Consider the signal

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right).$$

This signal is periodic with period N, and, as in Example 3.10, we can expand $x[n]$ directly in terms of complex exponentials to obtain

$$x[n] = 1 + \frac{1}{2j}[e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}] + \frac{3}{2}[e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}]$$

$$+ \frac{1}{2}[e^{j(4\pi n/N+\pi/2)} + e^{-j(4\pi n/N+\pi/2)}].$$

Collecting terms, we find that

$$x[n] = 1 + \left(\frac{3}{2} + \frac{1}{2j}\right)e^{j(2\pi/N)n} + \left(\frac{3}{2} - \frac{1}{2j}\right)e^{-j(2\pi/N)n}$$

$$+ \left(\frac{1}{2}e^{j\pi/2}\right)e^{j2(2\pi/N)n} + \left(\frac{1}{2}e^{-j\pi/2}\right)e^{-j2(2\pi/N)n}$$

Thus the Fourier series coefficients for this example are

$$a_0 = 1,$$

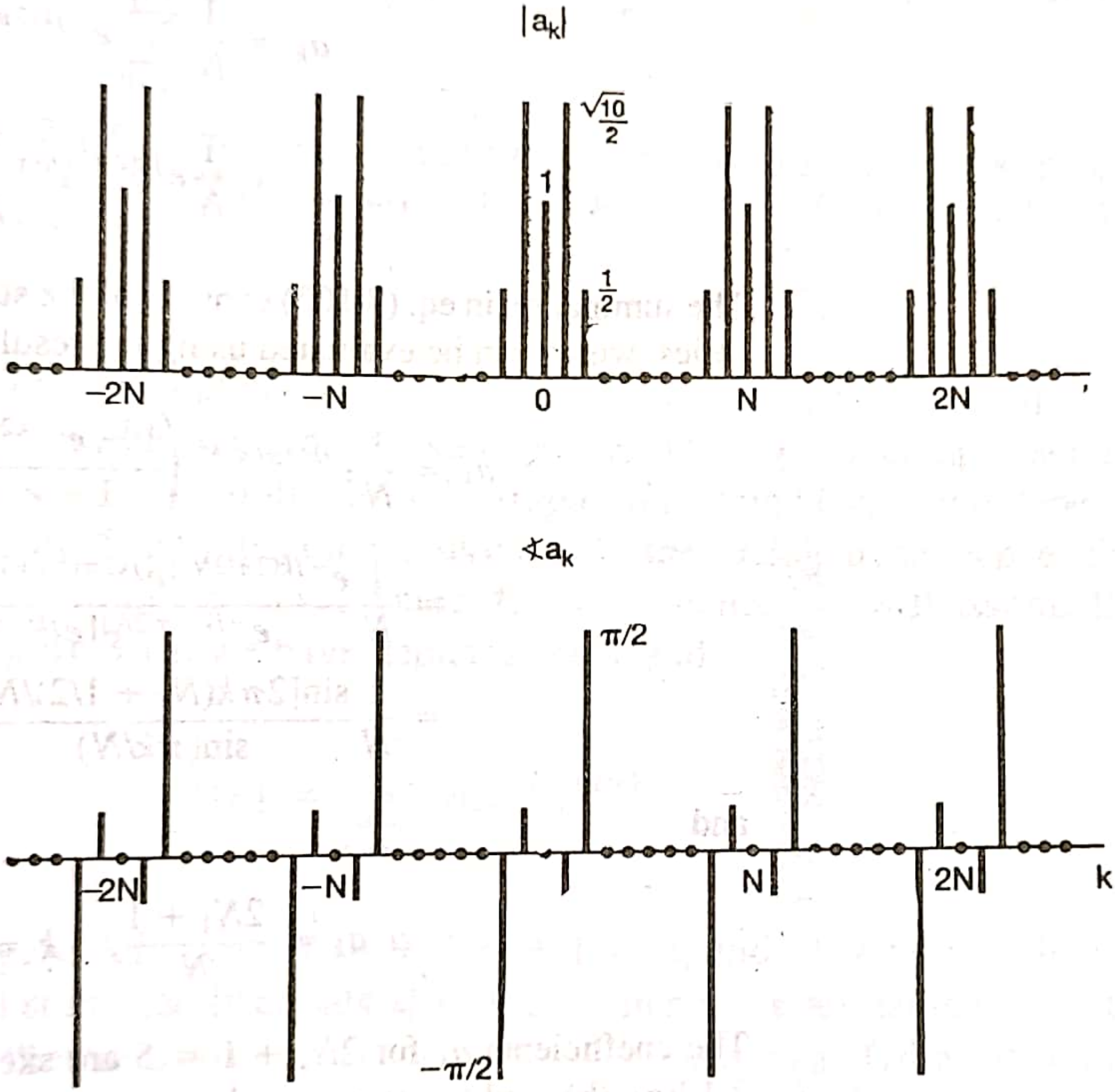
$$a_1 = \frac{3}{2} + \frac{1}{2j} = \frac{3}{2} - \frac{1}{2}j,$$

$$a_{-1} = \frac{3}{2} - \frac{1}{2j} = \frac{3}{2} + \frac{1}{2}j,$$

$$a_2 = \frac{1}{2}j,$$

$$a_{-2} = -\frac{1}{2}j,$$

with $a_k = 0$ for other values of k in the interval of summation in the synthesis equation (3.94). Again, the Fourier coefficients are periodic with period N, so, for example, $a_N = 1$, $a_{3N-1} = \frac{3}{2} + \frac{1}{2}j$, and $a_{2-N} = \frac{1}{2}j$. In Figure 3.15(a) we have plotted the real and imaginary parts of these coefficients for $N = 10$, while the magnitude and phase of the coefficients are depicted in Figure 3.15(b).



$|a_k|$
$\frac{\sqrt{10}}{2}$
1
$\frac{1}{2}$
−2N
−N
0
N
2N
$\sphericalangle a_k$
π/2
−π/2
k

2.3.8 The Unit Step Response of an LTI System

Up to now, we have seen that the representation of an LTI system in terms of its unit impulse response allows us to obtain very explicit characterizations of system properties. Specifically, since $h[n]$ or $h(t)$ completely determines the behavior of an LTI system, we have been able to relate system properties such as stability and causality to properties of the impulse response.

There is another signal that is also used quite often in describing the behavior of LTI systems: the *unit step response*, $s[n]$ or $s(t)$, corresponding to the output when $x[n] = u[n]$ or $x(t) = u(t)$. We will find it useful on occasion to refer to the step response, and therefore, it is worthwhile relating it to the impulse response. From the convolution-sum representation, the step response of a discrete-time LTI system is the convolution of the unit step with the impulse response; that is,

$$s[n] = u[n] * h[n].$$

However, by the commutative property of convolution, $s[n] = h[n] * u[n]$, and therefore, $s[n]$ can be viewed as the response to the input $h[n]$ of a discrete-time LTI system with unit impulse response $u[n]$. As we have seen in Example 2.12, $u[n]$ is the unit impulse response of the accumulator. Therefore,

$$s[n] = \sum_{k=-\infty}^{n} h[k]. \tag{2.91}$$

From this equation and from Example 2.12, it is clear that $h[n]$ can be recovered from $s[n]$ using the relation

$$h[n] = s[n] - s[n-1]. \tag{2.92}$$

That is, the step response of a discrete-time LTI system is the running sum of its impulse response [eq. (2.91)]. Conversely, the impulse response of a discrete-time LTI system is the first difference of its step response [eq. (2.92)].

Similarly, in continuous time, the step response of an LTI system with impulse response $h(t)$ is given by $s(t) = u(t) * h(t)$, which also equals the response of an integrator [with impulse response $u(t)$] to the input $h(t)$. That is, the unit step response of a continuous-time LTI system is the running integral of its impulse response, or

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau, \tag{2.93}$$

and from eq. (2.93), the unit impulse response is the first derivative of the unit step response,[1] or

$$h(t) = \frac{ds(t)}{dt} = s'(t). \tag{2.94}$$

Therefore, in both continuous and discrete time, the unit step response can also be used to characterize an LTI system, since we can calculate the unit impulse response from it. In Problem 2.45, expressions analogous to the convolution sum and convolution integral are derived for the representations of an LTI system in terms of its unit step response.

$$y[n] = \frac{1}{2}(x[n] + x[n-1]). \qquad (3.138)$$

[11]Specifically each image in Figure 3.24(b) is the magnitude of the two-dimensional gradient of its counterpart image in Figure 3.24(a) where the gradient of $f(x, y)$ is defined as

$$\left[\left(\frac{\partial f(x, y)}{\partial x}\right)^2 + \left(\frac{\partial f(x, y)}{\partial y}\right)^2\right]^{1/2}$$

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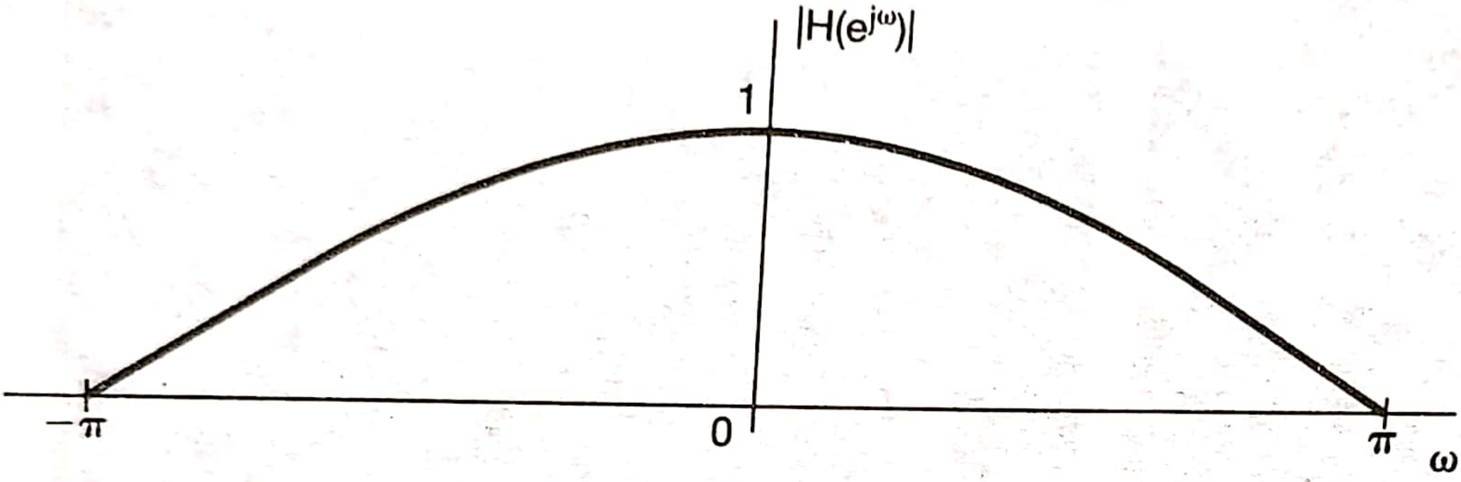
In this case $h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$, and from eq. (3.122), we see that the frequency response of the system is

$$H(e^{j\omega}) = \frac{1}{2}[1 + e^{-j\omega}] = e^{-j\omega/2} \cos(\omega/2). \tag{3.139}$$

The magnitude of $H(e^{j\omega})$ is plotted in Figure 3.25(a), and $\sphericalangle H(e^{j\omega})$ is shown in Figure 3.25(b). As discussed in Section 1.3.3, low frequencies for discrete-time complex exponentials occur near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$, and high frequencies near $\omega = \pm\pi, \pm 3\pi, \ldots$. This is a result of the fact that $e^{j(\omega+2\pi)n} = e^{j\omega n}$, so that in discrete time we need only consider a 2π interval of values of ω in order to cover a complete range of distinct discrete-time frequencies. As a consequence, any discrete-time frequency responses $H(e^{j\omega})$ must be periodic with period 2π, a fact that can also be deduced directly from eq. (3.122).

For the specific filter defined in eqs. (3.138) and (3.139), we see from Figure 3.25(a) that $|H(e^{j\omega})|$ is large for frequencies near $\omega = 0$ and decreases as we increase $|\omega|$ toward π, indicating that higher frequencies are attentuated more than lower ones. For example, if the input to this system is constant—i.e., a zero-frequency complex exponential



(a)

$$y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1]), \tag{3.158}$$

so that each output $y[n]$ is the average of three consecutive input values. In this case,

$$h[n] = \frac{1}{3}[\delta[n+1] + \delta[n] + \delta[n-1]],$$

and thus, from eq. (3.122), the corresponding frequency response is

$$H(e^{j\omega}) = \frac{1}{3}[e^{j\omega} + 1 + e^{-j\omega}] = \frac{1}{3}(1 + 2\cos\omega). \tag{3.159}$$

The magnitude of $H(e^{j\omega})$ is sketched in Figure 3.35. We observe that the filter has the general characteristics of a lowpass filter, although, as with the first-order recursive filter, it does not have a sharp transition from passband to stopband.

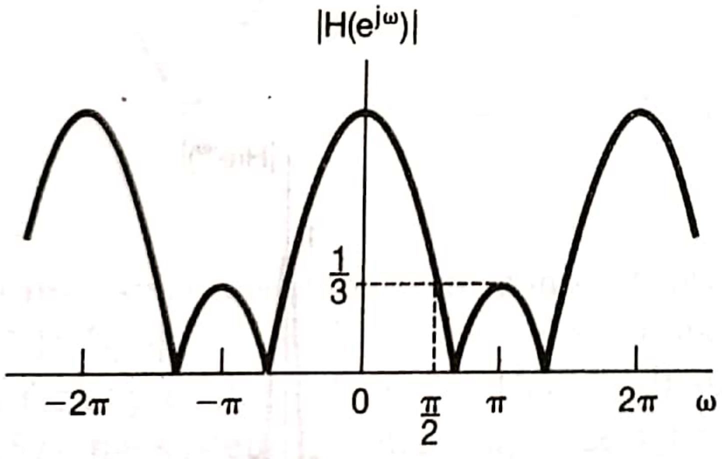


Figure 3.35 Magnitude of the frequency response of a three-point moving-average lowpass filter.

Nonrecursive filters can also be used to perform highpass filtering operations. To illustrate this, again with a simple example, consider the difference equation

$$y[n] = \frac{x[n] - x[n-1]}{2}. \tag{3.163}$$

For input signals that are approximately constant, the value of $y[n]$ is close to zero. For input signals that vary greatly from sample to sample, the values of $y[n]$ can be expected to have larger amplitude. Thus, the system described by eq. (3.163) approximates a highpass filtering operation, attenuating slowly varying low-frequency components and passing rapidly varying higher frequency components with little attenuation. To see this more precisely we need to look at the system's frequency response. In this case, $h[n] = \frac{1}{2}\{\delta[n] - \delta[n-1]\}$, so that direct application of eq. (3.122) yields

$$H(e^{j\omega}) = \frac{1}{2}[1 - e^{-j\omega}] = je^{j\omega/2}\sin(\omega/2). \tag{3.164}$$

In Figure 3.37 we have plotted the magnitude of $H(e^{j\omega})$, showing that this simple system approximates a highpass filter, albeit one with a very gradual transition from passband to stopband. By considering more general nonrecursive filters, we can achieve far sharper transitions in lowpass, highpass, and other frequency-selective filters.

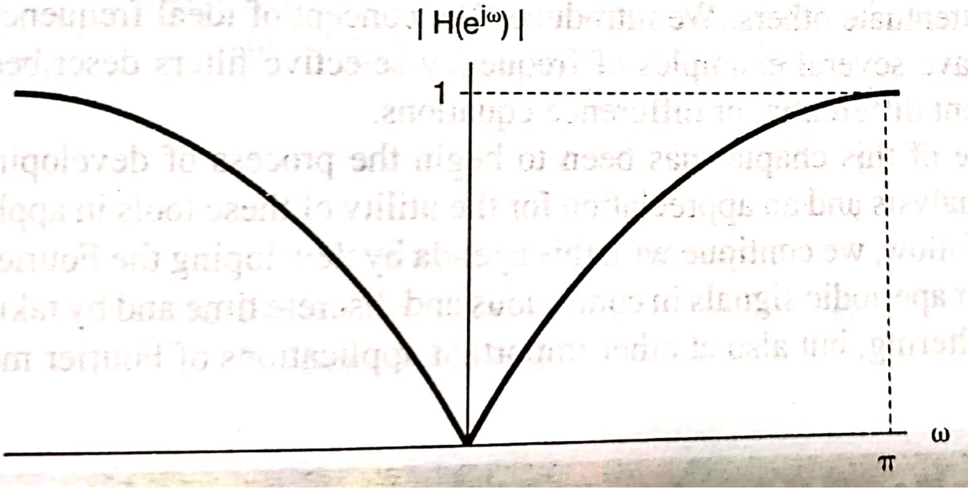


Figure 3.37 Frequency response of a simple highpass filter.

A signal that we will find extremely useful in our analysis of sampling systems in Chapter 7 is the impulse train

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

which is periodic with period T, as indicated in Figure 4.14(a). The Fourier series coefficients for this signal were computed in Example 3.8 and are given by

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} \delta(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}.$$

That is, every Fourier coefficient of the periodic impulse train has the same value, $1/T$. Substituting this value for a_k in eq. (4.22) yields

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right).$$

Thus, the Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2\pi/T$, as sketched in Figure 4.14(b). Here again, we see an illustration of the inverse relationship between the time and the frequency domains. As the spacing between the impulses in the time domain (i.e., the period) gets longer, the spacing between the impulses in the frequency domain (namely, the fundamental frequency) gets smaller.

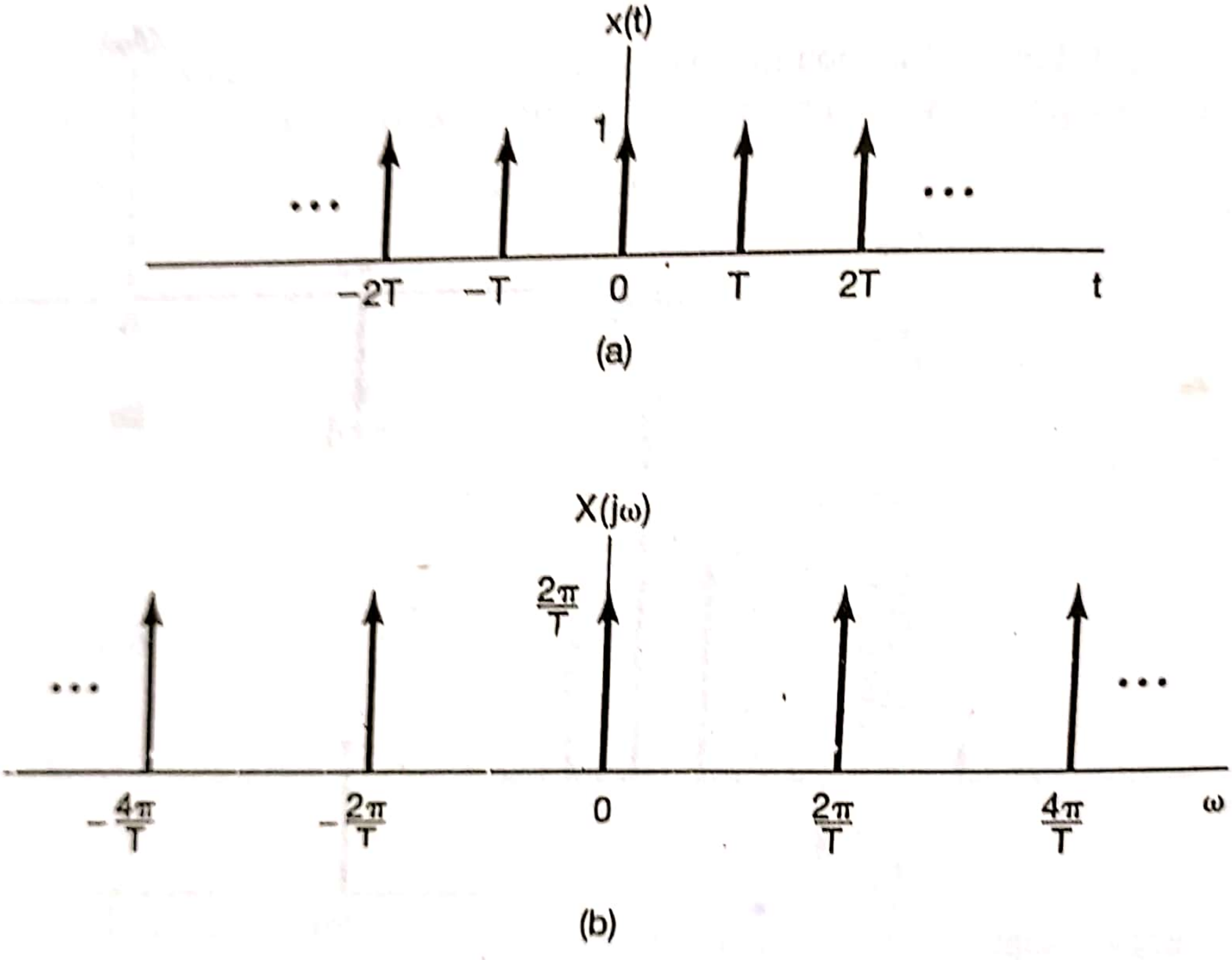


Figure 4.14 (a) Periodic impulse train; (b) its Fourier transform.

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The discrete-time counterpart of the periodic impulse train of Example 4.8 is the sequence

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN], \tag{5.25}$$

as sketched in Figure 5.11(a). The Fourier series coefficients for this signal can be calculated directly from eq. (3.95):

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

Choosing the interval of summation as $0 \leq n \leq N-1$, we have

$$a_k = \frac{1}{N}. \tag{5.26}$$

Using eqs. (5.26) and (5.20), we can then represent the Fourier transform of the signal as

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right), \tag{5.27}$$

which is illustrated in Figure 5.11(b).

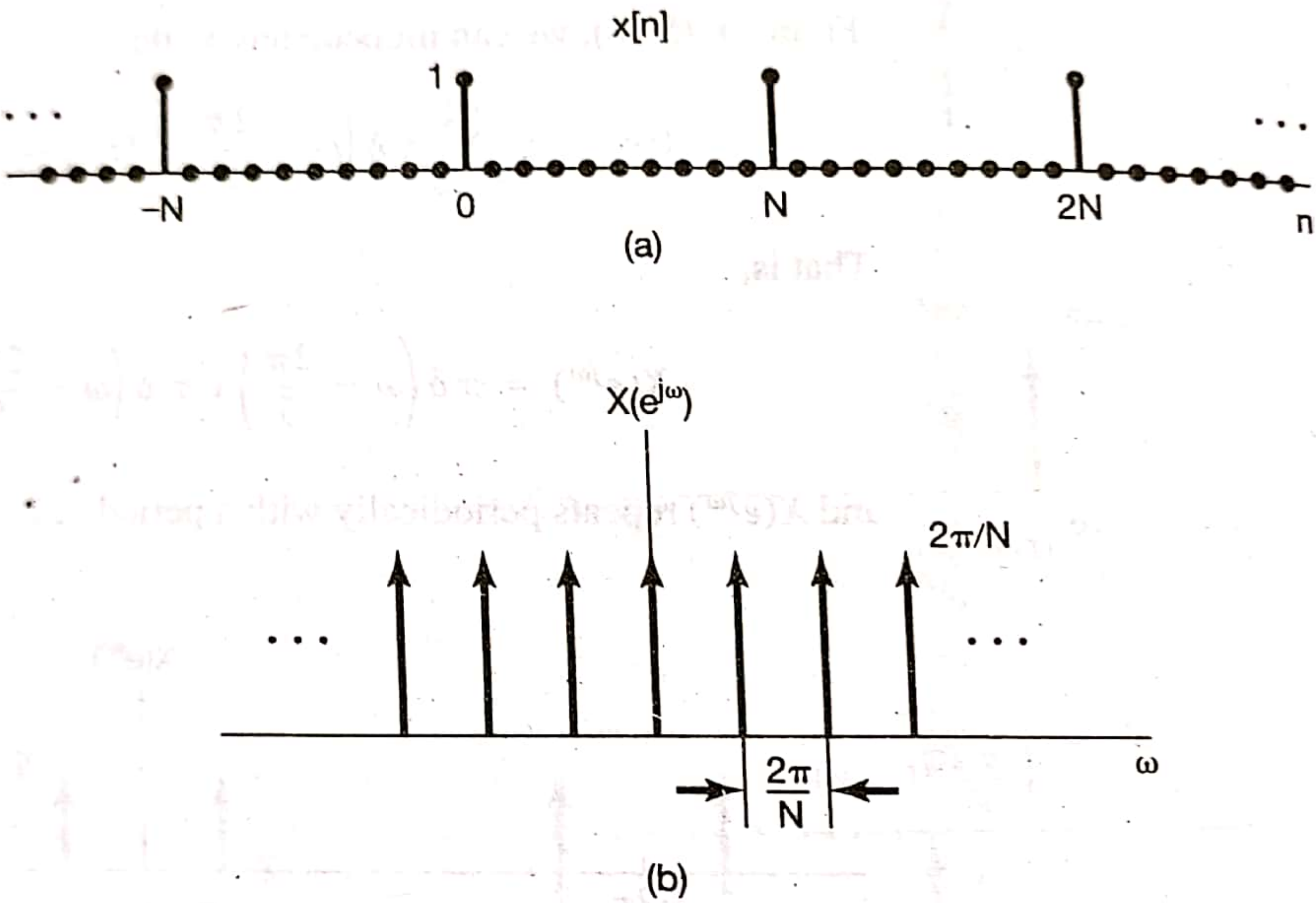

Figure 5.11 (a) Discrete-time periodic impulse train: (b) its Fourier transform.

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Example 10.1

Consider the signal $x[n] = a^n u[n]$. Then, from eq. (10.3),

$$X(z) = \sum_{n=-\infty}^{+\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n.$$

For convergence of $X(z)$, we require that $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$. Consequently, the region of convergence is the range of values of z for which $|az^{-1}| < 1$, or equivalently, $|z| > |a|$.

Then

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|. \tag{10.9}$$

Thus, the z-transform for this signal is well-defined for any value of a, with an ROC determined by the magnitude of a according to eq. (10.9). For example, for $a = 1$, $x[n]$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1.$$

We see that the z-transform in eq. (10.9) is a rational function. Consequently, just as with rational Laplace transforms, the z-transtorm can be characterized by its zeros (the roots of the numerator polynomial) and its poles (the roots of the denominator polynomial). For this example, there is one zero, at $z = 0$, and one pole, at $z = a$. The pole-zero plot and the region of convergence for Example 10.1 are shown in Figure 10.2 for a value of a between 0 and 1. For $|a| > 1$, the ROC does not include the unit circle, consistent with the fact that, for these values of a, the Fourier transform of $a^n u[n]$ does not converge.

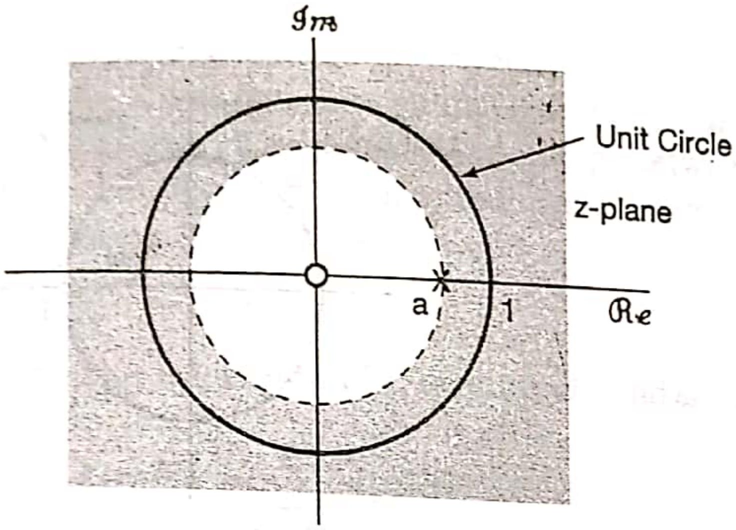

Figure 10.2 Pole-zero plot and region of convergence for Example 10.1 for $0 < a < 1$.

Example 10.2

Now let $x[n] = -a^n u[-n-1]$. Then

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} \\ &= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n. \end{aligned} \tag{10.10}$$

If $|a^{-1}z| < 1$, or equivalently, $|z| < |a|$, the sum in eq. (10.10) converges and

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| < |a|. \tag{10.11}$$

The pole-zero plot and region of convergence for this example are shown in Figure 10.3 for a value of a between 0 and 1.

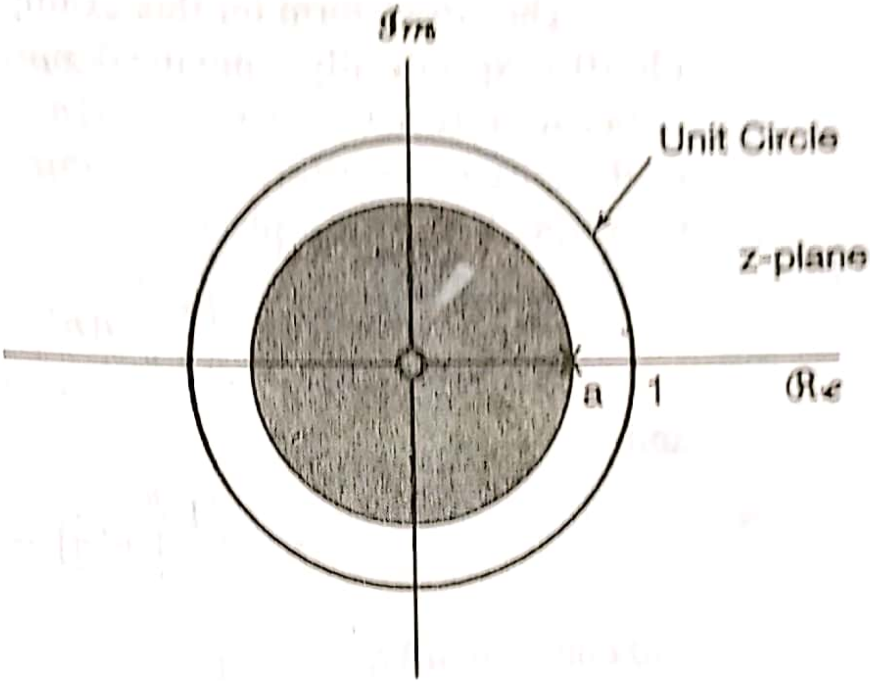

Figure 10.3 Pole-zero plot and region of convergence for Example 10.2 for $0 < a < 1$.

There is a result that does closely parallel eq. (5.43), however. Let k be a positive integer, and define the signal

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases} \tag{5.44}$$

As illustrated in Figure 5.13 for $k = 3$, $x_{(k)}[n]$ is obtained from $x[n]$ by placing $k-1$ zeros between successive values of the original signal. Intuitively, we can think of $x_{(k)}[n]$ as a slowed-down version of $x[n]$. Since $x_{(k)}[n]$ equals 0 unless n is a multiple of k, i.e., unless $n = rk$, we see that the Fourier transform of $x_{(k)}[n]$ is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}.$$

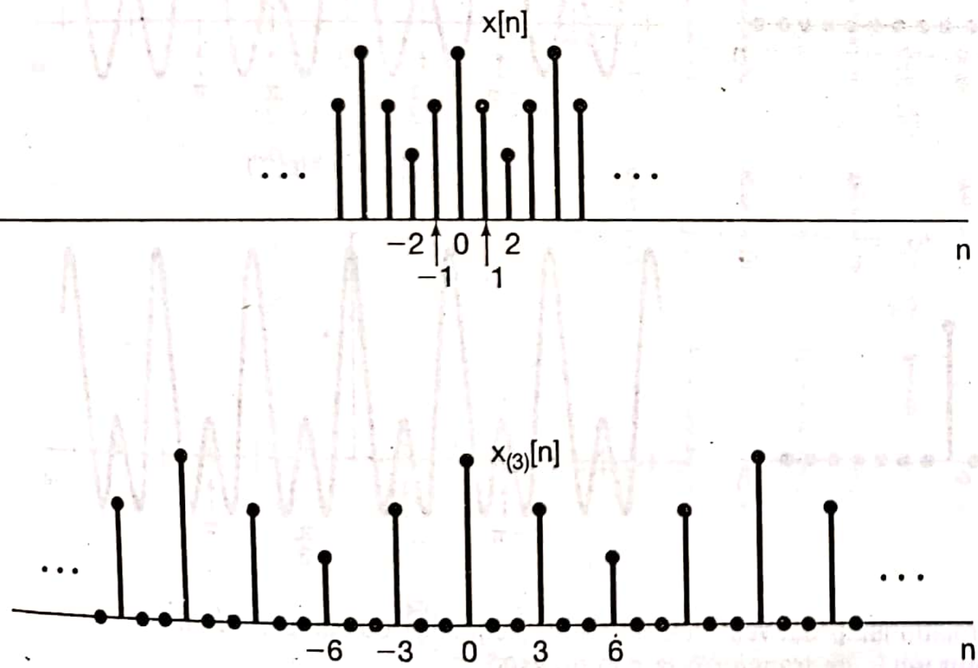

Figure 5.13 The signal $x_{(3)}[n]$ obtained from $x[n]$ by inserting two zeros between successive values of the original signal.

Furthermore, since $x_{(k)}[rk] = x[r]$, we find that

$$X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega}).$$

That is,

$$\boxed{x_{(k)}[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{jk\omega}).} \tag{5.45}$$

Note that as the signal is spread out and slowed down in time by taking $k > 1$, its Fourier transform is compressed. For example, since $X(e^{j\omega})$ is periodic with period 2π, $X(e^{jk\omega})$ is periodic with period $2\pi/k$. This property is illustrated in Figure 5.14 for a rectangular pulse.