Subject : Digital Communication

Code : UEC 639

Credit : 4

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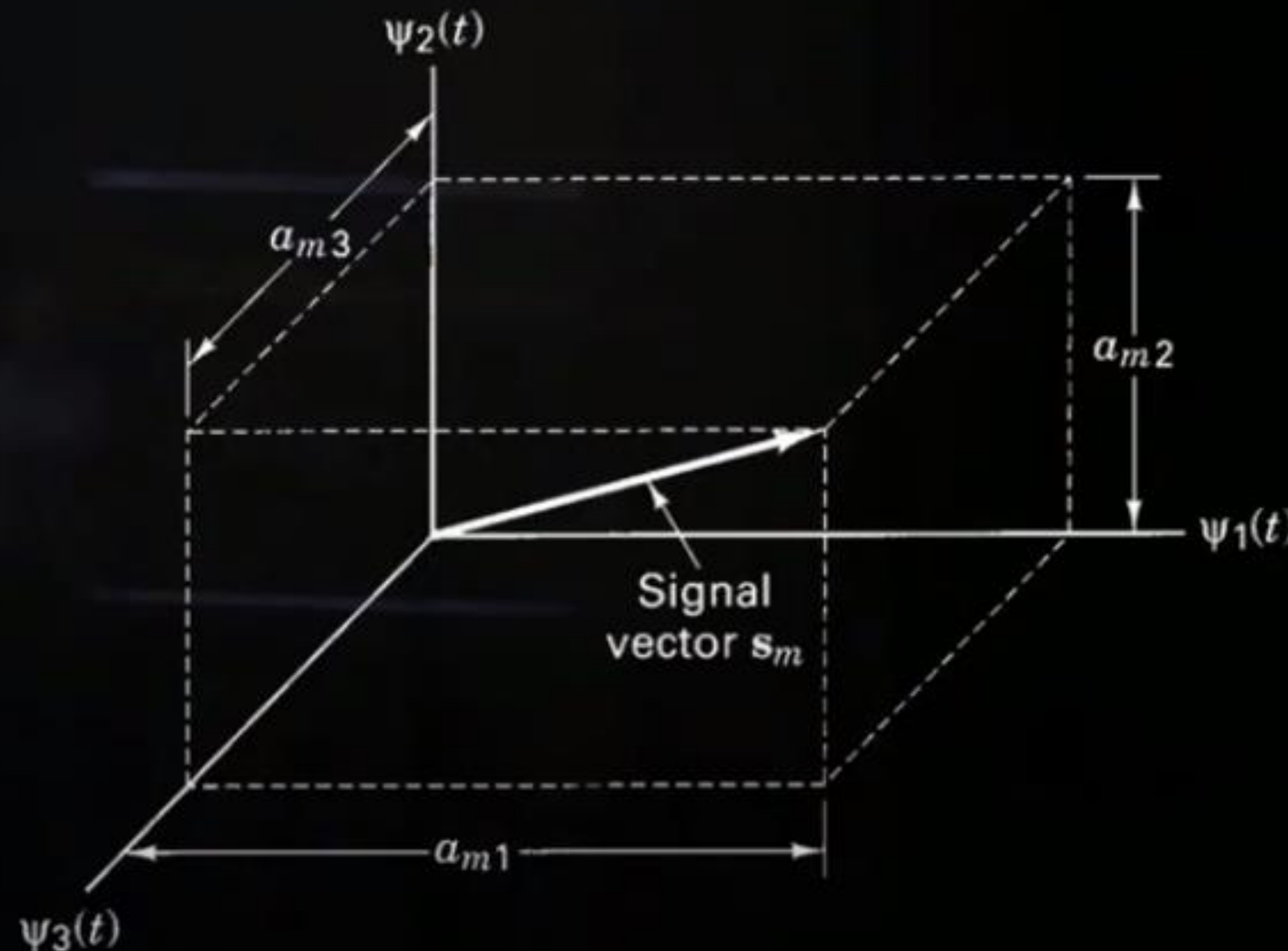
Signal space

- Motivation
  - Mathematics to attack ML & MAP
  - Decision rules
  - Geometric intuition
  - Error probability calculation
- Vector space
  - M symbols can be represented in a finite vector space (at most M dimensional)
  - Noise has a space of ∞ dimension

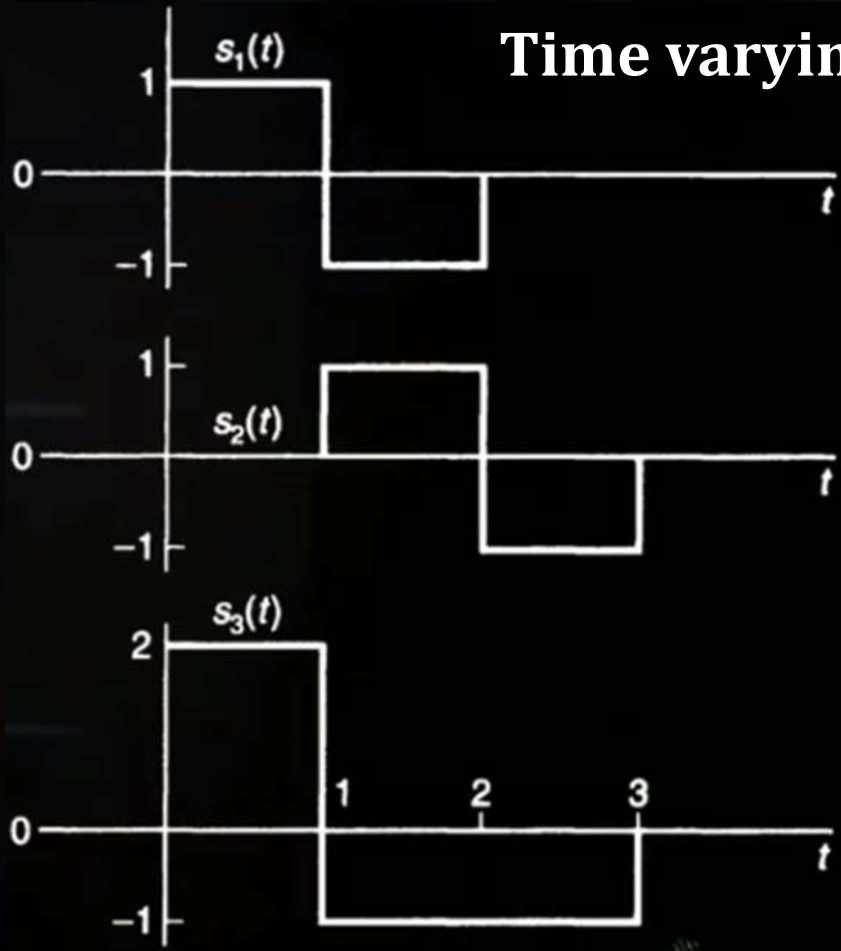
Vector space



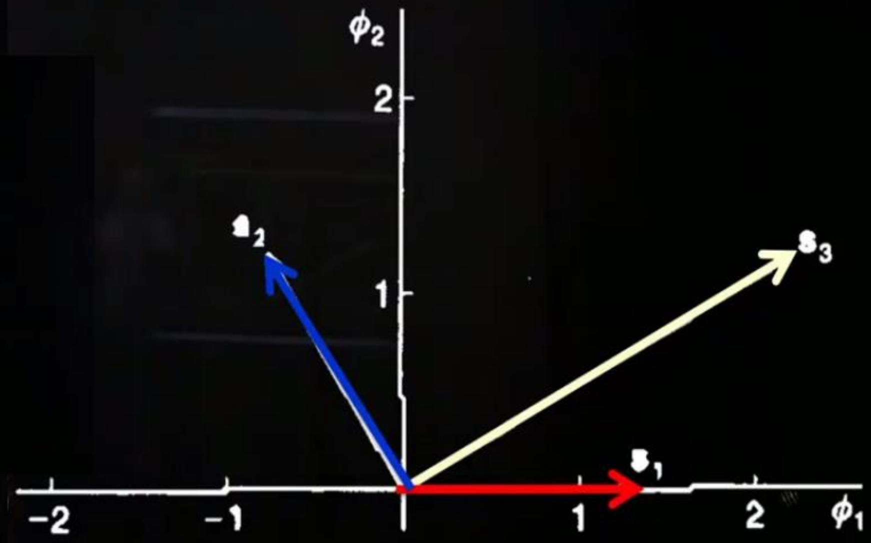
NOTE: Need to device a tool which help us to represent signal waveforms in terms of vectors.

Example

Time varying signals: *s1(t), s2(t), s3(t)*



Solution



Vector representation

Orthonormal basis

- Basis vectors

$$\{\psi_1(t), \psi_2(t), \dots \psi_N(t)\}$$

$$\langle \psi_i(t), \psi_j(t) \rangle = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

- Inner product

$$\langle \psi_i(t), \psi_j(t) \rangle \triangleq \int_0^T \psi_i(t) \psi_j(t) \, dt$$

M symboles $\Rightarrow \quad N \leq M$

N= # Basis vector
M= # symbols

Explanation: The standard inner product for vectors in R^n is

$$\langle x, y \rangle = x^T y \qquad x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n ,$$

Inner Product (real vectors)

$$x = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \ \text{[5x1]} \qquad y = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix} \ \text{[5x1]} \qquad x^T = \begin{bmatrix} 1 & -1 & 0 & 2 & 4 \end{bmatrix} \ \text{[1x5]}$$

Example If $\mathbf{x} = (1, -1, 0, 2, 4)$ and $\mathbf{y} = (2, 1, 1, 3, 0)$ in $\mathbb{R}^5$, then

$$x^T y = \underbrace{\begin{bmatrix} 1 & -1 & 0 & 2 & 4 \end{bmatrix}}_{\text{[1x5]}} \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}}_{\text{[5x1]}}$$

$$\langle \mathbf{x}, \mathbf{y} \rangle = (1)(2) + (-1)(1) + (0)(1) + (2)(3) + (4)(0) = 7, \ \text{[1x1]}$$

The **norm** of **x** is

$$||\mathbf{x}|| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

$$< x, x > = x^T x \qquad < y, y > = y^T y$$

$$x = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \qquad y = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}$$

$$||\mathbf{x}|| = \sqrt{1^2 + (-1)^2 + 0^2 + 2^2 + 4^2} = \sqrt{22},$$

$$||\mathbf{y}|| = \sqrt{2^2 + 1^2 + 1^2 + 3^2 + 0^2} = \sqrt{15}$$

$$\|v\| = Length = \sqrt{v.v} \Leftrightarrow \sqrt{\int_{-\infty}^{\infty} v(t)v^*(t)dt} = \sqrt{\int_{-\infty}^{\infty} |v(t)|^2 dt} = \sqrt{Energy}$$

Inner product (complex vectors)

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T \boldsymbol{v}^*$$

$$\text{if } z = a + ib, \text{ then } z^* = a - ib$$

Example If $\mathbf{u} = (1 + 2i, 2 - 3i)$ and $\mathbf{v} = (2 - i, 3 + 4i)$, find $\langle \mathbf{u}, \mathbf{v} \rangle$ and $||\mathbf{u}||$.

$$\langle u, v \rangle = \begin{bmatrix} 1 + 2i & 2 - 3i \end{bmatrix} \begin{bmatrix} 2 + i \\ 3 - 4i \end{bmatrix}$$

$$\langle \mathbf{u}, \mathbf{v} \rangle = (1 + 2i)(2 + i) + (2 - 3i)(3 - 4i) = 5i - 6 - 17i = -6 - 12i,$$

$$||\mathbf{u}|| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \sqrt{(1 + 2i)(1 - 2i) + (2 - 3i)(2 + 3i)} = \sqrt{5 + 13} = 3\sqrt{2}.$$

A set of *n*-orthonormal vectors in an *n*-dimensional inner product space which forms a basis is said to form an **orthonormal basis**.

Example of Orthonormal vectors, in R^3

$$S_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad S_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Exercise: Test these vectors for their orthonormal characteristic.

Signal vector

- Signal in vector form
  - $s_i(t) = \sum_{j=1}^{N} a_{i,j}\psi_j(t)$
  - a_{ij} are the coefficients of $s_i(t)$ in signal space
  - $\underline{s}_i = [a_{i1}\ a_{i2} \cdots a_{iN}]$
  - $a_{ik} = \langle s_i(t), \psi_k(t) \rangle$
  - $\|s_i(t)\|^2 = \sum_{j=1}^{N} a_{ij}^2$

Signal vector

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  - $a_{ik} = \langle s_i(t), \psi_k(t) \rangle$
  - $\|s_i(t)\|^2 = \sum_{j=1}^{N} a_{ij}^2$

$$< \sum_{j=1}^{N} a_{ij}\psi_j(t), \psi_k(t) >$$
$$= \sum_{j=1}^{N} a_{ij} < \psi_j(t), \psi_k(t) >$$
$$= a_{i,k}$$

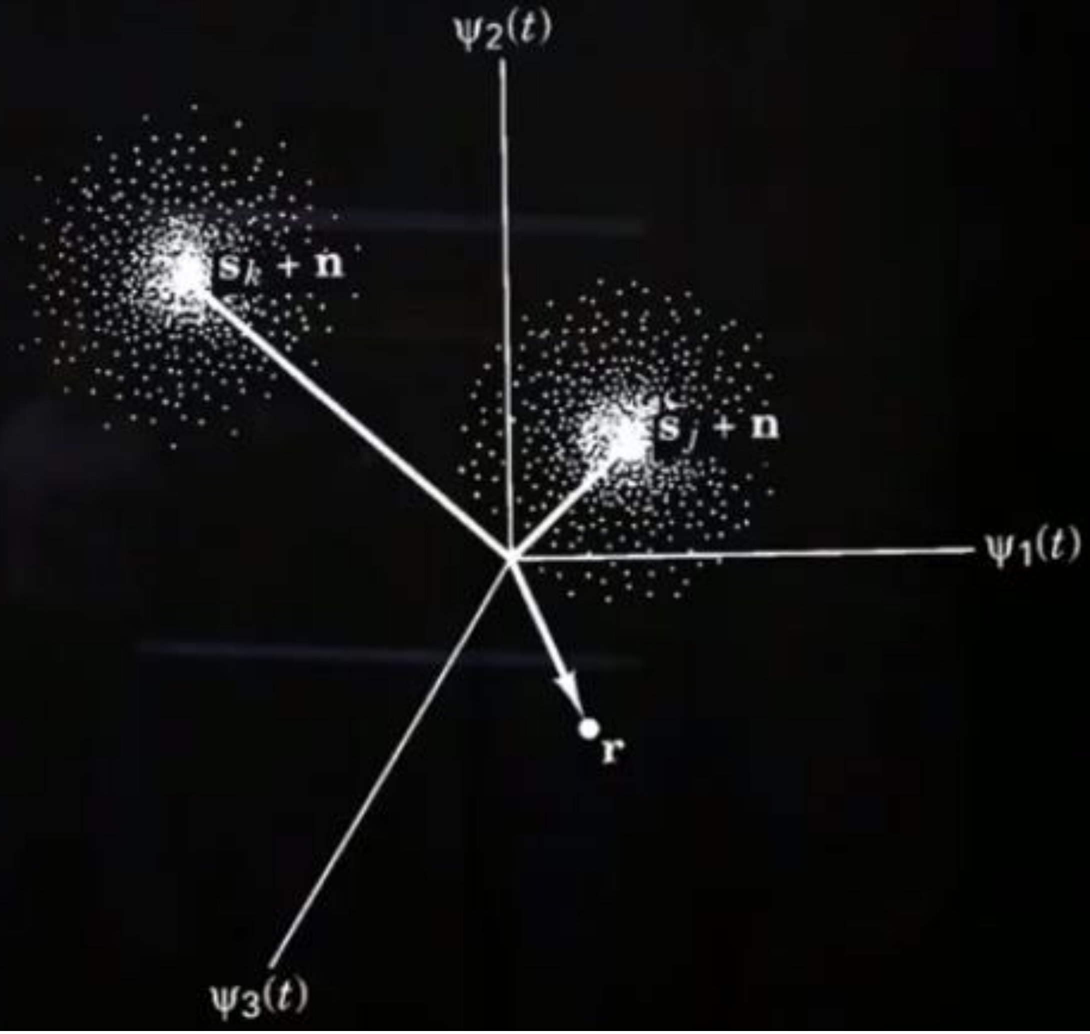
$$\|v\| = Length = \sqrt{v.v} \Leftrightarrow \sqrt{\int_{-\infty}^{\infty} v(t)v^*(t)dt} = \sqrt{\int_{-\infty}^{\infty} |v(t)|^2 dt} = \sqrt{Energy}$$

Orthonormal basis

- Signal energy
  - $$\langle s_i(t), s_i(t) \rangle = \int_0^T s_i^2(t)dt$$
    $$= \|s_i(t)\|^2$$
  - Average signal energy for a constellation of *M* symbols

$$E_s = \frac{1}{M}\sum_{i=1}^{M} \|s_i(t)\|^2$$

Noise - vector space



Basis extension

- Noise can fall into a larger space than the signal space
- Infinite dimension
  - Form a basis with the same first *N* vectors

$$\{\psi_1(t), \psi_2(t) \ldots, \psi_N(t), \psi_{N+1}(t), \ldots\}$$

$$\underline{n} = [n_1\ n_2 \ldots n_N\ n_{N+1} \ldots] \qquad n_i = \langle n(t), \psi_i(t) \rangle$$

Basis extension

- Symbol
  - Only non-zero coefficients are in the signal space
  - $\underline{s}_i = [a_{i1}\ a_{i2} \ldots a_{iN}\ 0\ 0 \ldots]$

Received signal

- $r(t) = s_i(t) + n(t)$
- vector form

$$\underline{r} = \underline{s}_i + \underline{n}$$

How to generate orthonormal basis for signal representation?

Gram Schmidt Process

Signal Space Representation of Waveforms

- Let $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ are "N" orthonormal basis functions

- Each signal in the set $\{s_i(t), i = 1,2,..,M\}$ can be represented using these "N" orthonormal basis functions in the similar way a vector is represented in N-dimensional vector space.

$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t) \ldots\ldots\ldots + s_{1N}\phi_N(t)$$

$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t) \ldots\ldots\ldots + s_{2N}\phi_N(t)$$

$$v = ax + by + cz$$

Vector in 3D space

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\ \phi_j(t)\,; \quad \begin{matrix} 0 \le t \le T \\ i = 1,2,\ldots,M \end{matrix}$$

where the coefficients of the expansion are defined by

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)\,; \quad \begin{matrix} i = 1,2,\ldots,M \\ j = 1,2,\ldots,N \end{matrix}$$

Notations

Compact

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\ \phi_j(t);\quad \begin{matrix} 0 \le t \le T \\ i = 1,2,\ldots,M \end{matrix}$$

Matrix

$$S_i(t) = \begin{bmatrix} S_{11} & S_{12} & S_{13} & \cdots & S_{1N} \\ S_{21} & S_{22} & S_{23} & \cdots & S_{2N} \\ S_{31} & S_{32} & S_{33} & \cdots & S_{3N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{M1} & S_{M2} & S_{M3} & \cdots & S_{MN} \end{bmatrix} \begin{bmatrix} \emptyset_1 \\ \emptyset_2 \\ \emptyset_2 \\ \vdots \\ \emptyset_N \end{bmatrix}$$

Equations

$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t) \ldots\ldots\ldots + s_{1N}\phi_N(t)$$

$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t) \ldots\ldots\ldots + s_{2N}\phi_N(t)$$

$$s_M(t) = s_{M1}\phi_1(t) + s_{M2}\phi_2(t) \ldots\ldots\ldots + s_{MN}\phi_N(t)$$

Gram Schmidt Process

- Generate an orthonormal basis from M signals $\{s_i(t)\}_{i=1}^{M}$

  - 1- $\psi_1(t) = \dfrac{1}{\sqrt{E_1}} s_1(t)$

where $E_1 = \int_0^T s_1^2(t)\,dt$

Gram Schmidt Stage 2

- 2- we remove the part of the second signal that falls into the signal space of $s_1(t)$
  [i.e. the projection of $s_2(t)$ on $s_1(t)$]

$$\theta_2(t) = s_2(t) - \langle s_2(t), \psi_1(t) \rangle \psi_1(t)$$

$$E_2 = \int_0^T \theta_2^2(t)\, dt$$

$$\psi_2(t) = \frac{\theta_2(t)}{\sqrt{E_2}}$$

Gram Schmidt i^{th} step

- we remove the part of $s_i(t)$ which can be represented by the basic vectors already found

$$\theta_i(t) = s_i(t) - \sum_{k=1}^{i-1} \langle s_i(t), \psi_k(t) \rangle \psi_k(t)$$

$$E_i = \int_0^T \theta_i^2(t)\,dt \qquad \psi_i(t) = \frac{\theta_i(t)}{\sqrt{E_i}}$$

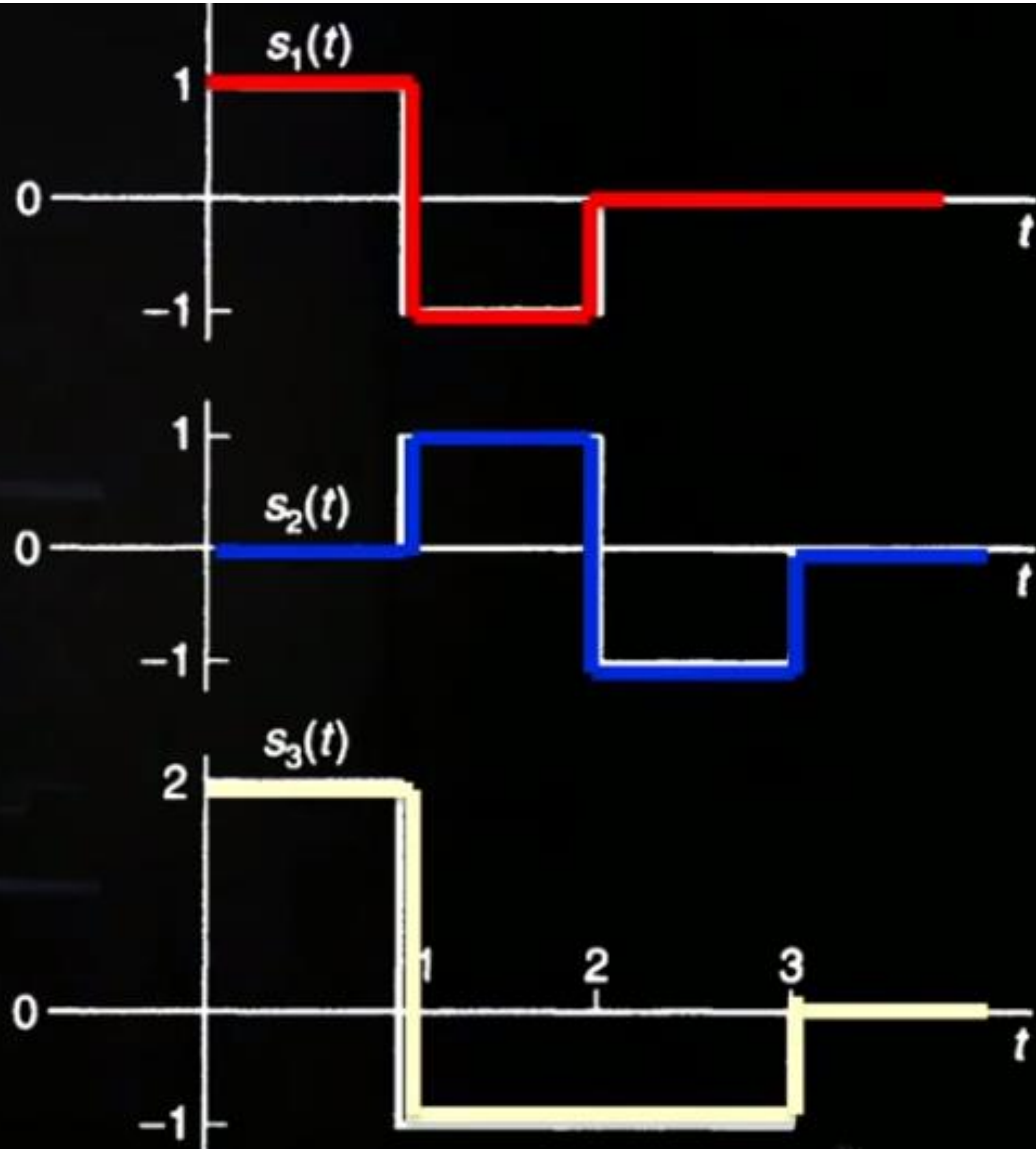
- When $\psi_i(t) = 0$ there is no other basis vector at this stage
- The process stops at the latest when $i=M$

EXAMPLE: Evaluate the basis vector for the given signals?

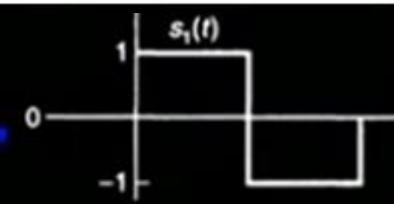
$$s_1(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$$

$$s_2(t) = \begin{cases} 0 & 0 < t < 1 \\ 1 & 1 < t < 2 \\ -1 & 2 < t < 3 \end{cases}$$

$$s_3(t) = \begin{cases} 2 & 0 < t < 1 \\ -1 & 1 < t < 3 \end{cases}$$



$$s_1(t) = \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$$



Step 1

$$\psi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$$E_1 = \int_0^3 s_1^2(t)\,dt = \int_0^1 (1)^2\,dt + \int_1^2 (-1)^2\,dt$$
$$= 2$$

$$\psi_1(t) = \frac{s_1(t)}{\sqrt{2}} = \begin{cases} 1/\sqrt{2} & 0 < t < 1 \\ -1/\sqrt{2} & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$$

Step 2a

$$s_2(t) = \begin{cases} 0 & 0 < t < 1 \\ 1 & 1 < t < 2 \\ -1 & 2 < t < 3 \end{cases} \qquad \psi_1(t) = \begin{cases} 1/\sqrt{2} & 0 < t < 1 \\ -1/\sqrt{2} & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$$

$$\theta_2(t) = s_2(t) - \langle s_2(t), \psi_1(t) \rangle \psi_1(t)$$

$$\begin{aligned} \langle s_2(t), \psi_1(t) \rangle &= \int_0^3 s_2(t)\psi_1(t)\,dt \\ &= \int_0^1 0 \cdot \frac{1}{\sqrt{2}}\,dt + \int_1^2 1 \cdot \frac{-1}{\sqrt{2}}\,dt + \int_2^3 -1 \cdot 0\,dt \\ &= \frac{-1}{\sqrt{2}} \end{aligned}$$

$$\theta_2(t) = s_2(t) - \frac{-1}{\sqrt{2}}\psi_1(t)$$

Step 2b

$s_2(t)$ $\qquad \frac{1}{\sqrt{2}}\psi_1(t)$

$$\theta_2(t) = \begin{cases} 0 & 0<t<1 \\ 1 & 1<t<2 \\ -1 & 2<t<3 \end{cases} + \begin{cases} 1/2 & 0<t<1 \\ -1/2 & 1<t<2 \\ 0 & 2<t<3 \end{cases}$$

$$= \begin{cases} 1/2 & 0<t<2 \\ -1 & 2<t<3 \end{cases}$$

$$E_2 = \int_0^3 \theta_2^2(t)\,dt = \int_0^2 \left(\frac{1}{2}\right)^2 dt + \int_2^3 (-1)^2\,dt = \frac{2}{4} + 1 = \frac{3}{2}$$

$$\psi_2(t) = \theta_2(t)\cdot\sqrt{\frac{2}{3}} = \begin{cases} 1/\sqrt{6} & 0<t<2 \\ -2/\sqrt{6} & 2<t<3 \end{cases}$$

Step 3a

$$\psi_1(t) = \begin{cases} 1/\sqrt{2} & 0 < t < 1 \\ -1/\sqrt{2} & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases}$$

$$\theta_3(t) = s_3(t) - \langle s_3(t), \psi_1(t) \rangle \psi_1(t) - \langle s_3(t), \psi_2(t) \rangle \psi_2(t)$$

$$\langle s_3(t), \psi_1(t) \rangle = \frac{1}{\sqrt{2}} \int_0^1 2 \cdot 1 dt + \frac{1}{\sqrt{2}} \int_1^2 (-1) \cdot (-1) dt + \int_2^3 (-1) \cdot 0 dt$$

$$= \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$\psi_2(t) = \begin{cases} 1/\sqrt{6} & 0 < t < 2 \\ -2/\sqrt{6} & 2 < t < 3 \end{cases}$$

Step 3a

$$\theta_3(t) = s_3(t) - \langle s_3(t), \psi_1(t) \rangle \psi_1(t) - \langle s_3(t), \psi_2(t) \rangle \psi_2(t)$$

$$\langle s_3(t), \psi_1(t) \rangle = \frac{1}{\sqrt{2}} \int_0^1 2 \cdot 1 dt + \frac{1}{\sqrt{2}} \int_1^2 (-1) \cdot (-1) dt + \int_2^3 (-1) \cdot 0 dt$$

$$= \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$\langle s_3(t), \psi_2(t) \rangle = \int_0^1 2 \cdot \frac{1}{\sqrt{6}} dt + \int_1^2 (-1) \cdot \frac{1}{\sqrt{6}} dt + \int_2^3 (-1) \left(-\frac{2}{\sqrt{6}} \right) dt$$

$$= \frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} = \frac{3}{\sqrt{6}}$$

Step 3b

$$\theta_3(t) = s_3(t) - \langle s_3(t), \psi_1(t) \rangle \psi_1(t) - \langle s_3(t), \psi_2(t) \rangle \psi_2(t)$$

$$\theta_3(t) = \begin{cases} 2 & 0 < t < 1 \\ -1 & 1 < t < 2 \\ -1 & 2 < t < 3 \end{cases} - - \frac{3}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{cases} 1 & 0 < t < 1 \\ -1 & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases} - \frac{3}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} \begin{cases} 1 & 0 < t < 1 \\ 1 & 1 < t < 2 \\ -2 & 2 < t < 3 \end{cases}$$

$$= \begin{cases} 2 - 3/2 - 3/6 \\ -1 + 3/2 - 3/6 = 0 \\ -1 - 0 + 6/6 \end{cases}$$

$$\psi_3(t) = 0$$

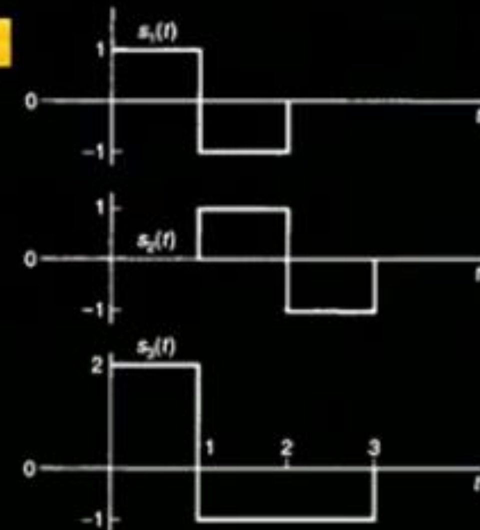
Result

- Two dimensional basis

$$\psi_1(t) = \begin{cases} 1/\sqrt{2} & 0 < t < 1 \\ -1/\sqrt{2} & 1 < t < 2 \\ 0 & 2 < t < 3 \end{cases} \qquad \psi_2(t) = \begin{cases} 1/\sqrt{6} & 0 < t < 2 \\ -2/\sqrt{6} & 2 < t < 3 \end{cases}$$

$$\psi_1(t)=\begin{cases}1/\sqrt{2} & 0<t<1\\ -1/\sqrt{2} & 1<t<2\\ 0 & 2<t<3\end{cases}\qquad \psi_2(t)=\begin{cases}1/\sqrt{6} & 0<t<2\\ -2/\sqrt{6} & 2<t<3\end{cases}$$

Result



- Signal coefficients
  - $\langle s_1(t),\psi_1(t)\rangle = \dfrac{E_1}{\sqrt{E_1}} = \sqrt{E_1} = \sqrt{2}$

    $\langle s_1(t),\psi_2(t)\rangle = 0$ by construction
  - $\langle s_2(t),\psi_1(t)\rangle = -\,{}^{1}\!/\!_{\sqrt{2}}$ see step 2a

    $\langle s_2(t),\psi_2(t)\rangle = \int_1^2 (1)\left({}^{1}\!/\!_{\sqrt{6}}\right)dt + \int_2^3 (-1)\left(-\,{}^{2}\!/\!_{\sqrt{6}}\right)dt = {}^{3}\!/\!_{\sqrt{6}}$
  - $\langle s_3(t),\psi_1(t)\rangle = {}^{3}\!/\!_{\sqrt{2}}$ see step 3a

    $\langle s_3(t),\psi_2(t)\rangle = {}^{3}\!/\!_{\sqrt{6}}$

Solution

$$\underline{s}_1 = \left[\sqrt{2} \quad 0\right]$$

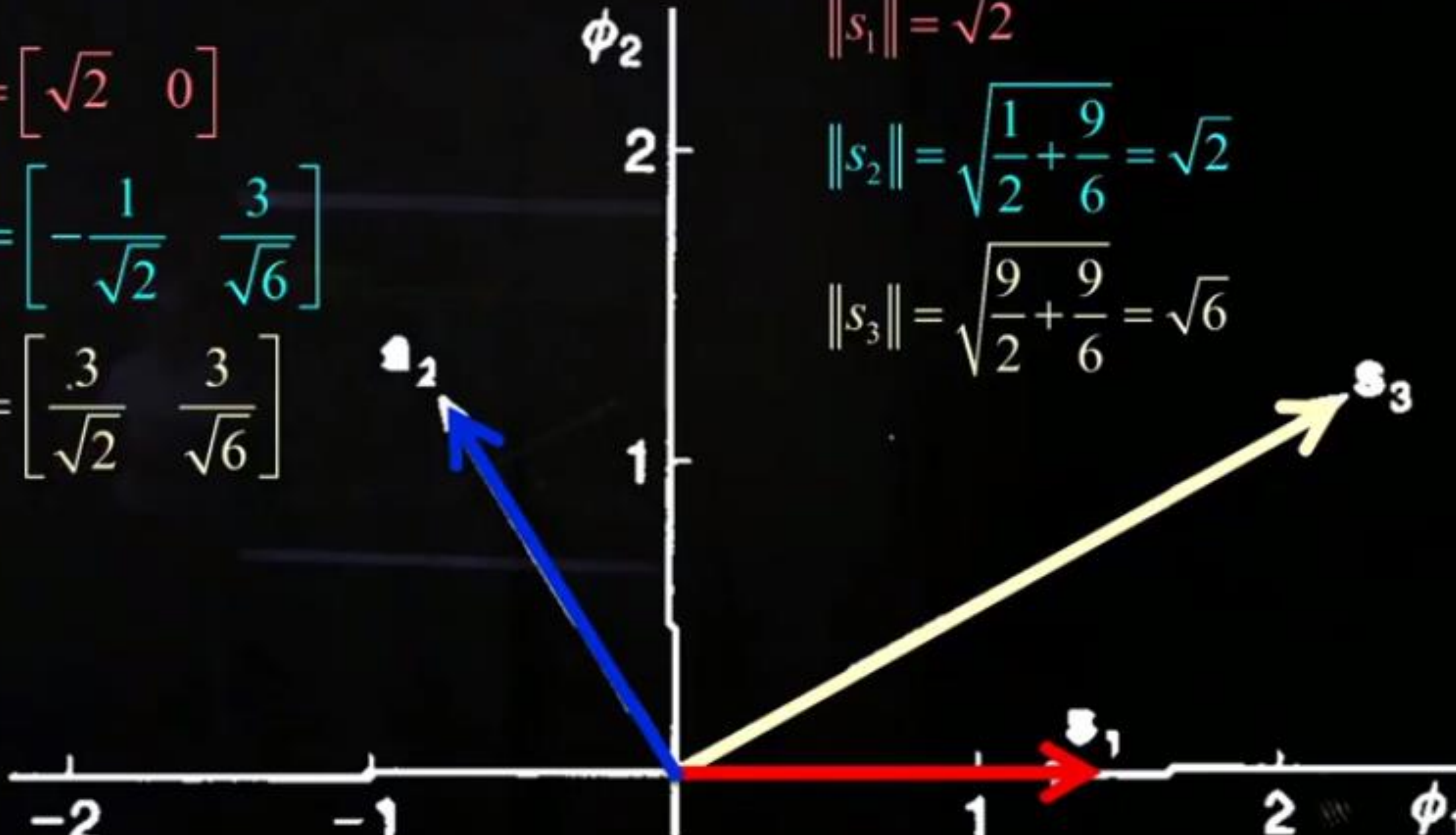
$$\underline{s}_2 = \left[-\frac{1}{\sqrt{2}} \quad \frac{3}{\sqrt{6}}\right]$$

$$\underline{s}_3 = \left[\frac{3}{\sqrt{2}} \quad \frac{3}{\sqrt{6}}\right]$$

$$\|s_1\| = \sqrt{2}$$

$$\|s_2\| = \sqrt{\frac{1}{2}+\frac{9}{6}} = \sqrt{2}$$

$$\|s_3\| = \sqrt{\frac{9}{2}+\frac{9}{6}} = \sqrt{6}$$



Thanks !