Keldysh Field Theory

1 Mathematical Preliminaries

1.1 Closed time contour

The density matrix evolves in time according to the Von Neumann equation as

$$\partial_t \hat{\rho}(t) = -i[\hat{H}(t), \hat{\rho}(t)], \tag{1}$$

which can be solved by taking

$$\hat{\rho}(t) = \hat{\mathcal{U}}_{t,-\infty} \hat{\rho}(-\infty)[\hat{\mathcal{U}}_{t,-\infty}]^{\dagger}. \tag{2}$$

Where the unitary evolution operator obeys

$$\partial_t \hat{\mathcal{U}}_{t,t'} = -i\hat{H}(t)\ \hat{\mathcal{U}}_{t,t'}; \quad \partial_{t'} \hat{\mathcal{U}}_{t,t'} = i\hat{\mathcal{U}}_{t,t'}\ \hat{H}(t'). \tag{3}$$

The time evolution operator can be written as

$$\begin{aligned} \hat{\mathcal{U}}_{t,t'} &= \lim_{N\to\infty} e^{-i\hat{H}(t-\delta_t)\delta_t} e^{-i\hat{H}(t-2\delta_t)\delta_t} ... e^{-i\hat{H}(t')\delta_t} \\ &= \mathbb{T}\mathrm{exp}\left(-i \int_{t'}^{t} \hat{H}(t) dt \right). \end{aligned} \tag{4}$$

The expectation value of any general operator is given as

$$\langle \hat{\mathcal{O}} \rangle(t) \equiv \frac{\mathrm{Tr}\{\hat{\mathcal{O}}\hat{\rho}(t)\}}{\mathrm{Tr}\{\hat{\rho}(t)\}} = \frac{1}{\mathrm{Tr}\{\hat{\rho}(t)\}} \mathrm{Tr}\{\hat{\mathcal{U}}_{-\infty,t} \hat{\mathcal{O}} \hat{\mathcal{U}}_{t,-\infty} \hat{\rho}(-\infty)\}. \tag{5}$$

Using the below two equations the general contour can be taken from t $= -\infty$ to t $= \infty$ and again back.

$$\hat{\mathcal{U}}_{t,+\infty} \hat{\mathcal{U}}_{+\infty,t} = \hat{1}, \quad \hat{\mathcal{U}}_{-\infty,t} \hat{\mathcal{U}}_{t,+\infty} = \hat{\mathcal{U}}_{-\infty,+\infty} \tag{6}$$

This gives

$$\langle \hat{\mathcal{O}} \rangle(t) = \frac{1}{\mathrm{Tr}\{\hat{\rho}(t)\}} \mathrm{Tr}\{\hat{\mathcal{U}}_{-\infty,+\infty} \hat{\mathcal{U}}_{+\infty,t} \hat{\mathcal{O}} \hat{\mathcal{U}}_{t,-\infty} \hat{\rho}(-\infty)\}. \tag{7}$$

1.2 Coherent states

A coherent state parametrized by a complex number ϕ is defined as the eigenstate of the annhilation operator as $\hat{b}|\phi\rangle = \phi|\phi\rangle$

1.3 Partition function

1.4 Gaussian like integrals

1.5 Going Green

2 Keldysh Sorcery

3 Keldysh Kung Fu