Solution. The spectrum of the analog signal is

$$X_a(F) = \frac{1}{1 + j2\pi F}$$

The exponential analog signal sampled at the rate of 20 samples per second yields the sequence

$$\begin{aligned} x(n) &= e^{-nT} = e^{-n/20}, \qquad n \geq 0 \\ &= (e^{-1/20})^n = (0.95)^n, \qquad n \geq 0 \end{aligned}$$

Now, let

$$x(n) = \begin{cases} (0.95)^n, & 0 \leq n \leq 99 \\ 0, & \text{otherwise} \end{cases}$$

The N-point DFT of the $L = 100$ point sequence is

$$\hat{X}(k) = \sum_{k=0}^{99} \hat{x}(n) e^{-j2\pi k/N}, \qquad k = 0, 1, \ldots, N-1$$

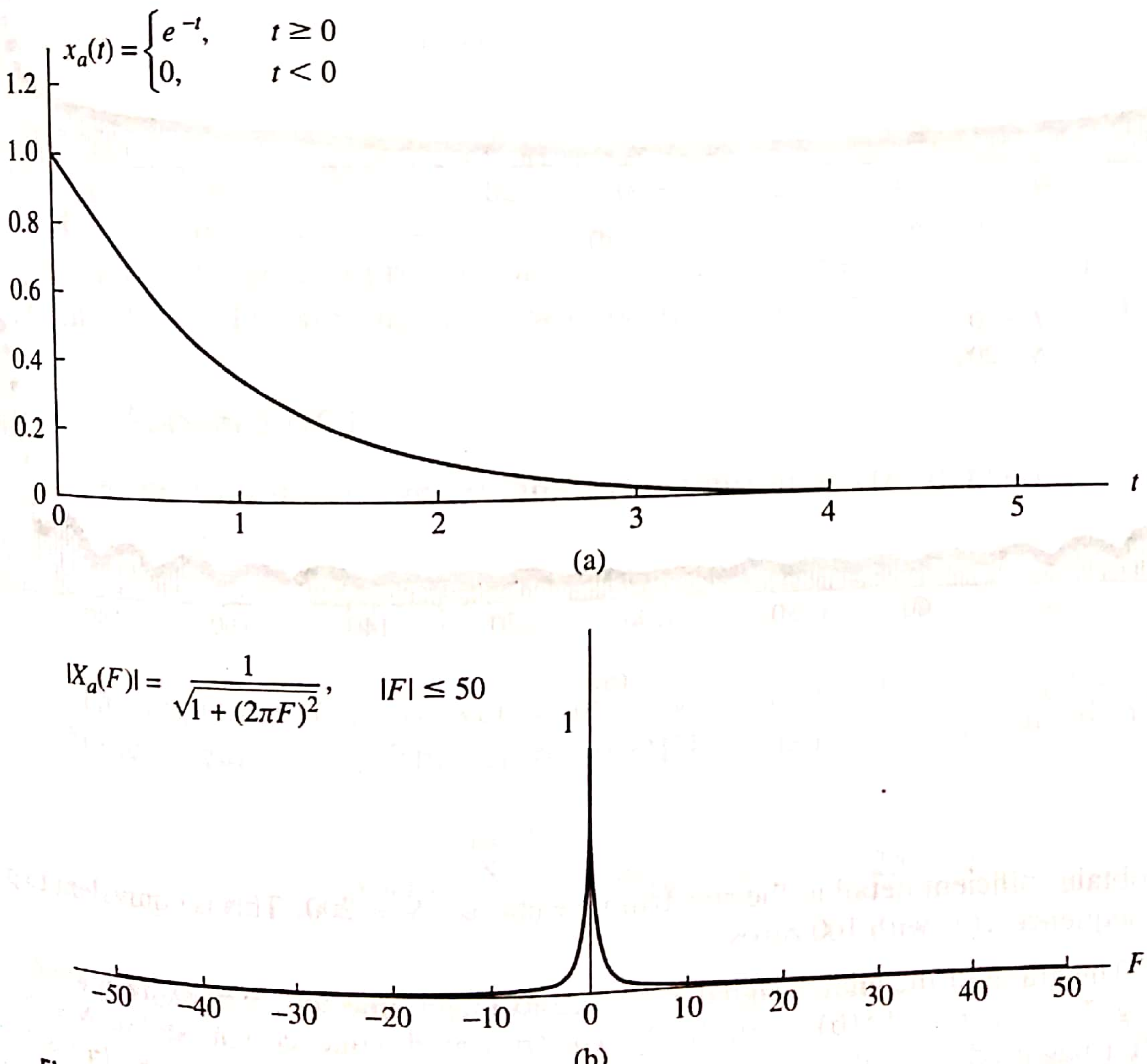


Figure 7.4.5 Effect of windowing (truncating) the sampled version of the analog signal in Example 7.4.1

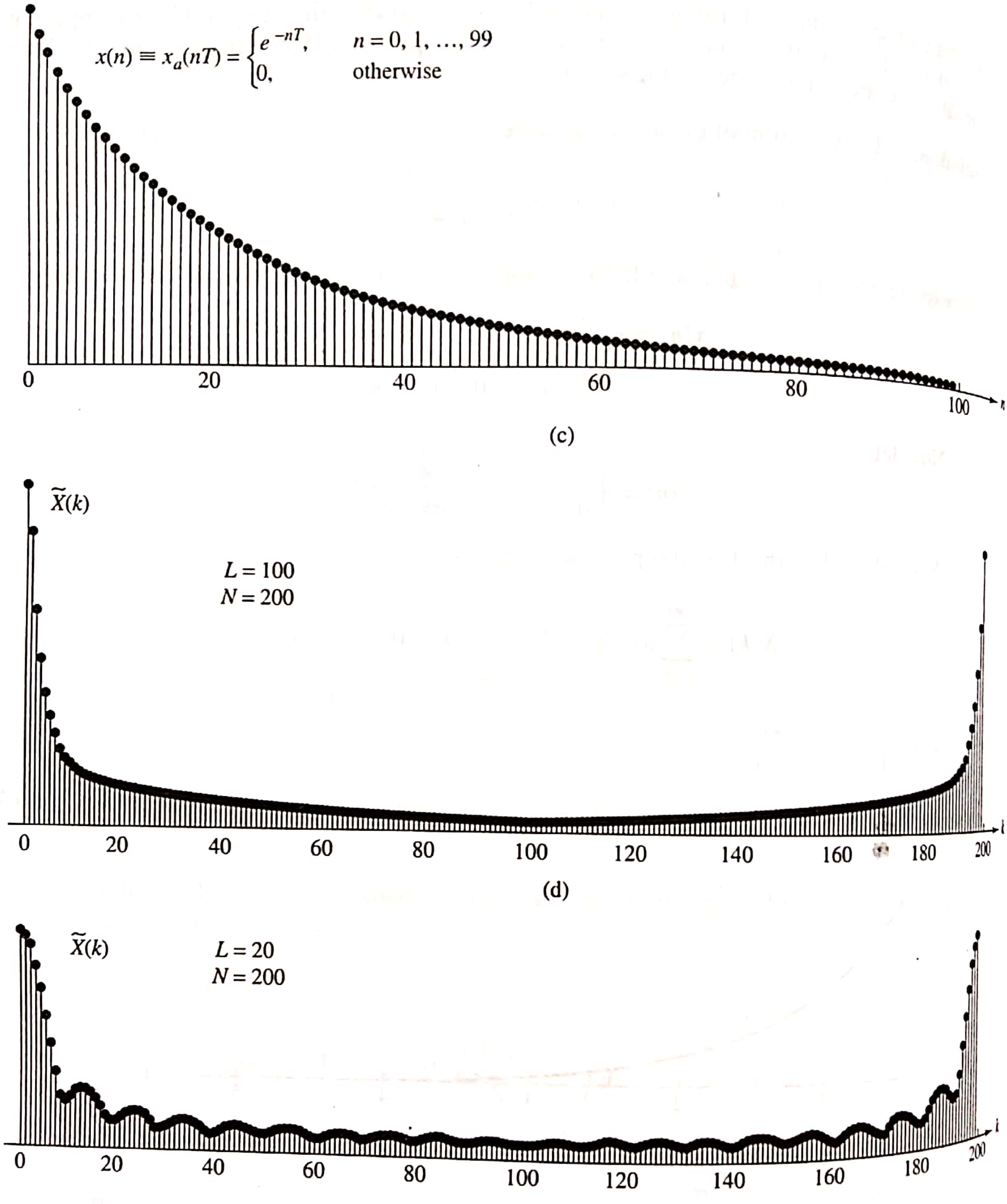


Figure 7.4.5 *Continued*

To obtain sufficient detail in the spectrum we choose $N = 200$. This is equivalent to padding the sequence $x(n)$ with 100 zeros.

The graph of the analog signal $x_a(t)$ and its magnitude spectrum $|X_a(F)|$ are illustrated in Fig. 7.4.5(a) and 7.4.5(b), respectively. The truncated sequence $x(n)$ and its $N = 200$ point DFT (magnitude) are illustrated in Fig. 7.4.5(c) and 7.4.5(d), respectively. In this case the DFT $\{X(k)\}$ bears a close resemblance to the spectrum of the analog signal. The effect of the window function is relatively small.

On the other hand, suppose that a window function of length $L = 20$ is selected. Then the truncated sequence $x(n)$ is given as

$$\hat{x}(n) = \begin{cases} (0.95)^n, & 0 \le n \le 19 \\ 0, & \text{otherwise} \end{cases}$$

Its $N = 200$-point DFT is illustrated in Fig. 7.4.5(e). Now the effect of the wider spectral window function is clearly evident. First, the main peak is very wide as a result of the wide spectral window. Second, the sinusoidal envelope variations in the spectrum away from the main peak are due to the large sidelobes of the rectangular window spectrum. Consequently, the DFT is no longer a good approximation of the analog signal spectrum.

In general, the unit sample response $h_d(n)$ obtained from (10.2.17) is infinite in duration and must be truncated at some point, say at $n = M - 1$, to yield an FIR filter of length M. Truncation of $h_d(n)$ to a length $M-1$ is equivalent to multiplying $h_d(n)$ by a "rectangular window," defined as

$$w(n) = \begin{cases} 1, & n = 0, 1, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases} \tag{10.2.19}$$

Thus the unit sample response of the FIR filter becomes

$$\begin{aligned} h(n) &= h_d(n)w(n) \\ &= \begin{cases} h_d(n), & n = 0, 1, \ldots, M-1 \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{10.2.20}$$

It is instructive to consider the effect of the window function on the desired frequency response $H_d(\omega)$. Recall that multiplication of the window function $w(n)$ with $h_d(n)$ is equivalent to convolution of $H_d(\omega)$ with $W(\omega)$, where $W(\omega)$ is the frequency-domain representation (Fourier transform) of the window function, that is

$$W(\omega) = \sum_{n=0}^{M-1} w(n)e^{-j\omega n} \tag{10.2.21}$$

Thus the convolution of $H_d(\omega)$ with $W(\omega)$ yields the frequency response of the (truncated) FIR filter. That is,

$$H(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\nu)W(\omega-\nu)d\nu \tag{10.2.22}$$

The Fourier transform of the rectangular window is

$$\begin{aligned} W(\omega) &= \sum_{n=0}^{M-1} e^{-j\omega n} \\ &= \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} = e^{-j\omega(M-1)/2}\,\frac{\sin(\omega M/2)}{\sin(\omega/2)} \end{aligned} \tag{10.2.23}$$

This window function has a magnitude response

$$|W(\omega)| = \frac{|\sin(\omega M/2)|}{|\sin(\omega/2)|}, \qquad \pi \le \omega \le \pi \tag{10.2.24}$$

and a piecewise linear phase

$$\Theta(\omega) = \begin{cases} -\omega\left(\dfrac{M-1}{2}\right), & \text{when } \sin(\omega M/2) \ge 0 \\ -\omega\left(\dfrac{M-1}{2}\right) + \pi, & \text{when } \sin(\omega M/2) < 0 \end{cases} \tag{10.2.25}$$

filter.

5.4.4 Notch Filters

A notch filter is a filter that contains one or more deep notches or, ideally, perfect nulls in its frequency response characteristic. Figure 5.4.8 illustrates the frequency response characteristic of a notch filter with nulls at frequencies ω_0 and ω_1. Notch filters are useful in many applications where specific frequency components must be eliminated. For example, instrumentation and recording systems require that the power-line frequency of 60 Hz and its harmonics be eliminated.

To create a null in the frequency response of a filter at a frequency ω_0, we simply introduce a pair of complex-conjugate zeros on the unit circle at an angle ω_0. That is,

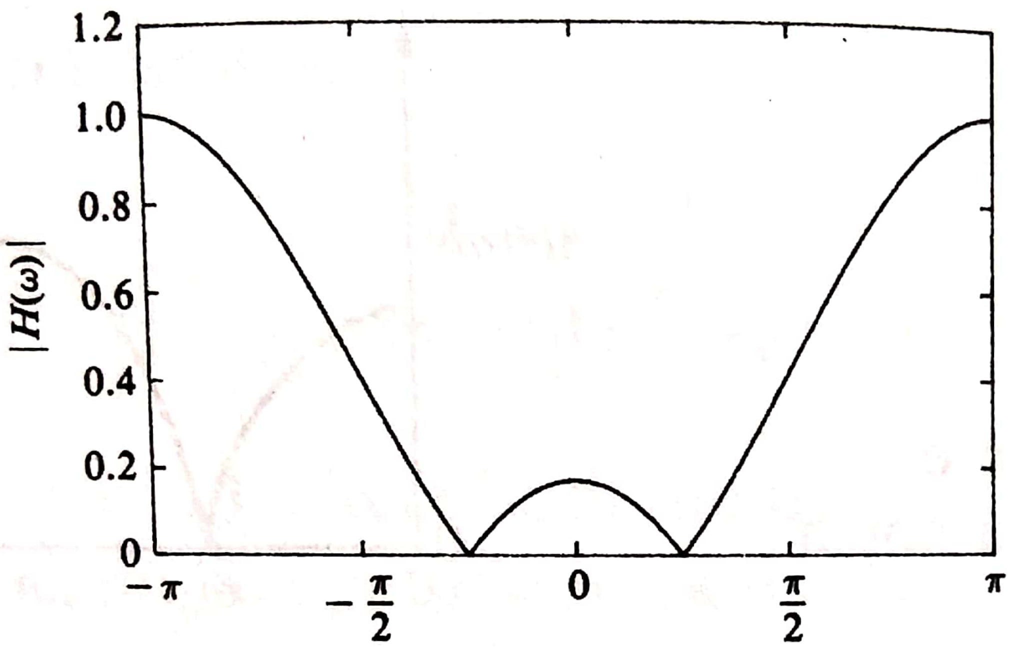
$$z_{1,2} = e^{\pm j\omega_0}$$

Thus the system function for an FIR notch filter is simply

$$\begin{aligned} H(z) &= b_0(1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1}) \\ &= b_0(1 - 2\cos\omega_0 z^{-1} + z^{-2}) \end{aligned} \tag{5.4.30}$$

As an illustration, Fig. 5.4.9 shows the magnitude response for a notch filter having a null at $\omega = \pi/4$.

The problem ...


1.2
1.0
0.8
0.6
0.4
0.2
0
$|H(\omega)|$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π