Topology

1 Illustrative example: Particle on a ring

For a particle on a ring the lagrangian is given as below:

$$\mathcal{L}(\phi, \dot{\phi}) = \frac{1}{2M}\dot{\phi}^2 + A\dot{\phi} \tag{1}$$

Here we are measuring coordinates in terms of the angular variable $\phi \in [0, 2\pi]$ and M is the moment of inertia. Note this gives the Euler-Lagrange equation as $\ddot{\phi} = 0$. Here A is a constant and we are adding just a total derivative term in the free particle lagrangian which does not change it. This gives the conjugate momenta $\hat{p}_\phi = \dot{\phi} + A$. This then gives the Hamiltonian as

$$\mathcal{H}(\phi, \hat{p}_\phi) = \hat{p}_\phi\dot{\phi} - \mathcal{L}(\phi, \dot{\phi}) = \frac{1}{2M}(\hat{p}_\phi - A)^2 \tag{2}$$

This is just like the Hamiltonian when the magnetic flux is A_ϕ. For a magnetic flux Φ inside the ring we can write it as

$$\oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{S} = \Phi. \tag{3}$$

This gives the magnetic potential as

$$A_\phi = \frac{\Phi}{2\pi r} \tag{4}$$

Now the Hamiltonian for the quantum version can be written as

$$\hat{\mathcal{H}} = \frac{1}{2}\Big(- i\hbar\frac{1}{r}\frac{\partial}{\partial\phi} - qA_\phi\Big)^2 = \frac{1}{2}(-i\partial_\phi - A)^2 \tag{5}$$

We have chosen the units so that all the constants are unity. Here A= Φ/Φ_0 where $\Phi_0 = \frac{hc}{e} = 2\pi$ represents the magnetic quantum flux. Periodicity implies $\psi(0) = \psi(2\pi)$. This gives us

$$\psi_n(\phi) = \frac{1}{\sqrt{2\pi}}\exp(in\phi), \quad \epsilon_n = \frac{1}{2}\Big(n - \frac{\Phi}{\Phi_0}\Big), \quad n \in \mathbb{Z}. \tag{6}$$

Let's reformulate this problem in terms of path integrals in imaginary time

$$Z = \int_{\phi(\beta)-\phi(0)\in 2\pi\mathbb{Z}} D\phi \; e^{-\int d\tau \; L(\phi,\dot{\phi})} \tag{7}$$

The Lagrangian is given by

$$L(\phi, \dot{\phi}) = \frac{1}{2}\dot{\phi}^2 - iA\dot{\phi} \tag{8}$$