EXAMPLE 7.1.3

Compute the DFT of the four-point sequence

$$x(n) = (0 \quad 1 \quad 2 \quad 3)$$

Solution. The first step is to determine the matrix $\mathbf{W}_4$. By exploiting the periodicity property of $\mathbf{W}_4$ and the symmetry property

$$W_N^{k+N/2} = -W_N^k$$

the matrix $\mathbf{W}_4$ may be expressed as

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Then

$$\mathbf{X}_4 = \mathbf{W}_4\mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

The IDFT of $\mathbf{X}_4$ may be determined by conjugating the elements in $\mathbf{W}_4$ to obtain $\mathbf{W}_4^*$ and then applying the formula (7.1.28).

EXAMPLE 7.2.1

Perform the circular convolution of the following two sequences:

$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

Solution. Each sequence consists of four nonzero points. For the purposes of illustrating t[he] operations involved in circular convolution, it is desirable to graph each sequence as points [on] a circle. Thus the sequences $x_1(n)$ and $x_2(n)$ are graphed as illustrated in Fig. 7.2.2(a). We no[te] that the sequences are graphed in a counterclockwise direction on a circle. This establish[es] the reference direction in rotating one of the sequences relative to the other.

Now, $x_3(m)$ is obtained by circularly convolving $x_1(n)$ with $x_2(n)$ as specified by (7.2.3[9]). Beginning with $m = 0$ we have

$$x_3(0) = \sum_{n=0}^{3} x_1(n)x_2((-n))N$$

$x_2((-n))_4$ is simply the sequence $x_2(n)$ folded and graphed on a circle as illustrated in Fig. 7.2.2[(b)]. In other words, the folded sequence is simply $x_2(n)$ graphed in a clockwise direction.

The product sequence is obtained by multiplying $x_1(n)$ with $x_2((-n))_4$, point by poi[nt]. This sequence is also illustrated in Fig. 7.2.2(b). Finally, we sum the values in the produ[ct] sequence to obtain

$$x_3(0) = 14$$

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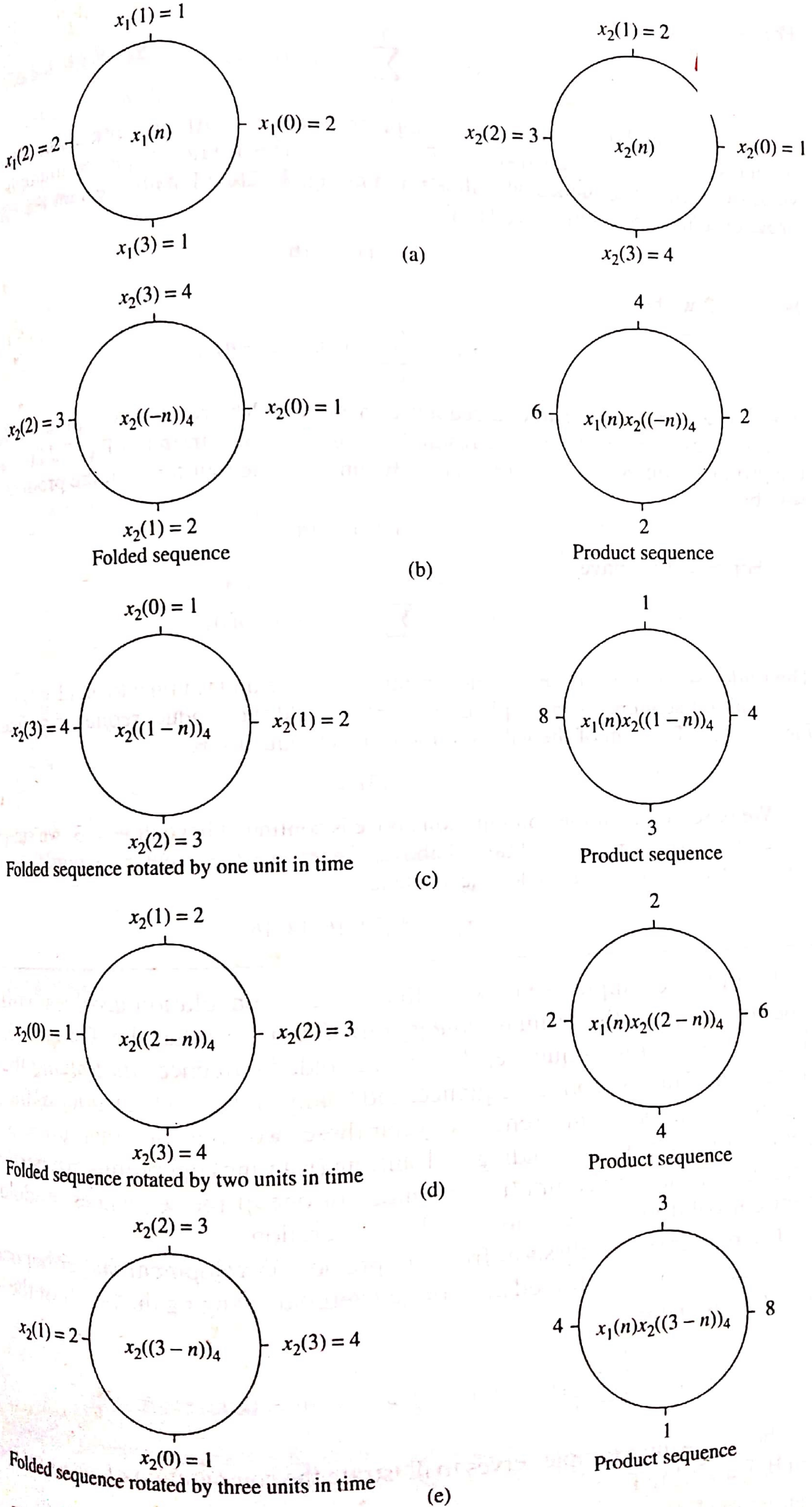

$x_1(1) = 1$
$x_1(2) = 2$
$x_1(n)$
$x_1(0) = 2$
$x_1(3) = 1$
$x_2(1) = 2$
$x_2(2) = 3$
$x_2(n)$
$x_2(0) = 1$
$x_2(3) = 4$
(a)
$x_2(3) = 4$
$x_2(2) = 3$
$x_2((-n))_4$
$x_2(0) = 1$
$x_2(1) = 2$
Folded sequence
4
6
$x_1(n)x_2((-n))_4$
2
2
Product sequence
(b)
$x_2(0) = 1$
$x_2(3) = 4$
$x_2((1-n))_4$
$x_2(1) = 2$
$x_2(2) = 3$
Folded sequence rotated by one unit in time
1
8
$x_1(n)x_2((1-n))_4$
4
3
Product sequence
(c)
$x_2(1) = 2$
$x_2(0) = 1$
$x_2((2-n))_4$
$x_2(2) = 3$
$x_2(3) = 4$
Folded sequence rotated by two units in time
2
2
$x_1(n)x_2((2-n))_4$
6
4
Product sequence
(d)
$x_2(2) = 3$
$x_2(1) = 2$
$x_2((3-n))_4$
$x_2(3) = 4$
$x_2(0) = 1$
Folded sequence rotated by three units in time
3
4
$x_1(n)x_2((3-n))_4$
8
1
Product sequence
(e)

Figure

For $m = 1$ we have

$$x_3(1) = \sum_{n=0}^{3} x_1(n)x_2((1-n))_4$$

It is easily verified that $x_2((1-n))_4$ is simply the sequence $x_2((-n))_4$ rotated counterclockwise by one unit in time as illustrated in Fig. 7.2.2(c). This rotated sequence multiplies $x_1(n)$ to yield the product sequence, also illustrated in Fig. 7.2.2(c). Finally, we sum the values in the product sequence to obtain $x_3(1)$. Thus

$$x_3(1) = 16$$

For $m = 2$ we have

$$x_3(2) = \sum_{n=0}^{3} x_1(n)x_2((2-n))_4$$

Now $x_2((2-n))_4$ is the folded sequence in Fig. 7.2.2(b) rotated two units of time in the counterclockwise direction. The resultant sequence is illustrated in Fig. 7.2.2(d) along with the product sequence $x_1(n)x_2((2-n))_4$. By summing the four terms in the product sequence, we obtain

$$x_3(2) = 14$$

For $m = 3$ we have

$$x_3(3) = \sum_{n=0}^{3} x_1(n)x_2((3-n))_4$$

The folded sequence $x_2((-n))_4$ is now rotated by three units in time to yield $x_2((3-n))_4$ and the resultant sequence is multiplied by $x_1(n)$ to yield the product sequence as illustrated in Fig. 7.2.2(e). The sum of the values in the product sequence is

$$x_3(3) = 16$$

We observe that if the computation above is continued beyond $m = 3$, we simply repeat the sequence of four values obtained above. Therefore, the circular convolution of the two sequences $x_1(n)$ and $x_2(n)$ yields the sequence

$$x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$$

From this example, we observe that circular convolution involves basically the same four steps as the ordinary *linear convolution* introduced in Chapter 2: *folding* (time reversing) one sequence, *shifting* the folded sequence, *multiplying* the two sequences to obtain a product sequence, and finally, *summing* the values of the product sequence. The basic difference between these two types of convolution is that, in circular convolution, the folding and shifting (rotating) operations are performed in a circular fashion by computing the index of one of the sequences modulo N. In linear convolution, there is no modulo N operation.

EXAMPLE 7.2.2

By means of the DFT and IDFT, determine the sequence $x_3(n)$ corresponding to the circular convolution of the sequences $x_1(n)$ and $x_2(n)$ given in Example 7.2.1.

Solution. First we compute the DFTs of $x_1(n)$ and $x_2(n)$. The four-point DFT of $x_1(n)$ is

$$X_1(k) = \sum_{n=0}^{3} x_1(n)e^{-j2\pi nk/4}, \qquad k = 0, 1, 2, 3$$

$$= 2 + e^{-j\pi k/2} + 2e^{-j\pi k} + e^{-j3\pi k/2}$$

Thus

$$X_1(0) = 6, \qquad X_1(1) = 0, \qquad X_1(2) = 2, \qquad X_1(3) = 0$$

The DFT of $x_2(n)$ is

$$X_2(k) = \sum_{n=0}^{3} x_2(n)e^{-j2\pi nk/4}, \qquad k = 0, 1, 2, 3$$

$$= 1 + 2e^{-j\pi k/2} + 3e^{-j\pi k} + 4e^{-j3\pi k/2}$$

Thus

$$X_2(0) = 10, \qquad X_2(1) = -2 + j2, \qquad X_2(2) = -2, \qquad X_2(3) = -2 - j2$$

When we multiply the two DFTs, we obtain the product

$$X_3(k) = X_1(k)X_2(k)$$

or, equivalently,

$$X_3(0) = 60, \qquad X_3(1) = 0, \qquad X_3(2) = -4, \qquad X_3(3) = 0$$

Now, the IDFT of $X_3(k)$ is

$$x_3(n) = \sum_{k=0}^{3} X_3(k)e^{j2\pi nk/4}, \ \frac{1}{4} \qquad n = 0, 1, 2, 3$$

$$= \frac{1}{4}(60 - 4e^{j\pi n})$$

Thus

$$x_3(0) = 14, \qquad x_3(1) = 16, \qquad x_3(2) = 14, \qquad x_3(3) = 16$$

which is the result obtained in Example 7.2.1 from circular convolution.

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Proof We can write $\tilde{r}_{xy}(l)$ as the circular convolution of $x(n)$ with $y^*(-n)$, that is,

$$\tilde{r}_{xy}(l) = x(l) \circledN y^*(-l)$$

Then, with the aid of the properties in (7.2.41) and (7.2.46), the N-point DFT of $\tilde{r}_{xy}(l)$ is

$$\tilde{R}_{xy}(k) = X(k)Y^*(k)$$

In the special case where $y(n) = x(n)$, we have the corresponding expression for the circular autocorrelation of $x(n)$,

$$\tilde{r}_{xx}(l) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \tilde{R}_{xx}(k) = |X(k)|^2 \tag{7.2.48}$$

Multiplication of two sequences. If

$$x_1(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X_2(k)$$

then

$$x_1(n)x_2(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} \frac{1}{N} X_1(k) \circledN X_2(k) \tag{7.2.49}$$

This property is the dual of (7.2.41). Its proof follows simply by interchanging the roles of time and frequency in the expression for the circular convolution of two sequences.

Parseval's Theorem. For complex-valued sequences $x(n)$ and $y(n)$, in general, if

$$x(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X(k)$$

and

$$y(n) \overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} Y(k)$$

then

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)Y^*(k) \tag{7.2.50}$$

Proof The property follows immediately from the circular correlation property in (7.2.47). We have

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \tilde{r}_{xy}(0)$$

and

$$\tilde{r}_{xy}(l) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{R}_{xy}(k) e^{j2\pi kl/N}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) Y^*(k) e^{j2\pi kl/N}$$

Hence (7.2.50) follows by evaluating the IDFT at $l = 0$.

The expression in (7.2.50) is the general form of Parseval's theorem. In the special case where $y(n) = x(n)$, (7.2.50) reduces to

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 \tag{7.2.51}$$

which expresses the energy in the finite-duration sequence $x(n)$ in terms of the frequency components $\{X(k)\}$.

The properties of the DFT given above are summarized in Table 7.2.