Example 10.25

Consider an LTI system for which the input $x[n]$ and output $y[n]$ satisfy the linear constant-coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1]. \qquad (10.102)$$

Applying the z-transform to both sides of eq. (10.102), and using the linearity property set forth in Section 10.5.1 and the time-shifting property presented in Section 10.5.2, we obtain

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z),$$

or

$$Y(z) = X(z)\left[\frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}\right]. \tag{10.103}$$

From eq. (10.96), then,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}. \tag{10.104}$$

This provides the algebraic expression for $H(z)$, but not the region of convergence. In fact, there are two distinct impulse responses that are consistent with the difference equation (10.102), one right sided and the other left sided. Correspondingly, there are two different choices for the ROC associated with the algebraic expression (10.104). One, $|z| > 1/2$, is associated with the assumption that $h[n]$ is right sided, and the other, $|z| < 1/2$, is associated with the assumption that $h[n]$ is left sided.

Consider first the choice of ROC equal to $|z| > 1/2$ Writing

$$H(z) = \left(1 + \frac{1}{3}z^{-1}\right)\frac{1}{1 - \frac{1}{2}z^{-1}},$$

we can use transform pair 5 in Table 10.2, together with the linearity and time-shifting properties, to find the corresponding impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1].$$

For the other choice of ROC, namely, $|z| < 1/2$ we can use transform pair 6 in Table 10.2 and the linearity and time-shifting properties, yielding

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[-n].$$

In this case, the system is anticausal ($h[n] = 0$ for $n > 0$) and unstable.

then

$$\boxed{x_{(k)}[n] \overset{z}{\longleftrightarrow} X(z^k), \quad \text{with ROC} = R^{1/k}.} \tag{10.77}$$

That is, if z is in the ROC of $X(z)$, then the point $z^{1/k}$ is in the ROC of $X(z^k)$. Also, if $X(z)$ has a pole (or zero) at $z = a$, then $X(z^k)$ has a pole (or zero) at $z = a^{1/k}$.

The interpretation of this result follows from the power-series form of the z-transform, from which we see that the coefficient of the term z^{-n} equals the value of the signal at time n. That is, with

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n},$$

it follows that

$$X(z^k) = \sum_{n=-\infty}^{+\infty} x[n](z^k)^{-n} = \sum_{n=-\infty}^{+\infty} x[n]z^{-kn}. \tag{10.78}$$

Examining the right-hand side of eq. (10.78), we see that the only terms that appear are of the form z^{-kn}. In other words, the coefficient of the term z^{-m} in this power series equals 0 if m is not a multiple of k and equals $x[m/k]$ if m is a multiple of k. Thus, the inverse transform of eq. (10.78) is $x_{(k)}[n]$.

10.5.6 Conjugation

If

$$x[n] \overset{z}{\longleftrightarrow} X(z), \quad \text{with ROC} = R, \tag{10.79}$$

then

$$\boxed{x^*[n] \overset{z}{\longleftrightarrow} X^*(z^*), \quad \text{with ROC} = R.} \tag{10.80}$$

Consequently, if $x[n]$ is real, we can conclude from eq. (10.80) that

$$X(z) = X^*(z^*).$$

then

$$x[n-n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0}X(z), \quad \text{with ROC} = R\text{, except for the possible addition or deletion of the origin or infinity.} \tag{10.72}$$

Because of the multiplication by z^{-n_0}, for $n_0 > 0$ poles will be introduced at $z = 0$, which may cancel corresponding zeros of $X(z)$ at $z = 0$. Consequently, $z = 0$ may be a pole of $z^{-n_0}X(z)$ while it may not be a pole of $X(z)$. In this case the ROC for $z^{-n_0}X(z)$ equals the ROC of $X(z)$ but with the origin deleted. Similarly, if $n_0 < 0$, zeros will be introduced at $z = 0$, which may cancel corresponding poles of $X(z)$ at $z = 0$. Consequently, $z = 0$ may be a zero of $z^{-n_0}X(z)$ while it may not be a pole of $X(z)$. In this case $z = \infty$ is a pole of $z^{-n_0}X(z)$, and thus the ROC for $z^{-n_0}X(z)$ equals the ROC of $X(z)$ but with the $z = \infty$ deleted.

10.5.6 Conjugation

If

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \quad \text{with ROC} = R, \tag{10.79}$$

then

$$\boxed{x^*[n] \overset{Z}{\longleftrightarrow} X^*(z^*), \quad \text{with ROC} = R.} \tag{10.80}$$

Consequently, if $x[n]$ is real, we can conclude from eq. (10.80) that

$$X(z) = X^*(z^*).$$

Thus, if $X(z)$ has a pole (or zero) at $z = z_0$, it must also have a pole (or zero) at the complex conjugate point $z = z_0^*$. For example, the transform $X(z)$ for the real signal $x[n]$ in Example 10.4 has poles at $z = (1/3)e^{\pm j\pi/4}$

Example 10.6

Consider the signal

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1,\ a > 0 \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} a^n z^{-n} \\ &= \sum_{n=0}^{N-1} (az^{-1})^n \\ &= \frac{1-(az^{-1})^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}. \end{aligned} \tag{10 28}$$

Since $x[n]$ is of finite length, it follows from Property 3 that the ROC includes the entire z-plane except possibly the origin and/or infinity. In fact, from our discussion of Property 3, since $x[n]$ is zero for $n < 0$, the ROC will extend to infinity. However, since $x[n]$ is nonzero for some positive values of n, the ROC will not include the origin. This is evident from eq. (10.28), from which we see that there is a pole of order $N-1$ at $z = 0$. The N roots of the numerator polynomial are at

$$z_k = ae^{j(2\pi k/N)}, \quad k = 0, 1, \ldots, N-1. \tag{10.29}$$

The root for $k = 0$ cancels the pole at $z = a$. Consequently, there are no poles other than at the origin. The remaining zeros are at

$$z_k = ae^{j(2\pi k/N)}, \quad k = 1, \ldots, N-1. \tag{10.30}$$

The pole-zero pattern is shown in Figure 10.9.

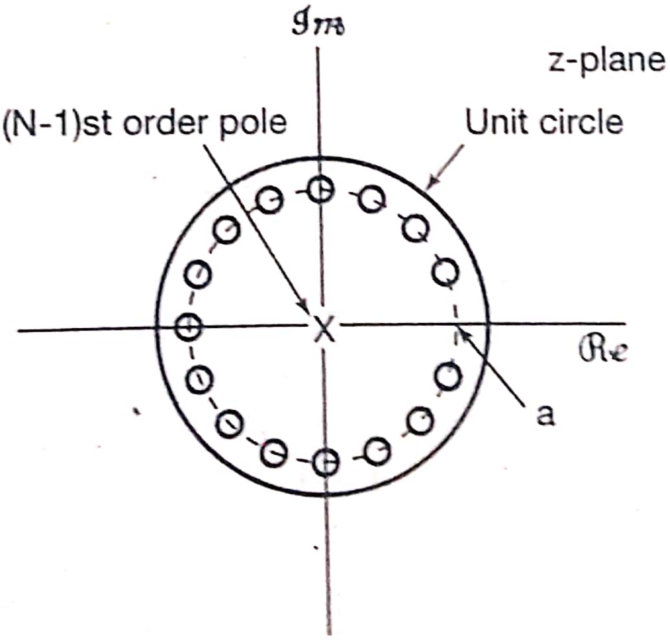

Figure 10.9 Pole-zero pattern for Example 10.6 with $N = 16$ and $0 < a < 1$. The region of convergence for this example consists of all values of z except $z = 0$.

Example 10.7

If

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \qquad \text{with ROC} = R,$$

then

$$\boxed{x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0}X(s), \quad \text{with ROC} = R.} \tag{9.87}$$

9.5.3 Shifting in the s-Domain

If

$$x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s), \qquad \text{with ROC} = R,$$

then

$$\boxed{e^{s_0 t}x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s - s_0), \quad \text{with ROC} = R + \mathcal{R}e\{s_0\}.} \tag{9.88}$$

That is, the ROC associated with $X(s - s_0)$ is that of $X(s)$, shifted by $\mathcal{R}e\{s_0\}$. Thus, for any value s that is in R, the value $s + \mathcal{R}e\{s_0\}$ will be in R_1. This is illustrated in Figure 9.23. Note that if $X(s)$ has a pole or zero at $s = a$, then $X(s-s_0)$ has a pole or zero at $s - s_0 = a$—i.e., $s = a + s_0$.

An important special case of eq. (9.88) is when $s_0 = j\omega_0$—i.e., when a signal $x(t)$ is used to modulate a periodic complex exponential $e^{j\omega_0 t}$. In this case, eq. (9.88) becomes

$$e^{j\omega_0 t}x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s - j\omega_0), \quad \text{with ROC} = R. \tag{9.89}$$

The right-hand side of eq. (9.89) can be interpreted as a shift in the s-plane parallel to the $j\omega$-axis. That is, if the Laplace transform of $x(t)$ has a pole or zero at $s = a$, then the Laplace transform of $e^{j\omega_0 t}x(t)$ has a pole or zero at $s = a + j\omega_0$.

9.5.7 Differentiation in the Time Domain

If

$$x(t) \xleftrightarrow{\mathcal{L}} X(s), \qquad \text{with ROC} = R,$$

then

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} sX(s), \qquad \text{with ROC containing } R.} \tag{9.98}$$

This property follows by differentiating both sides of the inverse Laplace transform as expressed in equation (9.56). Specifically, let

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds.$$

Then

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}ds. \tag{9.99}$$

Consequently, $dx(t)/dt$ is the inverse Laplace transform of $sX(s)$. The ROC of $sX(s)$ includes the ROC of $X(s)$ and may be larger if $X(s)$ has a first-order pole at $s = 0$ that is canceled by the multiplication by s. For example, if $x(t) = u(t)$, then $X(s) = 1/s$, with an ROC that is $\mathcal{Re}\{s\} > 0$. The derivative of $x(t)$ is an impulse with an associated Laplace transform that is unity and an ROC that is the entire s-plane.

9.5.8 Differentiation in the s-Domain

Differentiating both sides of the Laplace transform equation (9.3), i.e.,

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt,$$

we obtain

$$\frac{dX(s)}{ds} = \int_{-\infty}^{+\infty} (-t)x(t)e^{-st}dt.$$

Consequently, if

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s), \qquad \text{with ROC} = R,$$

then

$$\boxed{-tx(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{dX(s)}{ds}, \qquad \text{with ROC} = R.} \tag{9.100}$$

The next two examples illustrate the use of this property.

5.3.4 Conjugation and Conjugate Symmetry

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

$$\boxed{x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega}).} \tag{5.35}$$

Also, if $x[n]$ is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is,

$$\boxed{X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n] \text{real}].} \tag{5.36}$$

From this, it follows that $\mathcal{R}e\{X(e^{j\omega})\}$ is an even function of ω and $\mathcal{I}m\{X(e^{j\omega})\}$ is an odd function of ω. Similarly, the magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function. Furthermore,

$$\mathcal{E}v\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e\{X(e^{j\omega})\}$$

and

$$\mathcal{O}d\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} j\mathcal{I}m\{X(e^{j\omega})\},$$

where $\mathcal{E}v$ and $\mathcal{O}d$ denote the even and odd parts, respectively, of $x[n]$. For example, if $x[n]$ is real and even, its Fourier transform is also real and even. Example 5.2 illustrates this symmetry for $x[n] = a^{|n|}$.