Subject : Digital Communication

Code : UEC 639

Credit : 4

Dr. Amit Mishra

Signal space analysis

- Received signal
  - Time domain

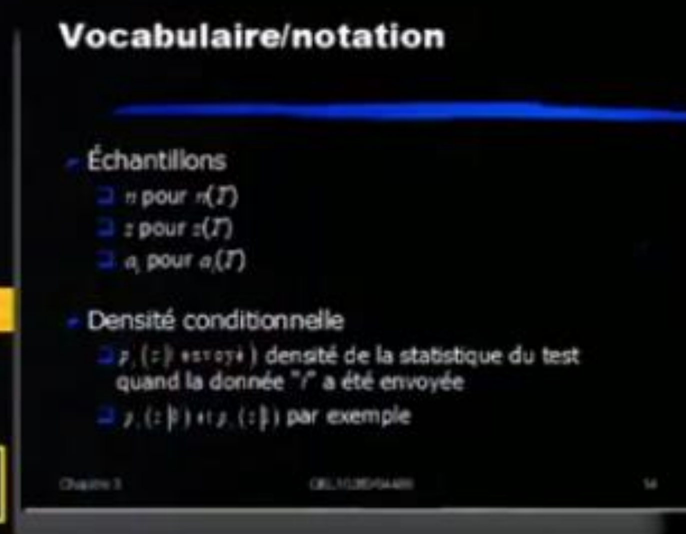
$$r(t) = s(t) + n(t)$$

  - Vector form

$$\underline{r} = \underline{s} + \underline{n}$$

- How to choose a transmitted signal from an observation of $\underline{r}$
- MAP Maximize "a posteriori" probability
  - ML Maximize the Likelihood
- **We need to analyze the noise ...**

Signal space



$$r(t)=\sum_{j=1}^{\infty} r_j \psi_j(t) \qquad \underline{r}=\left[r_1\, r_2 \ldots r_n\, r_{n+1} \cdots\right]$$

$$n(t)=\sum_{j=1}^{\infty} n_j \psi_j(t) \qquad \underline{n}=\left[n_1\, n_2 \ldots n_n\, n_{n+1} \cdots\right]$$

$$s_i(t)=\sum_{j=1}^{\infty} s_{ij} \psi_j(t) \qquad \underline{s}_i=\left[s_{i,1} s_{i,2} \ldots s_{i,n} 0\; 0 \ldots\right]$$

$$\begin{aligned}\Pr\left(r(t) \middle| s_i(t)\right) &= \Pr\left(r(t)=s_i(t)+n(t)\right)\\ &= \Pr\left(\underline{r}=\underline{s}_i+\underline{n}\right)\end{aligned}$$

Noise in signal space

- Noise vector

Coefficients of noise vector

$$\underline{n} = [n_1 \; n_2 \ldots] \qquad n_j = \int_0^T n(t)\psi_j(t)dt$$

- Recall $n(t)$ Gaussian process
  - $En(t) = 0$ Mean

    Variance

    Autocorrelation
  - $\mathrm{PSD}[n(t)] = G_N(f) = \frac{N_0}{2} \quad \forall f \qquad R_N(\tau) = \frac{N_0}{2}\delta(\tau)$
  - $En(t)n(t+\tau) = 0 \quad \tau \neq 0$

- $n(t)$ Gaussian

  $\Rightarrow n_j$ Gaussian (integration is a linear operation)

Noise analysis

- Expected value

$$En_j = E\int_0^T n(t)\psi_j(t)dt$$

$$= \int_0^T E\{n(t)\}\psi_j(t)\,dt$$

$$= \int_0^T 0\cdot\psi_j(t)\,dt = 0$$

Noise analysis

- Variance

$$\mathrm{Var}[X] \stackrel{\mathrm{def}}{=} E[(X - E[X])^2]$$

$$En_j^2 = E\int_0^T n(t)\psi_j(t)dt \cdot \int_0^T n(z)\psi_j(z)dz$$

$$= E\int_0^T\int_0^T n(t)n(z)\psi_j(t)\psi_j(z)dzdt$$

$$= \int_0^T\int_0^T E\{n(t)n(z)\}\psi_j(t)\psi_j(z)dzdt$$

$$= \int_0^T\int_0^T \frac{N_0}{2}\delta(t-z)\psi_j(t)\psi_j(z)dzdt$$

$$= \frac{N_0}{2}\int_0^T \psi_j^2(t)dt = \frac{N_0}{2}$$

In statistics, a **variance** is the **spread of a data** set around its mean value, while a **covariance** is the measure of the **directional relationship** between two random variables.

Independence of coefficients

- Covariance, *i.e.*, suppose $i \neq j$

$$E n_i n_j = E\int_0^T\int_0^T n(t)n(z)\psi_i(t)\psi_j(z)\,dtdz$$

$$=\int_0^T\int_0^T E\{n(t)n(z)\}\psi_i(t)\psi_j(z)\,dtdz$$

$$=\int_0^T\int_0^T \frac{N_0}{2}\delta(t-z)\psi_i(t)\psi_j(z)\,dtdz$$

The $\delta(t-z)$ only valid at $t=z$

$$=\frac{N_0}{2}\int_0^T\psi_i(t)\psi_j(t)\,dt$$

$$=\frac{N_0}{2}\langle\psi_i(t)\psi_j(t)\rangle=0$$

Independence of coefficients

- The coefficients are not correlated for white noise in an orthonormal base
- $$E n_i n_j = \frac{N_0}{2} \left\langle \psi_i(t) \psi_j(t) \right\rangle = 0$$
- For Gaussian random variables, uncorrelated $\Rightarrow$ Independent

Noise Summary

- Gaussian of zero mean
- $$n_j \sim N\left(0, \frac{N_0}{2}\right) \qquad \sigma^2 = \frac{N_0}{2}$$
- Independent $\quad En_jn_i = 0 \text{ for } j \neq i$
- Vector density

  =joint density of coefficients

iid: Independent Identically Distributed.

Noise density

For two or more independent signals / vectors the joint *pdf* is expressed as product of the *pdf* of all the signals / vectors.

- By independence
- $$P_{\underline{n}}(\underline{n}) = \prod_{j=1}^{\infty} P_{n_j}(n_j) = \prod_{j=1}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-n_j^2/2\sigma^2}$$

$$p\big(r(t)\,|\,s_i(t)\big) = p\big(r(t) = s_i(t) + n(t)\big)$$

$$= p\big(\underline{r} = \underline{s}_i + \underline{n}\big)$$

$$= p\big(\underline{n} = \underline{r} - \underline{s}_i\big)$$

$$= \prod_{j=1}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-(r_j - s_{j,i})^2/2\sigma^2}$$

$$\underline{S}_i = \begin{bmatrix} S_{1,i} & S_{2,i} & S_{3,i} & \cdots \end{bmatrix}$$

Detectors

- ML (Likelihood)
  - Maximize the likelihood, i.e. choosing the $s_i(t)$ which maximizes the likelihood of receiving $r(t)$

$$s_i(\mathrm{t}) \text{ such that } \arg\max_j P\left(\underline{r} \mid \underline{s}_j\right) = i$$

- MAP (*a posteriori*)
  - Choose transmitted signal to maximize the "*a posteriori*" *probability*

$$s_i \text{ such that } \arg\max_j P\left(\underline{r} \mid \underline{s}_j\right) \underbrace{\boxed{P\left(\underline{s}_j\right)}}_{\textit{a priori} \text{ probability}} = i$$

MAP choice

$$Arg \max_j P(\underline{r}|\underline{s}_j)P(\underline{s}_j) = Arg \max_j P(\underline{r} = \underline{s}_j + \underline{n})P(\underline{s}_j)$$

$$= Arg \max_j P(\underline{n} = \underline{r} - \underline{s}_j)P(\underline{s}_j)$$

$$= Arg \max_j \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-(r_k - s_{j,k})/2\sigma^2} P(\underline{s}_j)$$

We can discard $\sqrt{2\pi\sigma}$, as it is a constant and does not contribute any information

$$= Arg \max_j \prod_{k=1}^{\infty} e^{-(r_k - s_{j,k})^2/2\sigma^2} P(\underline{s}_j)$$

In two or more exponential functions power multiplication can be replaced by addition

$$= Arg \max_j e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{\infty}(r_k - s_{j,k})^2} P(\underline{s}_j)$$

MAP choice

$$Arg\ \max_j P\left(\underline{r}\middle|\underline{s}_j\right)P\left(\underline{s}_j\right) = Arg\ \max_j e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{\infty}\left(r_k - s_{j,k}\right)^2} P\left(\underline{s}_j\right)$$

Taking Log

$$= Arg\ \max_j \ln\left[e^{-\frac{1}{2\sigma^2}\sum_{k=1}^{\infty}\left(r_k - s_{j,k}\right)^2} P\left(\underline{s}_j\right)\right]$$

$$= Arg\ \max_j \left[-\frac{1}{2\sigma^2}\sum_{k=1}^{\infty}\left(r_k - s_{j,k}\right)^2 + \ln P\left(\underline{s}_j\right)\right]$$

We can discard $2\sigma^2$, as it is a constant and does not contribute any information

$$= Arg\ \max_j \left[-\sum_{k=1}^{\infty}\left(r_k - s_{j,k}\right)^2 + 2\sigma^2 \ln P\left(\underline{s}_j\right)\right]$$

Convert function from max to min

$$= Arg\ \min_j \left[\sum_{k=1}^{\infty}\left(r_k - s_{j,k}\right)^2 - 2\sigma^2 \ln P\left(\underline{s}_j\right)\right]$$

MAP choice

- We want

$$\underset{j}{Arg\ \min}\left[\sum_{k=1}^{\infty}\left(r_k - s_{j,k}\right)^2 - 2\sigma^2 \ln P\left(\underline{s}_j\right)\right]$$

- Theorem of irrelevance
  - $s_{j,k} = 0 \quad \forall \ k > N$
  - Elements for $k>N$ in the sum do not change the minimization

$$\underset{j}{Arg\ \min}\left[\sum_{k=1}^{\infty}\left(r_k - s_{j,k}\right)^2 - 2\sigma^2 \ln P\left(\underline{s}_j\right)\right]$$

$$= \underset{j}{Arg\ \min}\left[\sum_{k=1}^{N}\left(r_k - s_{j,k}\right)^2 - 2\sigma^2 \ln P\left(\underline{s}_j\right)\right]$$

Represent energy

$$= \underset{j}{Arg\ \min}\left[\left\|\underline{r} - \underline{s}_j\right\|^2 - 2\sigma^2 \ln P\left(\underline{s}_j\right)\right]$$

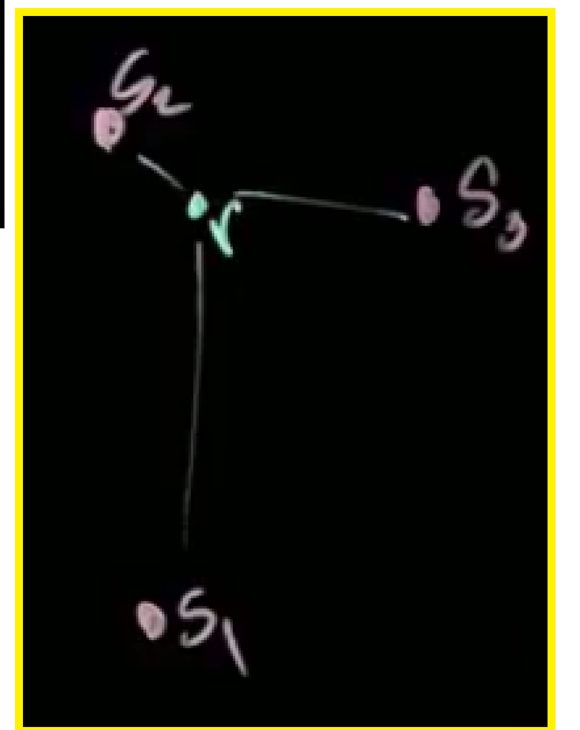
norm in the finite basis of the signal space

MAP choice

Choose the signal that is closer to the received signal modulo a weighting for a priori probability

$$Arg \min_{j} \left[\left\| r - \underline{s}_j \right\|^2 - 2\sigma^2 \ln P\left(\underline{s}_j \right) \right]$$

Explanation:



MAP choice

Choose the signal that is closer to the received signal modulo a weighting for a priori probability

$$Arg\ \min_j \left[\left\| r - \underline{s}_j \right\|^2 - 2\sigma^2 \ln P\left(\underline{s}_j\right) \right]$$

ML choice

Choose the signal closest to the received signal

$$Arg\ \min_j \left[\left\| r - \underline{s}_j \right\|^2 \right]$$

Vector space receiver

- MAP
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$
- ML
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2$

What is the structure of the test statistics??

Vector space receiver

- MAP
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$$

- ML
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2$$

Vector spac

doesn't change the minimization

identical for all i

- MAP
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$$

- ML
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2$$

Vector space re

known: calculated in advance
- not at each reception

- MAP
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$$

- ML
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2$$

Vector space receiver

only calculated at the receiver

- MAP
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$$

- ML
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2$$

Vector space receiver

correlator

- MAP
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$

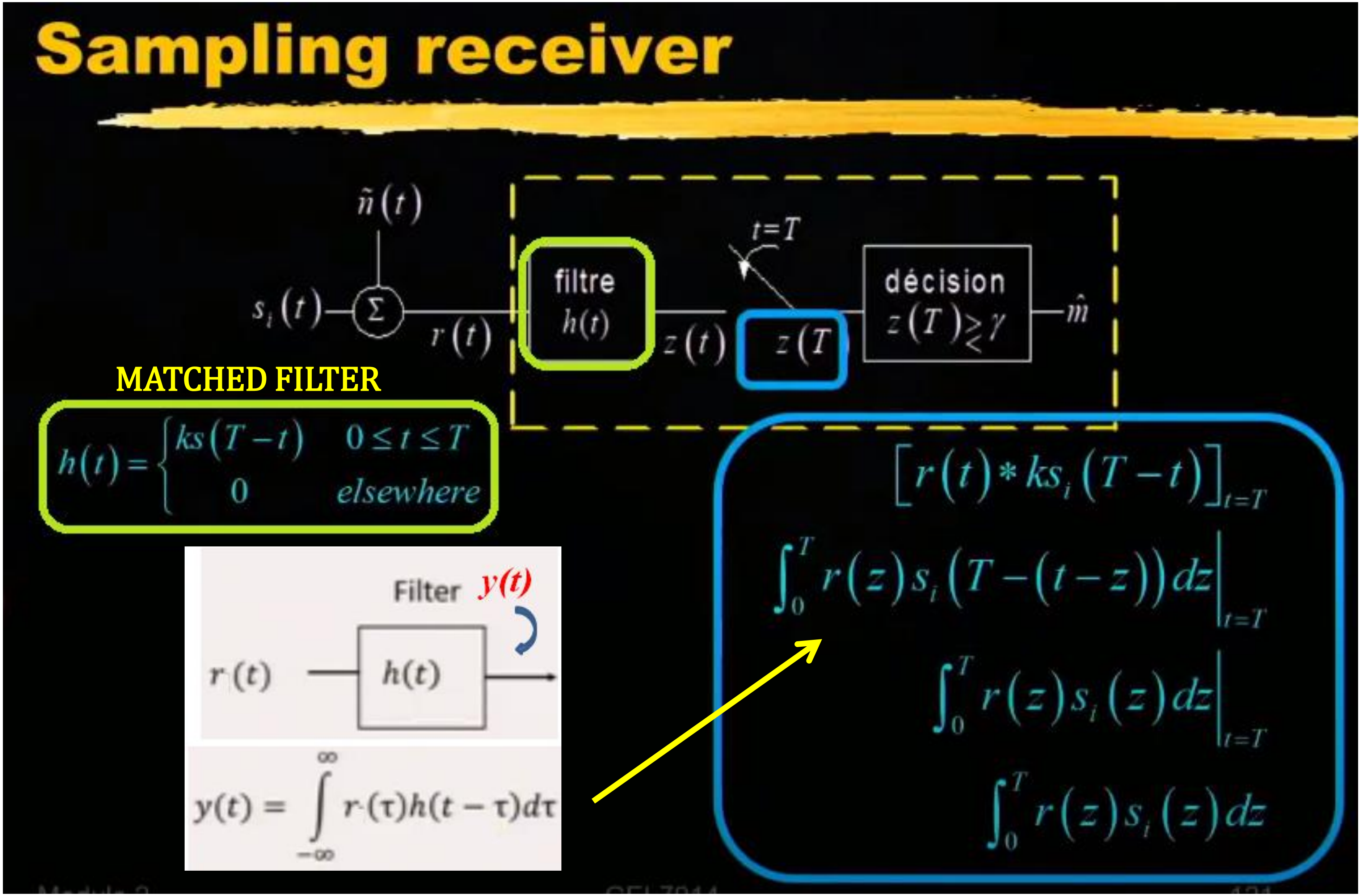
$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2 - N_0 \ln P(\mathbf{s}_i)$$

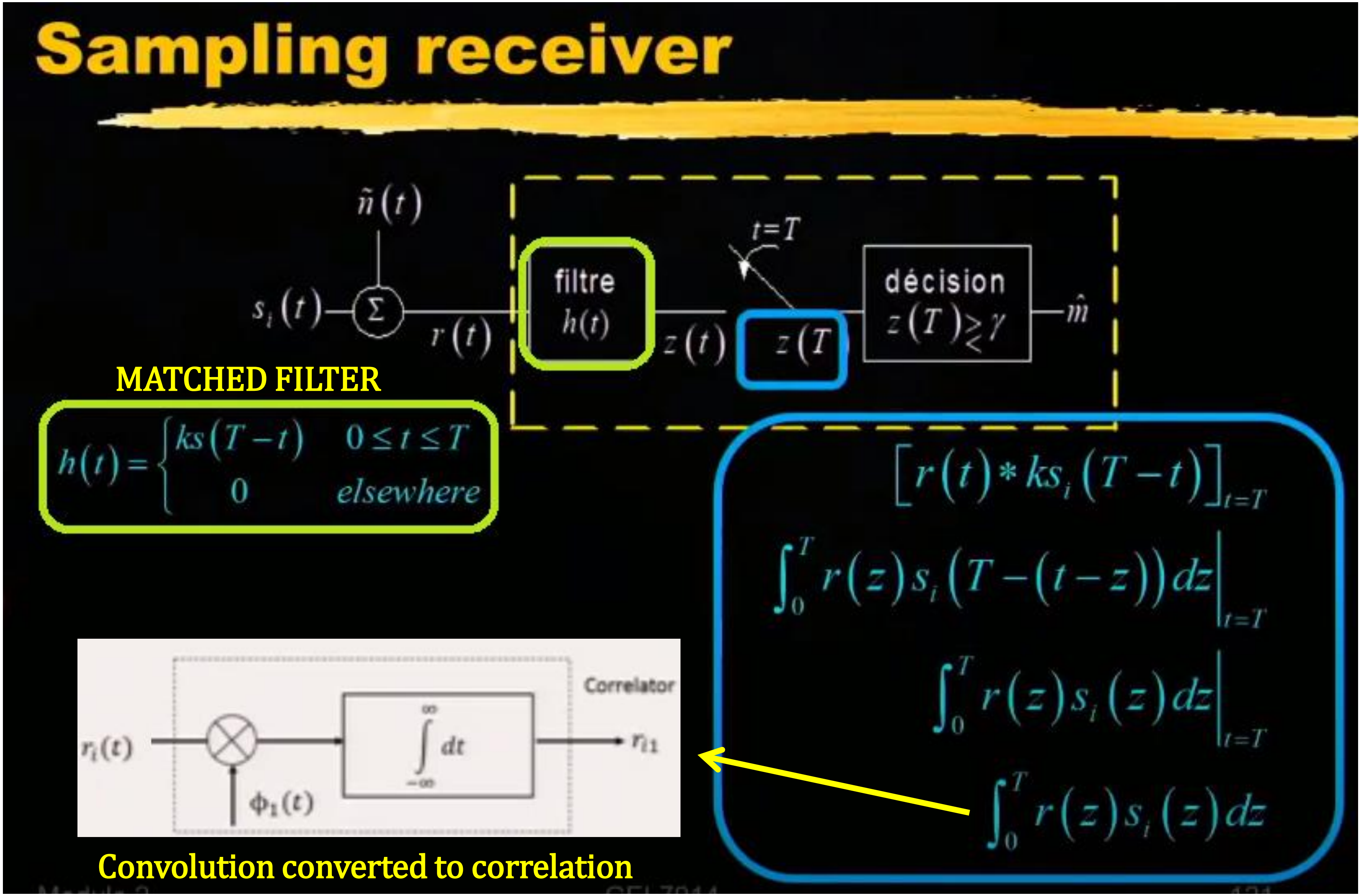
- ML
  - i that minimizes $\|\mathbf{r}-\mathbf{s}_i\|^2$

$$\|\mathbf{r}\|^2 - 2\langle \mathbf{r}, \mathbf{s}_i \rangle + \|\mathbf{s}_i\|^2$$

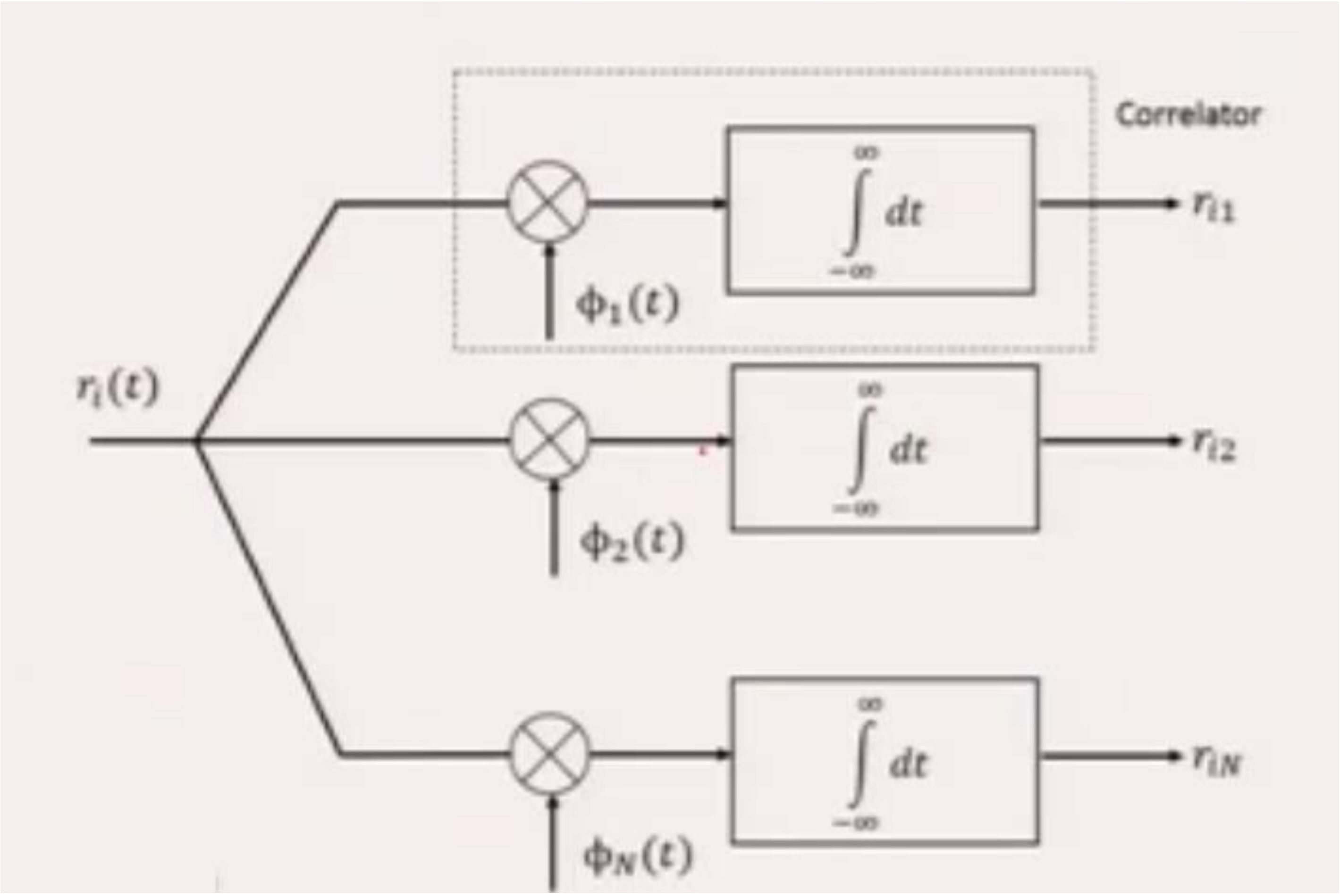
Two approaches

- Linear sampling receiver
  - fix the type of receiver
  - probability theory for decision rule
  - maximize the SNR with a **matched filter**
- Signal space analysis
  - representation in a vector space
  - probability theory for decision rule
  - structure is a **correlator**


Sampling receiver
$\tilde{n}(t)$
$s_i(t)$
Σ
$r(t)$
filtre
$h(t)$
$z(t)$
$t=T$
$z(T)$
décision
$z(T) \gtrless \gamma$
$\hat{m}$
MATCHED FILTER
$h(t) = \begin{cases} ks(T-t) & 0 \le t \le T \\ 0 & elsewhere \end{cases}$
Filter
$y(t)$
$r(t)$
$h(t)$
$y(t) = \int_{-\infty}^{\infty} r(\tau)h(t-\tau)d\tau$
$\left[r(t) * ks_i(T-t)\right]_{t=T}$
$\int_0^T r(z)s_i(T-(t-z))dz\Big|_{t=T}$
$\int_0^T r(z)s_i(z)dz\Big|_{t=T}$
$\int_0^T r(z)s_i(z)dz$

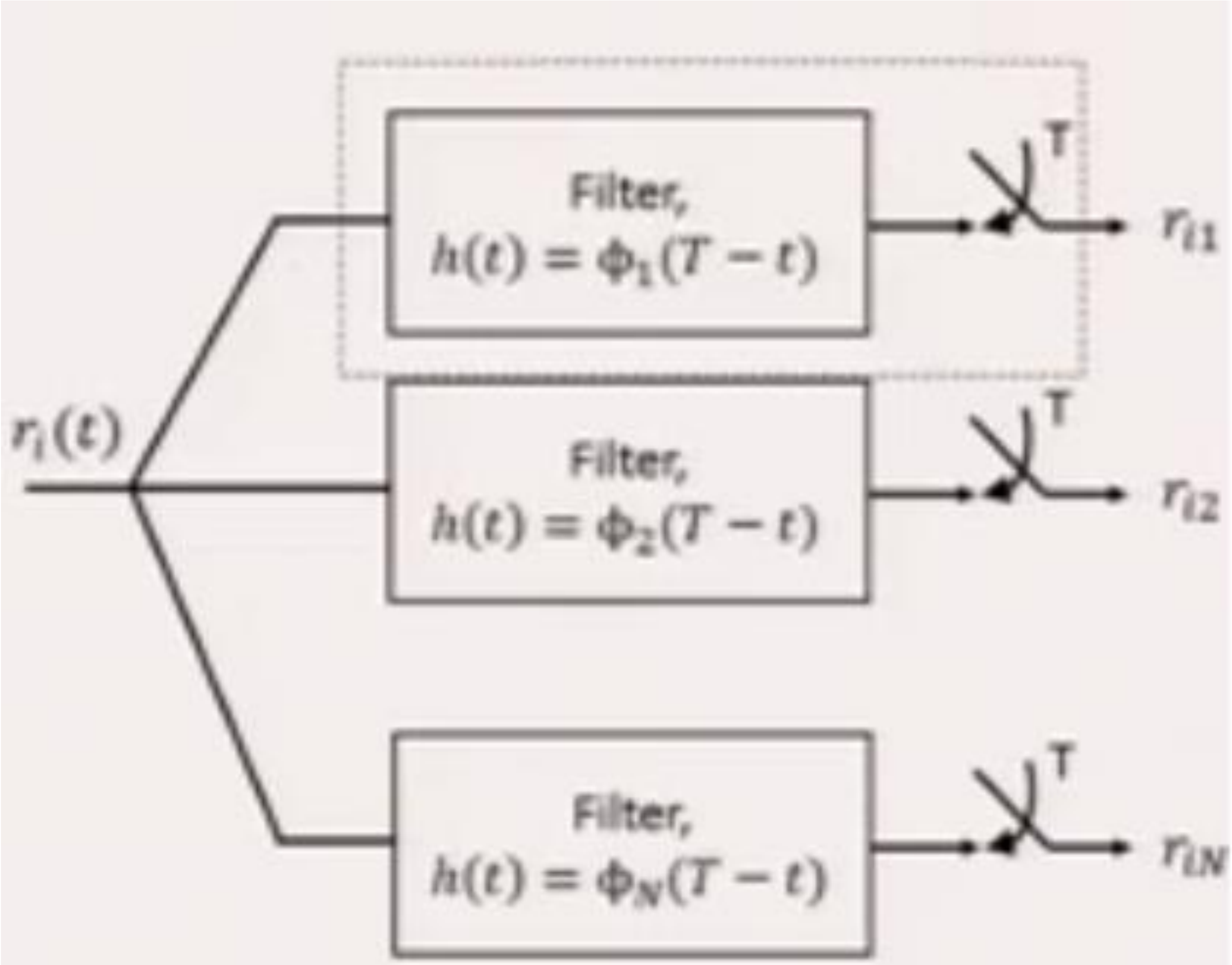


Elaborate further?



Correlator

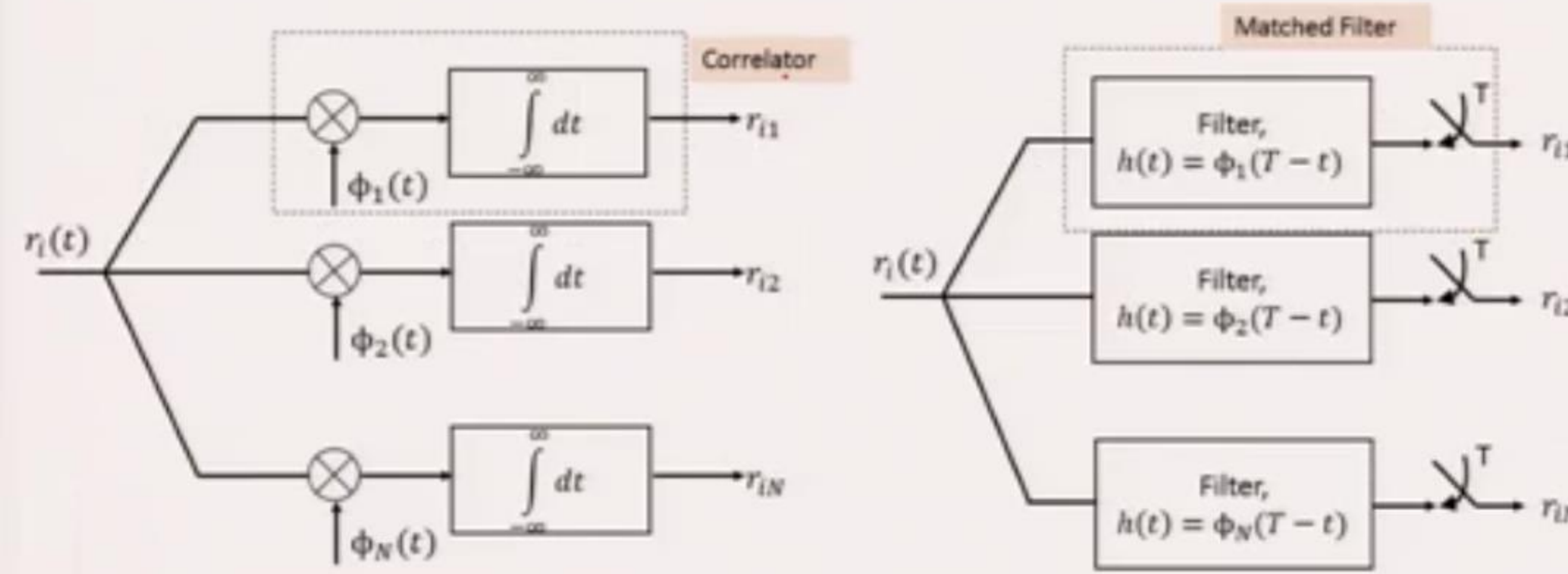
$$r_{i1} = \int_{-\infty}^{\infty} r_i(t)\phi_1(t)dt$$



Matched Filter

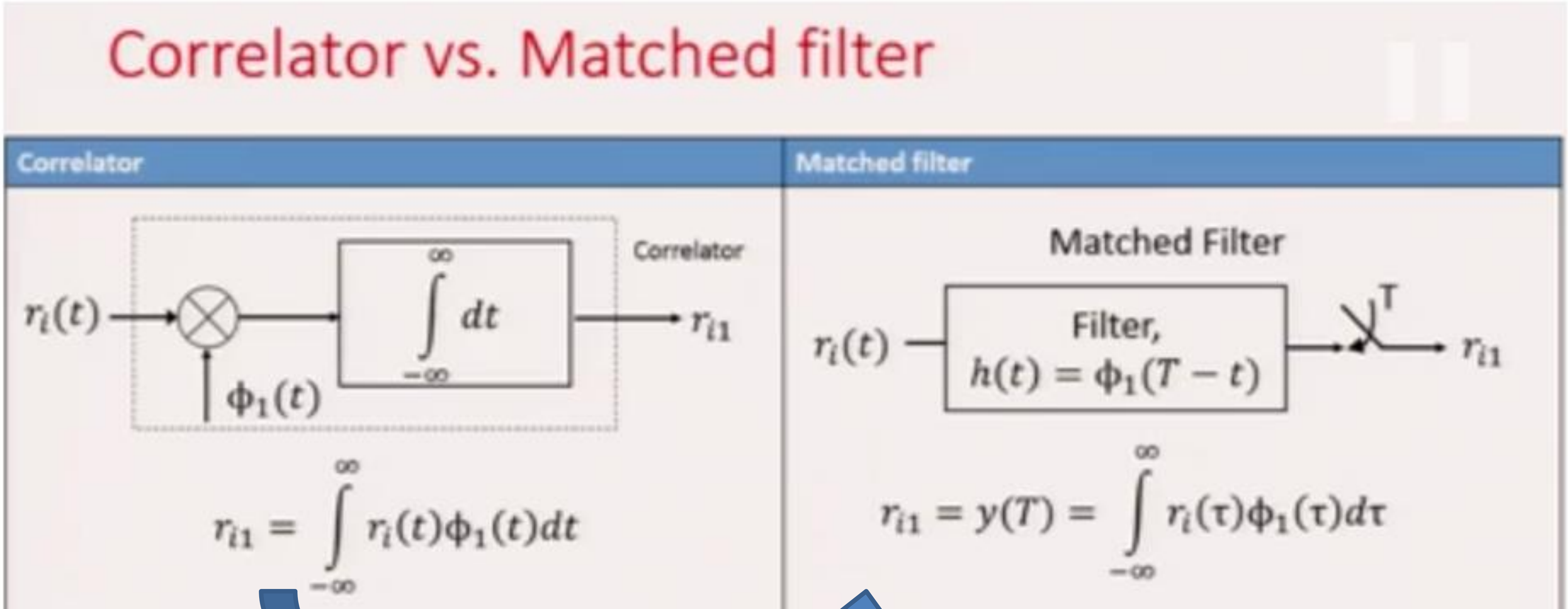
$$r_{i1} = y(T) = \int_{-\infty}^{\infty} r_i(\tau)\phi_1(\tau)d\tau$$

Correlator vs. Matched Filter



$$r_{i1} = \int_{-\infty}^{\infty} r_i(t)\phi_1(t)dt$$

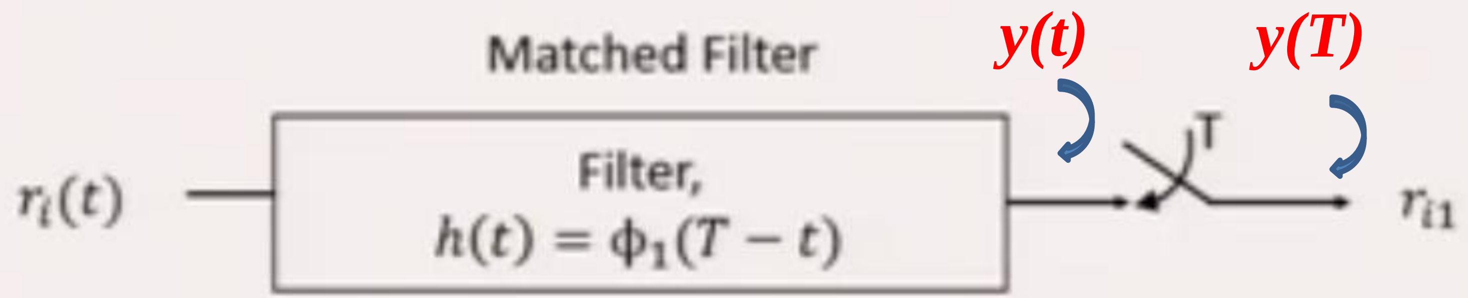
$$r_{i1} = y(T) = \int_{-\infty}^{\infty} r_i(\tau)\phi_1(\tau)d\tau$$



Equivalent

How?

Matched filter



$$y(t) = \int_{-\infty}^{\infty} r_i(\tau) h(t - \tau) d\tau$$

From the convolution formula

$$= \int_{-\infty}^{\infty} r_i(\tau)\, \phi_1\, (T - (t - \tau)) d\, \tau$$

$$= \int_{-\infty}^{\infty} r_i(\tau)\, \phi_1\, (T - t + \tau) d\, \tau$$

At *t*=*T*,

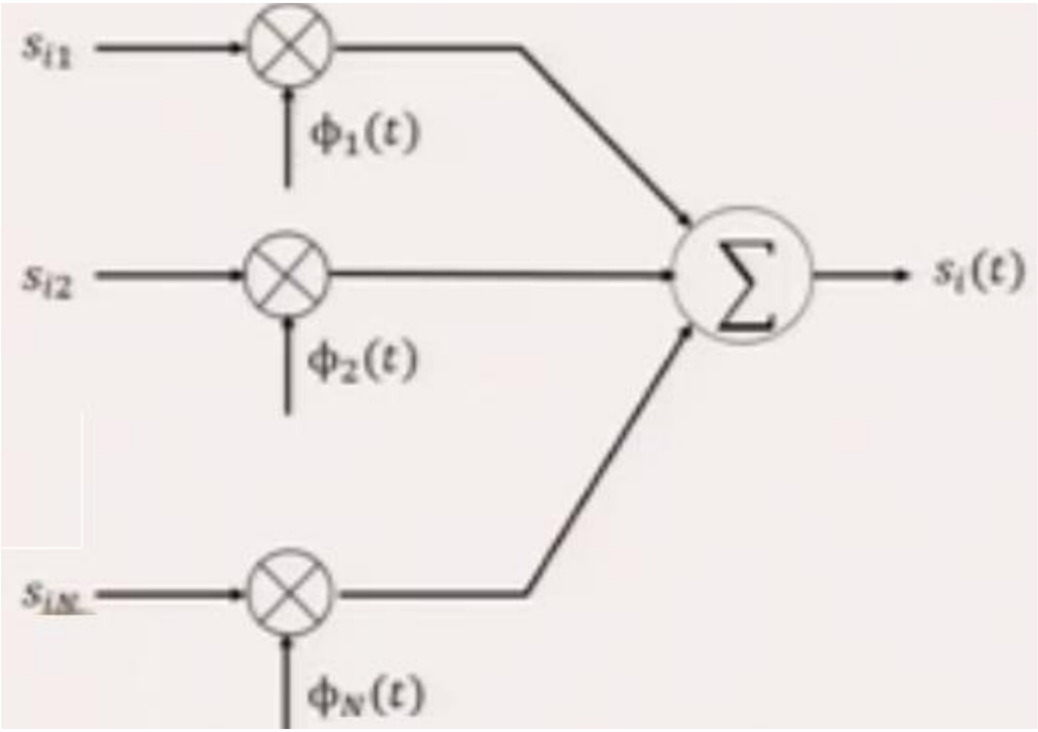
$$r_{i1} = y(T) = \int_{-\infty}^{\infty} r_i(\tau) \phi_1(\tau) d\tau$$

The result is same as Correlator filter

Transmitter Design (Sequence to waveforms)

Sequence of symbols:
Real numbers or Quantized real numbers

Transmitter



$$S_i(t) = \begin{bmatrix} S_{11} & S_{12} & S_{13} & \cdots & S_{1N} \\ S_{21} & S_{22} & S_{23} & \cdots & S_{2N} \\ S_{31} & S_{32} & S_{33} & \cdots & S_{3N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{M1} & S_{M2} & S_{M3} & \cdots & S_{MN} \end{bmatrix} \begin{bmatrix} \emptyset_1 \\ \emptyset_2 \\ \emptyset_2 \\ \vdots \\ \emptyset_N \end{bmatrix}$$

$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t) \ldots\ldots\ldots + s_{1N}\phi_N(t)$$

$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t) \ldots\ldots\ldots + s_{2N}\phi_N(t)$$

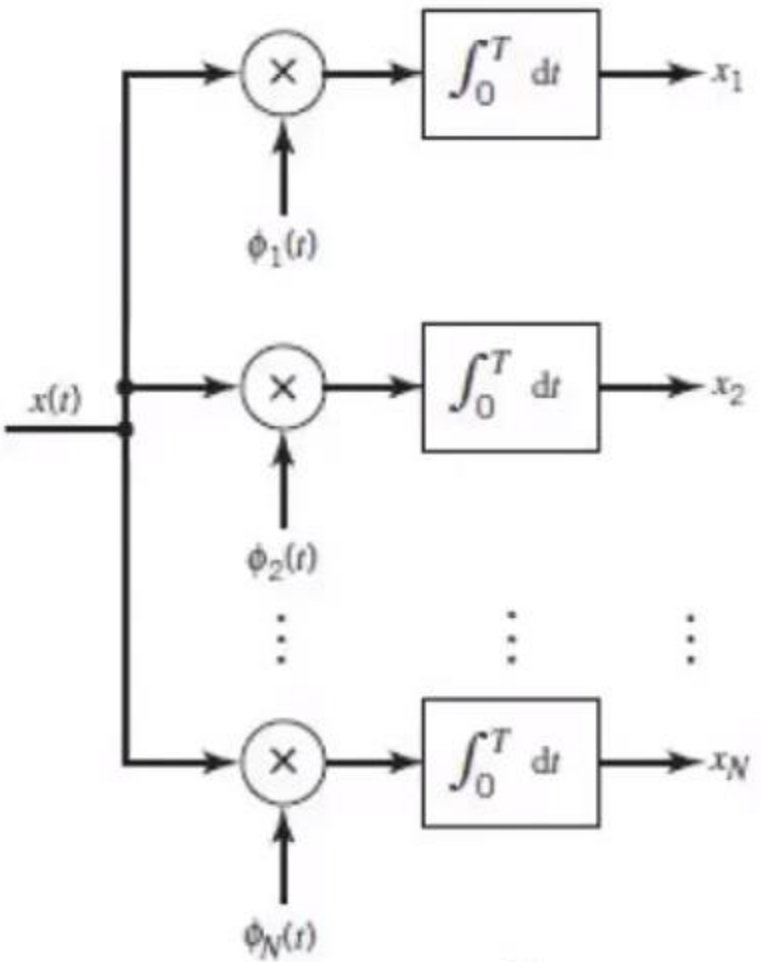
$$s_M(t) = s_{M1}\phi_1(t) + s_{M2}\phi_2(t) \ldots\ldots\ldots + s_{MN}\phi_N(t)$$

Receiver Design (Waveform to sequences)

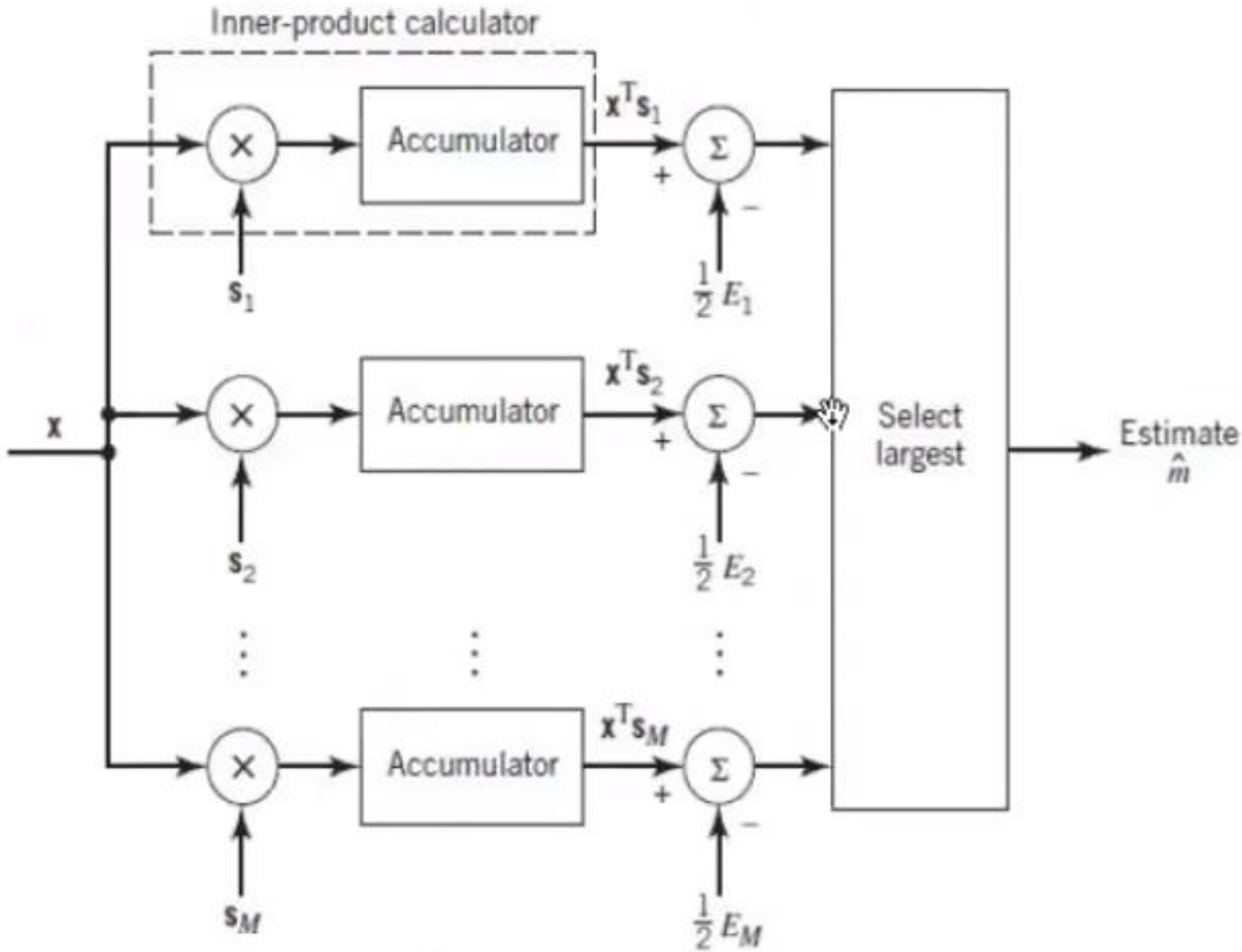
- Correlation receiver contains two parts:
  Detector
  Vector Receiver

- Detector converts signal x(t) to observation vector x as per the equation

$$X_j = \int_0^T X(t)\phi_j(t)dt$$

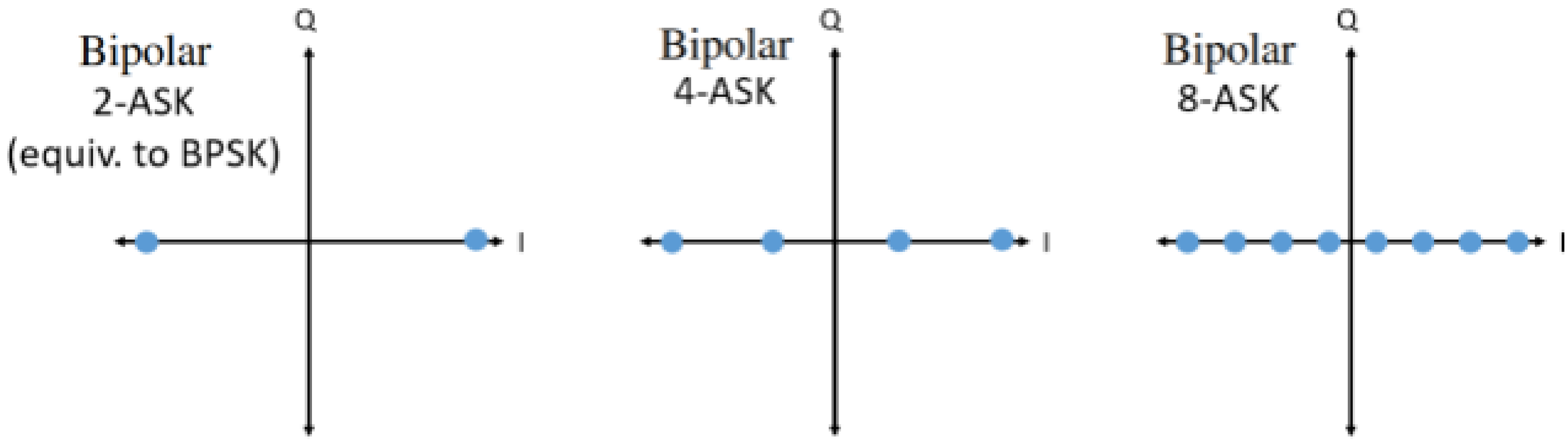


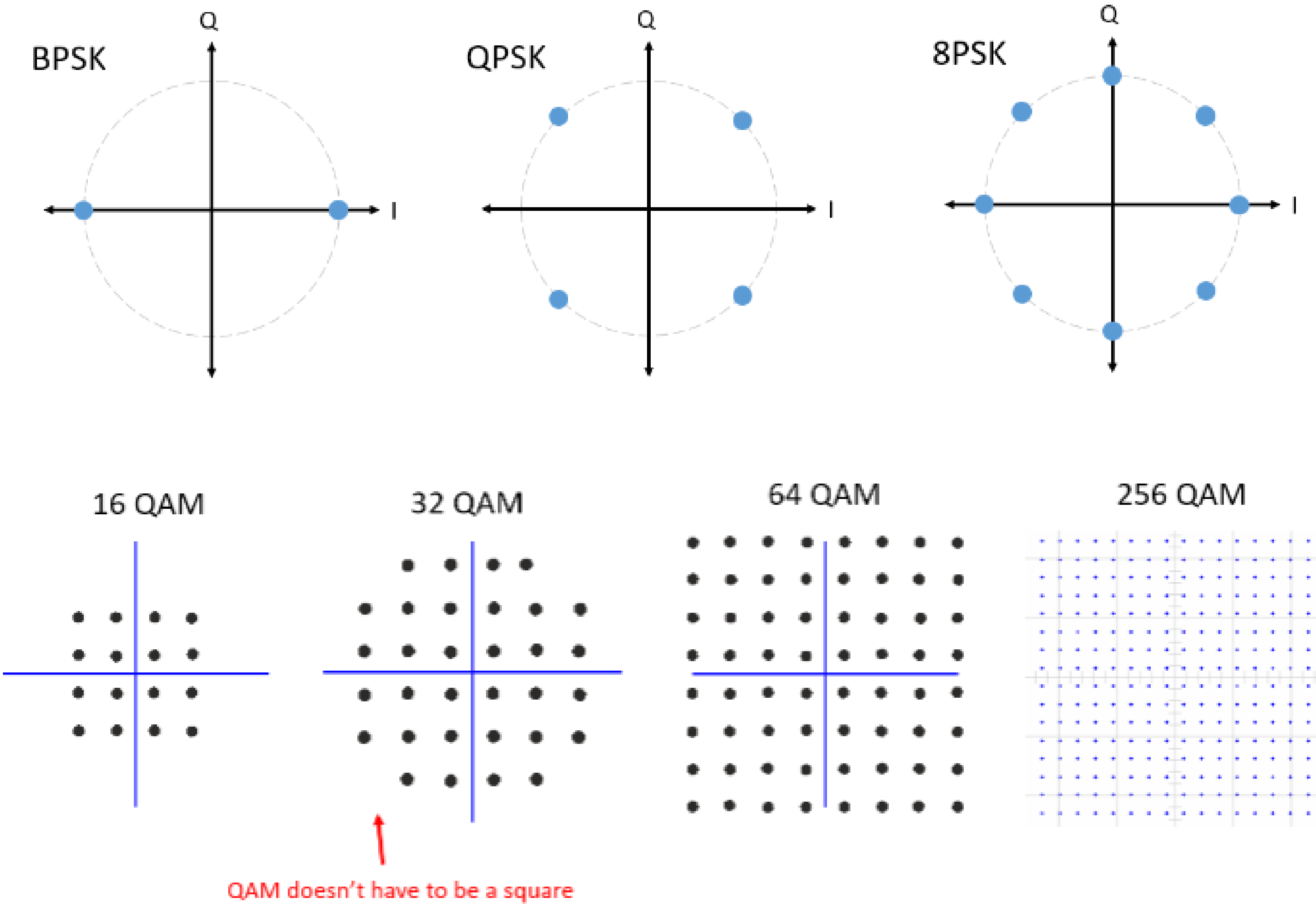
- Vector receiver converts x to estimate $\widehat{m}$

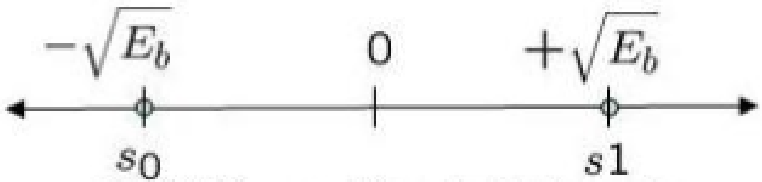


Signal Constellation

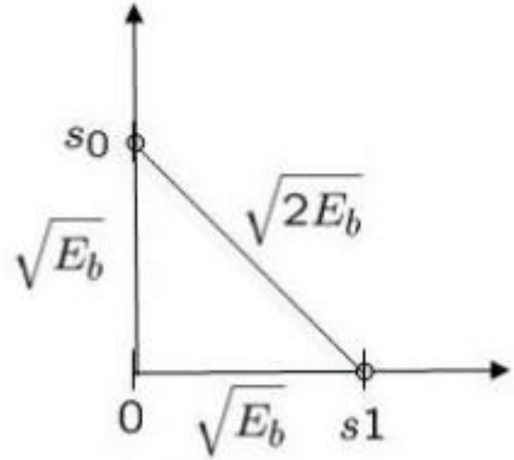
A signal constellation is the physical diagram used to describe all the possible symbols used by a signaling system to transmit data and is an aid to designing better communications systems.




BPSK
Q
I
QPSK
Q
I
8PSK
Q
I
16 QAM
32 QAM
64 QAM
256 QAM
QAM doesn't have to be a square



BPSK – antipodal signals



FSK – orthogonal signals

Thanks !