ 1) Given it is linear transformation

$T(\alpha\bar{u} + \beta\bar{v}) = T(\alpha\bar{u}) + T(\beta\bar{v})$ [1st condition]

$= \alpha T(\bar{u}) + \beta T(\bar{v})$ [2nd condition]

$\to$ criterion is formed.

2) Given criterion.

$T(\alpha\bar{u} + \beta\bar{v}) = \alpha T(\bar{u}) + \beta T(\bar{v})$ $\quad \alpha, \beta$ are any scalars.

Let $\alpha = 1$; $\beta = 1$.

$T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v}) \Rightarrow$ first condition.

Let $\beta = 0$

$T(\alpha\bar{u}) = \alpha T(\bar{u}) \Rightarrow$ 2nd condition.

Properties of linear transformation :

1. $T(-\bar{x}) = -T(\bar{x})$ $[\alpha = -1]$

2. $T(\bar{\phi}) = \bar{\phi}$

proof: $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$

$\bar{u} = -\bar{v}$

$\Rightarrow T(\bar{u}-\bar{u}) = T(\bar{u}) - T(\bar{u})$

$T(\bar{\phi}) = \bar{\phi}.$

3. $T(\bar{u}-\bar{v}) = T(\bar{u}) - T(\bar{v})$

$[\alpha = 1 ; \beta = -1]$ & $T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v}).$

4. Identity transformation

$T(\bar{x}) = \bar{x}.$

5. Null transformation.

$T(\bar{x}) = \bar{\phi}$ [Whatever may be $\bar{x}$; it is mapped to $\bar{\phi}$ (null)]

→ Bijective / Non-singular transformation:

[Transformation is one-one and onto]

1. $T(\bar{x}_1) = T(\bar{x}_2) \Rightarrow \bar{x}_1 = \bar{x}_2$

2. $\forall \bar{x}_2 \in W_m ; \exists \bar{x}_1 \in V_m \mid T(\bar{x}_1) = \bar{x}_2.$

for every (for all) (such that)

→ Projection of $\bar{b}$ on $\bar{a}$: $(\bar{p})$

$$\bar{p} = \frac{\bar{a}^T \cdot \bar{b}}{\bar{a}^T \cdot \bar{a}} \cdot \bar{a}$$

→

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

(x_1, x_2, x_3) is coordinate of $\bar{x}$ w.r.t basis $\left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$

Theorem 1:

Let $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ is a basis for the vector space V_n. Let $\{\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n\}$ be some given elements in W_m. Then, there exists one and only one linear transformation $T: V_n \longrightarrow W_m$ such that $T(\bar{x}_i) = \bar{z}_i, \forall i$. $\begin{pmatrix} T(\bar{x}_1) : \bar{z}_1 \\ T(\bar{x}_2) : \bar{z}_2 \\ \vdots \end{pmatrix}$.

(First we have to prove that such linear transformation exists and then we have to prove that it is unique.)

Proof: Let us consider a vector $\bar{x}$ of n-dimension from V_n.

As $\bar{x}$ is from V_n; it can be represented as a linear combination of $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\}$ (basis).

$$\bar{x} = \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 + \ldots + \alpha_n \bar{x}_n.$$

Let us suppose a ~~linear~~ transformation in such a way that it maps to $\bar{z}$ from W_m.

(consider)

$T(\bar{x}) = \bar{z}$

$T(\bar{x}) = \alpha_1 \bar{z}_1 + \alpha_2 \bar{z}_2 + \ldots + \alpha_n \bar{z}_n$

[$\bar{z}_1, \bar{z}_2, \ldots \bar{z}_n$ are from W_m. Their linear combination is also from W_m]. $(\bar{z})$

We have to prove that this transformation is linear.

$$\text{Let } \bar{u} = \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 + \ldots + \alpha_n \bar{x}_n$$

$$\bar{v} = \beta_1 \bar{x}_1 + \beta_2 \bar{x}_2 + \ldots + \beta_n \bar{x}_n.$$

$$T(\bar{u}) = \alpha_1 \bar{z}_1 + \ldots + \alpha_n \bar{z}_n$$

$$T(\bar{v}) = \beta_1 \bar{z}_1 + \ldots + \beta_n \bar{z}_n.$$

we can prove

$T(\bar{u}+\bar{v}) = T(\bar{u}) + T(\bar{v})$ } satisfied

$T(\alpha\bar{u}) = \alpha T(\bar{u})$ } Hence, the transformation is linear.

$$T(\bar{u}+\bar{v}) = T(\alpha_1 \bar{x}_1 + \ldots + \alpha_n \bar{x}_n + \beta_1 \bar{x}_1 + \beta_2 \bar{x}_2 + \ldots + \beta_n \bar{x}_n) = T((\alpha_1+\beta_1)\bar{x}_1 + (\alpha_2+\beta_2)\bar{x}_2 + \ldots + (\alpha_n+\beta_n)\bar{x}_n)$$

$$= (\alpha_1+\beta_1)\bar{z}_1 + (\alpha_2+\beta_2)\bar{z}_2 + \ldots + (\alpha_n+\beta_n)\bar{z}_n.$$

$$= \alpha_1 \bar{z}_1 + \alpha_2 \bar{z}_2 + \ldots + \alpha_n \bar{z}_n + \beta_1 \bar{z}_1 + \beta_2 \bar{z}_2 + \ldots \beta_n \bar{x}_n$$

$$T(\bar{u}) + T(\bar{v}) = \alpha_1 \bar{z}_1 + \ldots + \alpha_n \bar{z}_n + \beta_1 \bar{z}_1 + \ldots + \beta_n \bar{z}_n.$$

$$T(\alpha\bar{u}) = T(\alpha\alpha_1 \bar{x}_1 + \alpha\alpha_2 \bar{x}_2 + \ldots + \alpha\alpha_n \bar{x}_n) = \alpha\alpha_1 \bar{z}_1 + \ldots + \alpha\alpha_n \bar{z}_n.$$

$$\alpha T(\bar{u}) = \alpha\alpha_1 \bar{z}_1 + \ldots + \alpha\alpha_n \bar{z}_n.$$

Now we have to prove that this is unique.

Let us assume that it is not unique.

Let us suppose $T'(x)$ is another linear transformation which maps $\bar{x}$ (in V_n) to $\bar{z}$ (in W_m).

$$T'(\bar{x}) = T'(\alpha_1\bar{x}_1 + \alpha_2\bar{x}_2 + \dots + \alpha_n\bar{x}_n)$$

$$= \cancel{T'(\alpha_1\bar{x}_1)}$$

$$= \alpha_1 T'(\bar{x}_1) + \alpha_2 T'(\bar{x}_2) + \dots + \alpha_n T'(\bar{x}_n)$$ [As the transformation is linear

$$= \alpha_1\bar{z}_1 + \alpha_2\bar{z}_2 + \dots + \alpha_n\bar{z}_n = T(\bar{x}).$$

$\therefore$ The linear transformation is unique.

⇒ Kernel of a linear tranformation kernel(T):

$$T: V_n \longrightarrow W_m$$

denoted by KerT

$$\ker T \subseteq V_n$$

$$T(\bar{x}) = \bar{\phi} \quad \forall x \in \ker T$$

→ (kerT) It is a set of all elements in V_n which are mapped to null vector.

Ex: $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 - 3x_2 + x_3.$

$$\ker T = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} \dots$$

Theorem: KerT is always a subspace of V_n.

(KerT is always a vectorspace).

Rank T = dimension (W_m)

Nullity T = dimension (ker T)

Rank + Nullity = dimension (V_n).

*) Given linear transformation (whose definition is not given)

[Pg. 171]

$$T : V_3 \to V_2$$

Basis of V_3 $\begin{pmatrix}1\\1\\-1\end{pmatrix}$ $\begin{pmatrix}4\\1\\1\end{pmatrix}$ $\begin{pmatrix}1\\-1\\2\end{pmatrix}$

$\downarrow$ $\downarrow$ $\downarrow$

Elements in V_2 $\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$ $\begin{pmatrix}1\\1\end{pmatrix}$

$T(\bar{x}) = \begin{pmatrix}\cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\end{pmatrix} = ?$

(From Theorem 1, we know T is unique)

Sol: Let V_3 be $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

It can be represented by linear combination of basis :

(HW)

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \alpha_1\begin{pmatrix}1\\1\\-1\end{pmatrix} + \alpha_2\begin{pmatrix}4\\1\\1\end{pmatrix} + \alpha_3\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

Solve to get

$\left.\begin{array}{l}\alpha_1 = ?\\ \alpha_2 = ?\\ \alpha_3 = ?\end{array}\right\}$ in terms of x_1, x_2, x_3

$$\alpha_1 = 3x_1 - 7x_2 - 5x_3$$

$$\alpha_2 = -x_1 + 3x_2 + 2x_3$$

$$\alpha_3 = 2x_1 - 5x_2 - 3x_3$$

$$T(\bar{x}) = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$= T\left(\alpha_1\begin{pmatrix}1\\1\\-1\end{pmatrix} + \alpha_2\begin{pmatrix}4\\1\\1\end{pmatrix} + \alpha_3\begin{pmatrix}1\\-1\\2\end{pmatrix}\right)$$

$$= \alpha_1 T\begin{pmatrix}1\\1\\-1\end{pmatrix} + \alpha_2 T\begin{pmatrix}4\\1\\1\end{pmatrix} + \alpha_3\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$= \alpha_1\begin{pmatrix}1\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\end{pmatrix} + \alpha_3\begin{pmatrix}1\\1\end{pmatrix}$$

Solve

$\left[\because T\begin{pmatrix}1\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \; T\begin{pmatrix}4\\1\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}, \; T\begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}\right]$

Theorem : A linear transformation $T : V \longrightarrow W$ is non-singular (bijective) iff $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$ is a basis in V implies $\{T(\bar{x}_1), T(\bar{x}_2), \dots, T(\bar{x}_n)\}$ is a basis in W. (Here dimension of V and W is same).

(Proof in MB book)

→ Composite transformation of T and T'.

$$\left[\begin{array}{l} T : V \longrightarrow V' \\ T' : V' \longrightarrow W \end{array}\right] \Rightarrow \text{[illegible]}$$

$$T' \circ T(\bar{x}) = T'(T(\bar{x}))$$

→ **Inverse mapping :-**

T is a linear transformation. Another transformation T^{-1} (if exists) is said of inverse of transformation T if $T \circ T^{-1} = T^{-1} \circ T = I$; where I is identity transformation.

($T(\bar{x}) = \bar{x} \ \forall \bar{x}$)

Theorem : A linear transformation $T : V \rightarrow W$ has an inverse iff T is non-singular (bijective).

Theorem (i): If $X = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$ and $X' = \{\bar{x}'_1, \bar{x}'_2, \dots, \bar{x}'_n\}$ are the basis of a vector space V_n, then, there exists B such that

$$\bar{x}'_i = b_{1i}\bar{x}_1 + b_{2i}\bar{x}_2 + \dots + b_{ni}\bar{x}_n$$

Here B is called 'transformation matrix'.

i.e,

$$\bar{x}'_i = [\bar{x}_1 \ \ \bar{x}_2 \ \dots \ \bar{x}_n] \begin{bmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix}$$

$$\Rightarrow \bar{x}' \Rightarrow [\bar{x}_1' \quad \bar{x}_2' \; \cdots\cdots \; \bar{x}_n'] = [\bar{x}_1 \quad \bar{x}_2 \quad \bar{x}_3 \; \cdots\cdots \; \bar{x}_n] \begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix}$$

β = transformed basis

α = earlier basis

$\beta = \alpha B$.

Theorem (ii):

In the earlier case, if $\bar{x} = \alpha_1 \bar{x}_1 + \alpha_2 \bar{x}_2 + \cdots\cdots + \alpha_n \bar{x}_n$ and

a vector in space V_n.

$\bar{x} = \alpha_1' \bar{x}_1' + \alpha_2' \bar{x}_2' + \cdots\cdots + \alpha_n' \bar{x}_n'$

then, $u' = B^{-1} u$ $\quad \left[\begin{array}{l} B \text{ is transformation matrix} \\ u = [\alpha_1 \, \alpha_2 \cdots\cdots \alpha_n] \\ u' = [\alpha_1' \, \alpha_2' \cdots\cdots \alpha_n'] \end{array} \right]$