(c) Suppose that the signal is sampled at the rate $F_s = 75$ Hz. What is the discrete-time signal obtained after sampling?

(d) What is the frequency $0 < F < F_s/2$ of a sinusoid that yields samples identical to those obtained in part (c)?

Solution.

(a) The frequency of the analog signal is $F = 50$ Hz. Hence the minimum sampling rate required to avoid aliasing is $F_s = 100$ Hz.

(b) If the signal is sampled at $F_s = 200$ Hz, the discrete-time signal is

$$x(n) = 3\cos\frac{100\pi}{200}n = 3\cos\frac{\pi}{2}n$$

(c) If the signal is sampled at $F_s = 75$ Hz, the discrete-time signal is

$$\begin{aligned} x(n) &= 3\cos\frac{100\pi}{75}n = 3\cos\frac{4\pi}{3}n \\ &= 3\cos\left(2\pi - \frac{2\pi}{3}\right)n \\ &= 3\cos\frac{2\pi}{3}n \end{aligned}$$

(d) For the sampling rate of $F_s = 75$ Hz, we have

$$F = fF_s = 75f$$

The frequency of the sinusoid in part (c) is $f = \frac{1}{3}$. Hence

$$F = 25 \text{ Hz}$$

Clearly, the sinusoidal signal

$$\begin{aligned} y_a(t) &= 3\cos 2\pi F t \\ &= 3\cos 50\pi t \end{aligned}$$

sampled at $F_s = 75$ samples/s yields identical samples. Hence $F = 50$ Hz is an alias of $F = 25$ Hz for the sampling rate $F_s = 75$ Hz.

Solution.

(a) The frequencies existing in the analog signal are

$$F_1 = 1 \text{ kHz}, \qquad F_2 = 3 \text{ kHz}, \qquad F_3 = 6 \text{ kHz}$$

Thus $F_{max} = 6$ kHz, and according to the sampling theorem,

$$F_s > 2F_{max} = 12 \text{ kHz}$$

The Nyquist rate is

$$F_N = 12 \text{ kHz}$$

(b) Since we have chosen $F_s = 5$ kHz, the folding frequency is

$$\frac{F_s}{2} = 2.5 \text{ kHz}$$

and this is the maximum frequency that can be represented uniquely by the sampled signal. By making use of (1.4.2) we obtain

$$\begin{aligned} x(n) &= x_a(nT) = x_a\left(\frac{n}{F_s}\right) \\ &= 3\cos 2\pi\left(\tfrac{1}{5}\right)n + 5\sin 2\pi\left(\tfrac{3}{5}\right)n + 10\cos 2\pi\left(\tfrac{6}{5}\right)n \\ &= 3\cos 2\pi\left(\tfrac{1}{5}\right)n + 5\sin 2\pi\left(1 - \tfrac{2}{5}\right)n + 10\cos 2\pi\left(1 + \tfrac{1}{5}\right)n \\ &= 3\cos 2\pi\left(\tfrac{1}{5}\right)n + 5\sin 2\pi\left(-\tfrac{2}{5}\right)n + 10\cos 2\pi\left(\tfrac{1}{5}\right)n \end{aligned}$$

Finally, we obtain

$$x(n) = 13\cos 2\pi\left(\tfrac{1}{5}\right)n - 5\sin 2\pi\left(\tfrac{2}{5}\right)n$$

The same result can be obtained using Fig. 1.4.4. Indeed, since $F_s = 5$ kHz, the folding frequency is $F_s/2 = 2.5$ kHz. This is the maximum frequency that can be represented uniquely by the sampled signal. From (1.4.17) we have $F_0 = F_k - kF_s$. Thus F_0 can be obtained by subtracting from F_k an integer multiple of F_s such that $-F_s/2 \leq F_0 \leq F_s/2$. The frequency F_1 is less than $F_s/2$ and thus it is not affected by aliasing. However, the other two frequencies are above the folding frequency and they will be changed by the aliasing effect. Indeed,

$$F_2' = F_2 - F_s = -2 \text{ kHz}$$

$$F_3' = F_3 - F_s = 1 \text{ kHz}$$

From (1.4.5) it follows that $f_1 = \frac{1}{5}$, $f_2 = -\frac{2}{5}$, and $f_3 = \frac{1}{5}$, which are in agreement with the result above.

(c) Since the frequency components at only 1 kHz and 2 kHz are present in the sampled signal, the analog signal we can recover is

$$ya(t) = 13\cos 2000\pi t - 5\sin 4000\pi t$$

which is obviously different from the original signal $x_a(t)$. This distortion of the original analog signal was caused by the aliasing effect, due to the low sampling rate used.

EXAMPLE 7.1.1

Consider the signal

$$x(n) = a^n u(n), \qquad 0 < a < 1$$

The spectrum of this signal is sampled at frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. Determine the reconstructed spectra for $a = 0.8$ when $N = 5$ and $N = 50$.

Solution. The Fourier transform of the sequence $x(n)$ is

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - ae^{-j\omega}}$$

Suppose that we sample $X(\omega)$ at N equidistant frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$.
Thus we obtain the spectral samples

$$X(\omega k) \equiv X\left(\frac{2\pi k}{N}\right) = \frac{1}{1 - ae^{-j2\pi k/N}}, \qquad k = 0, 1, \ldots, N-1$$

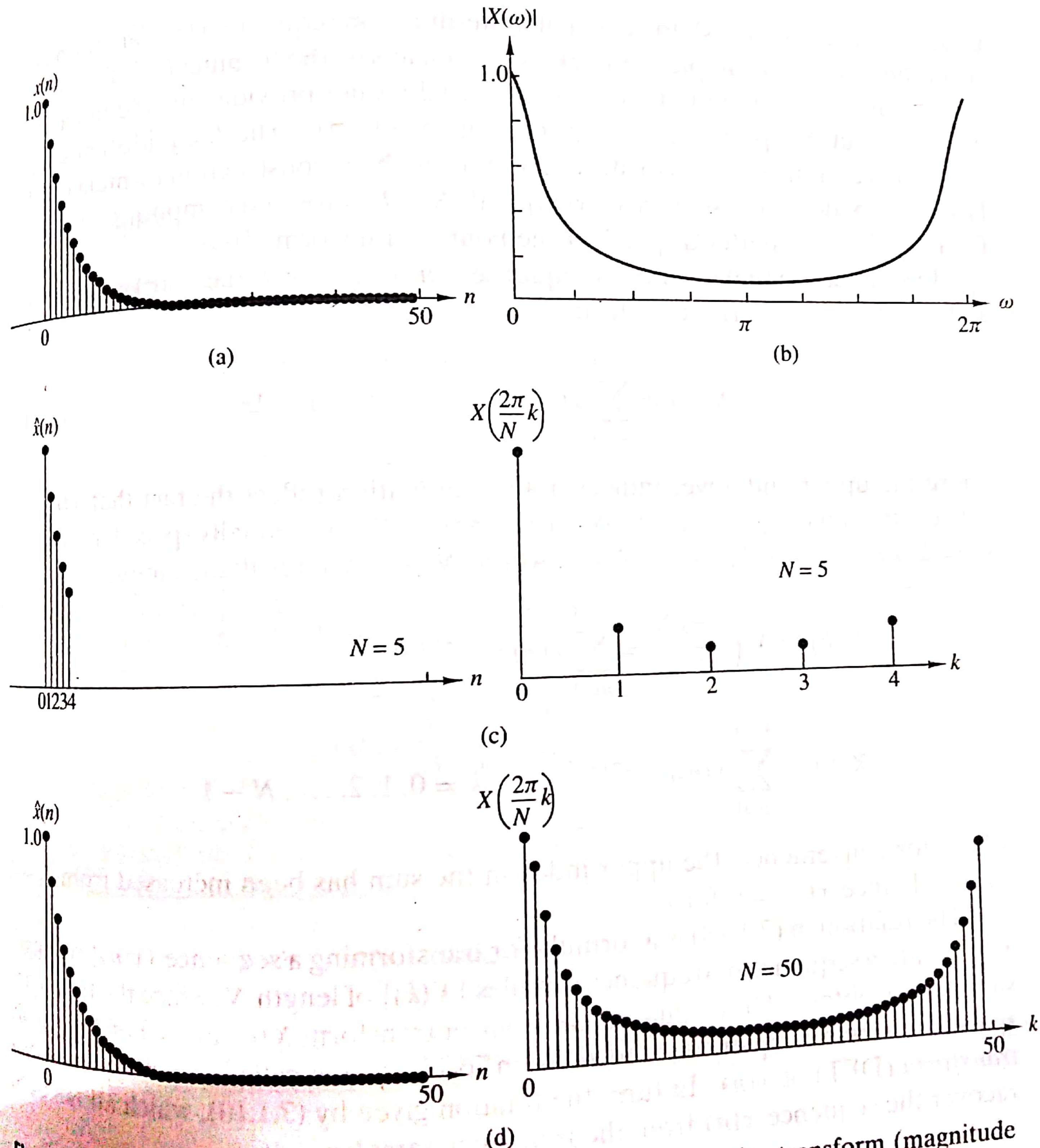

Figure 7.1.4 (a) Plot of sequence $x(n) = (0.8)^n u(n)$; (b) its Fourier transform (magnitude only); (c) effect of aliasing with $N = 5$; (d) reduced effect of aliasing with $N = 50$.

... as indicated by the relation in

$$h(t) = \begin{cases} 1 & 0 < t < T \\ 0 & t < 0, \text{ and } t > T \end{cases}$$

$$= \text{rect}\left(\frac{t}{T} - \frac{1}{2}\right) \tag{4.57}$$

From Eq. 4.1 the discrete-time signal $g_\delta(t)$, obtained by instantaneously sampling $g(t)$, is given by

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s) \tag{4.58}$$

Convolving $g_\delta(t)$ with the pulse $h(t)$, we get

$$g_\delta(t)\star h(t) = \int_{-\infty}^{\infty} g_\delta(\tau)h(t - \tau)d\tau$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} g(nT_s)\delta(\tau - nT_s)h(t - \tau)d\tau$$

$$= \sum_{n=-\infty}^{\infty} g(nT_s) \int_{-\infty}^{\infty} \delta(\tau - nT_s)h(t - \tau)d\tau \tag{4.59}$$

From the sifting property of the delta function, we have

$$g_\delta(t)\star h(t) = \sum_{n=-\infty}^{\infty} g(nT_s)h(t - nT_s) \tag{4.60}$$

Therefore, from Eqs. 4.56 and 4.60 it follows that $s(t)$ is mathematically equivalent to the convolution of $g_\delta(t)$, the instantaneously sampled version of $g(t)$, and the pulse $h(t)$, as shown by

$$s(t) = g_\delta(t)\star h(t) \tag{4.61}$$

Taking the Fourier transform of both sides of Eq. 4.61 and recognizing that the convolution of two time functions is transformed into the multiplication of their respective Fourier transforms, we get

$$S(F) = G_\delta(F)H(F) \tag{4.62}$$

where $S(F) = F[s(t)]$, $G_\delta(F) = F[g_\delta(t)]$, and $H(F) = F[h(t)]$. Therefore, substitution of Eq. 4.6 into Eq. 4.62 yields

$$S(F) = F_s \sum_{m=-\infty}^{\infty} G\left(F - mF_s\right) H(F) \tag{4.63}$$

where $G(F) = F[g(t)]$.

Finally, suppose that $g(t)$ is strictly band-limited and that the sampling rate F_s is greater than the Nyquist rate. Then, passing $s(t)$ through a low-pass reconstruction filter, we find that the spectrum of the resulting filter output is equal to $G(F)H(F)$. This is equivalent to passing the original analog signal $g(t)$ through a low-pass filter of transfer function $H(F)$.

From Eq. 4.57 we find that

$$H(F) = T\,\mathrm{sinc}(FT)\exp(-j\pi FT) \tag{4.64}$$

which is shown plotted in Fig. 4.15*b*. Hence, we see that by using flat-top samples, we have introduced *amplitude distortion* as well as a *delay* of $T/2$. This effect is similar to the variation in transmission with frequency that is caused by the finite size of the scanning aperture in television and facsimile. Accordingly, the distortion caused by lengthening the samples, as in Fig. 4.14, is referred to as the *aperture effect*.

This distortion may be corrected by connecting an *equalizer* in cascade with the low-pass reconstruction filter. The equalizer has the effect of decreasing the in-band loss of the reconstruction filter as the frequency increases in such a manner as to compensate for the aperture effect. Ideally, the amplitude response of the equalizer is given by

$$\frac{1}{|H(F)|} = \frac{1}{T\,\mathrm{sinc}(FT)} = \frac{1}{T}\frac{\pi FT}{\sin(\pi FT)}$$

The amount of equalization needed in practice is usually small.