Subject : Digital Communication

Code : UEC 639

Credit : 4

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Modulation

- Motivation
- Many modulation formats
  - PSK, FSK, ASK, QAM
- Important criteria
  - Spectral efficiency
  - BER vs. SNR
  - Complexity (Cost)

Demodulation / Detection

- Two types
  - Coherent detection
    - phase of the carrier known to the receiver
    - optimal receptor: correlator
    - Gaussian noise
  - Noncoherent detection
    - carrier phase unknown at receiver
    - optimal receiver : power detection (envelope detection)
    - non-Gaussian noise

Carrier

- Base band
  - no carrier
  - positive frequencies bandwidth
- Pass band
  - with carrier
  - bandwidth twice the base band

Modulation

- Reasons for carrier modulation
  - Antennas, FDMA, etc.
- Signal space analysis
  - Valid for carrier modulation
  - Basis vectors take into account modulation
  - Often $\psi_1 = \cos(\omega_0 t)$ $\psi_2 = \sin(\omega_0 t)$
    - Parties **in-phase** et **quadrature**

Signal space

- Binary case
  - Signal space - with good normalization
  - Distance between the two bits
  - Probability of error
- M-ary Case
  - Signal space
  - Distance between symbols

NOTE: **Derive the amplitude value of a transmitted signal in terms of its bit Energy.**

$$x(t) = A \cos 2\pi f_c t, \quad 0 \le t \le T_b$$

$E_b \leftarrow$ Energy per bit

$$E_b = \int_0^{T_b} x^2(t)\, dt$$

$$= \int_0^{T_b} A^2 \cos^2 2\pi f_c t \; dt$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$= \frac{A^2}{2}\int_0^{T_b} (1+\cos 4\pi f_c t)\, dt$$

$$= \frac{A^2}{2}\int_0^{T_b} (1+\cos 4\pi f_c t)\, dt$$

$$= \frac{A^2}{2} T_b + 0$$

$$E_b = \frac{A^2 T_b}{2}$$

$$\boxed{\sqrt{\frac{2E_b}{T_b}} = A}$$

Therefore,

$$x(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$$

PSK

2 dimensional space no matter how many symbols you have

- Phase shift keying $s(t)=\sqrt{\frac{2E}{T}}\cos\left(\omega_0 t+\phi_i(t)\right) \quad 0\le t\le T$

data

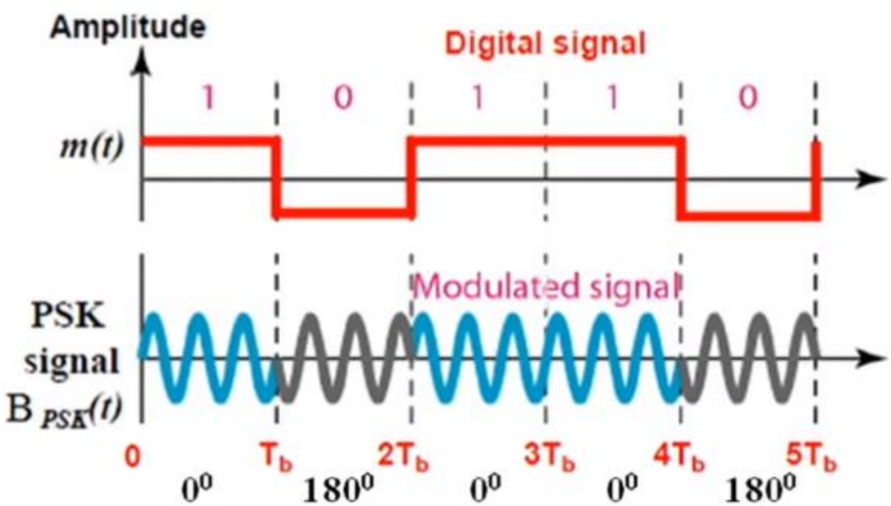
- Typically $\phi_i=\frac{2\pi(i-1)}{M} \quad i=1,2,\cdots,M$
- E energy/symbol, T symbol time
- Typically $\psi_1=\cos(\omega_0 t) \quad \psi_2=\sin(\omega_0 t)$
- Example: Antipodal (BPSK)

- In M-ary Phase Shift Keying (M-ary PSK), *M* different phase angles are used to represent M symbols.
- The signal is represented by:

  $$s_i(t) = A\cos(2\pi f_c t + \phi_i),\ 0 \le t \le T_s \ \text{ for } i = 1,2,\ldots,M$$

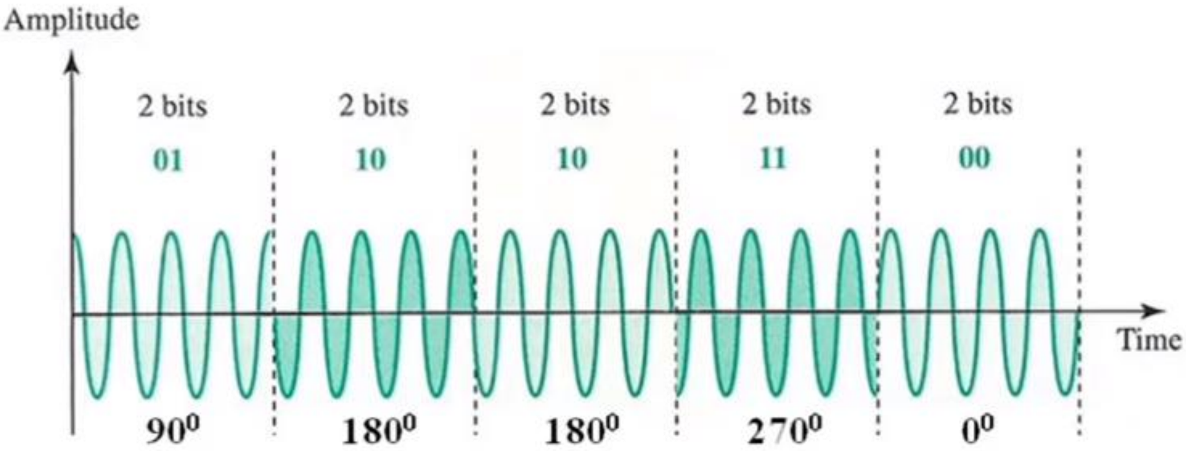
  where $\phi_i = \frac{2\pi}{M}(i-1) + \text{constant}$, for i= 1,2,…M
- $M = 4$ and the constant = 0, then four phases are 0, π/2, π, 3π/2.

  M= 4 and the constant = π/4, then four phases are π/4 , 3π/4 , 5π/4 , 7π/4
- Both 4-PSK are called Quadrature Phase Shift Keying or QPSK



BPSK : 2 Level

1 bit in 1 symbol duration



M-PSK : 4 Level

2 bit in 1 symbol duration

Fig. 6: BPSK and Mary-PSK Signal Representation

- Bandwidth of MPSK , $B = \frac{2}{Ts}$ Hz
- Bandwidth Efficiency , $\rho = \frac{log_2 M}{2}$ Bits/ s /Hz

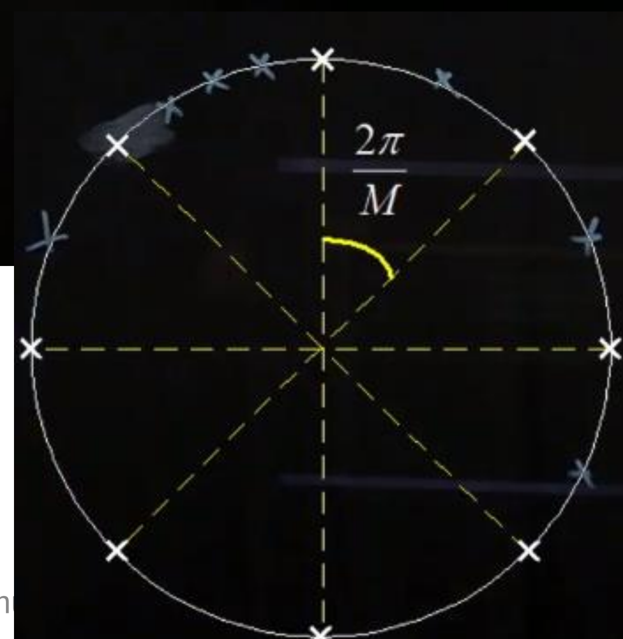
As M increases, Bandwidth Efficiency of Mary PSK increases

- Uses constant envelope so immune to noise.
- There is considerable reduction in bandwidth requirement.
- Better performance than ASK and FSK.
- Increase in probability of errors with increase in number of bits per symbol.
- Design of M-ary PSK modulator and demodulator is complex.

M-PSK

- Same distance from origin for the entire constellation

- Impact of larger M???



FSK

- Frequency shift keying $s_i(t) = \sqrt{\frac{2E}{T}} \cos(\omega_i t + \theta) \quad 0 \le t \le T$
- ω_i are M distinct frequencies

  E energy/symbol, T symbol time
- Typically $\left\{ \psi_i = \cos(\omega_i t) \right\}_{i=1}^{M}$ $\theta = 0$
- Typically M is a power of 2

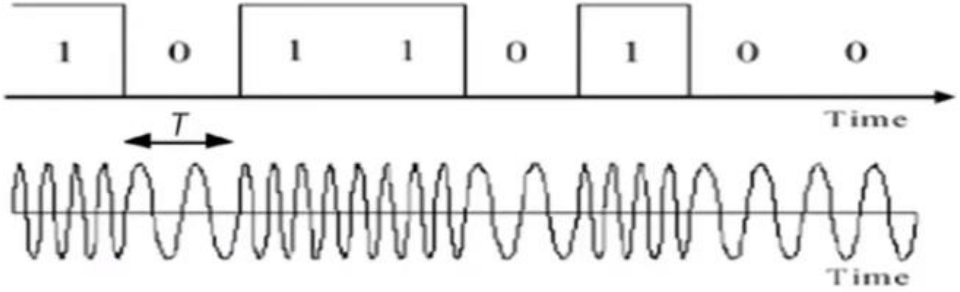
- In M-ary Frequency Shift Keying (M-ary FSK), there are *M* different carrier frequencies are used to represent M symbols.
- The signal is represented by:

$$s_i(t) = A\cos(2\pi f_{ci} t),\ \ 0 \le t \le T_s \ \ \text{for } i = 1,2,\ldots,M$$

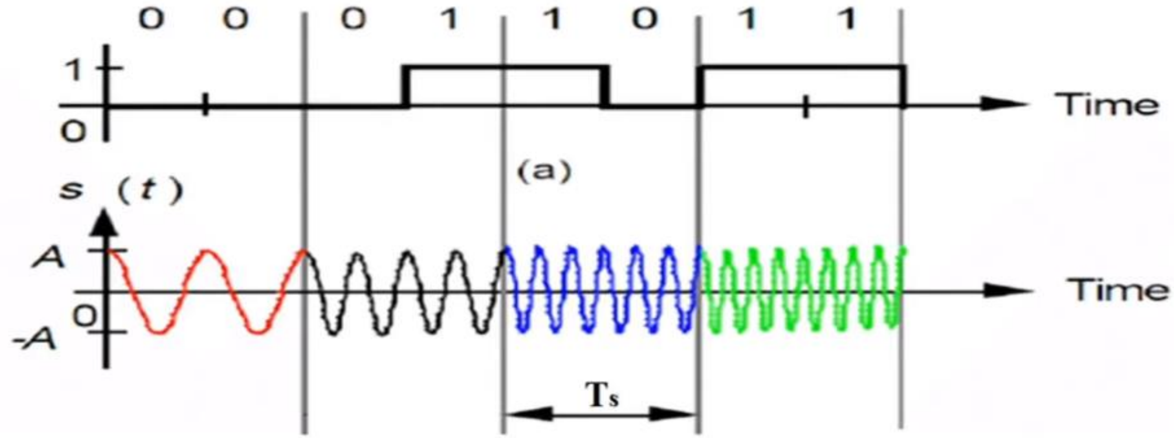
where M different frequencies are $f_{ci} = f_c + \left(i - \frac{M}{2}\right)\Delta f$, for i= 1,2,…M

Δf is difference between two adjacent frequencies.

- Frequencies are chosen such that they are orthogonal to each other and there is no interference between adjacent carriers.



BFSK : 2 Level
1 bit in 1 symbol duration



M-FSK : 4 Level
2 bit in 1 symbol duration

Fig. 4: BFSK and Mary-FSK Signal Representation

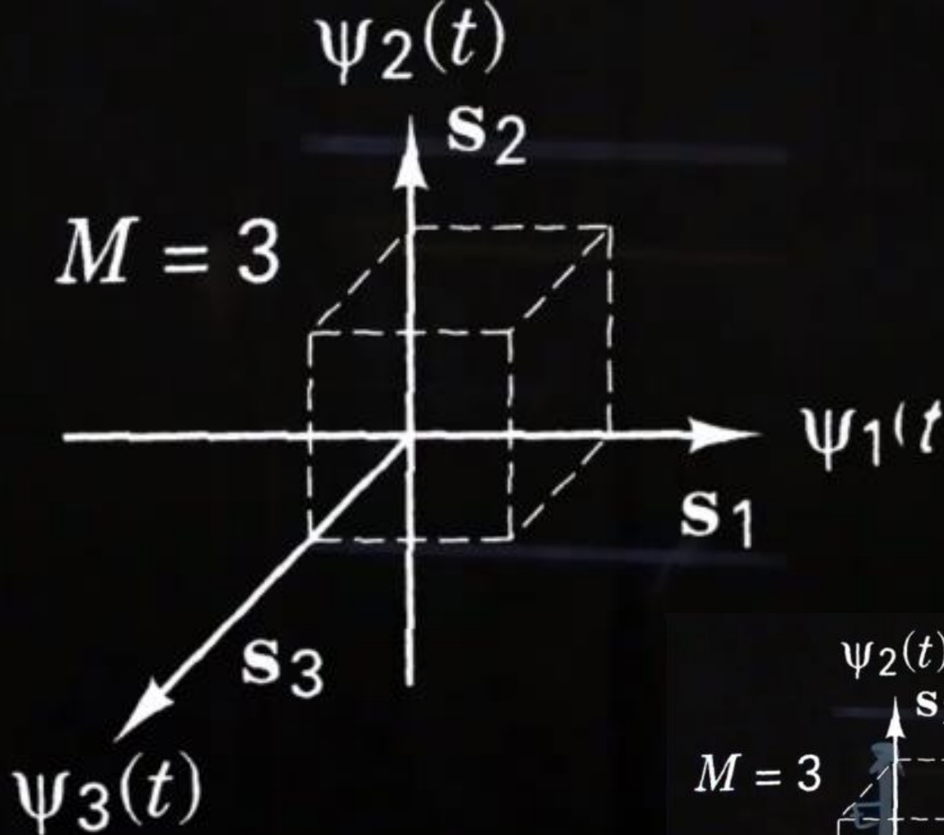
- Bandwidth of MFSK , $B = \frac{M}{2Ts}$ Hz

- Bandwidth Efficiency , $\rho = \frac{2log_2 M}{M}$ Bits/ s /Hz

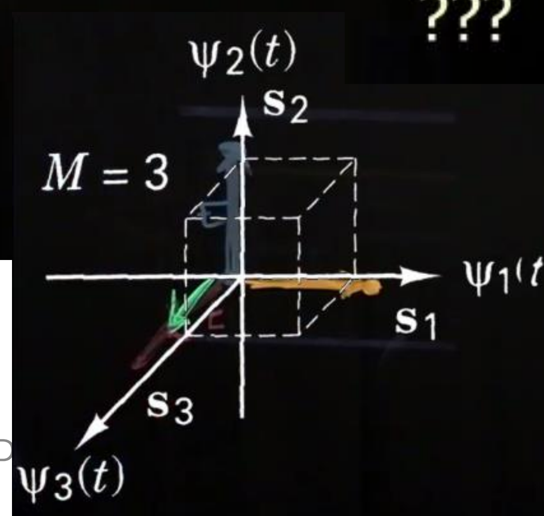
As M increases, Bandwidth Efficiency of MFSK decreases

- Better noise immunity than Mary-ASK.
- The transmitted M number of signals are equal in energy and duration.
- The signals are separated such that they are orthogonal to each other and therefore there is no crowding in the signal space.
- The bandwidth efficiency of M-ary FSK decreases and the power efficiency increases with the increase in M.

3-FSK



- Number of frequencies M
- Dimension of space is M
- Impact of larger M ???



ASK = PAM

- Amplitude shift keying

$$s_i(t) = \sqrt{\frac{2E_i}{T}} \cos \omega_c t \qquad i = 1, 2, \cdots, M$$

- Typically evenly spaced discrete levels
- Example: OOK
- Not high performance, so less popular

- The signal is represented by:

  $s_i(t) = A_i \cos(2\pi f_c t),\ 0 \le t \le T_s$ for $i = 1, 2, \ldots, M$

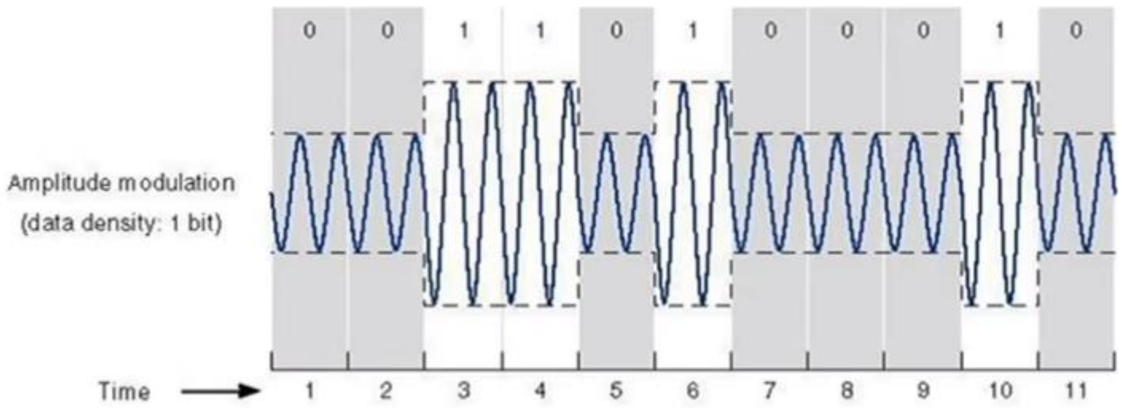
  $A_i = (2i - 1 - M)d$, where $2d$ is the difference between two consecutive signal amplitudes.

- Let $M = 4$ and $d = 1$, the four signal amplitudes will be $-3, -1, 1$ and 3 V.

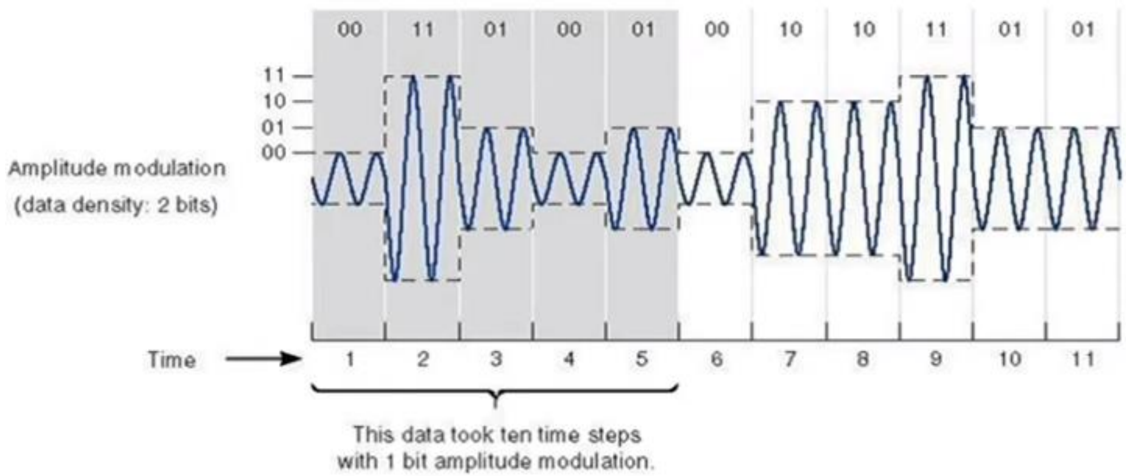
  The M-ASK signals will be: for $0 \le t \le T_s$

  $s_1(t) = \cos(2\pi f_c t)$, $\quad s_2(t) = -\cos(2\pi f_c t)$,

  $s_3(t) = 3\cos(2\pi f_c t)$, $\quad s_4(t) = -3\cos(2\pi f_c t)$.



BASK : 2 Level
1 bit in 1 symbol duration



M-ASK : 4 Level

2 bit in 1 symbol duration

Fig. 2: BASK and Mary ASK Signal Representation

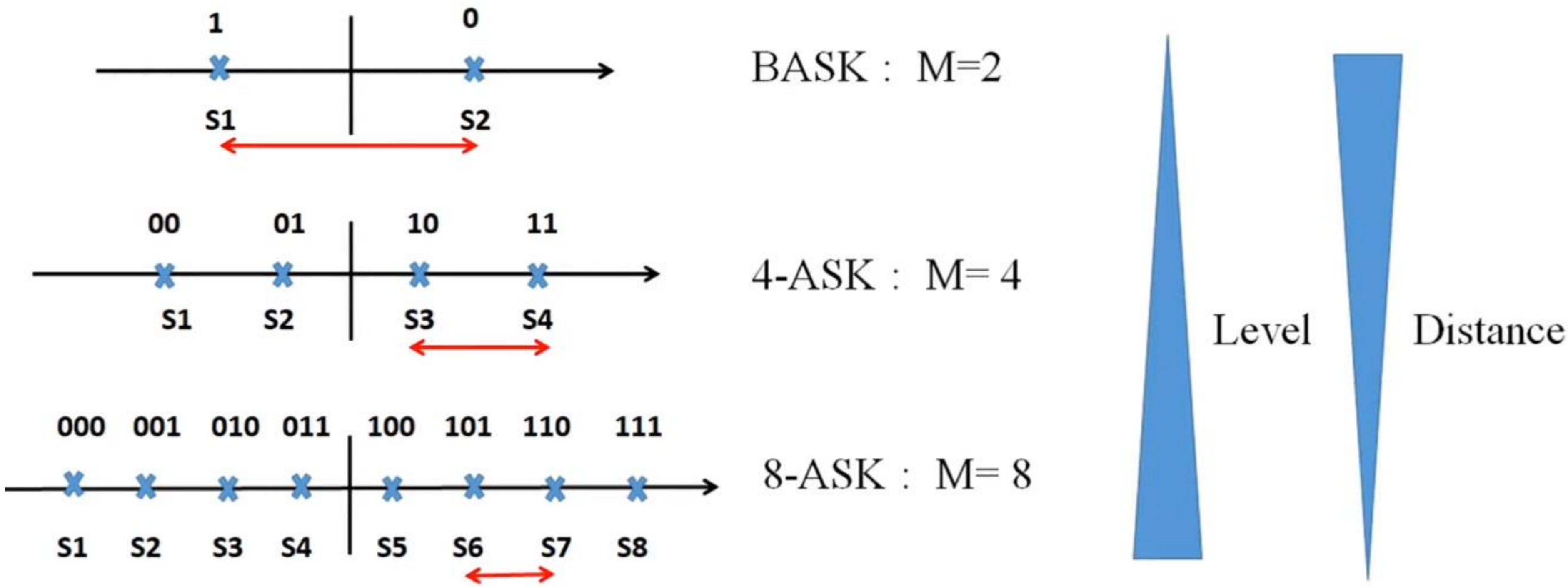


Fig. 3: Constellation Diagram of BASK and Mary-ASK Signal

APK or QAM

2 dimensional space no matter how many symbols you have

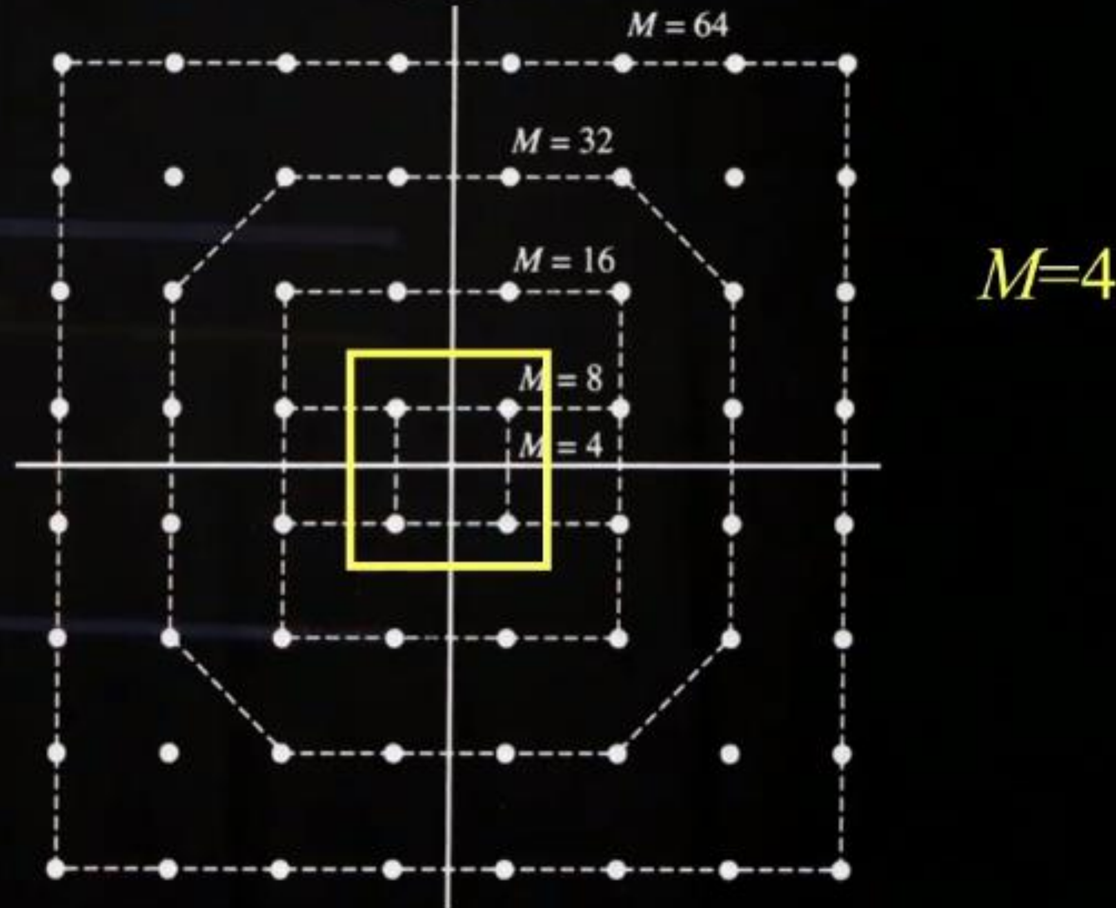
- Amplitude-phase modulation or Quadrature amplitude modulation

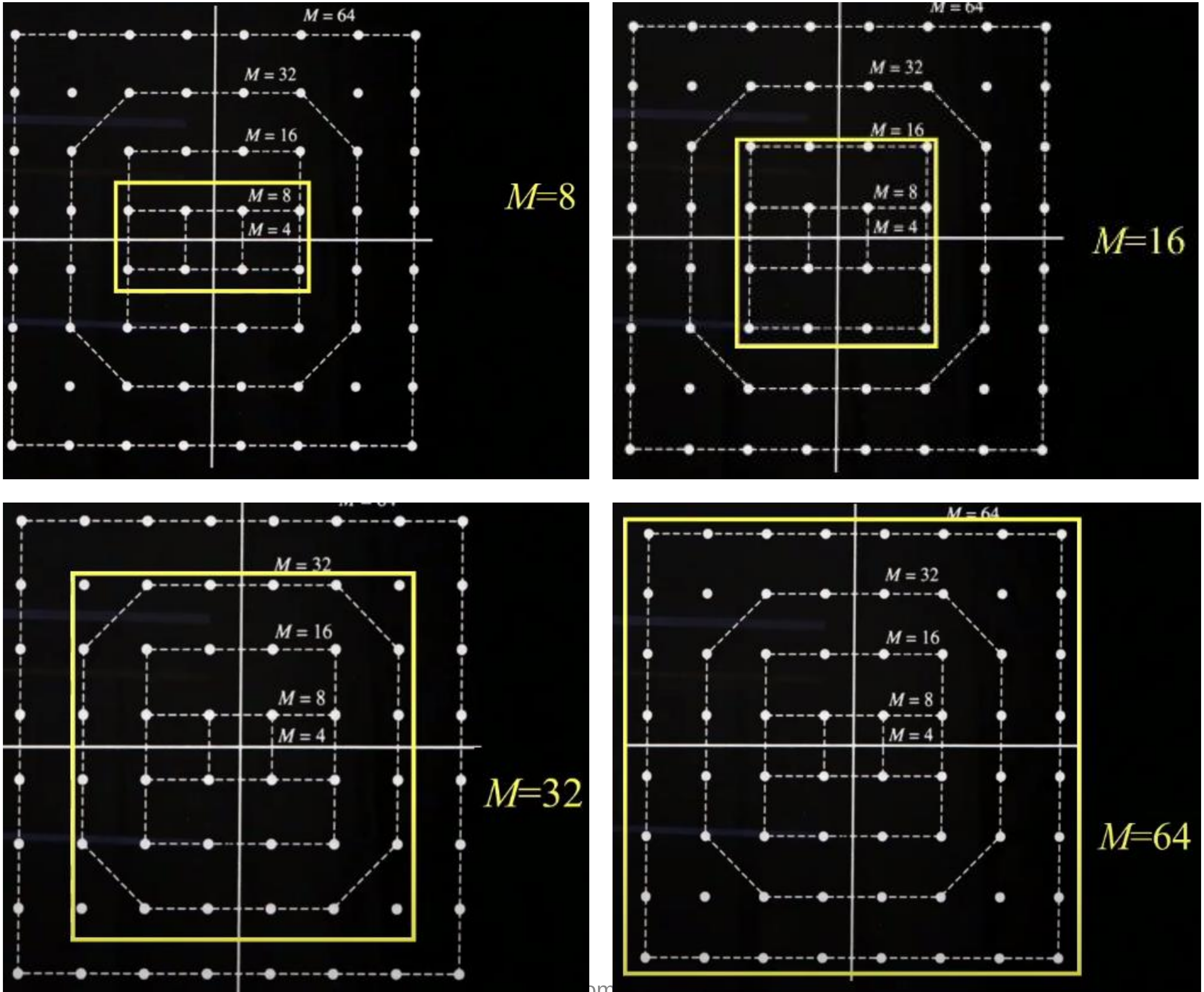
$$s_i(t) = \sqrt{\frac{2E_i}{T}} \cos\left(\omega_0 t + \phi_i(t)\right) \qquad i = 1, 2, \cdots, M$$

- Discrete levels
- Very BW efficient

unequal energies

QAM




M = 64
M = 32
M = 16
M = 8
M = 4
M=8
M=16
M=32
M=64

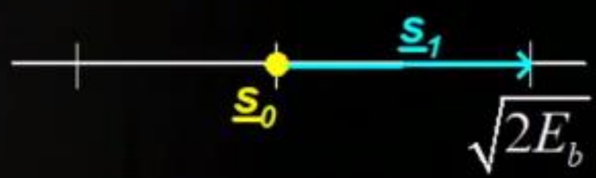
Probability of Error for Binary case

OOK vs. BPSK

- OOK: $s_1(t)$ NRZ, $s_0(t) = 0$

ASK

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$



- BPSK: $s_1(t)$ NRZ, $s_0(t) = -s_1(t)$

Antipodal

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

HOW ?

Noise... key for finding BER

- zero mean Gaussian

$$n_j \sim \left(0, \frac{N_0}{2}\right) \qquad \sigma^2 = \frac{N_0}{2}$$

- Independent coefficients

$$E n_j n_i = 0 \qquad i \neq j$$

- Probability density

$$p_{\underline{n}}\left(\underline{n}\right) = \prod_{j=1}^{\infty} p_{n_j}\left(n_j\right) = \prod_{j=1}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-n_j^2/2\sigma^2}$$

Binary case

- Case 1 Antipodal

BPSK

$$s_1(t) = \begin{cases} A & 0 \le t \le T \\ 0 & ailleurs \end{cases}$$

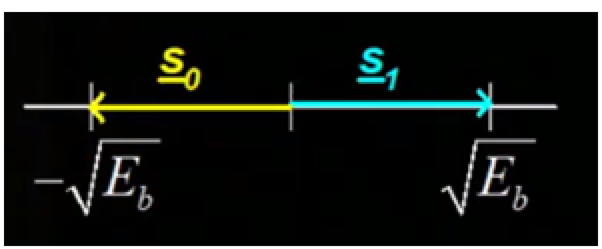
$$s_0(t) = -s_1(t)$$

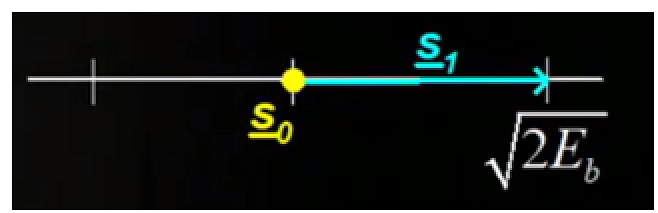
- Case 2 On-Off Keying

ASK

$$s_1(t) = \begin{cases} A & 0 \le t \le T \\ 0 & ailleurs \end{cases}$$

$$s_0(t) = 0$$





Gram-Schmidt Process

- Case 1 Antipodal

$$\psi_1(t) = \frac{1}{\sqrt{E_1}} s_1(t)$$

$$E_1 = \int_0^T s_1^2(t)dt = \int_0^T A^2 dt = A^2 T$$

$$\psi_1(t) = \frac{1}{A\sqrt{T}} s_1(t) = \begin{cases} 1/\sqrt{T} & 0 \le t \le T \\ 0 & ailleurs \end{cases}$$

- Case 2 On-Off Keying

$$\psi_1(t) = \frac{1}{\sqrt{E_1}} s_1(t)$$

$$E_1 = \int_0^T s_1^2(t)dt = \int_0^T A^2 dt = A^2 T$$

$$\psi_1(t) = \begin{cases} 1/\sqrt{T} & 0 \le t \le T \\ 0 & ailleurs \end{cases}$$

Signal coefficients

- Case 1 Antipodal

$$s_1(t) = \sqrt{E_1}\psi_1(t) = A\sqrt{T}\psi_1(t)$$
$$s_0(t) = -s_1(t) = -A\sqrt{T}\psi_1(t)$$
$$\underline{s}_1 = \left[A\sqrt{T}\right] \quad \underline{s}_0 = \left[-A\sqrt{T}\right]$$

- Case 2 On-Off Keying

$$s_1(t) = \sqrt{E_1}\psi_1(t) = A\sqrt{T}\psi_1(t)$$
$$s_0(t) = 0$$
$$\underline{s}_1 = \left[A\sqrt{T}\right] \quad \underline{s}_0 = [0]$$

Signal space is of unit dimension

One dimensional space

- Received signal $\underline{r} = [r]$

$$r = \langle r(t), \psi_1(t) \rangle = \frac{1}{A\sqrt{T}} \langle r(t), s_1(t) \rangle$$

$$= \frac{1}{A\sqrt{T}} \int_0^T r(t) s_1(t)\, dt$$

- Correlator version
  - Sampling receiver equivalent to matched filter

Signal space

- Case 1 Antipodal

$$\underline{s}_1 = \left[A\sqrt{T} \right] \quad \underline{s}_0 = \left[-A\sqrt{T} \right]$$

- Case 2 On-Off Keying

$$\underline{s}_1 = \left[A\sqrt{T} \right] \quad \underline{s}_0 = [0]$$

A fair comparison takes a normalization by the average energy of a bit

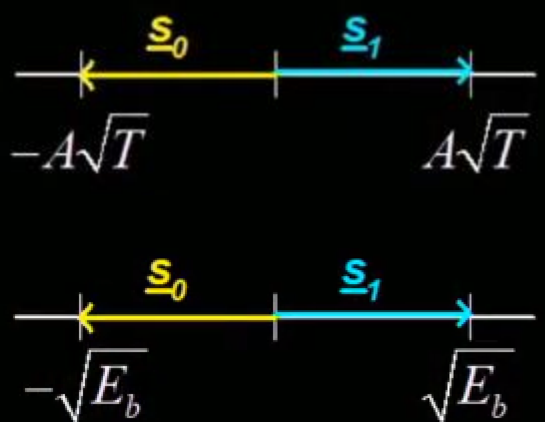
Normalization of E_b

- Case 1 Antipodal

$$E_1 = \int_0^T s_1^2(t)\,dt = \int_0^T A^2 dt = A^2 T$$

$$E_0 = \int_0^T s_0^2(t)\,dt = \int_0^T A^2 dt = A^2 T$$

$$E_b = \tfrac{1}{2}E_1 + \tfrac{1}{2}E_0 = A^2 T$$

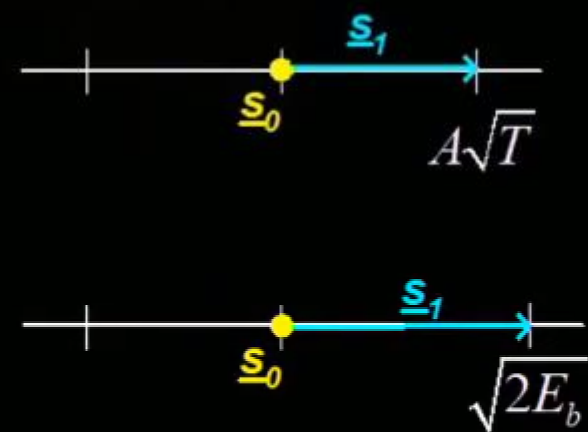


- Case 2 On-Off Keying

$$E_1 = \int_0^T s_1^2(t)\,dt = \int_0^T A^2 dt = A^2 T$$

$$E_0 = \int_0^T s_0^2(t)\,dt = 0$$

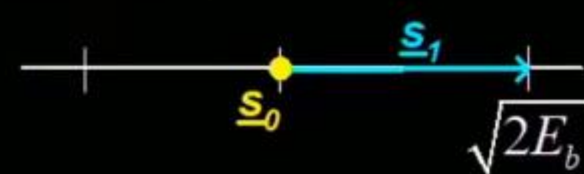
$$E_b = \tfrac{1}{2}E_1 + \tfrac{1}{2}0 = A^2 T/2$$



Distance as a function on E_b

- Case 1 Antipodal
  - Separation is $2\sqrt{E_b}$
  - Larger separation, so symbols easier to distinguish

- Case 2 On-Off Keying
  - Separation is $\sqrt{2E_b}$

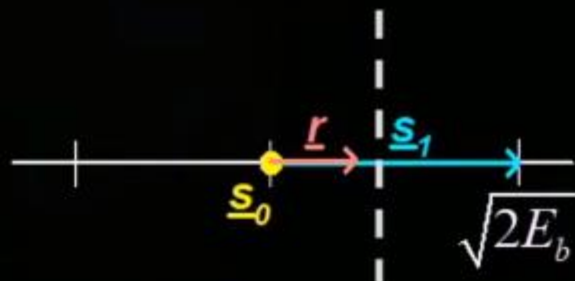


MAP Receiver

- Case 1 Antipodal
  - Choose the signal closest to received signal

$$r \ge 0 \;\Rightarrow\; \text{choose } s_1(t)$$
$$r \le 0 \;\Rightarrow\; \text{choose } s_0(t)$$

- Case 2 On-Off Keying
  - Choose the signal closest to received signal



$$r \ge \frac{\sqrt{E_b}}{\sqrt{2}} \;\Rightarrow\; \text{choose } s_1(t)$$
$$r \le \frac{\sqrt{E_b}}{\sqrt{2}} \;\Rightarrow\; \text{choose } s_0(t)$$

Calculate P_e

$r \geq 0 \Rightarrow$ choisir $s_1(t)$

$r \leq 0 \Rightarrow$ choisir $s_0(t)$

- Case 1 Antipodal

$$P_e = P(0 \text{ sent})P(r \geq 0 | 0 \text{ sent}) + P(1 \text{ sent})P(r \leq 0 | 1 \text{ sent})$$

Calculate P_e equal priors

- Case 1 Antipodal

$$P_e = P(0 \text{ sent})P(r \geq 0 | 0 \text{ sent}) + P(1 \text{ sent})P(r \leq 0 | 1 \text{ sent})$$

$$= \tfrac{1}{2}P\left(r \geq 0 | r = -\sqrt{E_b} + n\right) + \tfrac{1}{2}P\left(r \leq 0 | r = \sqrt{E_b} + n\right)$$

Calculate P_e

- Case 1 Antipodal

$$P_e = P(0\text{ sent})P(r \ge 0|0\text{ sent}) + P(1\text{ sent})P(r \le 0|1\text{ sent})$$

$$= \tfrac{1}{2}P\left(r \ge 0 \middle| r = -\sqrt{E_b} + n\right) + \tfrac{1}{2}P\left(r \le 0 \middle| r = \sqrt{E_b} + n\right)$$

$$= \tfrac{1}{2}P\left(-\sqrt{E_b} + n \ge 0\right) + \tfrac{1}{2}P\left(\sqrt{E_b} + n \le 0\right)$$

$$= \tfrac{1}{2}P\left(n \ge \sqrt{E_b}\right) + \tfrac{1}{2}P\left(n \le -\sqrt{E_b}\right)$$

$$= \tfrac{1}{2}P\left(n \ge \sqrt{E_b}\right) + \tfrac{1}{2}P\left(n \ge \sqrt{E_b}\right)$$

Sign changes

$$= P\left(n \ge \sqrt{E_b}\right) = \int_{\sqrt{E_b}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-n^2/2\sigma^2}\, dn$$

Calculate P_e

- Case 1 Antipodal

Quality factor

$$P_e = \int_{\sqrt{E_b}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-n^2/2\sigma^2} dn \qquad Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-x^2/2} dx$$

$$P_e = \int_{\sqrt{E_b}/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

Let, $u = \frac{n}{\sigma}$

$$= Q\left(\frac{\sqrt{E_b}}{\sigma}\right) = Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

SNR

$$\frac{E_b}{N_0} = \frac{average\ energy\ per\ bit}{PSD\ of\ Noise}$$

Calculate P_e

- Case 1 Antipodal
- Case 2 On–Off Keying

$$r \ge \frac{\sqrt{E_b}}{\sqrt{2}} \Rightarrow \text{choose } s_1(t) \qquad r \le \frac{\sqrt{E_b}}{\sqrt{2}} \Rightarrow \text{choose } s_0(t)$$

$$P_e = ½P\left(r \ge \sqrt{E_b/2} \middle| r = n\right) + ½P\left(r \le \sqrt{E_b/2} \middle| r = \sqrt{E_b} + n\right)$$

$$= P\left(n \ge \sqrt{E_b/2}\right) = \int_{\sqrt{E_b/2}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-n^2/2\sigma^2} du$$

$$= Q\left(\frac{\sqrt{E_b/2}}{\sigma}\right) = Q\left(\frac{\sqrt{E_b/2}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

OOK vs. BPSK

- **OOK:** $s_1(t)$ NRZ, $s_0(t) = 0$

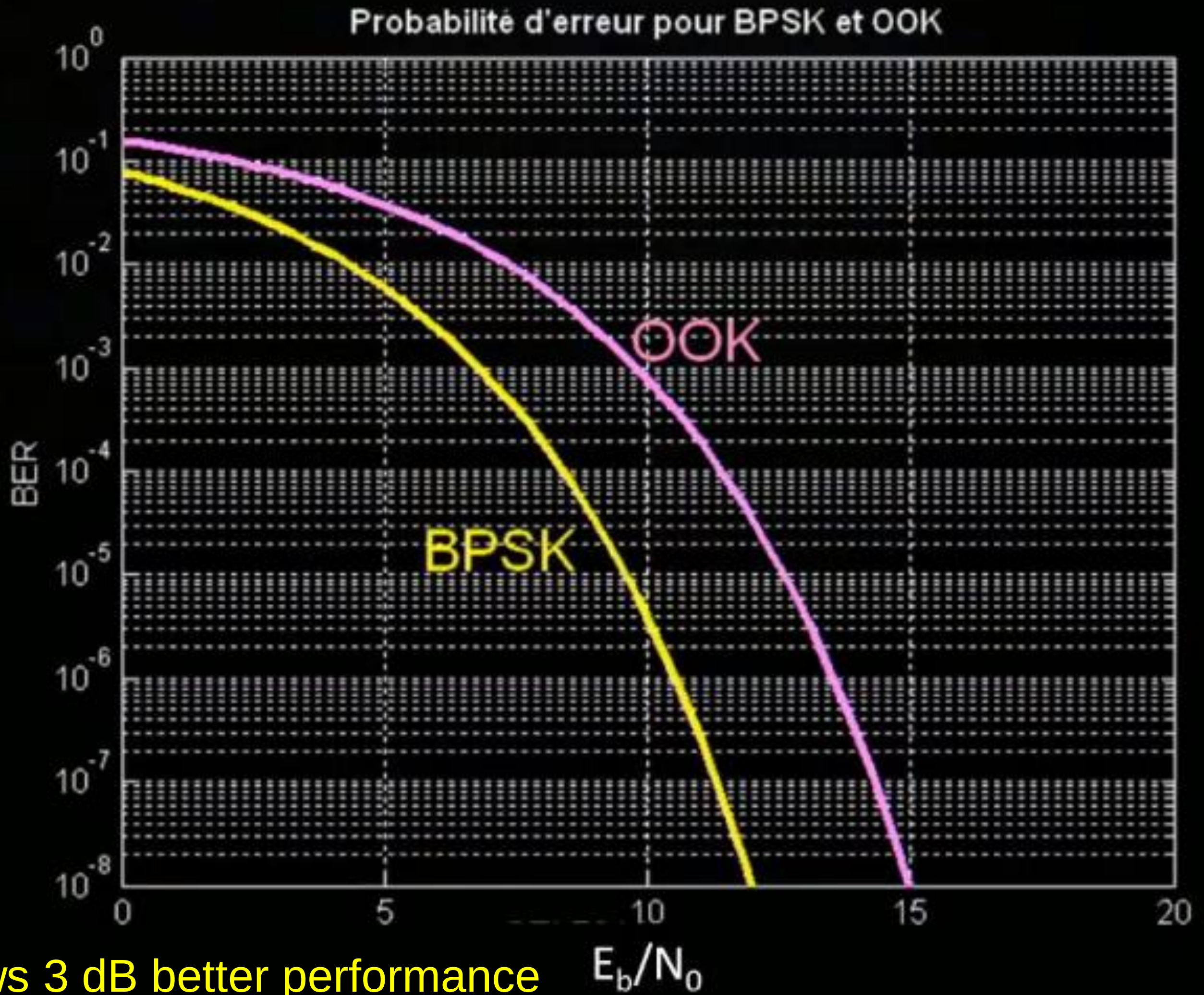
$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- **BPSK:** $s_1(t)$ NRZ, $s_0(t) = -s_1(t)$

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Large the SNR, lesser the value of Q function, better the signal detection, and it will improve be the performance of the system.

Probability of error



BPSK shows 3 dB better performance

Challenges for M-ary cases

- Choosing the symbol
  - MAP most probable (*a posteriori*), and
    ML most likely
  - modulation determines the form of the receiver
  - MAP/ML determines the receiver decision algorithm
- Analyzing the performance
  - exploit the results for the binary case
  - exploit the geometric interpretation of the receiver

Receivers

- Maximizes *a posteriori* probability (with *a priori* information)
  - i that minimizes $\left\| \mathbf{r} - \mathbf{s}_i \right\|^2 - N_0 \ln P(\mathbf{s}_i)$
- Maximizes the likelihood ratio
  - i that minimizes $\left\| \mathbf{r} - \mathbf{s}_i \right\|^2$

Receivers

- MAP
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$$

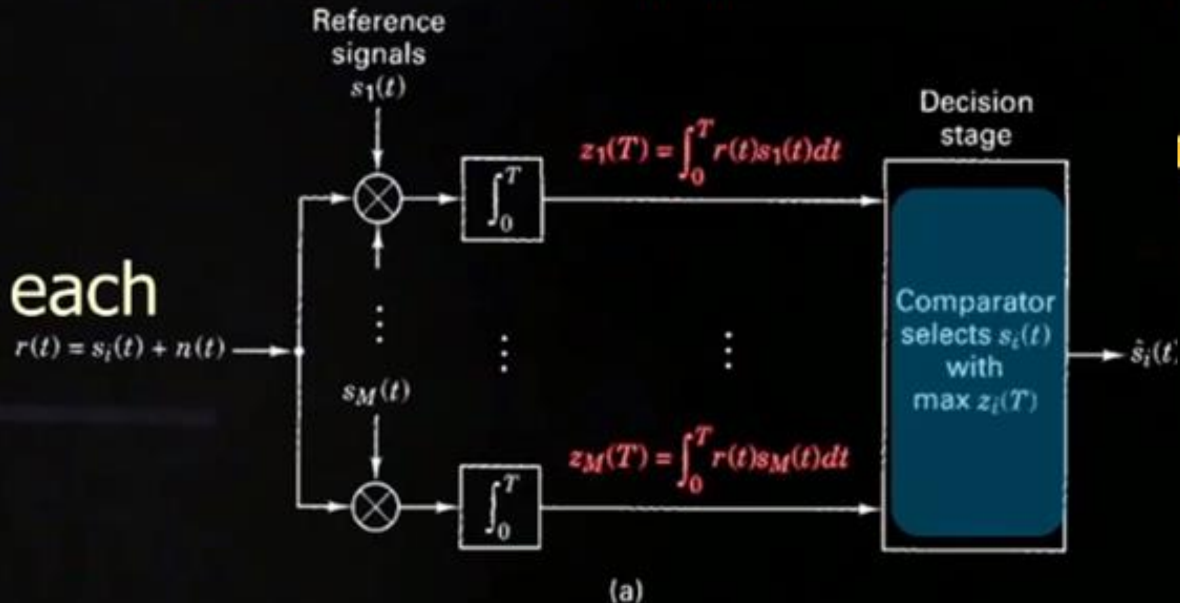
- ML
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2$$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$$

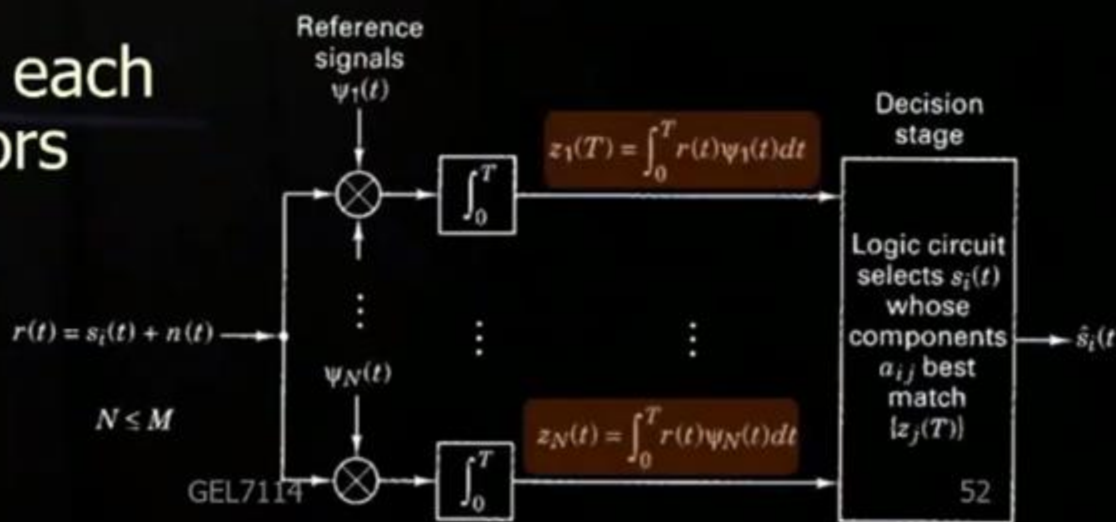
Receivers

- One branch for each of M symbols



- One branch for each of N basis vectors

  $N \leq M$



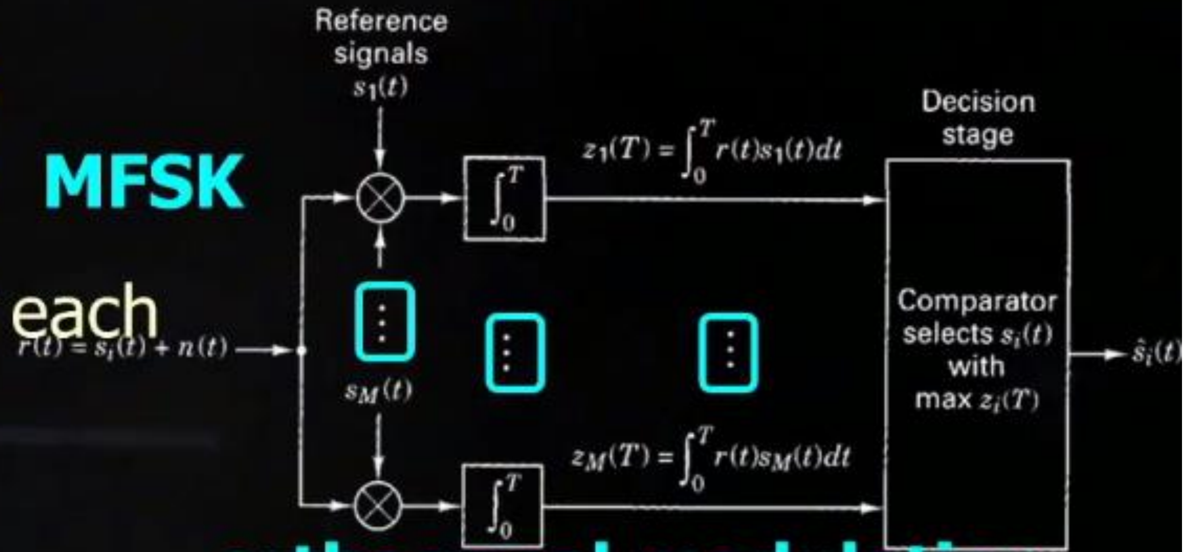
Receiver vs. modulation format

- MPSK
  - two branches: in-phase et quadrature
- MFSK
  - M branches: one for each symbol
  - basis vector = symbol
  - **orthogonal modulation**
- QAM
  - two branches: in-phase and quadrature

Receivers

MFSK

- One branch for each M symbols

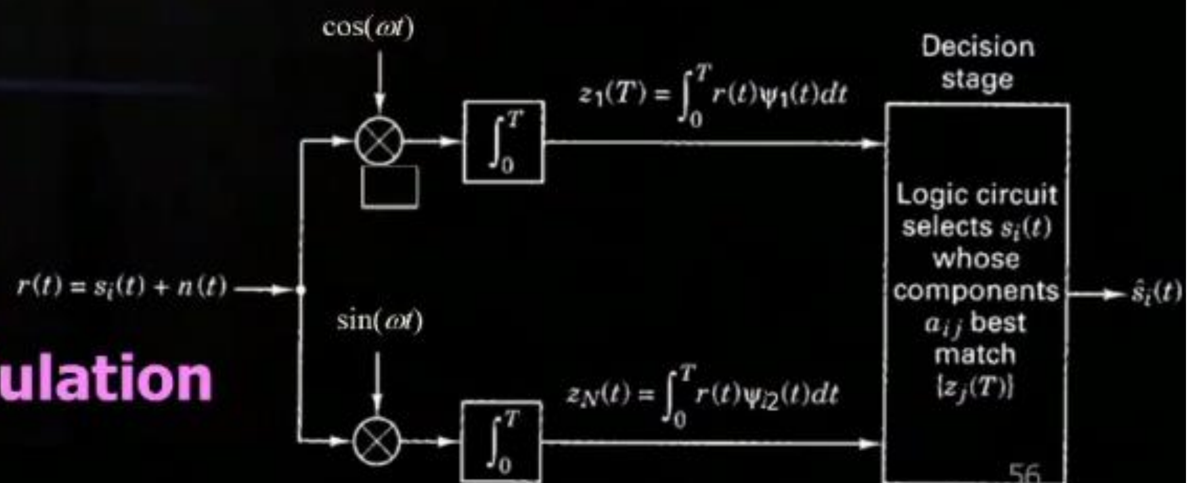


orthogonal modulation

- Two branches
- **QAM**

MPSK

Non orthogonal modulation



Decision algorithms

- Geometric interpretation
  - To reduce the complexity of the decision algorithm
  - To provide an intuition of how the receiver works
- i that minimizes

$$\left\| \mathbf{r} - \mathbf{s}_i \right\|^2$$

  - Equivalent to choosing the nearest symbol
  - Regions can be determined in advance to simplify the receiver

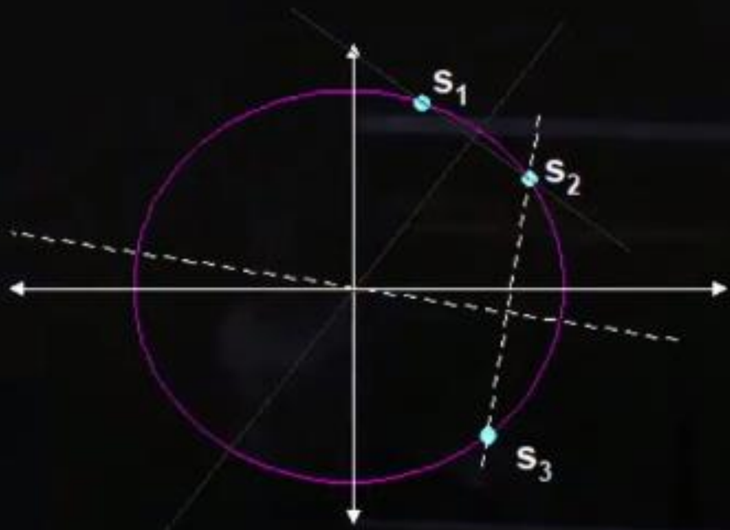
How to choose the nearest symbol directly from vector received?

Decision regions

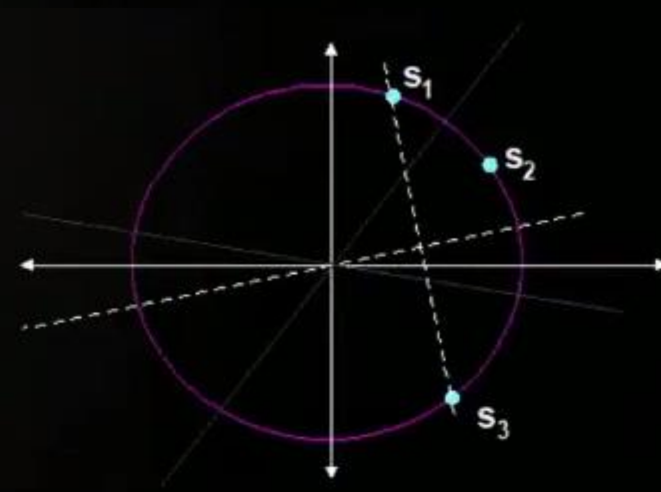
Constellation (equal energies)

Perpendicular bisector between $\mathbf{s}_1$ and $\mathbf{s}_2$

Decision regions

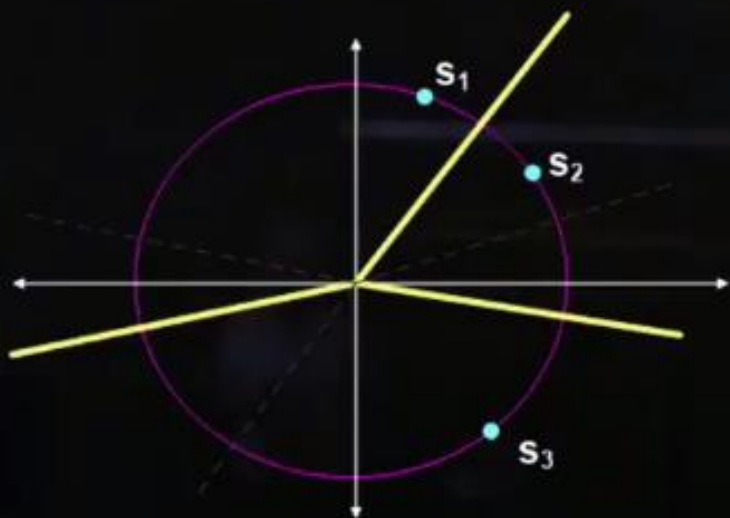


perpendicular bisector between $\mathbf{s}_3$ and $\mathbf{s}_2$

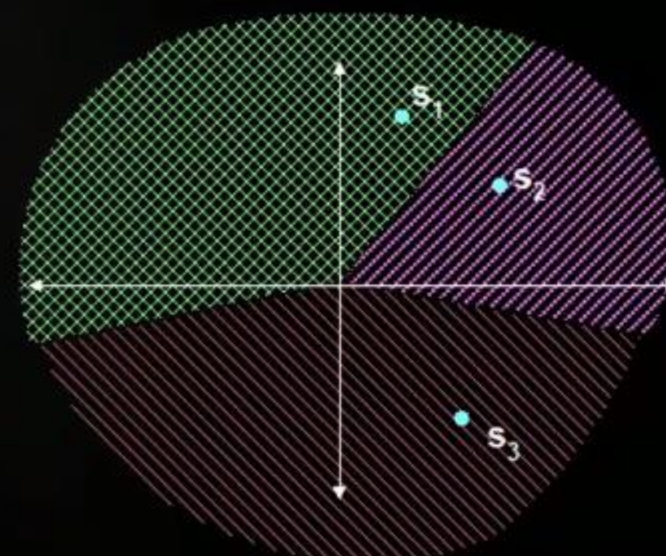


perpendicular bisector between $\mathbf{s}_1$ and $\mathbf{s}_3$

Decision regions



bissecteurs perpendiculaires



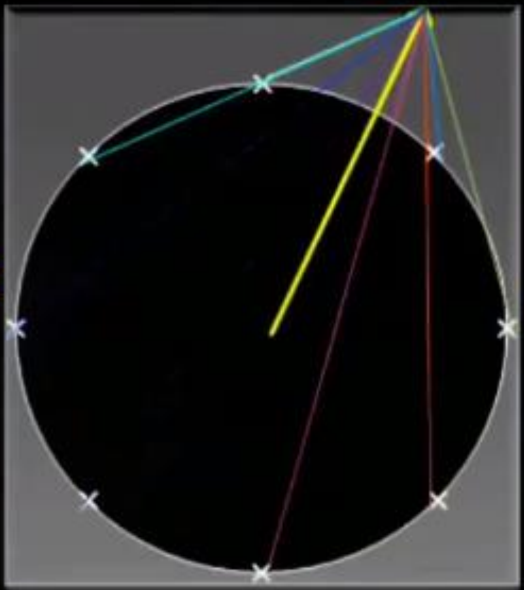
régions

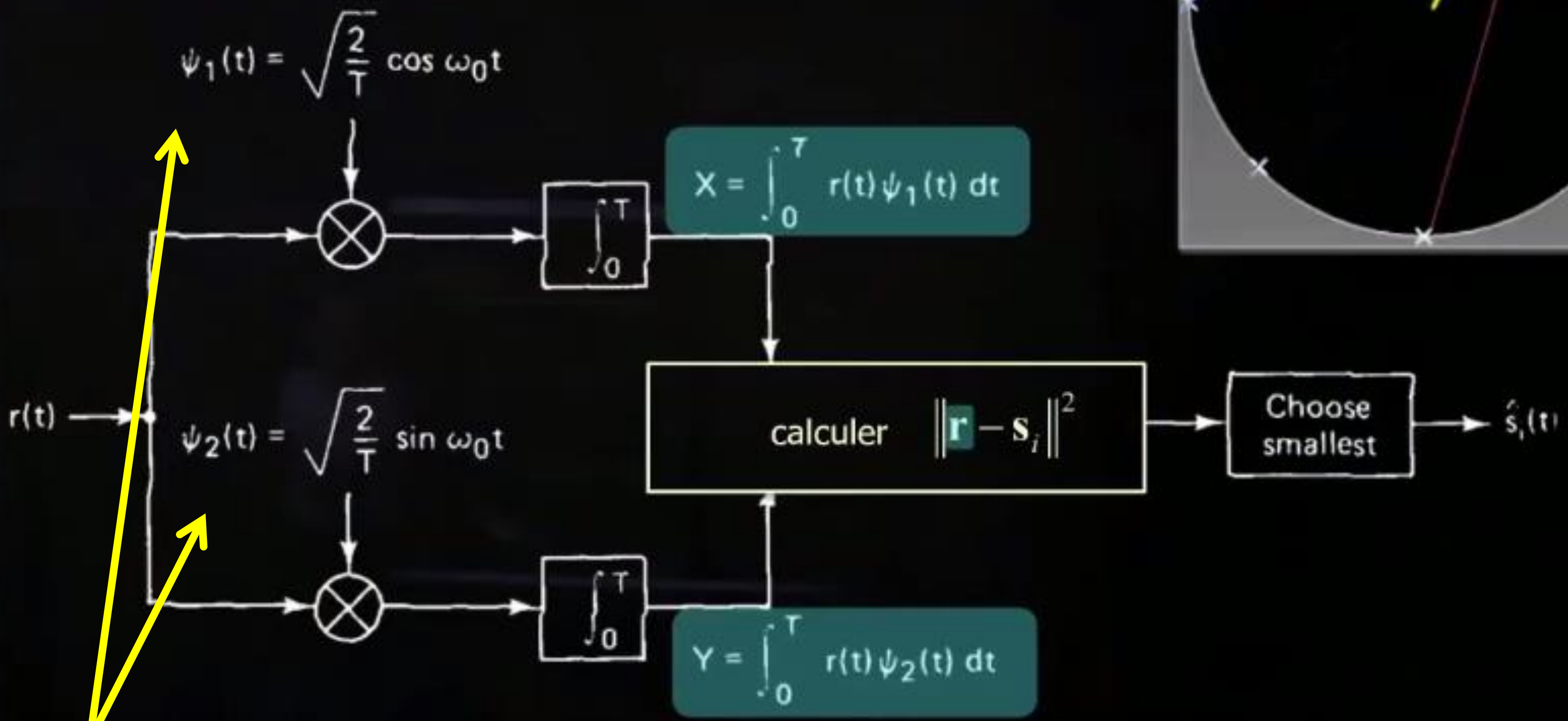
How to choose the nearest symbol directly from vector received? ⇒ **regions**

M-PSK

Example: M-ary PSK

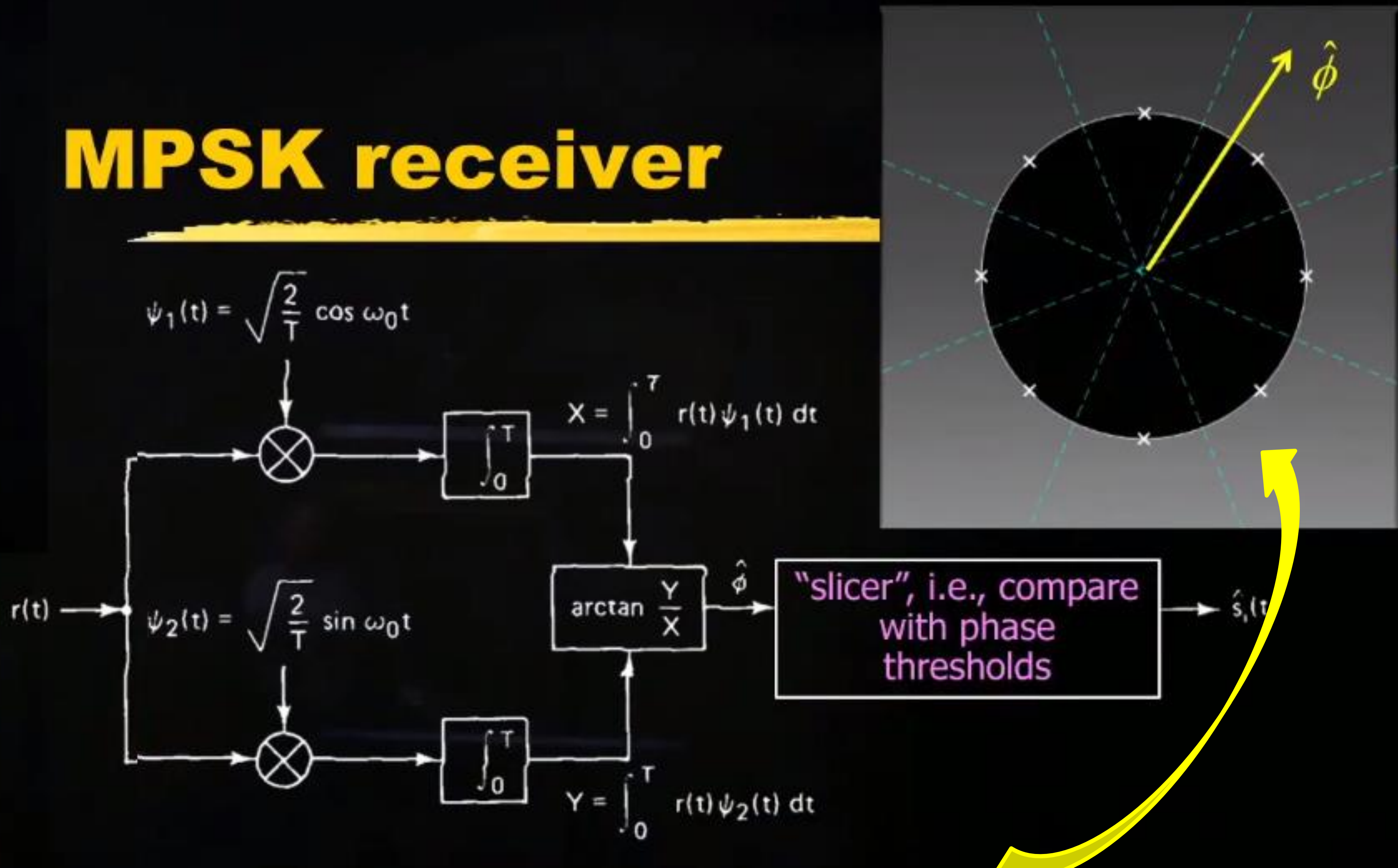
MPSK receiver





Complexity reduction: For M symbols, only two Correlator are sufficient due to basis vector.

MPSK receiver



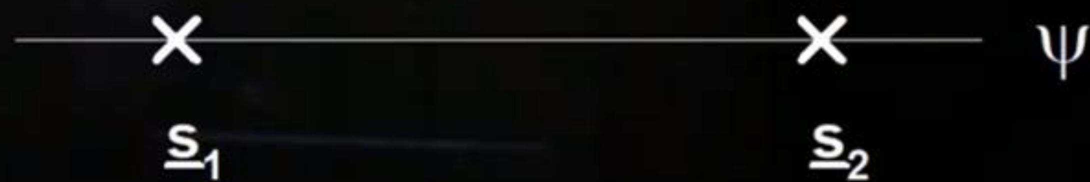
Complexity reduction: Now, Instead of calculating M distance vector corresponding to M symbols, we need to use only single phase of the received vector for detection of region it falls in.

Challenges for M-ary cases

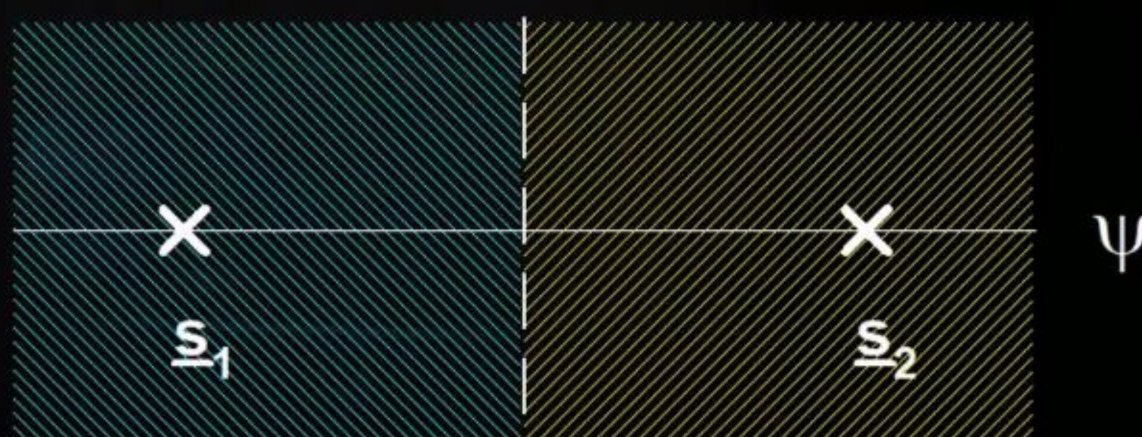
- Choosing the symbol
  - MAP most probable (*a posteriori*), and
    ML most likely
  - determines the form of the receiver
- Analyzing the performance
  - exploit the results for the binary case
  - exploit the geometric interpretation of the receiver

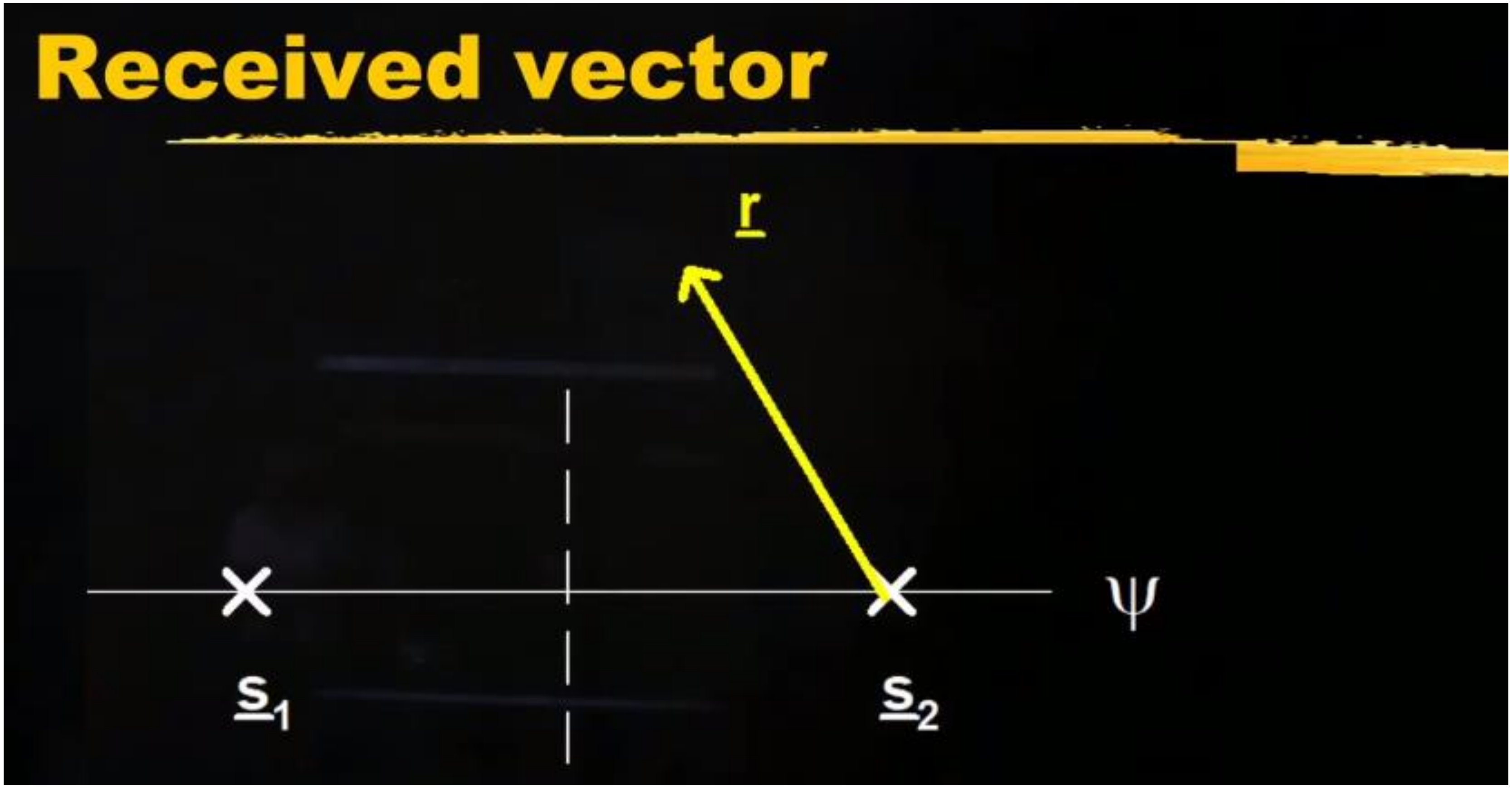
P_e from the signal space

- What can we learn from the binary case?
  - Signal space: two points



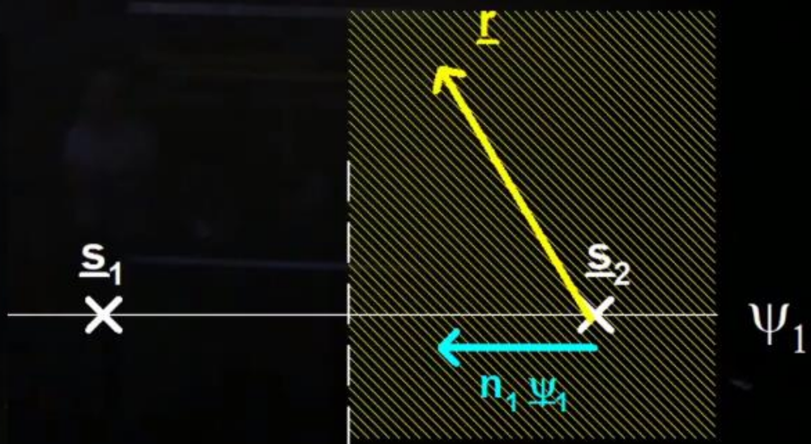
Decision regions



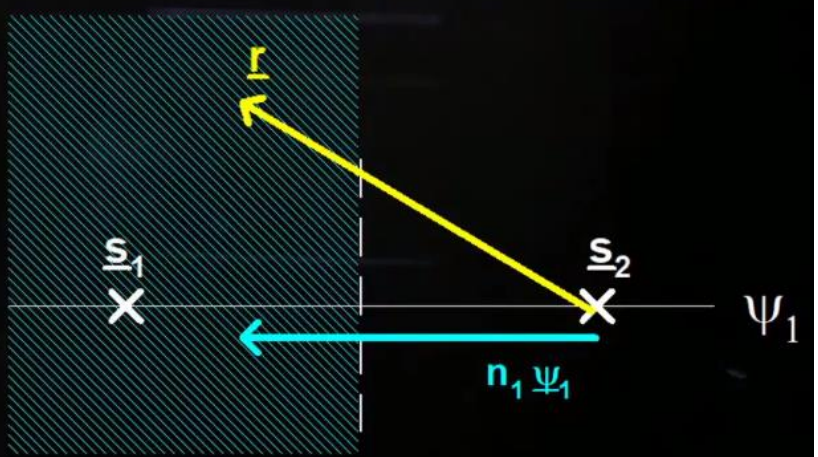

Received vector
r
ψ
s1
s2

Noise contribution

- Region two ⇒ choose s_2, no error



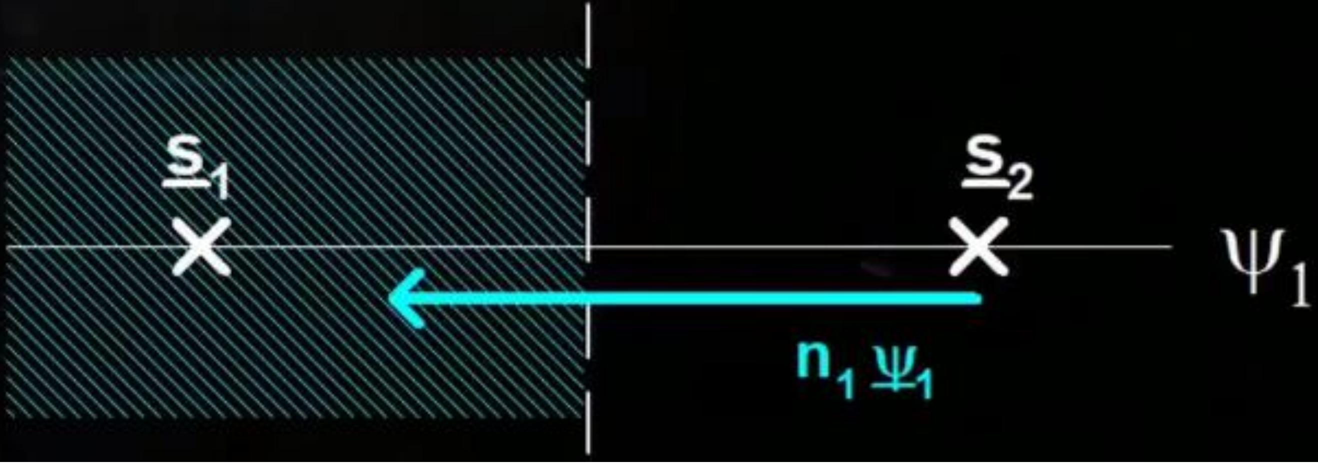
- Region one ⇒ choose s_1, **error**



Errors

- When does an error occur?
- When the noise vector exceeds the bisector …

$$n_1 > \frac{1}{2}\|\mathbf{s}_1 - \mathbf{s}_2\| = \frac{D}{2}$$



How to calculate P_e for the M-ary case now that we have the decision regions?

Minimal distance

- Definition $D_{\min} = \min_{i \neq k} \left\| \mathbf{s}_i - \mathbf{s}_k \right\|$
  - closer signals => more errors

Union bound

QPSK

- Quadrature phase shift keying

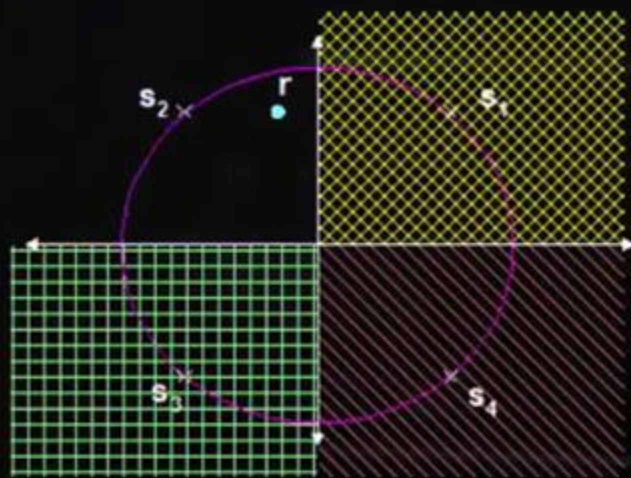
$$s(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_0 t + \phi_i(t)\right) \quad 0 \le t \le T \qquad \phi_i \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$$

- E energy/symbol, T symbol time
- Basis vectors

$$\psi_1 = \cos\left(\omega_0 t\right) \qquad \psi_2 = \sin\left(\omega_0 t\right)$$

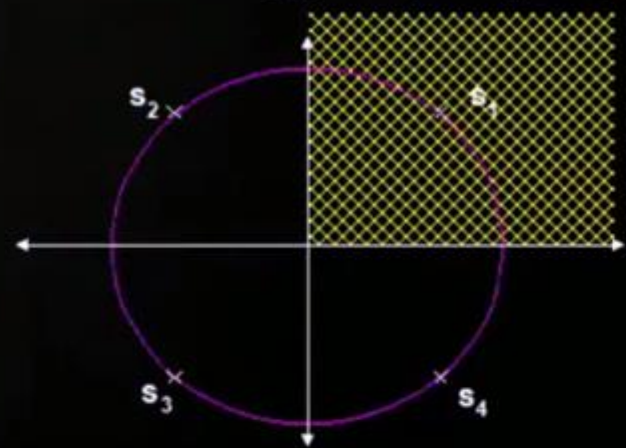
- Decision regions – four quadrants

Union bound



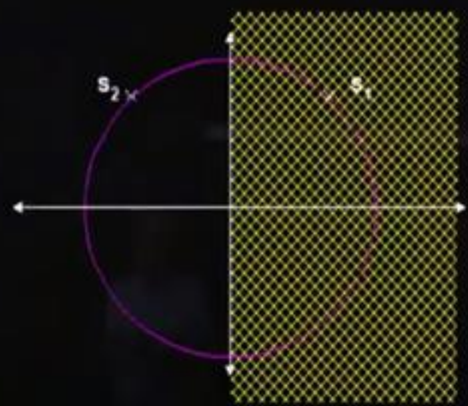
4 decision regions

$$p_e = P(\mathbf{r} \in S_2 \cup S_3 \cup S_4 | \mathbf{s_1} \text{ sent})$$



error region when
$\mathbf{s}_1$ was sent

Union bound



$$p_2\left(\mathbf{s}_2 \text{ chosen} \middle| \mathbf{s}_1 \text{ sent}\right) = P\left(\mathbf{r} \notin S_1 \middle| \mathbf{s}_1 \text{ sent}\right)$$

only s_1 and s_2 exist

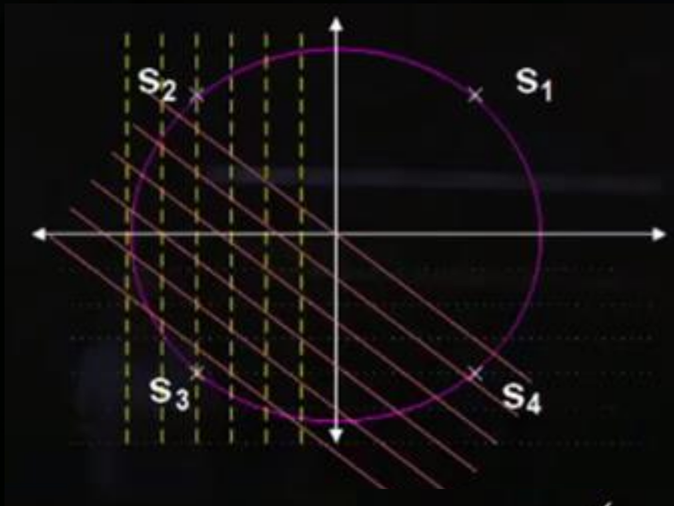
$$p_2\left(\mathbf{s}_4 \text{ chosen} \middle| \mathbf{s}_1 \text{ sent}\right) = P\left(\mathbf{r} \notin S_1 \middle| \mathbf{s}_1 \text{ sent}\right)$$

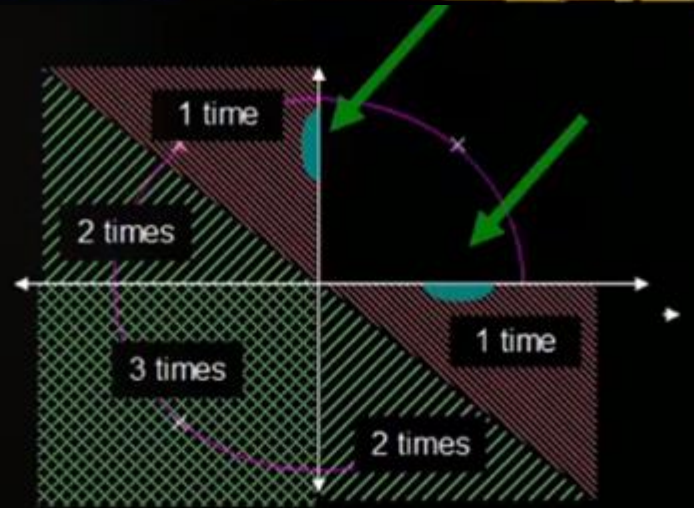
only s_1 and s_4 exist

$$P\left(\mathbf{r} \notin S_1 \middle| \mathbf{s}_1 \text{ sent}\right) = p_2\left(\mathbf{s}_3 \text{ choisi} \middle| \mathbf{s}_1 \text{ envoyé}\right)$$

only s_1 and s_3 exist

Union bound





$$p_e \le p_2(\mathbf{s}_2 \text{ chosen} | \mathbf{s}_1 \text{ sent}) + p_2(\mathbf{s}_3 \text{ chosen} | \mathbf{s}_1 \text{ sent}) + p_2(\mathbf{s}_4 \text{ chosen} | \mathbf{s}_1 \text{ sent})$$

UNION BOUND

$$\boxed{P_e(QPSK) \le 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(2\sqrt{\frac{E_b}{N_0}}\right)}$$

Approximated UNION BOUND

$$\boxed{P_e(QPSK) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)}$$

Union bound

For M-Ary modulation schemes

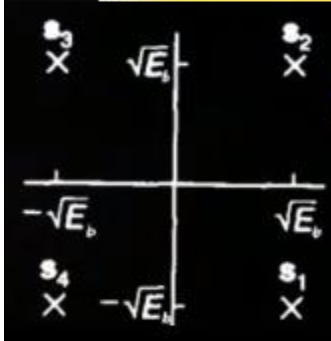
- Error probability dominated by the nearest points

$$P_e \approx \frac{2K}{M} Q\left(\frac{D_{\min}}{\sqrt{2N_0}}\right)$$

Approximated UNION BOUND

- K is the number of pairs of symbols separated by the minimum distance $D_{\min}$

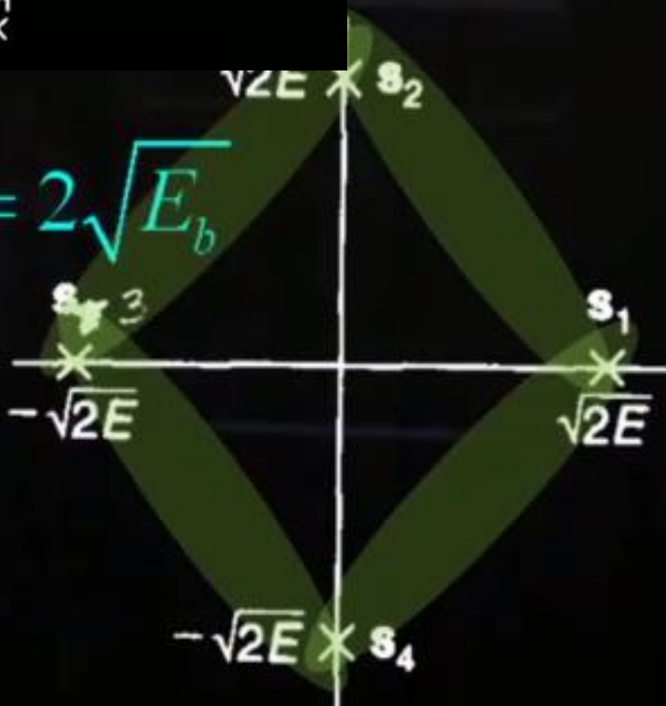
QPSK



$$D_{\min} = 2\sqrt{E_b}$$

$$P_e \approx \frac{2K}{M} Q\left(\frac{D_{\min}}{\sqrt{2N_0}}\right) = \frac{2K}{M} Q\left(\frac{2\sqrt{E_b}}{\sqrt{2N_0}}\right)$$

$$D_{\min} = 2\sqrt{E_b}$$



- $M=4$
- $K=4$

$$P_e \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

QPSK

- Error probability coming from the union bound

$$P_e(QPSK) \le 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(2\sqrt{\frac{E_b}{N_0}}\right)$$

- Approximation

$$P_e(QPSK) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- Exact error probability

$$P_e(QPSK) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

QPSK

0 dB

0.15963807454081

$$P_e(QPSK) \le 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(2\sqrt{\frac{E_b}{N_0}}\right)$$

0.15729920705029

$$P_e(QPSK) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

0.15111344691562

$$P_e(QPSK) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

QPSK

	5dB	10dB
$P_e(QPSK) \le 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(2\sqrt{\frac{E_b}{N_0}}\right)$	0.01190797987629	0.00000774421643
$P_e(QPSK) \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$	0.01190773429556	0.00000774421643
$P_e(QPSK) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$	0.01187228576154	0.00000774420144

Thanks !