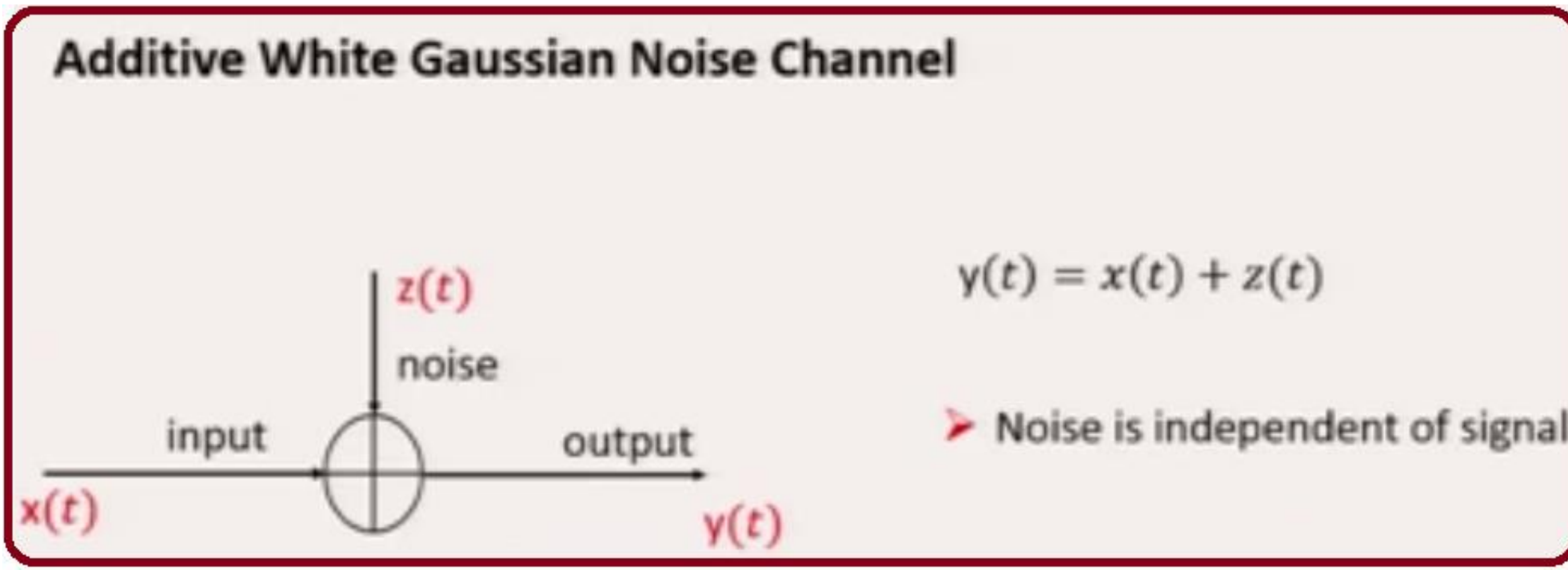
Subject : Digital Communication
Code : UEC 639
Credit : 4

Dr. Amit Mishra

Channel Modeling:

- ❖Additive White Gaussian Noise Channel
- ❖Linear Gaussian Channel
- ❖Linear Time varying Gaussian Channel



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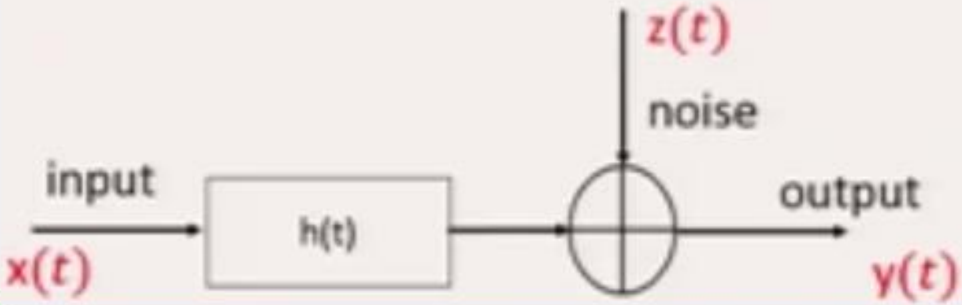
Linear Gaussian Channel

$$y(t) = x(t) * h(t) + z(t)$$

$$y(t) = \int x(\tau)h(t-\tau)d\tau + z(t)$$



- More general model
- Good model for wireline and LOS Communication

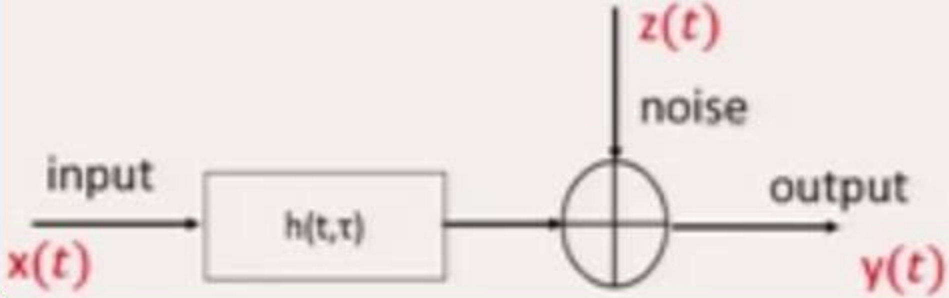
Continued...

Linear Time-Varying Gaussian Channel

$$y(t) = x(t) * h(t,\tau) + z(t)$$

$$y(t) = \int x(t-\tau)h(\tau,t)d\tau + z(t)$$

- Good model for wireless Communication



Probability Density Function of Gaussian random variable.

$$p(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right]$$

where μ and σ are the mean and standard deviation value of random variable z.

Thermal Noise

Thermal noise is caused by the thermal motion of electrons in all dissipative components—resistors, wires, and so on.
The same electrons that are responsible for electrical conduction are also responsible for thermal noise.

We can describe thermal noise as a zero-mean *Gaussian* random process. A Gaussian process $n(t)$ is a random function whose value n at any arbitrary time t is statistically characterized by the Gaussian probability density function

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma}\right)^2\right]$$

where σ^2 is the variance of n.

The *normalized* or *standardized Gaussian density function* of a zero-mean process is obtained by assuming that $\sigma = 1$.

This normalized pdf is shown sketched in Figure

$$p(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma}\right)^2\right]$$

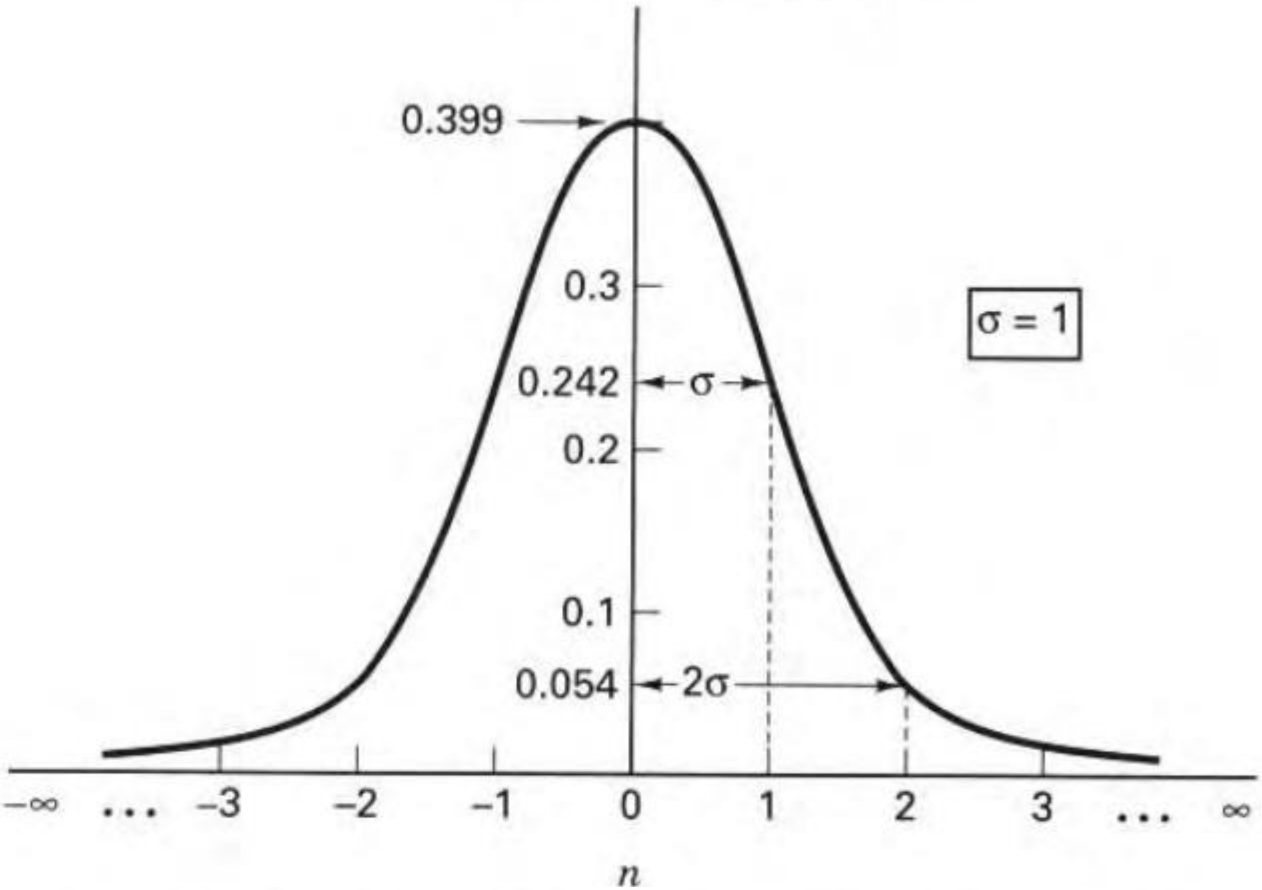


 Figure Normalized ($\sigma = 1$) Gaussian probability density function.

Example:

We will often represent a random signal as the sum of a Gaussian noise random variable and a dc signal. That is,

$$z = a + n$$

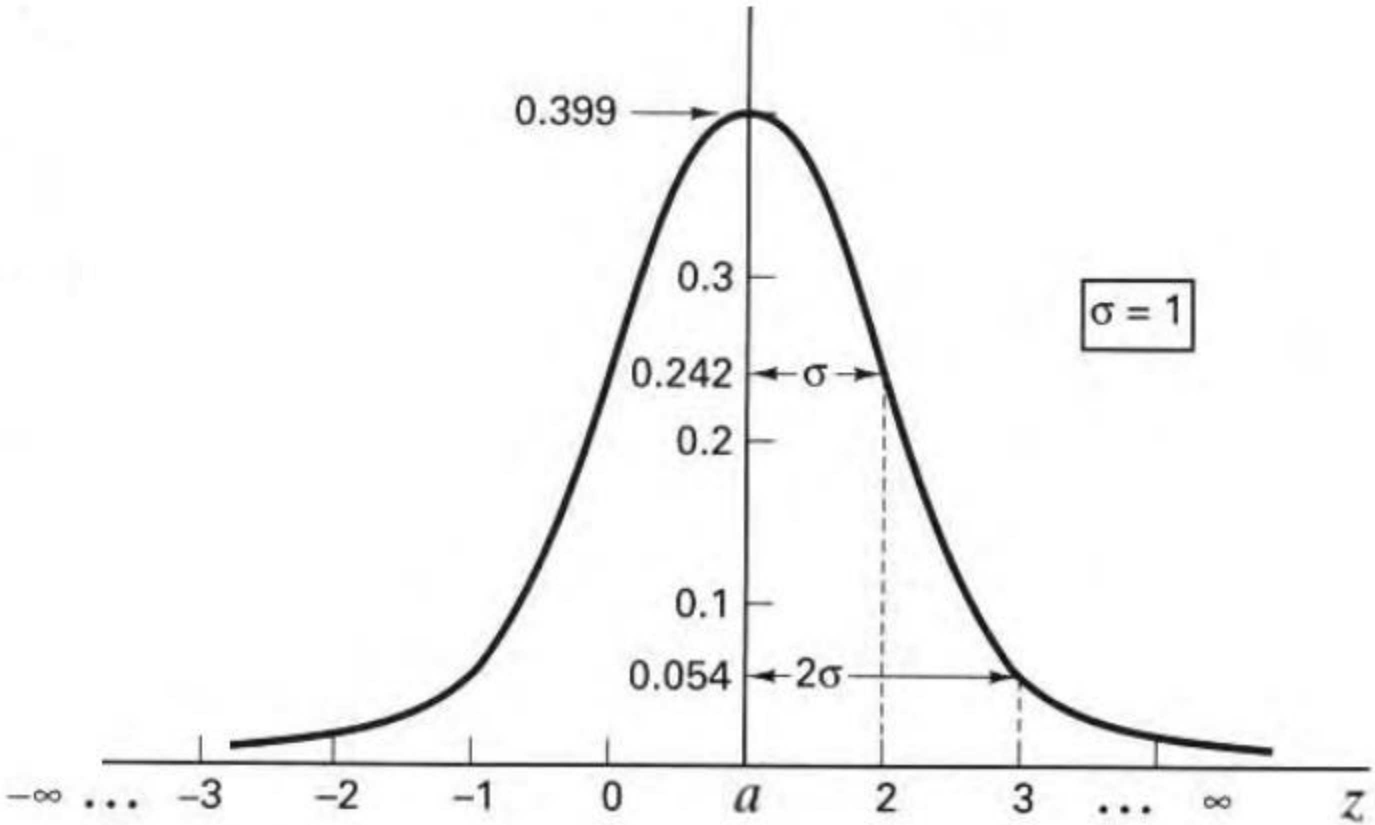
where z is the random signal, a is the dc component, and n is the Gaussian noise random variable.

The pdf $p(z)$ is then expressed as

$$p(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a}{\sigma}\right)^2\right]$$

where μ and σ are the mean and standard deviation value of random variable z.

$$p(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-a}{\sigma}\right)^2\right]$$



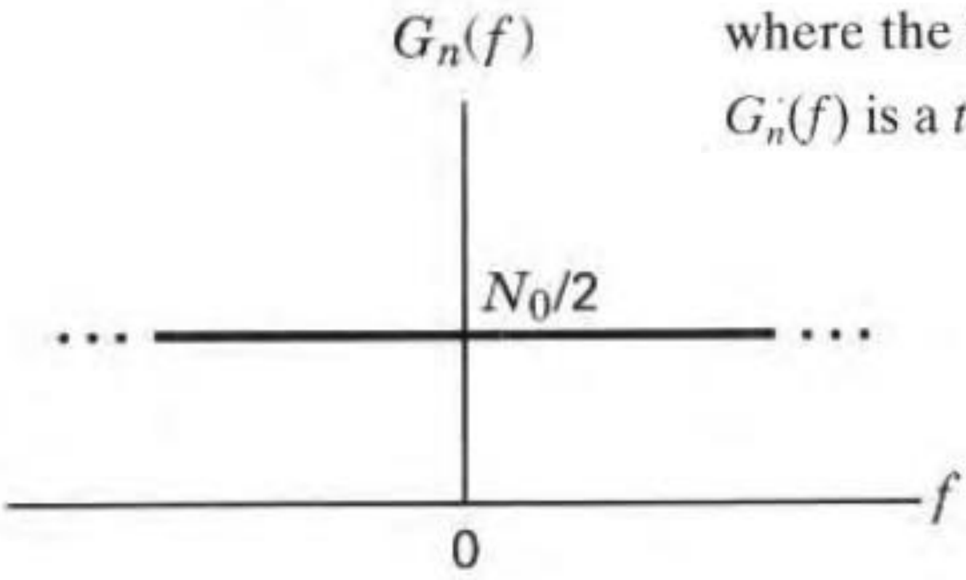
White Noise

The primary spectral characteristic of thermal noise is that its power spectral density is *the same* for all frequencies of interest in most communication systems; in other words, a thermal noise source emanates an equal amount of noise power per unit bandwidth at all frequencies—from dc to about 10^{12} Hz.

Therefore, a simple model for thermal noise assumes that its power spectral density $G_n(f)$ is flat for all frequencies, and is denoted as

Power spectral density of White Noise.

$$G_n(f) = \frac{N_0}{2} \qquad \text{watts/hertz}$$



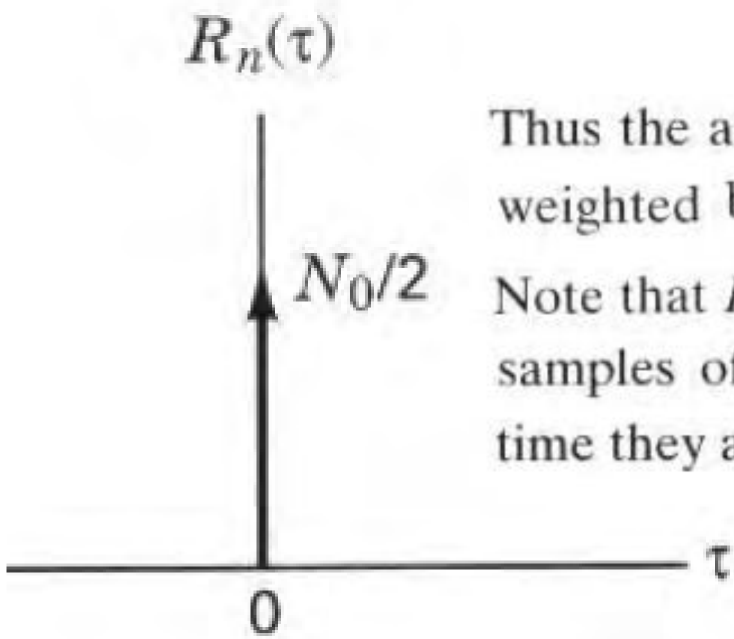
where the factor of 2 is included to indicate that $G_n(f)$ is a *two-sided* power spectral density.

The adjective "white" is used in the same sense as it is with white light, which contains equal amounts of all frequencies within the visible band of electromagnetic radiation.

Autocorrelation function

The autocorrelation function of white noise is given by the inverse Fourier transform of the noise power spectral density

$$R_n(\tau) = \mathcal{F}^{-1}\{G_n(f)\} = \frac{N_0}{2}\,\delta(\tau)$$



Thus the autocorrelation of white noise is a delta function weighted by the factor $N_0/2$ and occurring at $\tau = 0$,

Note that $R_n(\tau)$ is zero for $\tau \neq 0$; that is, any two different samples of white noise, no matter how close together in time they are taken, are uncorrelated.

Average Power

The average power P_n of white noise is *infinite* because its bandwidth is infinite.

This can be seen by

$$P_n = \int_{-\infty}^{\infty} \frac{N_0}{2} df = \infty$$

Filtered noise

- PSD at the output of the LTIS

$$G_Y(f) = G_X(f)\left|H_{LP}(f)\right|^2$$

- noise PSD at the input (*AWGN*) $$G_X(f) = \frac{N_0}{2}$$

- output PSD *colored noise*

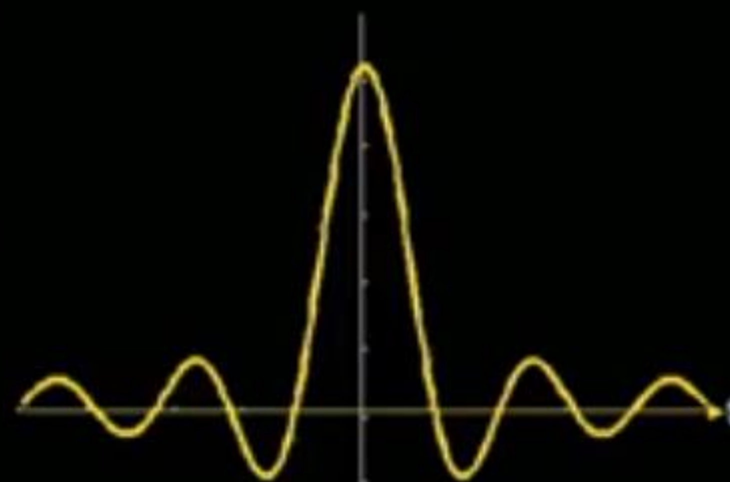
$$G_Y(f) = \begin{cases} \dfrac{N_0}{2} & |f| < f_u \\ 0 & elsewhere \end{cases}$$

Colored Noise

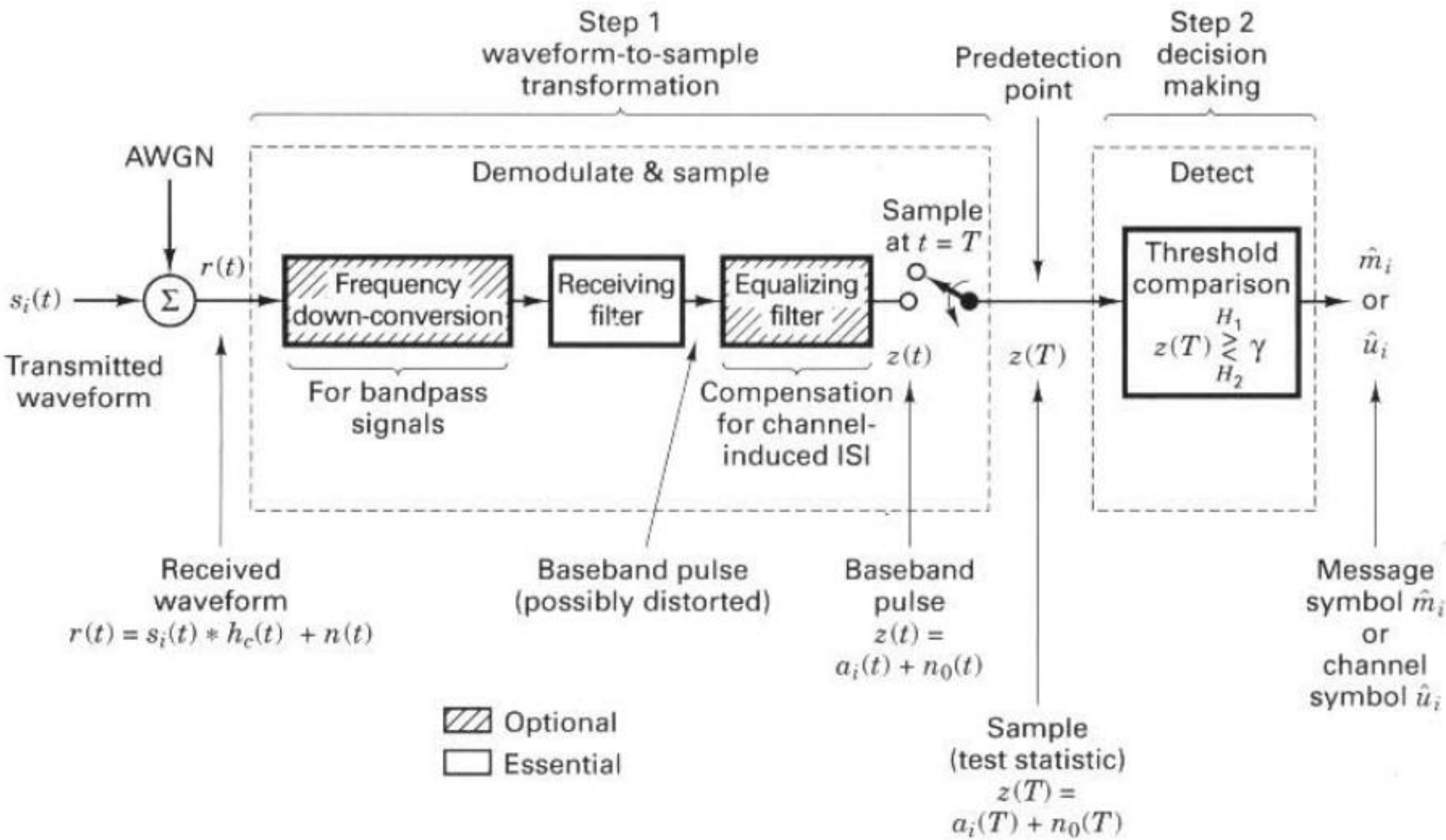
- Noise samples are correlated
- Finite power
- Autocorrelation function

$$R_Y(\tau) = N_0 f_u \,\mathrm{sinc}(2 f_u \tau)$$

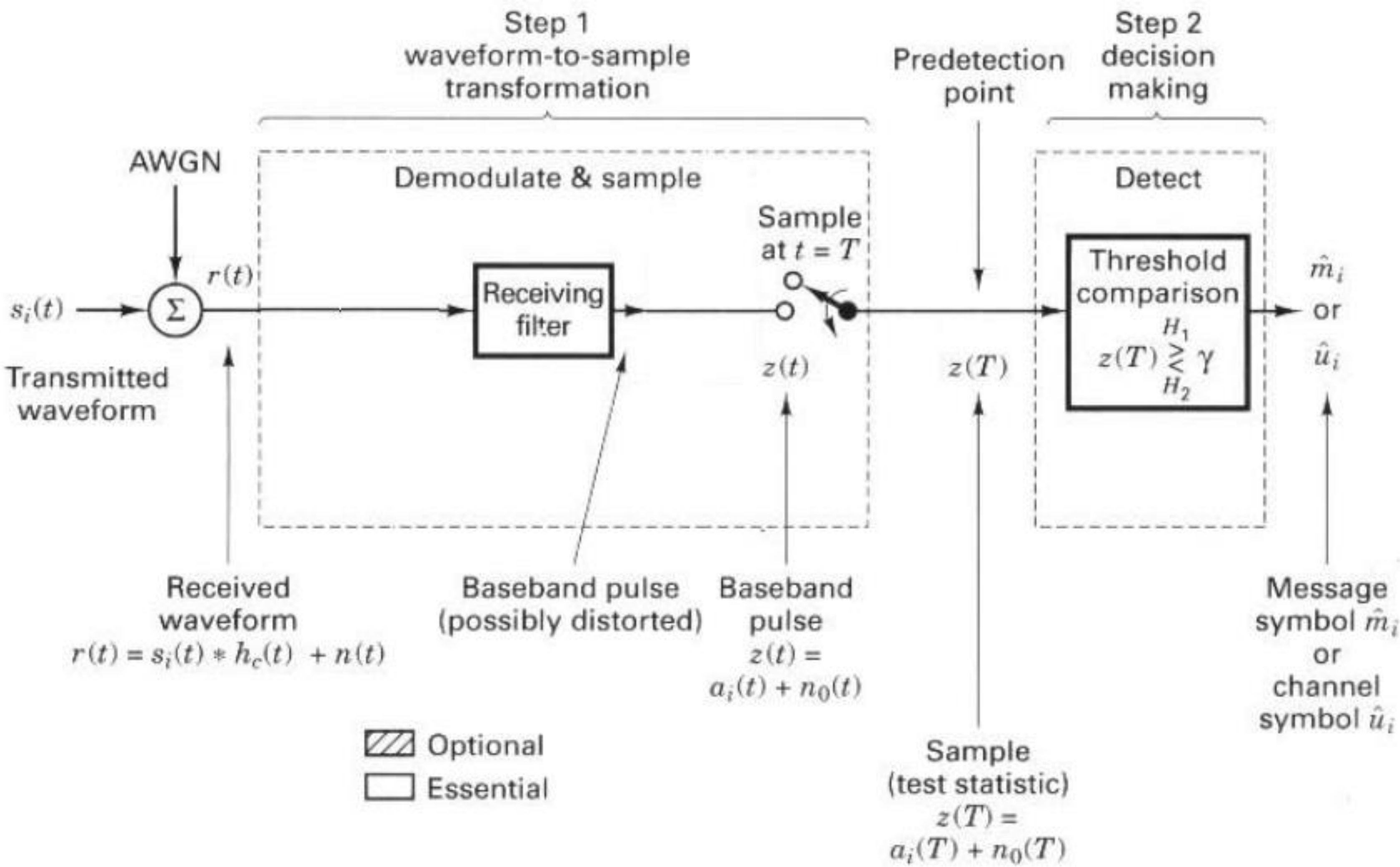
$$= N_0 f_u \frac{\sin 2\pi f_u \tau}{2\pi f_u \tau}$$



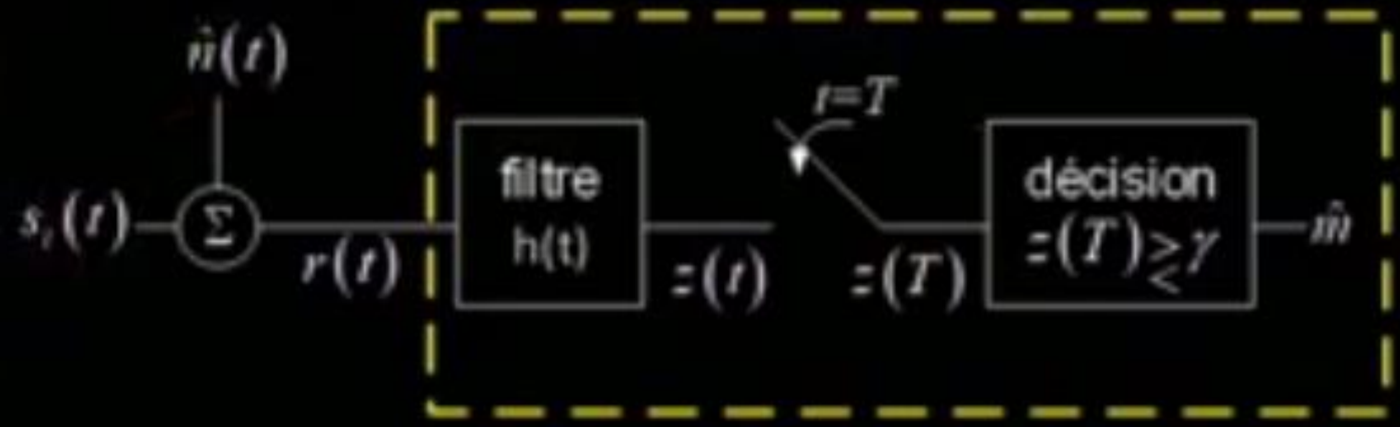
Detection of digital signal



Detection of digital signal: Two steps

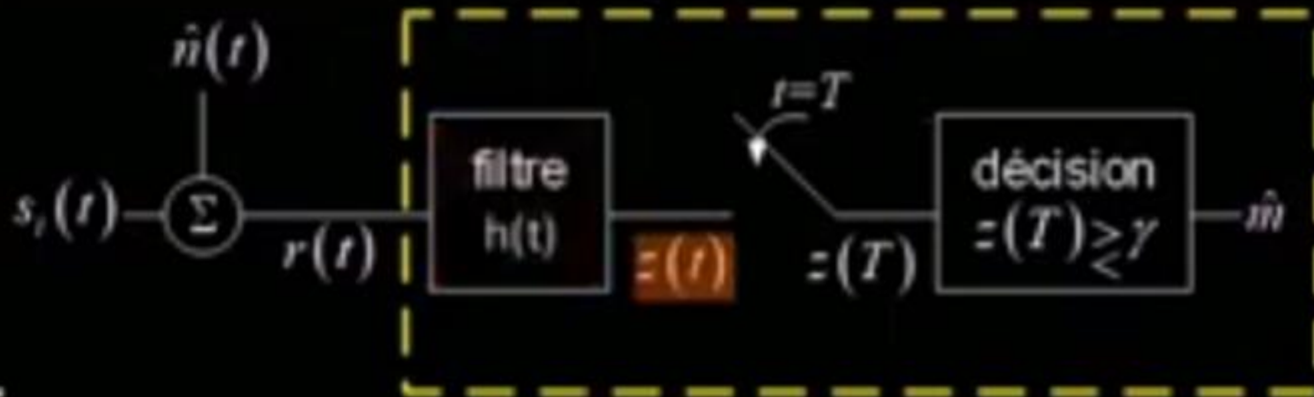


Sampling receiver



- Transmitted signal $s_1(t)$ or $s_0(t)$
- AWGN noise $\hat{n}(t)$
- Received signal $r(t) = s_i(t) + \hat{n}(t)$

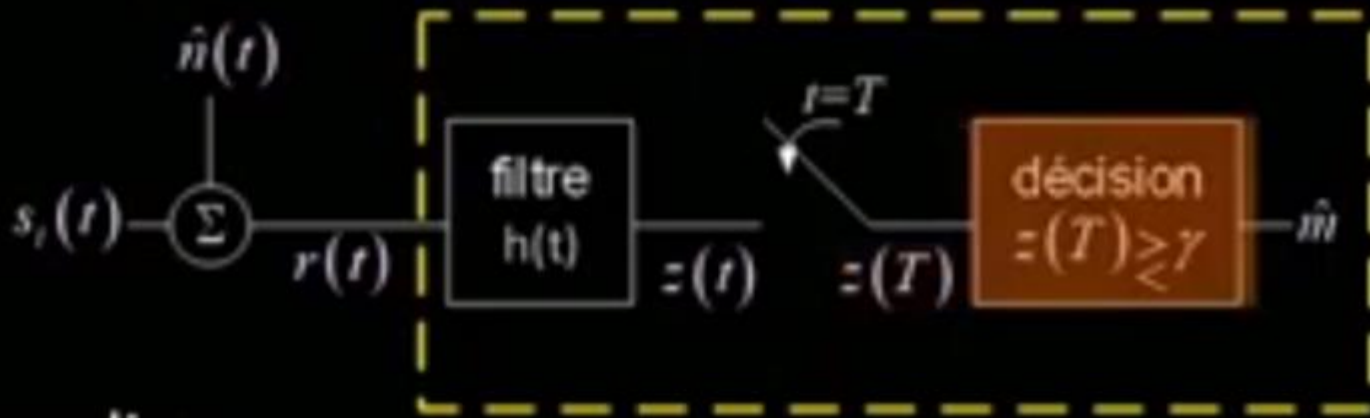
Filtering and sampling



- Filter
  - frequency response $H(f)$
  - impulse response $h(t)$

- Filter output
  - $$z(t) = r(t) * h(t) = s_i(t) * h(t) + \hat{n}(t) * h(t)$$
    $$= a_i(t) + n(t)$$

Test statistic



- After sampling
  - $z(T) = a_i(T) + n(T)$
  - Test statistic
- Detection
  - comparison of the test statistic with a threshold γ
  - decision or estimate of the data $\hat{m}$

Test statistic

- Part that comes from the signal
- $a_0(T) = s_0(t) * h(t)\big|_{t=T}$
  - $a_1(T) = s_1(t) * h(t)\big|_{t=T}$
  - deterministic : $a_0(T)$ ou $a_1(T)$ (two numbers)

- Part that comes from the noise
- $n(T) = \hat{n}(t) * h(t)\big|_{t=T}$
  - $\hat{n}(t)$ is a random Gaussian process
  - $\hat{n}(t) * h(t)$ is a random Gaussian process
  - $n(T)$ is a Gaussian random variable

Noise

- Density of sampled noise
  - $p_n(n) = \frac{1}{\sqrt{2\pi\sigma}} e^{-n^2/2\sigma^2}$ $\qquad n \sim N(0, \sigma^2)$
  - Gaussian of zero mean and variance σ^2

- Density of the test statistic
  - $z(T) = a_i(T) + n(T)$ $\qquad z(T) \sim \eta(a_i(T), \sigma^2)$
  - The sum of a Gaussian random variable and a constant is a Gaussian random variable with a shifted mean.

Vocabulary/notation

- Samples
  - n for $n(T)$
  - z for $z(T)$
  - a_i for $a_i(T)$
- Conditional density
  - $p_z(z|i \text{ sent})$ test statistic when the data "i" was sent
  - $p_z(z|0)$ et $p_z(z|1)$ for example

Vocabulary/notation

- Conditional probability
  - $\Pr\left(s_i \middle| z\right)$ is the probability that the data sent was "i" given the test statistic is z

Conditional densities

- Density of $z(T)$ if "0" was sent

$$p_z(z|0) = p(z = a_0 + n) = p_n(n = z - a_0)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-(z-a_0)^2 / 2\sigma^2}$$

- Density of $z(T)$ if "1" was sent

$$p_z(z|1) = p_Z(z = a_1 + n) = p_n(n = z - a_1)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} e^{-(z-a_1)^2 / 2\sigma^2}$$

- **Bayes Law**
  - **Maximum Aposteriori (MAP)**
  - **Maximum Likelihood (ML)**

Bayes Law

- Recall of this result in probability theory
- $$p(B)\,p(A|B) = p(B|A)\,p(A)$$

- In our application **A= *Si*, B= *z***

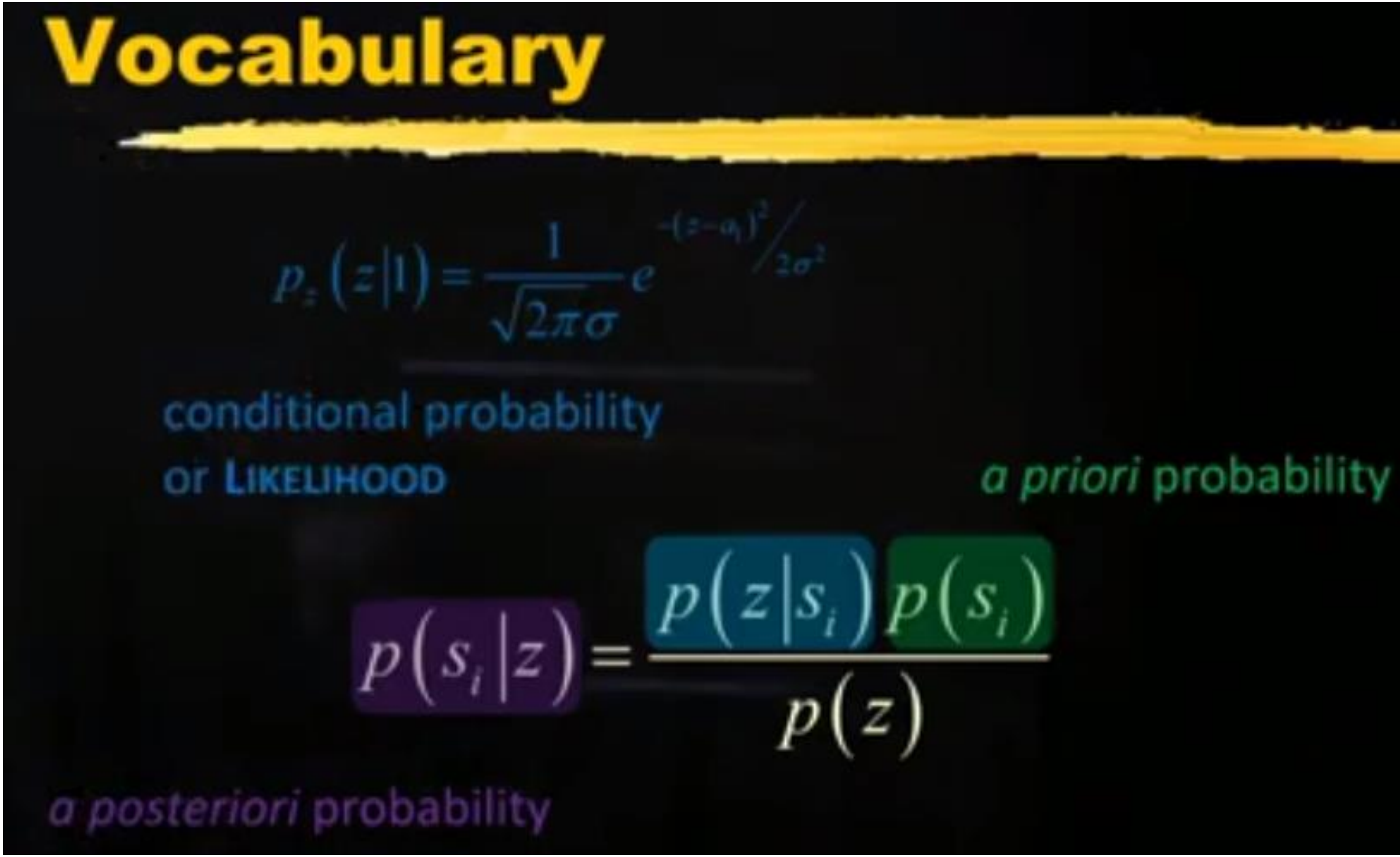
$$p(s_i|z) = \frac{p(z|s_i)\,p(s_i)}{p(z)}$$

Bayes Law

- "a priori" information
- $p(s_i)$ is the probability that s_i is sent
  - does not depend on the method of transmission or reception - it is a feature of the data
  - not always known

"a posteriori" Probability

- The "a posteriori" probability
  - The probability $\text{Prob}(s_i|z)$ that the data " i " was sent given that we measured the test statistic z
  - Direct function of our receiver



How to calculate probability from probability density function (pdf)
Link: https://www.youtube.com/watch?v=fol6lktcmAA

MAP rule

- Maximizes "a posteriori" probability
- A receiver that chooses the data that maximizes probability after an observation is called a "MAP receiver"

  - Choose s_i such that $p(s_i|z)$ is maximized

  - $s_i \ni \max_{j} p(s_j|z) = p(s_i|z)$

MAP Receiver

- Equivalent
  - Maximize $p(s_j|z) = \frac{p(z|s_j)\,p(s_j)}{p(z)}$
  - Maximize $p(z|s_j)\,p(s_j)$

- "a priori" probability
  - $p(s_j)$ side information that has nothing to do with the communication system
  - $p(s_j)$ not always known
  - weighting given to the Likelihood $p(z|s_j)$

ML Receiver

- Maximize the Likelihood (maximum Likelihood)
  - Choose " i " such that $p(z|s_i)$ is maximized
  - $s_i \ni \max_j p(z|s_j) = p(z|s_i)$

- Equivalence between ML & MAP
  - for the case of equal "a priori" probabilities, $p(s_i) = \frac{1}{M}$ the ML and MAP receivers are identical

ML binary example

- M-ary equation

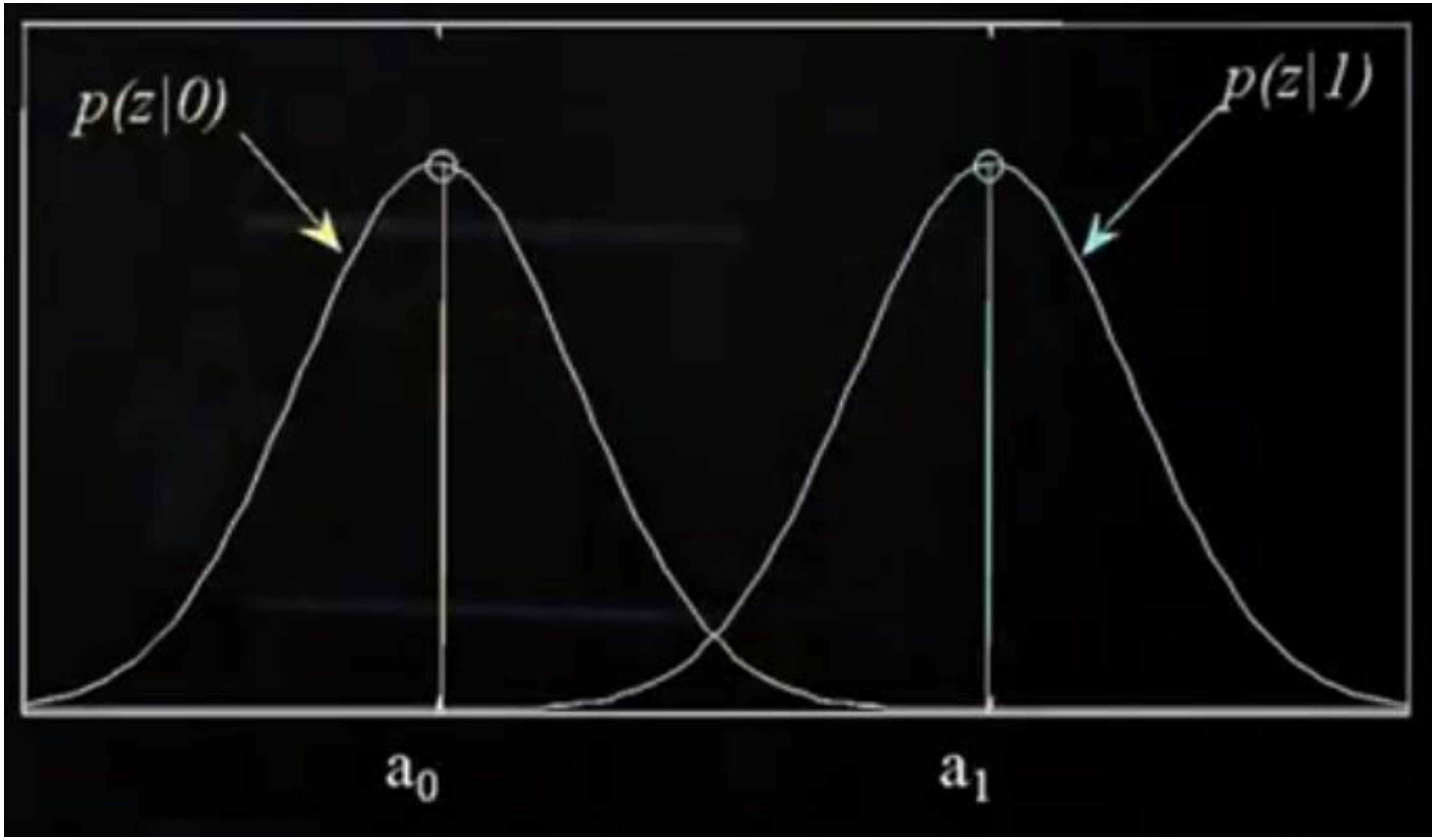
$$\arg \max_j p(z|s_j) = \arg \max_j e^{-(z-a_j)^2/2\sigma^2}$$

- Binary case
- Choose 1 if

$$e^{-(z-a_1)^2/2\sigma^2} > e^{-(z-a_0)^2/2\sigma^2}$$

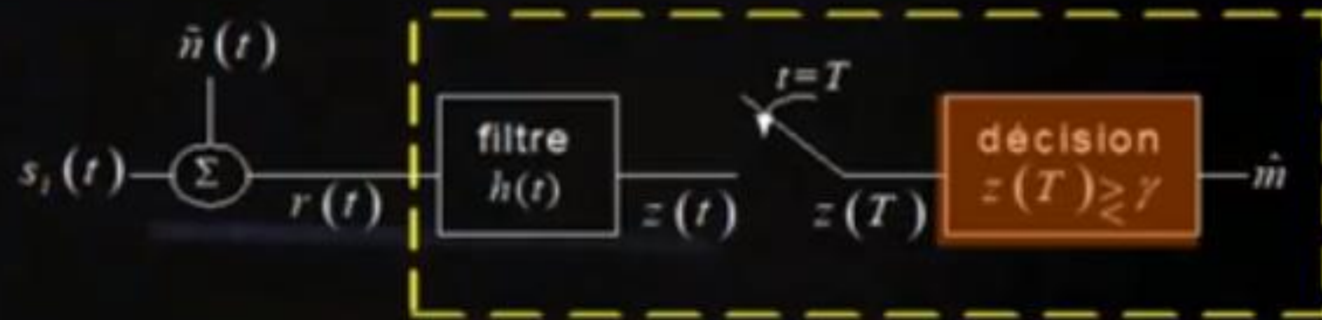
ML rule

Probability density function (Likelihood-pdf) of \`0\` and \`1\`



NOTE: At receiver, received signal will be Gaussian random variable with shifted mean due to \`0\` or \`1\`.

Filtering and sampling



- Output After sampling

signal

AWGN noise

- $z(T) = a_i(T) + n(T)$

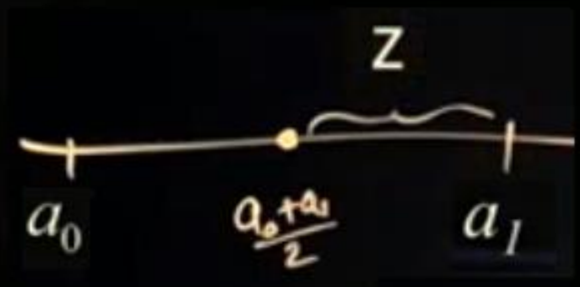
$$e^{-(z-a_1)^2/2\sigma^2} > e^{-(z-a_0)^2/2\sigma^2}$$

ML rule

ML rule, binary case

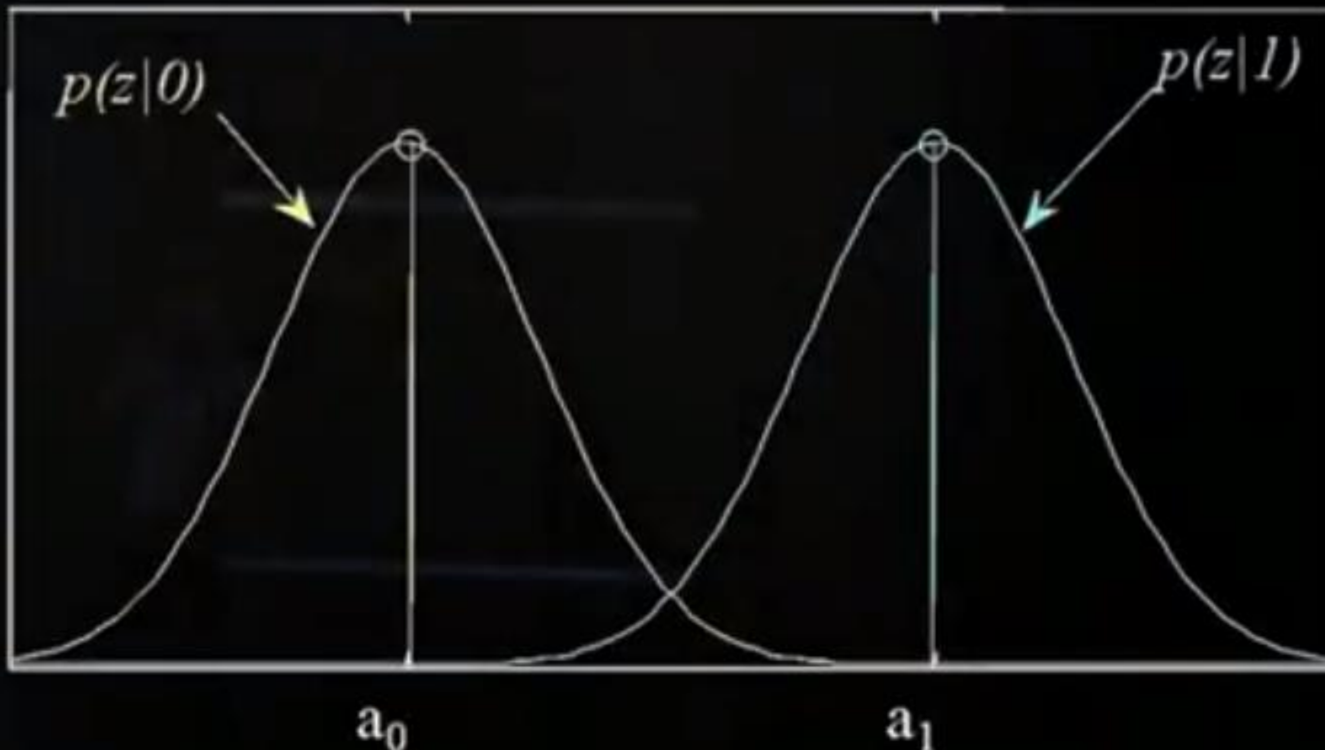
Choose 1 if ...

- $e^{-(z-a_1)^2/2\sigma^2} > e^{-(z-a_0)^2/2\sigma^2}$
- $-(z-a_1)^2/2\sigma^2 > -(z-a_0)^2/2\sigma^2$
- $(z-a_1)^2 < (z-a_0)^2$
- z closer to a_1



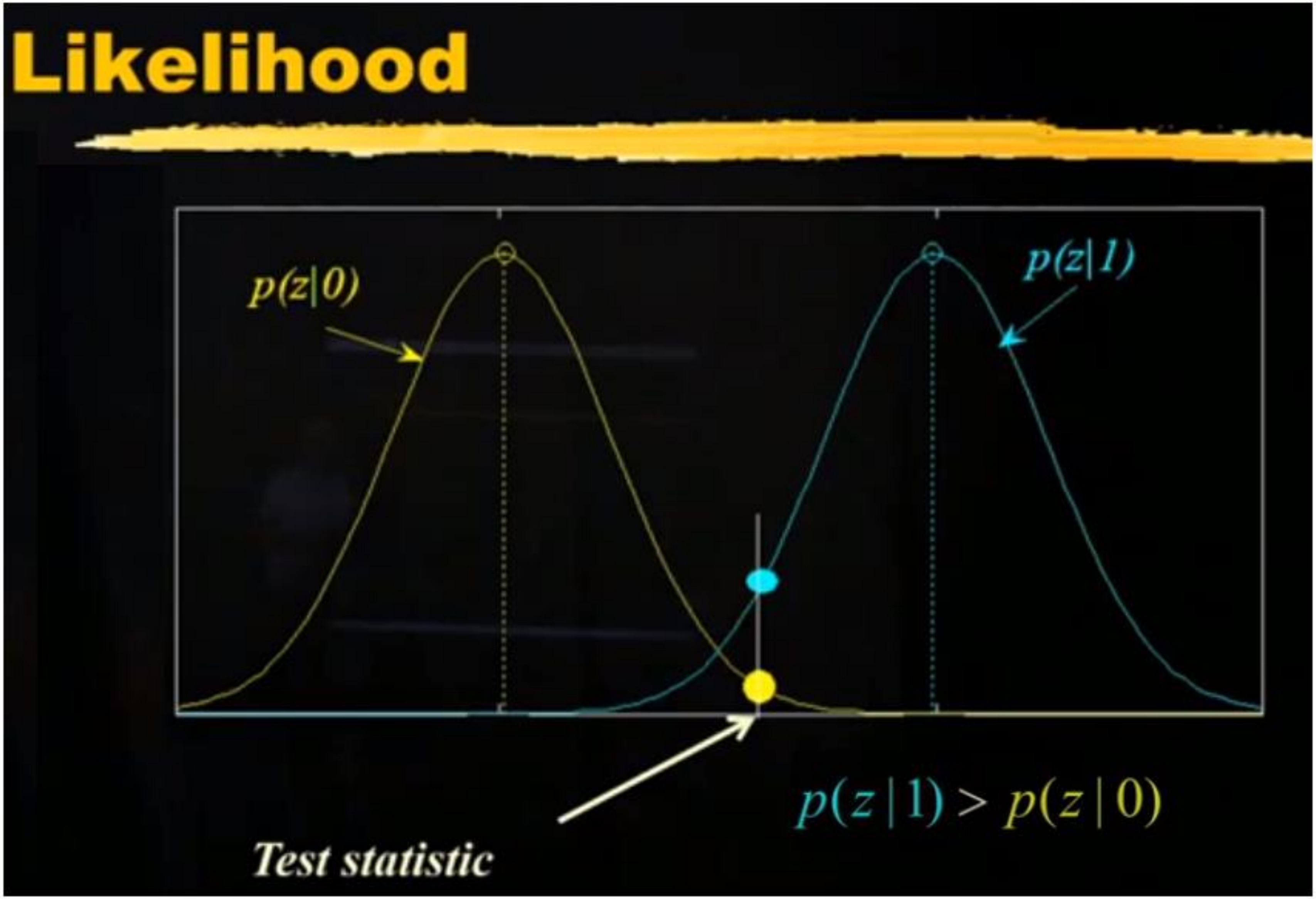
- $z > \dfrac{a_0+a_1}{2}$ assuming $a_1 > a_0$

Likelihood

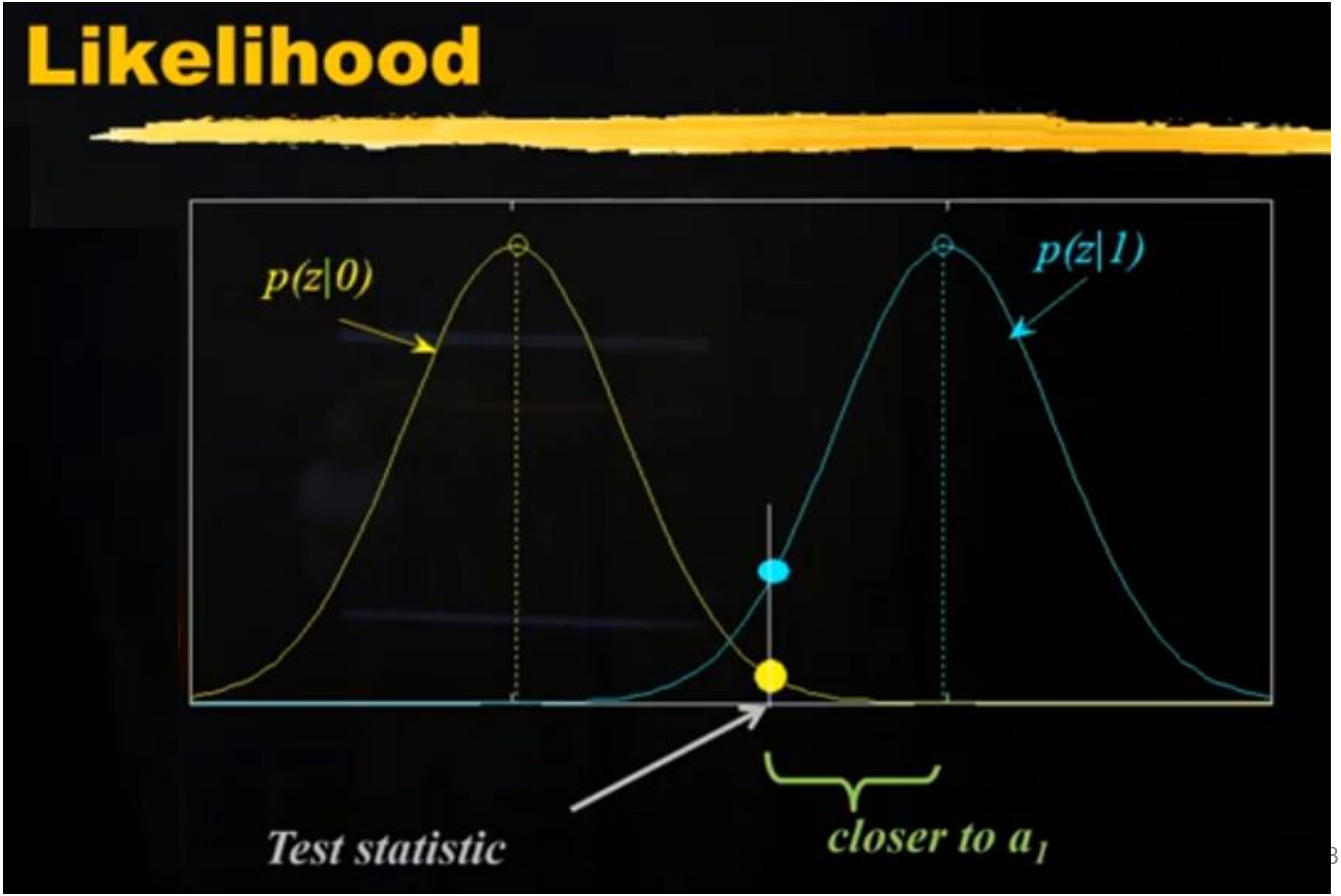


$$z(t) = s_i(t) * h_c(t) * h(t) + \hat{n}(t) * h(t)$$
$$= a_i(t) + n(t)$$

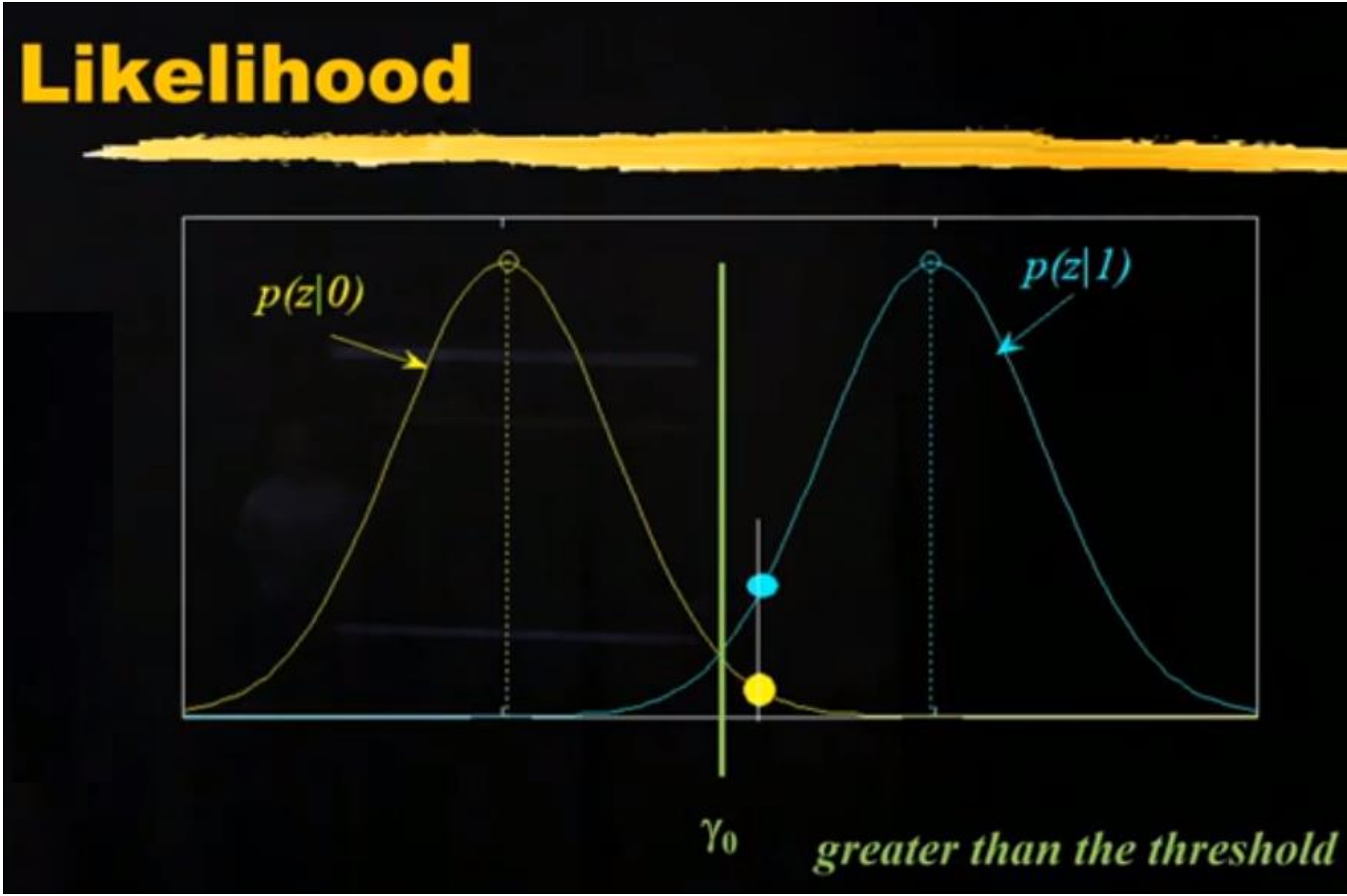
Descision-1: Based on probability



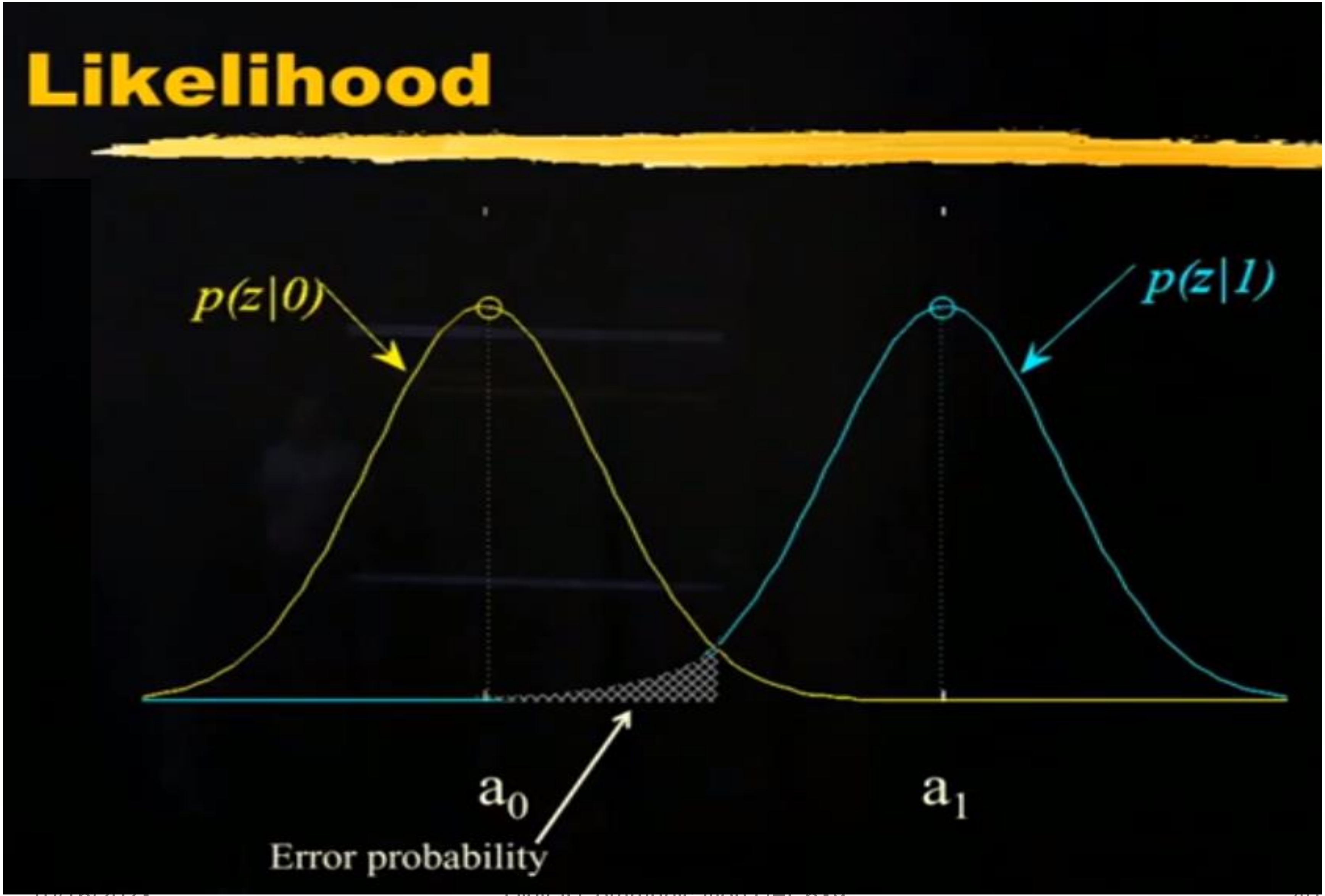
Descision-2: Based on value of `z` distance closure to mean `1` or `0`



Descision-3: Based on a threshold value (mid-point)



Error Probability



MAP binary example

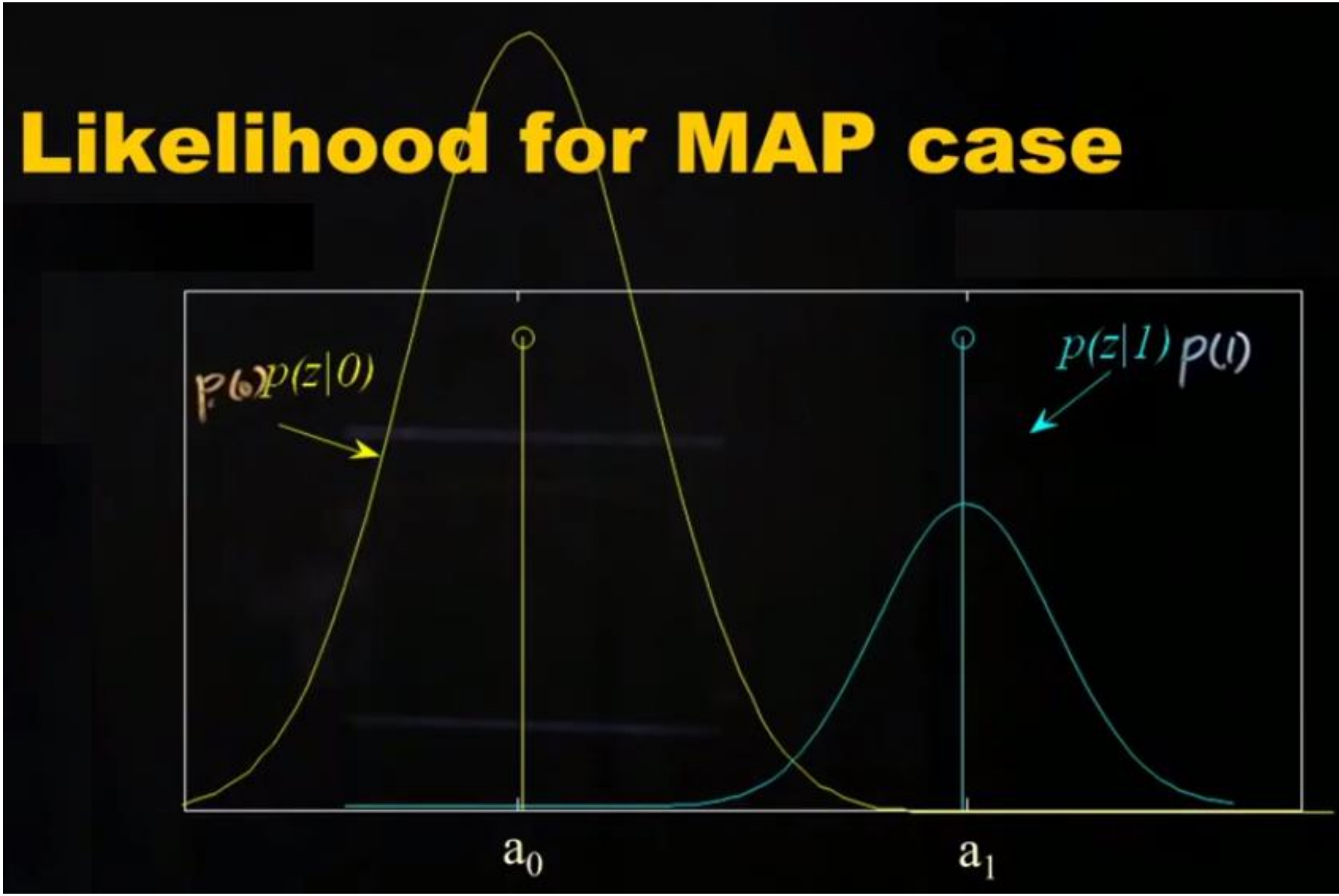
- M-ary equation

$$\arg \max_j p\left(z \middle| s_j\right) p\left(s_i\right) = \arg \max_j p\left(s_j\right) e^{-\left(z-a_j\right)^2 / 2\sigma^2}$$

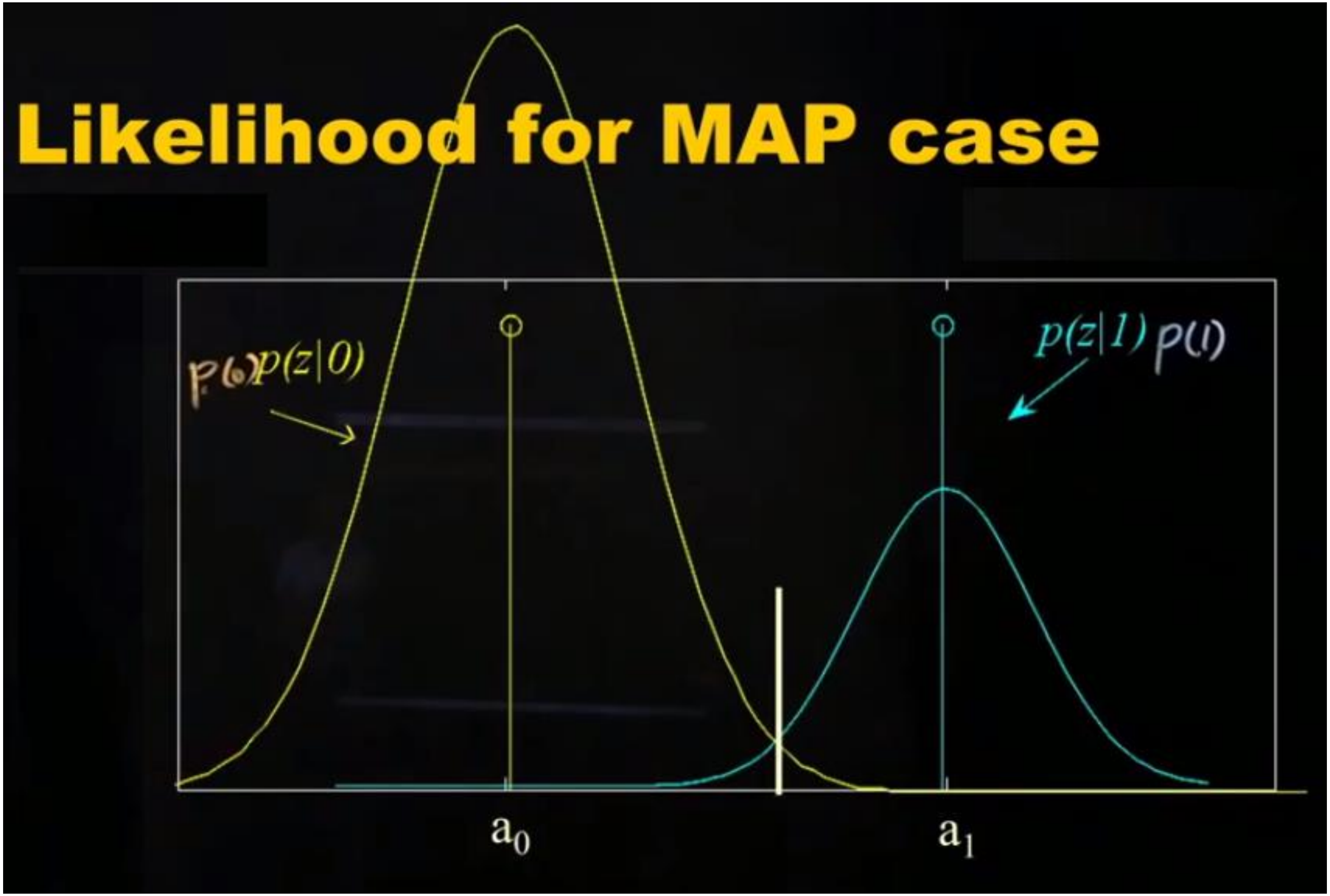
- Binary case

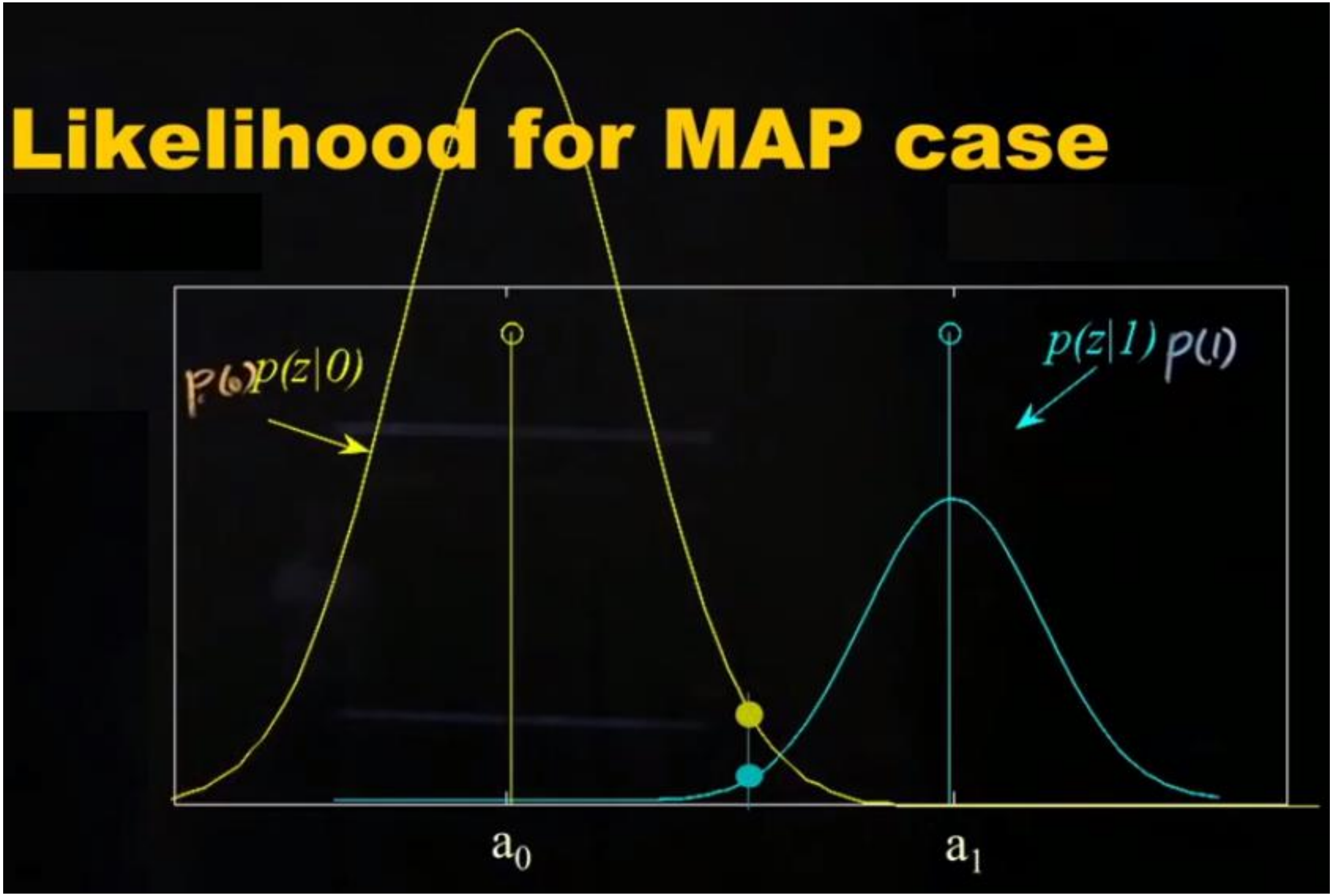
  Choose 1 if

$$p\left(s_1\right) e^{-\left(z-a_1\right)^2 / 2\sigma^2} > p\left(s_0\right) e^{-\left(z-a_0\right)^2 / 2\sigma^2}$$

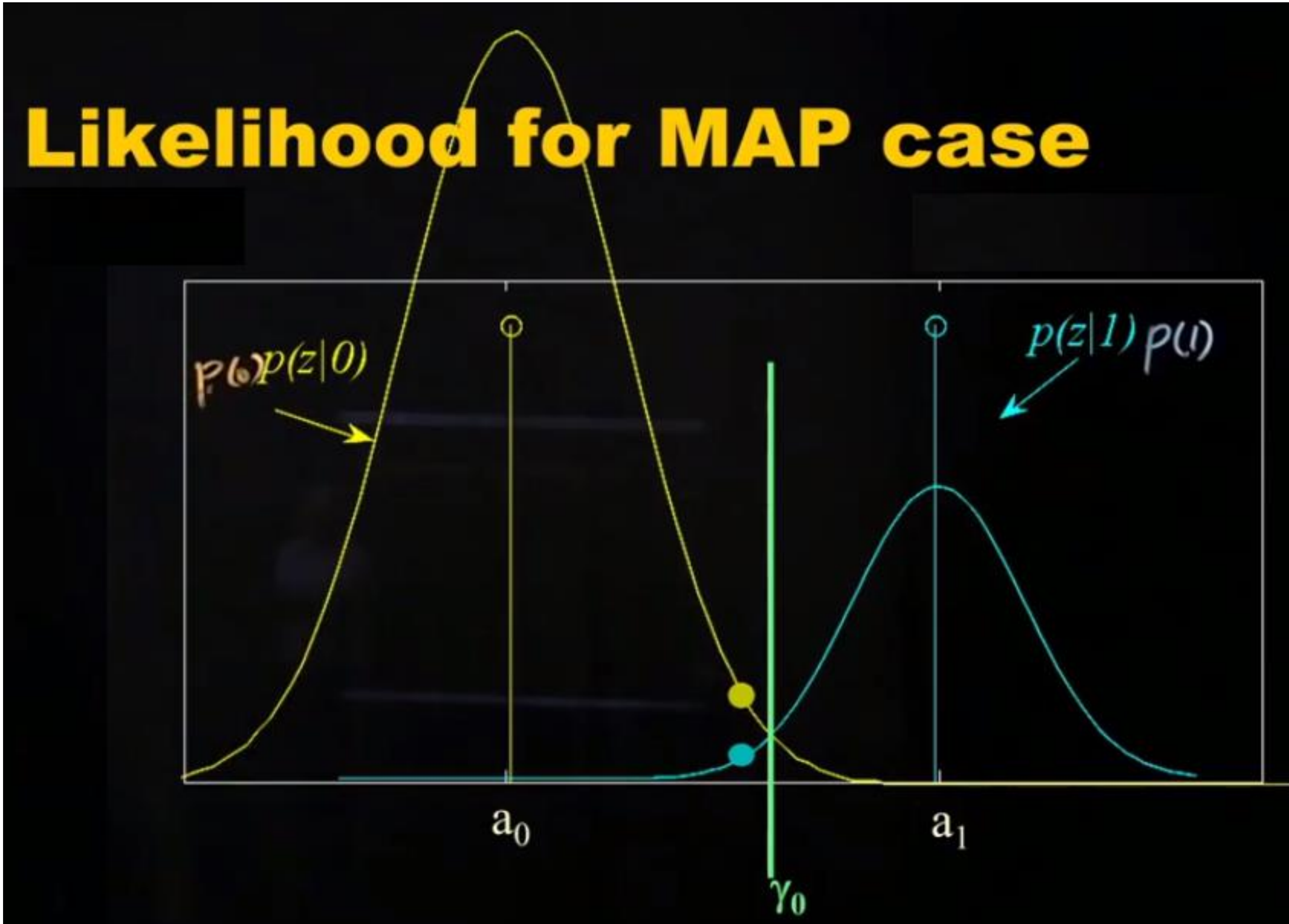

Likelihood for MAP case
p(0)p(z|0)
p(z|1) p(1)
a0
a1

Likelihood for MAP case

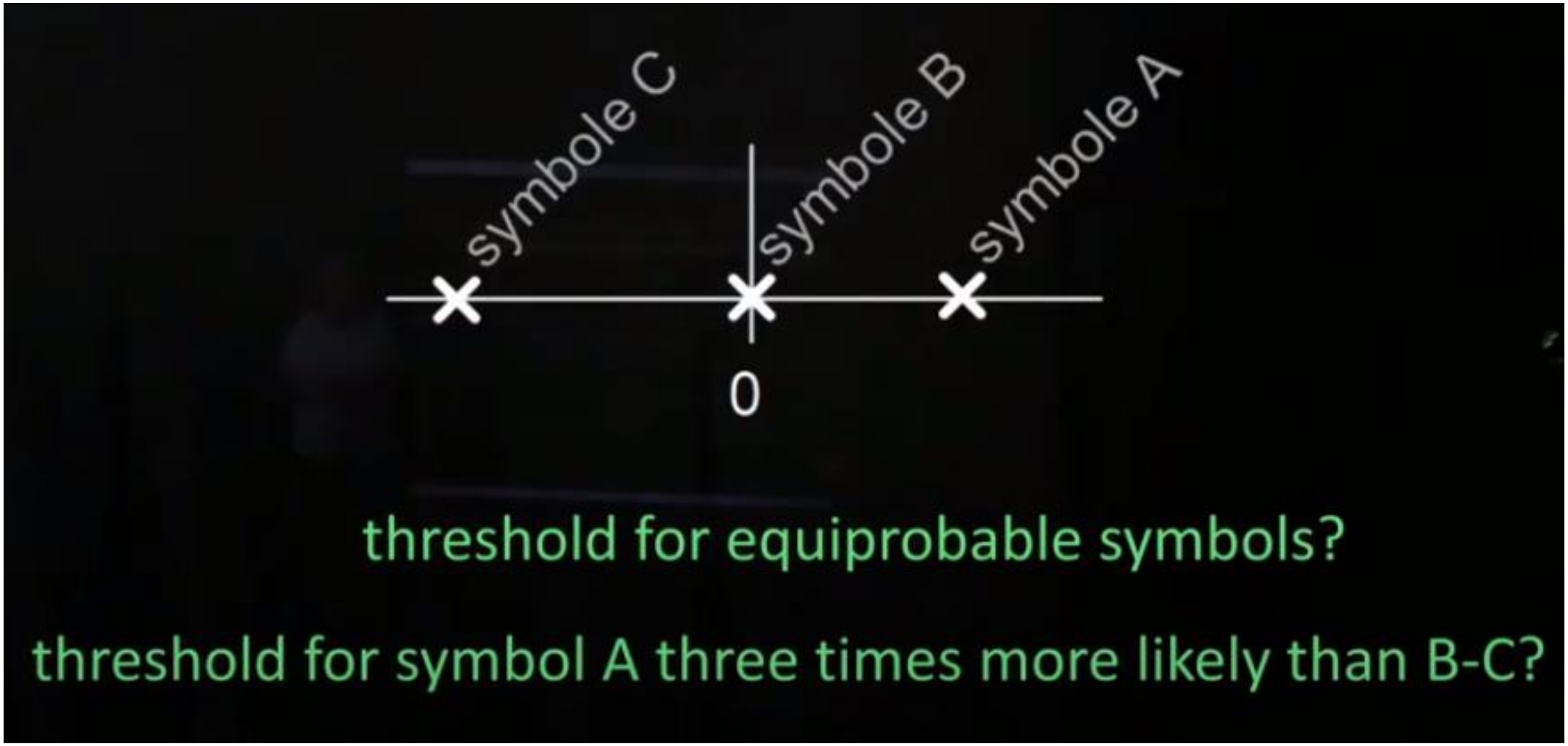



Likelihood for MAP case
P(0)p(z|0)
p(z|1) P(1)
a0
a1

Likelihood for MAP case

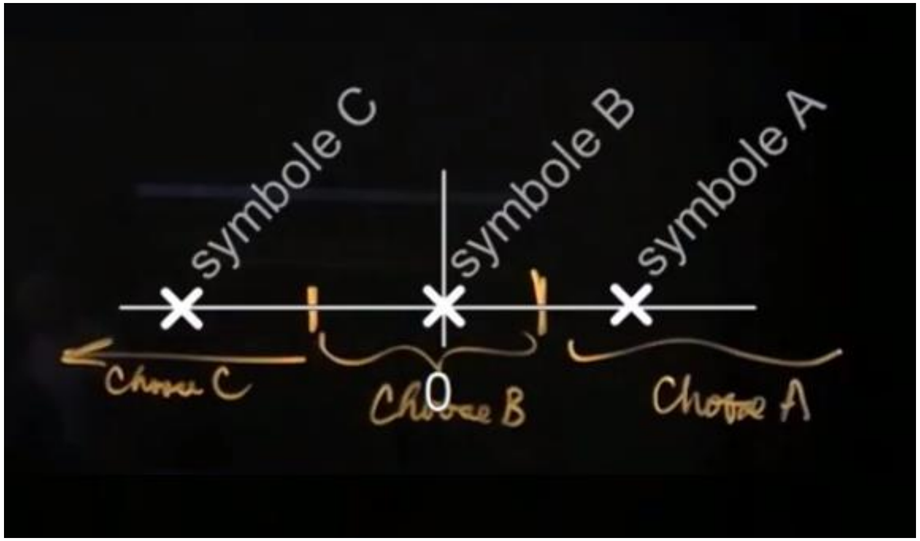


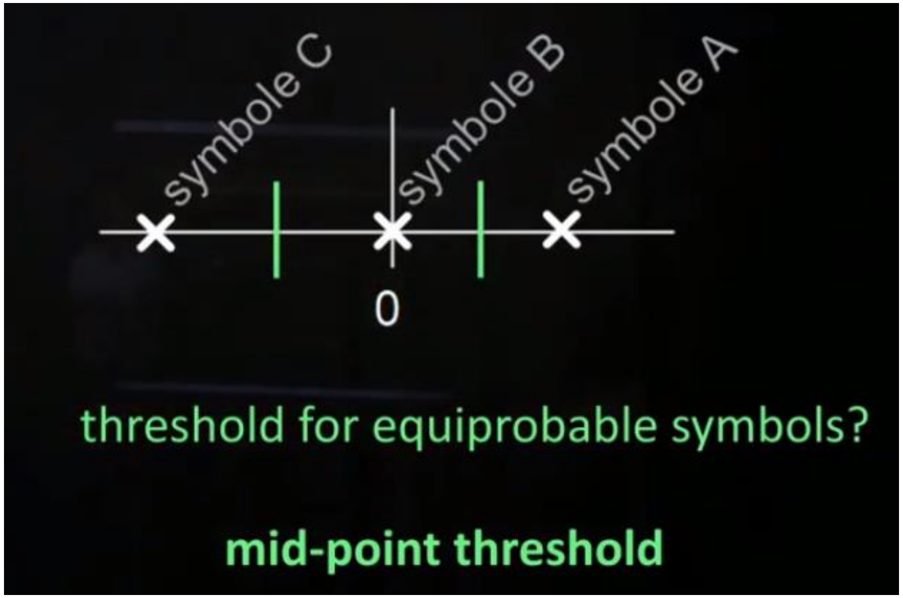
Example

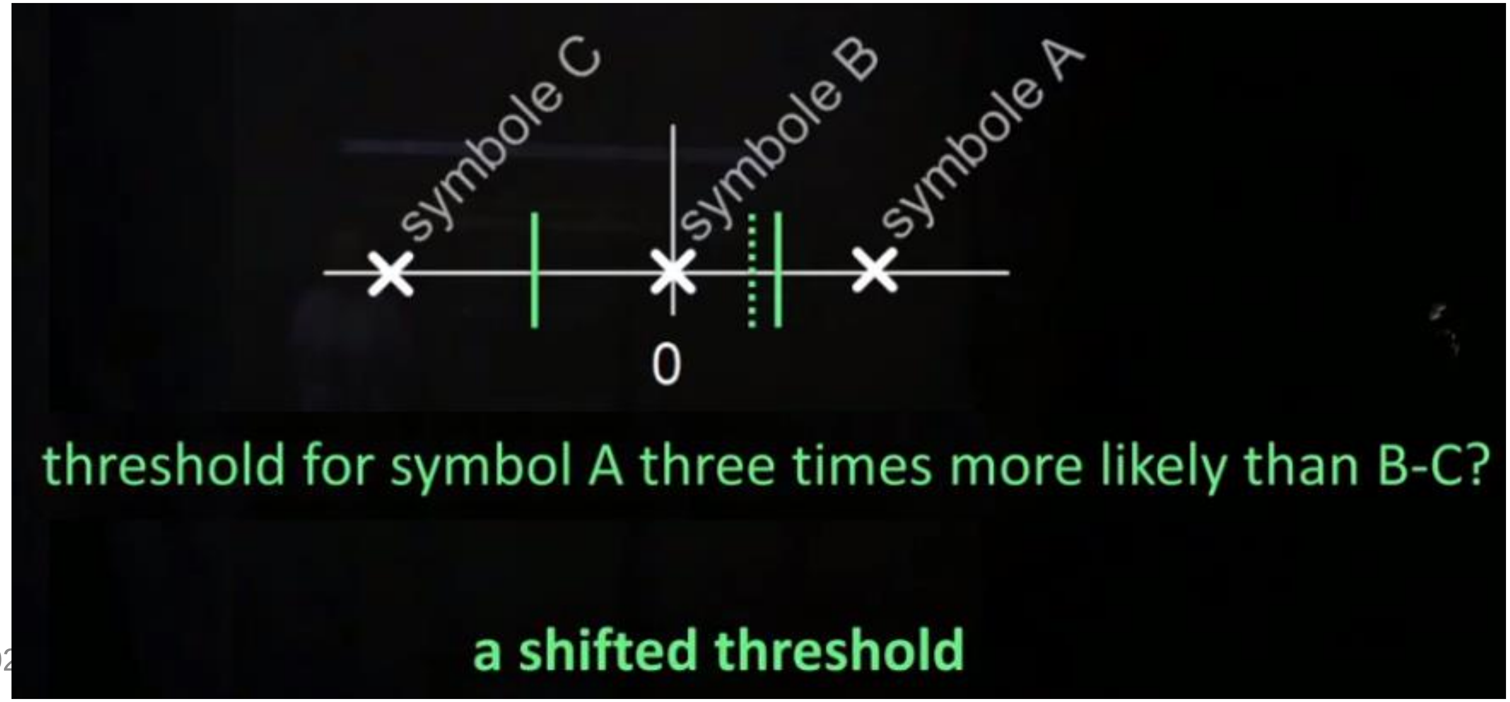


symbole C
symbole B
symbole A
0
threshold for equiprobable symbols?
threshold for symbol A three times more likely than B-C?

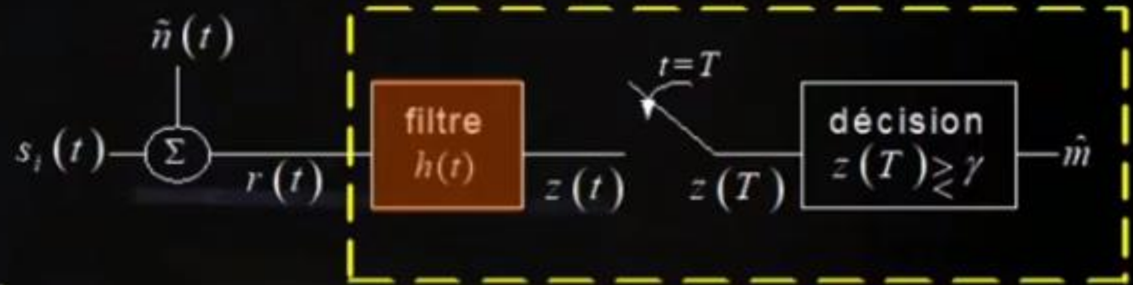
Solution


symbole C
symbole B
symbole A
0


symbole C
symbole B
symbole A
0
threshold for equiprobable symbols?
mid-point threshold


symbole C
symbole B
symbole A
0
threshold for symbol A three times more likely than B-C?
a shifted threshold

Filtering and sampling



- Filter
  - frequency response $H(f)$
- Output after sampling
  - $z(T) = a_i(T) + n(T)$

signal *noise*

How to choose $H(f)$?

SNR= Instantaneous power of Signal / Average power of noise

Instantaneous power

- Signal

inverse FT

$$a_i(T) = s(t) * h(t)\Big|_{t=T} = \int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi ft}\,df\Big|_{t=T} = \int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}\,df$$

- Instantaneous power

$$a_i^2(T) = \left|\int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}\,df\right|^2$$

PSD at filter output

- LTIS with frequency response $H(f)$
- input PSD $G_X(f)$

$$G_X(f) = \frac{N_0}{2}$$

- output PSD

$$G_Y(f) = G_X(f)\left|H(f)\right|^2$$

$$G_Y(f) = \frac{N_0}{2}\left|H(f)\right|^2$$

Average power

- Noise
  - Input power spectral density $$G_X(f) = \frac{N_0}{2}$$
  - Power spectral density after filtering
$$G_Y(f) = \frac{N_0}{2}\left|H(f)\right|^2$$
  - Average power
$$\int_{-\infty}^{\infty} G_Y(f)\,df = \frac{N_0}{2}\int_{-\infty}^{\infty}\left|H(f)\right|^2 df$$

Signal-to-noise ratio

- We want to choose $H(f)$ to maximize

$$\frac{\left|\int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}\,df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}\left|H(f)\right|^2 df}$$

Schwartz inequality

$$\left|\int_{-\infty}^{\infty} g_1(x)\, g_2(x)\, dx\right|^2 \le \int_{-\infty}^{\infty} \left|g_1(x)\right|^2 dx \cdot \int_{-\infty}^{\infty} \left|g_2(x)\right|^2 dx$$

- Equality for

$$g_1(x) = k g_2^*(x)$$

Inequality

$$\left|\int_{-\infty}^{\infty} g_1(x)\, g_2(x)\, dx\right|^2 \le \int_{-\infty}^{\infty} |g_1(x)|^2\, dx \cdot \int_{-\infty}^{\infty} |g_2(x)|^2\, dx$$

$$\frac{\left|\int_{-\infty}^{\infty} S(f)\, H(f)\, e^{j2\pi fT}\, df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\, df}$$

- Define

$$g_1 = H(f) \quad g_2 = S(f)\, e^{-j2\pi fT}$$

$$\left|\int_{-\infty}^{\infty} g_1(x)\, g_2(x)\, dx\right|^2 = \left|\int_{-\infty}^{\infty} S(f)\, H(f)\, e^{j2\pi fT}\, df\right|^2$$

$$\int_{-\infty}^{\infty} |g_1(x)|^2\, dx = \int_{-\infty}^{\infty} |H(f)|^2\, df$$

Signal-to-noise ratio

$$\frac{\left|\int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}\left|H(f)\right|^2 df}$$

$$\left|\int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}df\right|^2 \le \int_{-\infty}^{\infty}\left|H(f)\right|^2 df \cdot \int_{-\infty}^{\infty}\left|S(f)\right|^2 df$$

$$\frac{\left|\int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}df\right|^2}{\int_{-\infty}^{\infty}\left|H(f)\right|^2 df} \le \int_{-\infty}^{\infty}\left|S(f)\right|^2 df$$

Signal-to-noise ratio

$$\frac{\left|\int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}\,df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}\left|H(f)\right|^2 df}$$

$$\left|\int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}\,df\right|^2 \le \int_{-\infty}^{\infty}\left|H(f)\right|^2 df \cdot \int_{-\infty}^{\infty}\left|S(f)\right|^2 df$$

$$\frac{\left|\int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}\,df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}\left|H(f)\right|^2 df} \le \frac{2}{N_0}\int_{-\infty}^{\infty}\left|S(f)\right|^2 df$$

signal energy

continuing...

Schwartz inequality

$$\left|\int_{-\infty}^{\infty} g_1(x)\,g_2(x)\,dx\right|^2 \le \int_{-\infty}^{\infty}\left|g_1(x)\right|^2 dx \cdot \int_{-\infty}^{\infty}\left|g_2(x)\right|^2 dx$$

- Equality for

$$g_1(x) = k g_2^*(x)$$

$$\frac{\left|\int_{-\infty}^{\infty} S(f)H(f)e^{j2\pi fT}\,df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}\left|H(f)\right|^2 df} \le \frac{2}{N_0}\int_{-\infty}^{\infty}\left|S(f)\right|^2 df$$

- Equality for

$$g_1 = k g_2^*$$

$$H(f) = kS^*(f)\,e^{j2\pi fT}$$

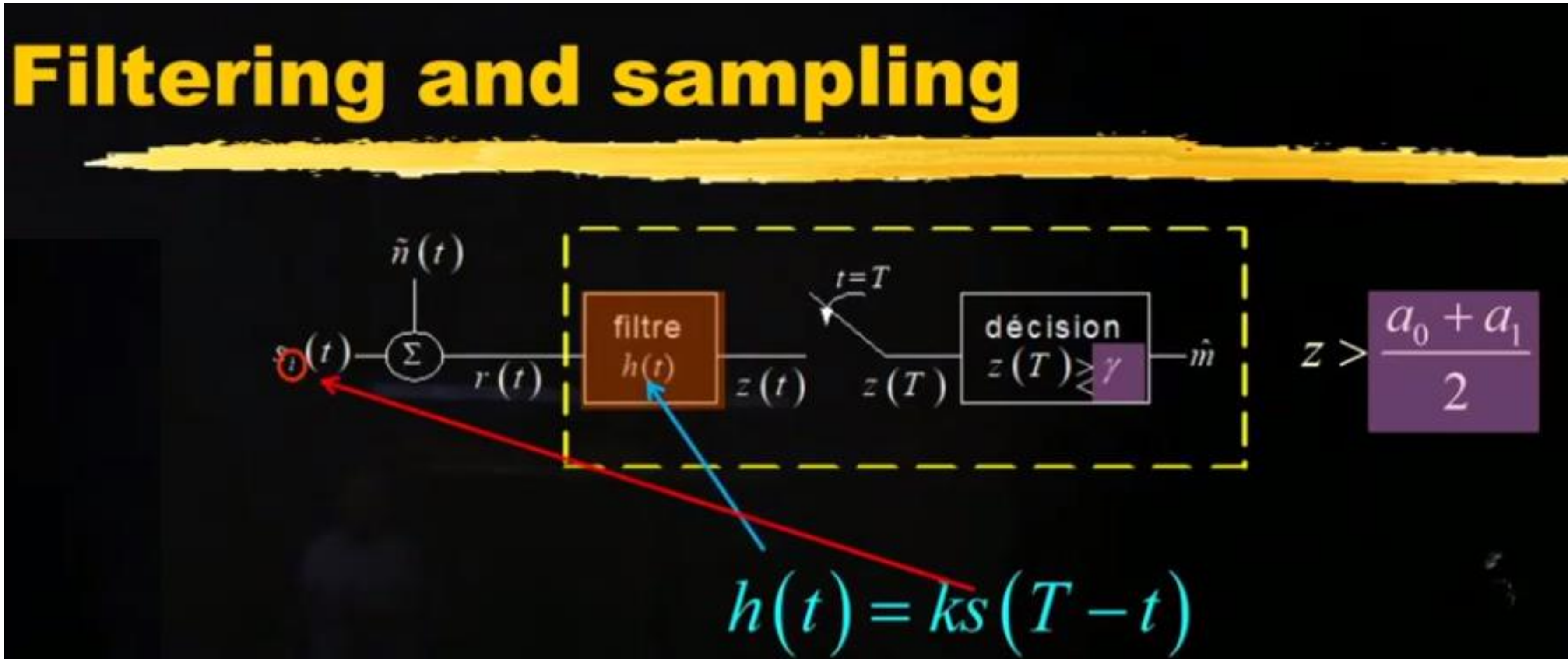
Matched filter

Matched filter

- Equality for $H(f) = kS^*(f)e^{j2\pi fT}$

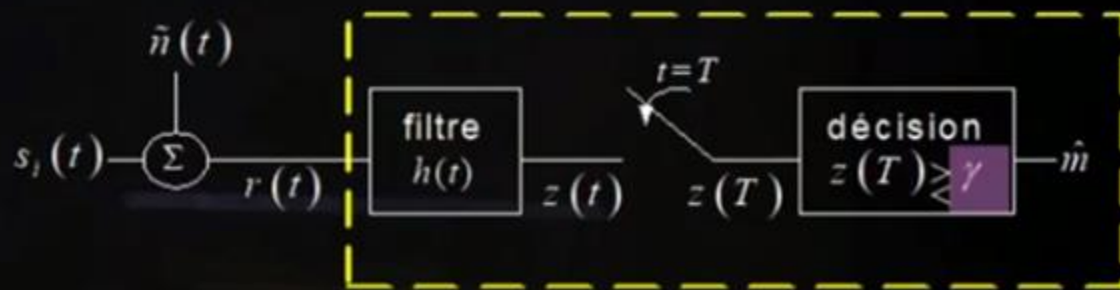
$$h(t) = TF^{-1}\left\{kS^*(f)e^{j2\pi fT}\right\} = \begin{cases} ks(T-t) & 0 \le t \le T \\ 0 & elsewhere \end{cases}$$

assuming $s(t) = 0$ for t other than $0 \le t \le T$

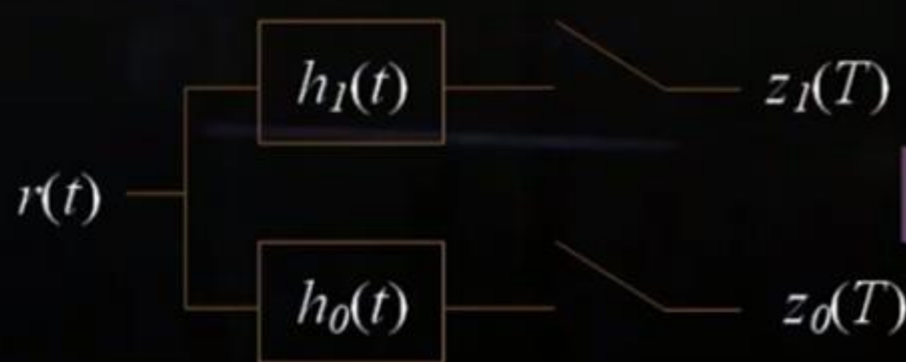


Filtering and sampling
$\tilde{n}(t)$
$s_i(t)$
Σ
$r(t)$
filtre
$h(t)$
$z(t)$
$t=T$
$z(T)$
décision
$z(T) \gtrless \gamma$
$\hat{m}$
$z > \frac{a_0 + a_1}{2}$
$h(t) = ks(T-t)$

Filtering and sampling

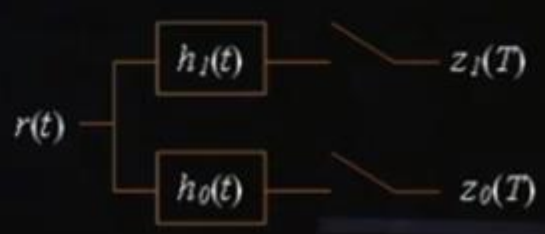


$$z > \frac{a_0 + a_1}{2}$$



choose the largest

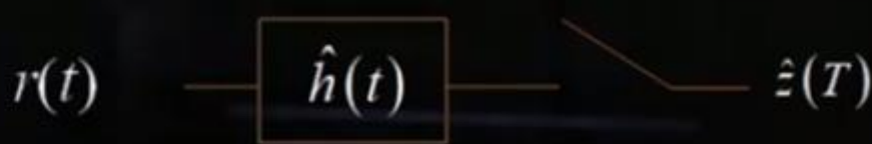
Filtering and sampling



choose the largest

$$\hat{h}(t) = h_1(t) - h_0(t)$$

$$\hat{z}(T) = z_1(T) - z_0(T)$$



If the signals are of equal energy...

Choose 1 if $\hat{z}(T) > 0$

- Maximize a posteriori probability (MAP)
- Maximize the Likelihood (ML)
- Use a matched filter to maximize the SNR
- The choice of waveform is arbitrary

- We only considered 2 symbols…

- Next: generalization M symbols & BER

Thanks !