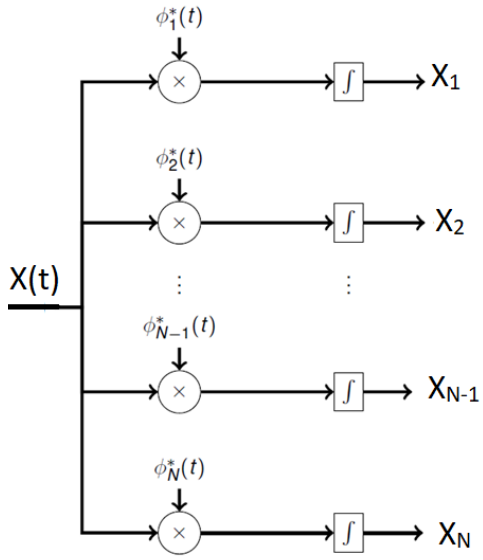
Thapar Institute of Engineering & Technology, Patiala

Department of Electronics and Communication Engineering

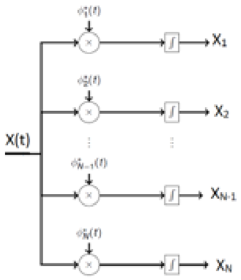
UEC639 – Digital Communication

B. E. (Third Year): Semester-VI (ECE)

Tutorial-7

Q1	Consider a random process X(t) given as input to the bank of correlators. $X(t) = s_i(t) + W(t) \qquad 0 \le t \le T \qquad i = 1,2,\dots,M$ The sample function of received random random process X(t) is defined as $x(t) = s_i(t) + w(t) \qquad 0 \le t \le T \qquad i = 1,2,\dots,M$ $s_i(t)$ --- Signal from transmitter $W(t)$-- White Gaussian Noise process of *zero mean* and power spectral density $N_0/2$ $w(t)$-- Sample function of $W(t)$ Determine the response of bank of Correlators to this random process X(t) Derive the mean and variance of output. 

Response of Bank of Correlators to Noisy Input



The output of correlator

$$X_j = \int_0^T X(t)\phi_j(t)dt = \int_0^T (s_i(t) + W(t))\phi_j(t)dt$$

$$X_j = \underbrace{\int_0^T s_i(t)\phi_j(t)dt}_{s_{ij}} + \underbrace{\int_0^T W(t)\phi_j(t)dt}_{W_j}$$

$$\boldsymbol{X_j = s_{ij} + W_j, \qquad j = 1, 2, \ldots, N}$$

- The output of correlator is differ from the signal $\boldsymbol{s_i}$ by the *noise vector* $\boldsymbol{w}$

$$x = s_i + w \quad i = 1,2,\ldots,M$$

$\boldsymbol{s_i}$ --- signal vector (have elements-$s_{i1}, s_{i2}, \ldots s_{iN}$)
The vector $\boldsymbol{x}$ --- sample value of the random vector $\boldsymbol{X}$
The vector $\boldsymbol{w}$ --- sample value of the random vector $\boldsymbol{W}$

$$X_j = s_{ij} + W_j, \qquad j = 1,2, \ldots, N$$

- Since received random process X(t) is Gaussian,
- Then each X_j is a Gaussian random variable.
- Hence each X_j is characterized completely by its mean and variance.

Mean of X_j ---- m_{X_j}

$$\begin{aligned} m_{xj} &= E[X_j] \\ &= E[S_{ij} + W_j] \\ &= s_{ij} + E[W_j] \\ &= s_{ij} \end{aligned}$$

Variance of X_j ---- $\sigma^2_{X_j}$

$$\text{Var}[X] = E[(X - E[X])^2]$$

$$\begin{aligned} \sigma^2{}_{xj} &= Var[X_j] \\ &= E[(X_j - S_{ij})^2] \\ &= E[W_j^2] \end{aligned}$$

$$\sigma^2{}_{xj} = E[W_j^2] \qquad W_j = \int_o^T W(t)\phi_j(t)dt$$

$$\sigma^2{}_{xj} = E\left[\int_0^T W(t)\varphi_j(t)dt \int_0^T W(u)\varphi_j(u)du\right]$$

$$= E\left[\int_0^T\int_0^T \varphi_j(t)\varphi_j(u)W(t)W(u)\,dt\,du\right]$$

$$\sigma^2{}_{xj} = \int_0^T\int_0^T \varphi_j(t)\varphi_j(u)E[W(t)W(u)]\,dt\,du$$

$$= \int_0^T\int_0^T \varphi_j(t)\varphi_j(u)R_w(t,u)\,dt\,du$$

$R_w(t,u)$ = autocorrelation function

$$R_w(t,u) = \frac{N_0}{2}\,\delta(t-u)$$

$$\sigma^2{}_{x\,j} = \frac{N_0}{2}\int_0^T\!\!\int_0^T \phi_j(t)\,\phi_j(u)\,\delta(t-u)\,dt\,du$$

$$= \frac{N_0}{2}\int_0^T \phi_j^2(t)\,dt$$

$$\sigma^2{}_{x\,j} = \frac{N_0}{2} \qquad \mathit{for\ all}\quad j$$

Q2 Derive the expression of likelihood function for AWGN channel.

Conditional Probability Density Function
(Likelihood Function)

- The conditional probability density function of the vector $\boldsymbol{X}$, *given that the signal* $\boldsymbol{s_i(t)}$ *or* correspondingly the symbol $\boldsymbol{m_i}$ was transmitted is represented by $\{f_X(x/m_i)\}$ and is defined as the product of the conditional ***pdf*** of its individual elements

$$f_X(\mathbf{x}/m_i) = \prod_{j=1}^{N} f_{X_j}(x_j/m_i), \qquad i = 1,2,\dots,M \qquad 3.47$$

Where vector $\mathbf{x}$ and scalar x_j are the sample values of the random vector $\mathbf{X}$ and random variable X_j

- This ***conditional pdf*** $\{f_X(x/m_i)\}$for each transmitted message $m_i, i = 1,2,\dots,M$ are called ***likelihood functions***
- These likelihood functions are also known as ***channel characterization function.***
- *Any channel whose likelihood function satisfied above expression is called a* ***memory less channel***

Likelihood function for AWGN channel

- Since X_j is a Gaussian random variables with mean values equal to s_{ij} variance equal to $N_0/2$

$$f_{X_j}(x_j/m_i) = \frac{1}{\sqrt{\pi N_0}}\exp\left[-\frac{1}{N_0}(x_j - s_{ij})^2\right], \qquad \begin{matrix} j = 1,2,\dots,N \\ i = 1,2,\dots,M \end{matrix} \qquad 3.48$$

$$f_X(\mathbf{x}/m_i) = \prod_{j=1}^{N} f_{X_j}(x_j/m_i), \qquad i = 1,2,\dots,M \qquad 3.47$$

After substituting eq. 3.48 into 3.47, we find the likelihood functions of AWGN channel

$$f_X(\boldsymbol{x}/m_i) = (\pi N_0)^{-\left(\frac{N}{2}\right)}\exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2\right], i = 1,2,\dots,M \qquad 3.49$$

Q3 Starting from MAP rule and by deriving the ML rule, determine the block diagram representation of vector receiver.

Refer Lecture Notes Available on LMS

<table>
<tr><td>Q4</td><td>Draw the block diagram representation of correlator receiver, including both detector and vector receiver diagrams. Also, explain its limitation.</td></tr>
<tr><td></td><td>Refer Lecture Notes available on LMS</td></tr>
<tr><td>Q5</td><td>Let X be a continuous random variable with the following PDF:

$$f_X(x) = \begin{cases} 2\,x & for\ 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

Also, suppose that

$$P_{Y/X}(y/x) \ = Geometric(x) = x\,(1-x)^{y-1} \quad for\ y = 1, 2, \ldots.$$

Find the MAP estimate of X given Y = 3;</td></tr>
<tr><td></td><td>We know that $Y \mid X = x \quad \sim \quad Geometric(x)$, so

$$P_{Y|X}(y|x) = x(1-x)^{y-1}, \quad \text{for } y = 1, 2, \cdots.$$

Therefore,

$$P_{Y|X}(3|x) = x(1-x)^2.$$

We need to find the value of $x \in [0, 1]$ that maximizes

$$\begin{aligned} P_{Y|X}(y|x) f_X(x) &= x(1-x)^2 \cdot 2x \\ &= 2x^2(1-x)^2. \end{aligned}$$

We can find the maximizing value by differentiation. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2(1-x)^2\right] = 2x(1-x)^2 - 2(1-x)x^2 = 0.$$

Solving for x (and checking for maximization criteria), we obtain the MAP estimate as

$$\hat{x}_{MAP} = \frac{1}{2}.$$</td></tr>
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