Keldysh Field Theory

1 Mathematical Preliminaries

1.1 Closed time contour

The density matrix evolves in time according to the Von Neumann equation as

$$\partial_t \hat{\rho}(t) = -i[\hat{H}(t), \hat{\rho}(t)], \tag{1}$$

which can be solved by taking

$$\hat{\rho}(t) = \hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)[\hat{\mathcal{U}}_{t,-\infty}]^{\dagger}. \tag{2}$$

Where the unitary evolution operator obeys

$$\partial_t \hat{\mathcal{U}}_{t,t'} = -i\hat{H}(t)\ \hat{\mathcal{U}}_{t,t'}; \quad \partial_{t'}\hat{\mathcal{U}}_{t,t'} = i\hat{\mathcal{U}}_{t,t'}\ \hat{H}(t'). \tag{3}$$

The time evolution operator can be written as

$$\begin{aligned}\hat{\mathcal{U}}_{t,t'} &= \lim_{N\to\infty} e^{-i\hat{H}(t-\delta_t)\delta_t} e^{-i\hat{H}(t-2\delta_t)\delta_t} ... e^{-i\hat{H}(t')\delta_t} \\ &= \mathbb{T}\mathrm{exp}\bigg(-i\int_{t'}^{t} \hat{H}(t)dt\bigg).\end{aligned} \tag{4}$$

The expectation value of any general operator is given as

$$\langle\hat{\mathcal{O}}\rangle(t) \equiv \frac{\mathrm{Tr}\{\hat{\mathcal{O}}\hat{\rho}(t)\}}{\mathrm{Tr}\{\hat{\rho}(t)\}} = \frac{1}{\mathrm{Tr}\{\hat{\rho}(t)\}}\mathrm{Tr}\{\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\}. \tag{5}$$

Using the below two equations the general contour can be taken from t = $-\infty$ to t = ∞ and again back.

$$\hat{\mathcal{U}}_{t,+\infty}\hat{\mathcal{U}}_{+\infty,t} = \hat{1}, \quad \hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{U}}_{t,+\infty} = \hat{\mathcal{U}}_{-\infty,+\infty} \tag{6}$$

This gives

$$\langle\hat{\mathcal{O}}\rangle(t) = \frac{1}{\mathrm{Tr}\{\hat{\rho}(-\infty)\}}\mathrm{Tr}\{\hat{\mathcal{U}}_{-\infty,+\infty}\hat{\mathcal{U}}_{+\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\}. \tag{7}$$

We introduce the generating(partition) function as

$$Z[V] \equiv \frac{\mathrm{Tr}\{\hat{\mathcal{U}}_C[V]\hat{\rho}(-\infty)\}}{\mathrm{Tr}\{\hat{\rho}(-\infty)\}} \tag{8}$$

where $\hat{\mathcal{U}}_C = \hat{\mathcal{U}}_{-\infty,+\infty}\hat{\mathcal{U}}_{+\infty,-\infty}$ and the hamiltonian for forward and bakward evolution are defined as $\hat{H}_V^{\pm}(t) \equiv \hat{H}(t) \pm \hat{\mathcal{O}}V(t)$. By taking the functional derivatives of the generating function we can calculate

$$\langle\hat{\mathcal{O}}\rangle(t) = (i/2)\frac{\delta Z[V]}{\delta V(t)}\bigg|_{V=0} \tag{9}$$

1.2 Coherent states

A coherent state parametrized by a complex number ϕ is defined as the eigenstate of the annhilation operator as $\hat{b}|\phi\rangle = \phi|\phi\rangle$. So

$$|\phi\rangle = \sum_{n=0}^{\infty}\frac{\phi^n}{\sqrt{n!}}|n\rangle = e^{\phi\hat{b}^\dagger}|0\rangle. \tag{10}$$

It follows that

$$\begin{aligned}
&\langle\phi|\phi'\rangle = e^{\phi\phi'},\\
&\hat{1} = \int d[\bar{\phi},\phi]\;\; e^{-|\phi|^2}|\phi\rangle\langle\phi|,\\
&Z[\bar{J},J] = \int d[\bar{\phi},\phi]\;\; e^{-\bar{\phi}\phi+\bar{\phi}J+\bar{J}\phi} = e^{\bar{J}J},\\
&\int d[\bar{\phi},\phi]\;\; e^{-|\phi|^2}\bar{\phi}^n\phi^{n'} = \frac{\partial^{n+n'}}{\partial J^n\partial\bar{J}^{n'}}Z[\bar{J},J]\bigg|_{\bar{J}=J=0} = n!\,\delta_{n,n'},\\
&\mathrm{Tr}\{\mathcal{O}\} = \int d[\bar{\phi},\phi]\;\; e^{-|\phi|^2}\langle\phi|\mathcal{O}|\phi\rangle,\\
&f(\rho) \equiv \langle\phi|\rho^{\hat{b}^\dagger\hat{b}}|\phi\rangle = e^{\bar{\phi}\phi'\rho}.
\end{aligned} \tag{11}$$

2 Bosonic Partition function

Simplest example of many body system: bosonic particles occupying a single quantum state with energy ω_0. So,

$$\hat{H}(\hat{b}^\dagger,\hat{b}) = \omega_0\hat{b}^\dagger\hat{b}. \tag{12}$$

Choose the initial density matrix be thermal density matrix

$$\hat{\rho}_0 = e^{-\beta(\hat{H}-\mu\hat{N})} = e^{-\beta(\omega_0-\mu)\hat{b}^\dagger\hat{b}}. \tag{13}$$

and

$$\mathrm{Tr}\{\hat{\rho}_0\} = \sum_{n=0}^{\infty} e^{-\beta(\omega_0-\mu)n} = [1-\rho(\omega_0)]^{-1}, \tag{14}$$

where $\rho(\omega_0) = e^{-\beta(\omega_0-\mu)}$. To calculate $\mathrm{Tr}\{\hat{\mathcal{U}}_C \hat{\rho}_0\}$ we divide the time contour $\mathcal{C}$ into 2N parts going from $t = -\infty$ to $+\infty$ and insert the identity in between. So the expression becomes

$$\langle\phi_{2N}|\hat{\mathcal{U}}_{-\delta t}|\phi_{2N-1}\rangle ... \langle\phi_{N+2}|\hat{\mathcal{U}}_{-\delta t}|\phi_{N+1}\rangle\langle\phi_{N+1}|\hat{1}|\phi_N\rangle\langle\phi_N|\hat{\mathcal{U}}_{+\delta t}|\phi_{N-1}\rangle ... \langle\phi_2|\hat{\mathcal{U}}_{+\delta t}|\phi_1\rangle\langle\phi_1|\hat{\rho}_0|\phi_{2N}\rangle. \tag{15}$$

Where each of the terms are given as

$$\langle\phi_j|\hat{\mathcal{U}}_{\pm\delta t}|\phi_{j-1}\rangle \approx \langle\phi_j|(1 \mp i\hat{H}(\hat{b}^\dagger, \hat{b})\delta t)|\phi_{j-1}\rangle \approx e^{\bar{\phi}_j\phi_{j-1}} e^{\mp i\hat{H}(\bar{\phi}_j,\phi_{j-1})\delta t} \tag{16}$$

2.1 Gaussian like integrals

2.2 Going Green